Groundwater Valuation with a Growing Population[†]

by

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Abstract

We study the optimal valuation of groundwater under different assumptions as to the evolution of the population. In the case of a population that is either constant or decreasing, the optimal consumption path always involves depletion of the groundwater stock without ever replenishing. With an increasing population however, it may involve refraining at times from consuming the totality of the surface water flow in order to restock in groundwater for future consumption. The aquifer then serves as a means to achieve welfare increasing intertemporal transfers of surface water. Therefore the aquifer itself, as distinct from the stock of water it serves to store, may have value and the marginal valuation of water when groundwater stocks are being drawn upon should, for this reason, differ at times from the marginal valuation of water when it is drawn strictly from surface water.

1 Introduction

A first source of value for groundwater stocks comes simply from the fact that they directly increase the limited supply of water. It has also been recognized that stocks of groundwater may have a buffer value, in that they may serve to smooth the stochastic supply of surface water (Tsur and Graham-Tomasi, 1991). Stocks of groundwater also derive value from the fact that they reduce future extraction costs: the greater the stock, the lower the pumping cost (see for instance the classic treatment by Burt, 1964a and 1964b). What we wish to stress in this paper is that with an increasing population, the aquifer itself, as distinct from the stock of water it serves to store, may have value and that the marginal valuation of water drawn from groundwater stocks should, for this reason, differ at times from the marginal valuation of water when it is drawn strictly from surface water. This in turn should be reflected in the pricing of surface and groundwater.

We will here assume away stock effects on costs as well any consideration of stochastic surface water supplies in order to focus on the effect of a growing population on the optimal consumption path. In the case of a population that is either constant or decreasing, the optimal consumption path always involves depletion of the groundwater stock without ever replenishing. With an increasing population, we show that it may involve refraining at times from consuming the totality of the surface water flow in order to restock in groundwater for future consumption. The aquifer may thus serve as a means to achieve welfare increasing intertemporal transfers of surface water and may, as such, acquire value as a reservoir.

In the next section we set down the model which we use and its assumptions. We then consider in section 3 the case of a nonincreasing population. Although the nature of the optimal path in this case reduces essentially to that of a fairly simple "mining" problem, its full derivation is useful in suggesting a method of constructing the optimal consumption path in the more complicated case of an increasing population. We deal with the case of an increasing population in section 4. In that section, we derive the price path which will decentralize the optimal consumption path and show how this price path should reflect the fact that, at times, water drawn from the stock of groundwater should be valued differently than water drawn strictly from the surface water flow. A brief conclusion follows.

2 The Model

Let $N(t)$ denote the population living over some territory at time t. We will assume $N(t)$ to be a continuous function of time and will consider alternatively the case where it is a nonincreasing function of time and that where it is an increasing function. In the latter case, we will assume $N(t)$ to be bounded from above. In all cases therefore, there exists some level towards which the population eventually converges.

Each person in this population is identical and consumes the same quantity of water, which we denote $c(t)$. The instantaneous utility derived by each person from this consumption is $U(c(t))$ and satisfies the following properties:

1. $U'(c(t)) > (=)\ 0$ if $c(t) < (\geq)\ \bar{c}$, $\bar{c}$ being some satiation level.

2. $\lim_{c \downarrow 0} U'(c) = \infty$.

3. $U''(c(t)) < 0\ \forall c < \bar{c}$.

The planner's objective is to maximize the sum over the population and over time of the future instantaneous utilities discounted at the rate r, namely

$$\mathcal{U} = \int_0^\infty e^{-rt} N(t) U(c(t)) dt. \tag{1}$$

Water can be supplied from either surface or underground sources. The surface source consists of a continual flow which we assume known and constant. Let α denote this rate of flow. By assuming this natural flow to be deterministic, we intentionally rule out any buffer value to the groundwater. Water can be drawn directly from the natural surface flow, at a rate to be determined and which we denote $x(t)$. Whatever surface water is not consumed is captured in the aquifer, provided it is not full. When it is, any excess water is simply wasted. The capacity of the aquifer is $\bar{S}$ and at any time t it holds a stock $S(t)$. Both $\bar{S}$ and $S(t)$ are known with certainty. Water is extracted from the aquifer at the rate $s(t)$. Therefore $c(t) = (x(t) + s(t))/N(t)$. Whether from the surface flow or from the stock, we assume that the water can be drawn costlessly. We thus rule out direct extraction cost considerations in the determination of the optimal stock to hold at any time and in particular any considerations of stock effects on costs.

The equation describing the evolution over time of the stock of groundwater can therefore be written

$$\dot{S}(t) = \begin{cases} \alpha - x(t) - s(t) & \text{if } S(t) < \bar{S} \\ \min\{\alpha - x(t) - s(t), 0\} & \text{if } S(t) = \bar{S} \end{cases} \tag{2}$$

The initial stock of groundwater is $S^0 \leq \bar{S}$ and given.

In order for there to be a meaningful economic problem, we must of course assume that the resource is scarce. In the case of a nonincreasing population, the resource is scarce only if

$$\alpha/N(0) < \bar{c}$$

and

$$S^0 < \int_0^\tau [N(t)\bar{c} - \alpha]dt$$

where τ is defined as

$$\tau = \begin{cases} \min\{t|\alpha/N(t) \geq \bar{c}\} & \text{if } \exists t : \alpha/N(t) > \bar{c} \\ +\infty & \text{otherwise} \end{cases}$$

Note that in the case of a constant population, the two conditions reduce to the first one.

In the case of an increasing population, the resource is scarce only if

$$\alpha/\bar{N} < \bar{c}$$

where $\bar{N}$ denotes the finite upper bound to the population.

The general decision problem can be stated as

$$\max_{\{\{x(t),s(t)\};t\geq 0\}} \mathcal{U} \tag{3}$$

subject to $c(t) = (x(t) + s(t))/N(t)$, the equation of motion given by (2) and the following constraints on the state and control variables

$$0 \leq S(t) \leq \bar{S}, \quad \text{and} \quad S(0) = S^0 \tag{4}$$

$$0 \leq x(t) \leq \alpha \tag{5}$$

$$s(t) \geq 0 \quad \text{and} \quad s(t) \leq \alpha - x(t) \;\; \text{if} \;\; S(t) = 0. \tag{6}$$

We will denote by an asterisk a solution path to this problem.

What at first sight appears to be a fairly straightforward optimal control problem is complicated by the fact that some constraints on the flow variables are contingent on the level of the stock variable, as is the form of the equation of motion (2). We show in the next section how the problem can be simplified to a form of "mining" problem in the case of a constant or of a decreasing population. Knowing how to characterize the solution in that case will then be helpful in constructing a solution for the more complicated case of an increasing population.

3 The Case of a Nonincreasing Population

Suppose that population is nonincreasing. The following four propositions will then hold:

Proposition 1 *If at some date $t' \in [0, \tau)$, $S(t') = \bar{S}$, then $c^*(t') > \alpha/N(t')$.*

Proposition 2 *If at some date $T \in [0, \tau]$, $S(T) = 0$, then it is optimal to set $c^*(t) = min\{\alpha/N(t), \bar{c}\}$ for all $t > T$.*

Proposition 3 *There exists a $T \in [0, \tau]$ such that $S(T) = 0$.*

Proposition 4 *For all $t \in [0, T]$, $c^*(t) \geq \alpha/N(t)$.*

The proof of each of those propositions is left to the Appendix.

By Proposition 1 we know that with a nonincreasing population, if at any time $t < \tau$ the aquifer is full, it is not optimal to keep it full. This means that $\min\{\alpha - x(t) - s(t), 0\} = \alpha - x(t) - s(t)$ and therefore (2) reduces to $\dot{S}(t) = \alpha - x(t) - s(t)$. We also know by Proposition 2 that if the stock of groundwater is exhausted at some date $T \leq \tau$, it should never be replenished. By Proposition 3 we know that it is in fact optimal to exhaust by some date $T \leq \tau$ the groundwater stock that was inherited at $t = 0$. It therefore follows that if we define

$$B(T) = \int_T^\infty e^{-rt} N(t) U(\min\{\alpha/N(t), \bar{c}\}) dt,$$

the planning problem can be simply rewritten

$$\max_{\{c(t);t\in[0,T]\},T} \int_0^T e^{-rt} N(t) U(c(t)) dt + B(T) \quad (7)$$

subject to

$$\dot{S}(t) = \alpha - x(t) - s(t), \quad (8)$$

$$S(T) = 0, \quad S(0) = S^0 \quad (9)$$

and

$$c(t) \geq 0. \quad (10)$$

Furthermore, since by Proposition 4 we know that with a nonincreasing population it will be optimal to always consume at least the surface water flow α, since the costs of drawing from the surface water or the groundwater is the same and since the consumer is indifferent between the two sources, we may set $x(t) = \alpha$. We then have $c(t) = (\alpha + s(t))/N(t)$, and determining $c(t)$ amounts to determining $s(t)$, the rate of exhaustion of the groundwater stock.

Let $\lambda(t)$ denote the discounted shadow value of a unit of water held in stock in the aquifer. The Hamiltonian associated with this problem is then written

$$H(t) = e^{-rt} N(t) U(c(t)) + \lambda(t)[\alpha - N(t)c(t)] \quad (11)$$

and, in addition to (8) and (9), the following conditions must hold along the optimal path:

$$e^{-rt}U'(c(t)) = \lambda(t) \tag{12}$$

$$\dot{\lambda}(t) = 0 \tag{13}$$

$$H(T) + \frac{dB(T)}{dT} \geq 0. \tag{14}$$

The transversality condition (14) serves in determining T. It must hold with equality whenever T is finite. From Proposition 3, this is obviously the case whenever τ is finite and, although the proof is rather involved, can be shown to be the case even when it is not (see Amigues, Gaudet and Moreaux, 1992). Taking T to be finite and remembering that $\min\{\alpha/N(T), \bar{c}\} = \alpha/N(T)$ since $T \leq \tau$ by Proposition 3, this condition is written

$$e^{-rT}N(T)U(c(T)) + \lambda(T)[\alpha - N(T)c(T)] - e^{-rT}N(T)U(\alpha/N(T)) = 0,$$

which implies

$$c(T) = \alpha/N(T) \quad \text{or, alternatively,} \quad s(T) = 0 \tag{15}$$

This guarantees that there is no discontinuity in $c^*(t)$ at T. If there were such a discontinuity, there would result a jump in $U'(c(t))$ at T and hence the possibility of increasing the value of the objective function by reducing consumption slightly before T in order to increase it immediately after T.

Those necessary conditions suggest the following procedure which could be followed for constructing the solution path and which will prove useful in the next section. Define

$$V(t) = e^{-rt}U'(\alpha/N(t)), \tag{16}$$

the discounted per capita marginal utility of water consumption when consuming exactly the flow of surface water. From (13) we note that $\lambda(t) = \lambda$, a constant. To any $\lambda \in (0, U'(\alpha/N(0)))$, associate the path $s(t;\lambda)$ defined as

$$s(t;\lambda) = \begin{cases} 0 & \text{if } V(t) < \lambda \\ \text{solution to (12)} & \text{if } V(t) \geq \lambda \end{cases} \tag{17}$$

From (15), $s(T;\lambda) = 0$. Let $T(\lambda)$ be the solution to this last condition and define

$$\Delta S(\lambda) = \int_0^{T(\lambda)} s(t;\lambda)dt,$$

the cumulative withdrawal of groundwater over the interval $[0, T(\lambda)]$. Then from (8) and (9) we must have

$$\Delta S = S^0.$$

Let λ^* denote the solution to this last equation. Then

$$T^* = T(\lambda^*)$$

$$s^*(t) = s(t;\lambda^*)$$

and the corresponding optimal per capita consumption path may be written

$$c^*(t) = \begin{cases} (\alpha + s^*(t))/N(t) & \text{for } t \in [0, T^*] \\ \min\{\alpha/N(t), \bar{c}\} & \text{for } t \in [T^*, \infty]. \end{cases}$$

The value λ^* is the value, discounted to $t = 0$, that must be assigned to a unit of groundwater in stock in the aquifer in order to maximize social welfare as defined by (1). The optimal consumption decision can be decentralized by the following instantaneous price path for water:

$$p(t) = \begin{cases} e^{rt}\lambda^* & \text{for } t \in [0, T^*] \\ e^{rt}V(t) & \text{for } t \in [T^*, \tau] \\ 0 & \text{for } t \in [\tau, \infty]. \end{cases}$$

Notice that from (15) and (12), $\lambda^* = V(T^*)$ and hence the current value of a unit of groundwater at the date of exhaustion of the stock is $e^{rT^*}\lambda^* = e^{rT^*}V(T^*) = U'(\alpha/N(T^*))$, assuring that there is no discontinuity in the price path at T^*. After having increased at the rate r, the current price reaches a maximum at T^*, after which it starts decreasing to reach zero at τ and beyond in the case of a decreasing population, or stays constant at $U'(\alpha/N(0))$ in the case of a stationary population.

A consumer whose instantaneous utility function were quasi-linear — of the form $U(c) + m$, m being the numéraire — would choose exactly $c^*(t)$, assuming of course his budget at each t was at least $p^*(t)c^*(t)$, thus generating a total instantaneous demand of $N(t)c^*(t)$. Assuming property rights to the resource to be well defined and the resource owners to be price-takers, $\{c^*(t); t \geq 0\}$ also yields one of many paths that would maximize profits given the price $p^*(t)$, while simultaneously exhausting the initial groundwater stock at T^*. The price path $\{p^*(t); t \geq 0\}$ could therefore theoretically be generated as the equilibrium price of a purely competitive market.

4 The Case of an Increasing Population

A critical property which served to simplify the problem in the case of a nonincreasing population is that $V(t)$, the per capita discounted marginal utility of water consumption when consuming exactly the flow of surface water, is a monotone decreasing function of time in that case. This being so, $V(T^*) > \lambda^*$ for all $t < T^*$ since $\lambda^* = V(T^*)$ and hence $s^*(t) > 0$ for all $t < T^*$, which means that the initial stock of groundwater is completely exhausted over the interval $[0, T^*]$. In the case of an increasing population, $V(t)$ will become monotone decreasing once the population has reached its stationary upper bound, but before that date its behavior over time can take many form, depending on the evolution of the population. What the actual solution path will look like will depend on the behavior of $V(t)$ and hence of $N(t)$. It may for instance become optimal to realize the depletion of the initial stock of groundwater over a number of separate and distinct intervals instead of over a single interval starting at initial time. It may also become optimal to refrain from consuming the whole surface water flow for some time in order to replenish the stock of groundwater.

In solving for the optimal program it now becomes important to take into account the different regimes under which we may theoretically find ourselves at any given time, namely the storage capacity of the aquifer is fully used up ($S(t) = \bar{S}$) or it is not ($S(t) < \bar{S}$) and, when it is, water usage exceeds the natural surface flow ($x(t) + s(t) > \alpha$) or it does not ($x(t) + s(t) \leq \alpha$).

It is useful to note that during any interval when the aquifer is being emptied (i.e., $x(t) + s(t) > \alpha$) or is being refilled (i.e., $x(t) + s(t) < \alpha$ and $S(t) < \bar{S}$), the evolution of the stock of groundwater can be written as in (8), as for the case of a nonincreasing population, and the equivalent of the necessary conditions (12) and (13) found for that case must again hold. From these it follows that over such an interval, the discounted value of per capita marginal utility of water consumption must be constant. Otherwise it would always be possible to improve on the consumption path by transferring some consumption from a date of low discounted marginal utility to one of high discounted marginal utility.

4.1 The Case Where $V(t)$ is First Increasing and Then Decreasing

If $V(t)$ were monotone decreasing, as is still theoretically possible even with an increasing population, then the analysis would be exactly as that of the previous section. The simplest case where it is not is that where $V(t)$ is at first increasing and then decreasing over time. Consider then such a case.

The first question to ask is whether a policy similar to that which was optimal in the case of a nonincreasing population can still be optimal. Such a policy would consist of exhausting the initial stock at an optimal rate without at any time reducing water consumption below the natural surface flow in order to replenish the stock of groundwater. We will distinguish between cases where such a policy remains optimal and those where it does not.

To any $\lambda \in (0, \max\{V(t), t \geq 0\})$ let us then associate the path $s(t; \lambda)$ of extraction of groundwater from the aquifer, where $s(t; \lambda)$ is defined exactly as in (17). Since $V(t)$ at first increases and then decreases, $V(t) = \lambda$ may now be satisfied at two different dates[1]. These two dates, which we will denote $T_1(\lambda)$ and $T_2(\lambda)$, define the interval over which $s(t; \lambda)$ is

[1] This reflects the fact that we will now need a transversality condition to determine the initial date as well as the terminal date of stock depletion.

positive, $T_1(\lambda)$ being the date at which we begin drawing down the initial stock and $T_2(\lambda)$ being the date at which we stop. Note that $s(T_1(\lambda);\lambda) = s(T_2(\lambda);\lambda) = 0$. From (12), we know that it must be the case that $s(t;\lambda)/N(t))$ is a continuous function of time, first increasing and then decreasing over the interval $[T_1(\lambda), T_2(\lambda)]$, reaching a maximum whenever $(V(t) - \lambda)e^{rt}$ reaches a maximum.

Clearly $s(t;\lambda)$ is a decreasing function of λ, as is $T_2(\lambda)$, while $T_1(\lambda)$ is an increasing function. It must therefore be the case that the cumulative withdrawal of groundwater over the interval $[T_1(\lambda), T_2(\lambda)]$, namely

$$\Delta S(\lambda) = \int_{T_1(\lambda)}^{T_2(\lambda)} s(t;\lambda)dt$$

is a decreasing function of λ, with

$$\lim_{\lambda \downarrow 0} \Delta S(\lambda) = +\infty \quad \text{and} \quad \lim_{\lambda \uparrow max\{V(t), t \geq 0\}} \Delta S(\lambda) = 0.$$

There will therefore exist a unique value of λ satisfying

$$\Delta S(\lambda) = S^0, \tag{18}$$

thus exhausting the initial stock. Denote this value $\hat{\lambda}$. The policy under consideration would then be $\{(x(t), s(t)), t \geq 0\} = \{(\alpha, s(t;\hat{\lambda})), t \geq 0\}$. It consists of consuming strictly the surface flow α over the interval $[0, T_1(\hat{\lambda}))$, of letting consumption exceed α by an amount $s(t;\hat{\lambda})$ over the interval $[T_1(\hat{\lambda}), T_2(\hat{\lambda})]$, thereby depleting the initial stock of groundwater, and of consuming exactly the natural surface flow thereafter[2].

A policy of this type will turn out to be optimal in two cases.

Case 1 The first case occurs when $\hat{\lambda} \leq V(0)$: at $t = 0$, the value of a marginal unit of groundwater is not greater than the marginal utility of water consumption when consuming the entire natural surface flow. This means that the initial stock of groundwater is relatively abondant, sufficiently so to make it unattractive to sacrifice some

[2]Note that $T_1(\hat{\lambda}) = 0$ if $\hat{\lambda}$ is sufficiently small, reflecting the fact that S^0 is sufficiently large.

of the surface flow in order to add to it, even though we may have $S^0 < \bar{S}$. In this case, $T_1(\hat{\lambda}) = 0$ and the optimal policy is to immediately begin depleting the stock of groundwater until it is fully depleted at $T_2(\hat{\lambda})$. Since by assumption $V(t)$ at first increases and then decreases, beyond $T_2(\hat{\lambda})$ it becomes monotone decreasing and therefore it is never desirable to replenish the groundwater stock beyond that date, so that $x(t) = \alpha$ is optimal for all $t > T_2(\hat{\lambda})$.

Case 2 The second case occurs when $\hat{\lambda} > V(0)$ and $S^0 = \bar{S}$. In this case, if the aquifer were not full, it would be possible to increase total welfare by consuming less than α over the interval $[0, T_1(\hat{\lambda}))$ in order to increase consumption over the interval $[T_1(\hat{\lambda}), T_2(\hat{\lambda})]$. Such a transfer is impossible, because the storage capacity is fully employed.

A policy of the type $\{(x(t), s(t)), t \geq 0\} = \{(\alpha, s(t; \hat{\lambda})), t \geq 0\}$ will not be optimal in all other cases, namely those characterized by $\hat{\lambda} > V(0)$ and $S^0 < \bar{S}$. It is then always the case that $T_1(\hat{\lambda}) > 0$ and for all $t < T_1(\hat{\lambda})$ the value of a unit of water in stock in the aquifer is greater than the discounted per capita marginal utility of consuming the entire surface flow. In that case it becomes desirable to set $x(t) < \alpha$ in order to increase the stock of groundwater, allowing greater future consumption. We must however take into account the possibility that the remaining storage capacity of the aquifer may be less than required.

Denote by T_0 the date at which refilling stops, having attained a stock $S(T_0)$ of groundwater held in the aquifer, and let $B_0(T_0, S(T_0))$ denote the value of the optimal program contingent on T_0 and $S(T_0)$. By construction, this optimal program is given by $\{(\alpha, s(t; \hat{\lambda})), t \geq T_0\}$ where $\hat{\lambda}$ is now the solution to $\Delta S(\lambda) = S(T_0)$ and $\hat{\lambda} = \partial B_0 / \partial S(T_0)$. Over the interval $[T_0, \infty]$ the solution can therefore be described by either of Case 1 or Case 2, with S^0 replaced by $S(T_0)$ and the initial date replaced by T_0. Define now the following problem:

$$\max_{\{c(t); t \in [0, T_0]\}, T_0, S(T_0)} \int_0^{T_0} e^{-rt} N(t) U(c(t)) dt + B_0(T_0, S(T_0)) \tag{19}$$

subject to

$$\dot{S}(t) = \alpha - x(t) - s(t), \tag{20}$$

$$S(0) = S^0, \quad S(T_0) \leq \bar{S}, \quad T_0 \leq T_1 \tag{21}$$

and

$$c(t) \geq 0. \tag{22}$$

We know that over the interval $[0, T_0]$, $s(t) = 0$ and hence $c(t) = x(t)/N(t)$. Determining $c(t)$ therefore amounts to determining $x(t)$, the rate of withdrawal from the natural surface flow.

Let $\gamma(t)$, $t \in [0, T_0]$ denote the discounted shadow value of a unit of water held in stock in the aquifer over this phase of stock replenishment. The Hamiltonian associated with this problem is then written

$$H_0(t) = e^{-rt}N(t)U(c(t)) + \gamma(t)[\alpha - N(t)c(t)] \tag{23}$$

and, in addition to (20) and (21), the following conditions must hold along the optimal path:

$$e^{-rt}U'(c(t)) = \gamma(t) \tag{24}$$

$$\dot{\gamma}(t) = 0 \tag{25}$$

along with the transversality conditions

$$\left[H_0(T_0) + \frac{\partial B_0}{\partial T_0}\right][T_0 - T_1] = 0, \quad H_0(T_0) + \frac{\partial B_0}{\partial T_0} \geq 0 \tag{26}$$

and

$$\left[\gamma(T_0) - \frac{\partial B_0}{\partial S(T_0)}\right]\left[S(T_0) - \bar{S}\right] = 0, \quad \gamma(T_0) - \frac{\partial B_0}{\partial S(T_0)} \leq 0. \tag{27}$$

which serve to determine T_0 and $S(T_0)$. Clearly $T_1 = T_0$ if $S(T_0) < \bar{S}$ and $T_1 > T_0$ only if $S(T_0) = \bar{S}$.

When $S(T_0) < \bar{S}$, then (27) reduces to

$$\gamma(T_0) - \frac{\partial B_0}{\partial S(T_0)} = 0. \tag{28}$$

We know that $\partial B_0/\partial S(T_0) = \hat{\lambda}$. Thus (28) says that there must be no discontinuity in the value imputed to a unit of groundwater in stock in this case.

When $S(T_0) = \bar{S}$ and binding, (26) reduces to

$$H_0(T_0) + \frac{\partial B_0}{\partial T_0} = 0. \tag{29}$$

The equivalent of Case 2 above then holds for $t \geq T_0$, which means that $x(t) = \alpha$ for $t \in (T_0, T_1)$ and therefore

$$\frac{\partial B_0}{\partial T_0} = e^{-rT_0} U(\alpha / N(T_0)).$$

Substituting into the transversality condition (29), we find that it reduces to $x(T_0) = \alpha$, assuring there is no discontinuity in the consumption path at T_0.

The construction of the solution path can proceed as follows. We know from (25) that $\gamma(t) = \gamma$, a constant. To any $\gamma \in (V(0), \hat{\lambda})$, associate the path $x(t; \gamma)$ of consumption from the surface flow defined by

$$x(t;\gamma) = \begin{cases} \text{solution to (24)} & \text{if } V(t) < \gamma \text{ and } S(t) < \bar{S} \\ \alpha & \text{otherwise} \end{cases} \tag{30}$$

Further define

$$\Delta S(\gamma) = \int_0^{T_0(\gamma)} (\alpha - x(t;\gamma)) dt,$$

where $T_0(\gamma)$ is the first date at which $V(t) = \gamma$. Clearly $\alpha - x(t;\gamma)$ is an increasing function of γ and so is $\Delta S(\gamma)$. The stock accumulated at $T_0(\gamma)$ is $S(T_0(\gamma)) = S^0 + \Delta S(\gamma)$. Let $\hat{\lambda}(\gamma)$ be the shadow value which will result in $S(T_0(\gamma))$ being depleted optimally without replenishment by withdrawing an amount $s(t, \hat{\lambda}(\gamma))$ over the interval $[T_1(\hat{\lambda}(\gamma)), T_2(\hat{\lambda}(\gamma))]$.

Then either $S^0 + \Delta S(\gamma) \leq \bar{S}$ with $\bar{S}$ not binding, in which case, by (28), γ^* must satisfy

$$\hat{\lambda}(\gamma) = \gamma, \tag{31}$$

or $S^0 + \Delta S(\gamma) = \bar{S}$ is binding, in which case γ^* is given by the solution to

$$S^0 + \Delta S(\gamma) = \bar{S}. \tag{32}$$

Note that in this last case $\gamma^* < \hat{\lambda}(\gamma^*)$.

We can now distinguish the following two additional cases, depending on whether or not the size of the aquifer eventually becomes a binding constraint:

Case 3 This case occurs when $\hat{\lambda} > V(0)$, $S^0 < \bar{S}$ and $\bar{S}$ is never binding. We then have $T_0(\gamma^*) = T_1(\hat{\lambda}(\gamma^*)) = T_1(\gamma^*)$. During a first phase $[0, T_1(\gamma^*)]$, the optimal program consists of consuming the surface water at the rate $x(t;\gamma^*)$ and stocking it at the rate $\alpha - x(t;\gamma^*)$. This first phase is immediately followed by a phase where the stock of groundwater is drawn down at the rate $s(t;\gamma^*)$, resulting in a discounted marginal utility of consumption equal to γ^*. This second phase covers the interval $[T_1(\gamma^*), T_2(\gamma^*)]$. At $T_2(\gamma^*)$, the initial stock of groundwater plus the replenishment of the first phase are completely depleted. Follows a third and final phase, beyond $T_2(\gamma^*)$, during which the natural surface flow is consumed indefinitely.

Case 4 This case occurs when $\hat{\lambda} > V(0)$, $S^0 < \bar{S}$ and $\bar{S}$ becomes binding. We then have $T_0(\gamma^*) < T_1(\hat{\lambda}(\gamma^*))$. The replenishment phase ends at $T_0(\gamma^*)$ when the aquifer is full. It is however not optimal to begin drawing down the groundwater stock immediately as in the previous case, since it will be worth more at a later date. Follows therefore a phase covering the interval $[T_0(\gamma^*), T_1(\hat{\lambda}(\gamma^*))]$ during which only the surface water flow is consumed and the aquifer is kept full for future consumption. All through the next phase, the aquifer is being emptied, with consumption equal to $\alpha + s(t;\hat{\lambda}(\gamma^*))$. At $T_2(\hat{\lambda}(\gamma^*))$, the aquifer is empty and from then on consumption is equal to the flow of surface water.

The above four cases cover all the possible configurations for the optimal consumption path when $V(t)$ first increases and then decreases. The full solution for the optimal program, which captures all four cases, can be written $\{(x^*(t), s^*(t)), t \geq 0\} = \{(x(t;\gamma^*), s(t;\hat{\lambda}(\gamma^*)), t \geq 0\}$, with $x(t;\gamma)$ and $s(t;\lambda)$ defined respectively by (17) and (30).

Let $T_0^* = T_0(\gamma^*)$, $T_1^* = T_1(\hat{\lambda}(\gamma^*))$ and $T_2^* = T_2(\hat{\lambda}(\gamma^*))$. Then the corresponding total water consumption path may be written

$$c^*(t) = \begin{cases} x^*(t)/N(t) & \text{for } t \in [0, T_0^*] \\ \alpha/N(t) & \text{for } t \in [T_0^*, T_1^*] \\ (\alpha + s^*(t))/N(t) & \text{for } t \in [T_1^*, T_2^*] \\ \alpha/N(t) & \text{for } t \in [T_2^*, \infty] \end{cases}$$

A current price path by which this consumption path can be decentralized is

$$p(t) = \begin{cases} e^{rt}\gamma^* & \text{for } t \in [0, T_0^*] \\ e^{rt}V(t) & \text{for } t \in [T_0^*, T_1^*] \\ e^{rt}\hat{\lambda}(\gamma^*) & \text{for } t \in [T_1^*, T_2^*] \\ e^{rt}V(t) & \text{for } t \in [T_2^*, \infty]. \end{cases}$$

We may have $T_0^* = 0$, which means that the behavior of the water consumption path is as described by either Case 1 ($T_1^* = T_0^* = 0$) or Case 2 ($T_1^* > T_0^* = 0$). When $T_0^* > 0$, it is as described by either Case 3 ($T_1^* = T_0^* > 0$) or Case 4 ($T_1^* > T_0^* > 0$), with $\hat{\lambda}(\gamma^*) = \gamma^*$ if and only if we have Case 3. In all cases, current instantaneous price is a monotone increasing function of time. In the more elaborate Case 4, price at first increases at the rate r during the stock replenishment phase, increases as does $U'(\alpha/N(t))$ from T_0^* to T_1^*, with consumption equal to the surface water flow and the aquifer kept full for future consumption, again increases at the rate r while the aquifer is being emptied and finally follows the path of $U'(\alpha/N(t))$ beyond T_2^* when consumption once more becomes equal to the surface water flow, with the aquifer now kept empty.

The important feature to notice about this price path is that it does not always attach the same discounted value to a unit of water consumption when consumption draws on the stock of groundwater as when it draws strictly from surface water. This is true of both cases 2 and 4. In Case 2, the discounted marginal value of water is $V(t)$ for $t < T_1^*$ which is less than $\hat{\lambda}$, its discounted marginal value over the interval $[T_1^*, T_2^*]$. This characteristic is still more striking in Case 4. In this case, the marginal unit of water is valued at γ^* when the aquifer is being refilled at the rate $\alpha - x^*(t)$ over the interval $[0, T_0^*]$ and is valued at $\hat{\lambda}(\gamma^*) > \gamma^*$ when the groundwater in the aquifer is being consumed at the rate $s^*(t)$ over the interval $[T_1^*, T_2^*]$. The difference between these two discounted values represents the discounted value of the aquifer as a reservoir[3], γ^* being the discounted value of a unit of groundwater in stock. The optimal pricing rule requires that consumers be charged the value of the aquifer as a reservoir when groundwater is being consumed. Of course, when the full capacity of the aquifer is never required, as in cases 1 and 3, the aquifer itself, as distinct from the stock of groundwater it contains, has no value and the marginal unit of water is valued equally whether the groundwater stock is bring drawn upon or not[4].

4.2 Generalization to More Complex Evolutions of $V(t)$

With an increasing population, the behavior over time of the per capita marginal utility derived from consuming exactly the flow of surface water, i.e., the behavior of the function $V(t)$, can obviously be more complex than the one we chose to analyze here. However, whatever this behavior, the solution path could always be constructed from the approach just described, which essentially consists in using the idea that whatever the stock of groundwater at any given time, the optimal path requires that we behave optimally from there on. This is in fact simply an application of the optimality principle. A proper treatment of the transversality conditions then assures an optimal linkage of the sequence of phases and regimes that can arise.

It is worth noting that contrary to what will necessarily be the case under the behavior assumed so far for the function $V(t)$, it may well be optimal, when the behavior of $V(t)$ is slightly more complex, to end a depletion phase before the stock has been completely

[3]It is in fact the discounted shadow value of the binding constraint which the size of the aquifer imposes on the stock of groundwater.

[4]An alternative way of viewing this is that in cases such as 2 and 4, where the size of the aquifer becomes a binding constraint, welfare could be increased by building an artificial reservoir provided its cost did not exceed the gross addition to welfare. The price of groundwater should then reflect the marginal cost of this reservoir.

exhausted. To illustrate this, suppose for instance that the evolution of population is such that the increasing and then decreasing stages of the function $V(t)$ are preceded by a decreasing stage. Conceivably, a path such as that described in Case 4 — a phase during which the aquifer is filled up, followed by a phase during which the aquifer is kept full for future consumption while all the surface water flow is being consumed, followed by a phase during which the aquifer is completely depleted, followed by the inevitable final phase during which the natural surface water flow is being forever consumed — could still be optimal. A requirement would be that $V(0) \leq \gamma^*$. Suppose however this requirement is not satisfied. Then for the above sequence of phases to remain part of the optimal path, it would have to be immediately preceded either by a phase during which the aquifer is completely depleted plus one during which it is kept empty by consuming the entire flow of surface water, or by a single phase during which the aquifer is only partially depleted. Both of these possibilities can be shown to be optimal for the proper S^0 and evolution of $N(t)$. The latter provides an example where the depletion of the initial stock of groundwater is spread over distinct and seperate intervals, punctuated by a period of replenishment.

Other than to illustrate that possible occurrence, no further insight can be gained here by going through in detail the various possible path configurations that can arise under a function $V(t)$ which first decreases, then increases and then decreases[5] and no new types of phases and linkages between them can arise from even more complicated ones.

5 Conclusion

In the absence of stock effects on the costs of extraction or of buffer considerations in the face of an uncertain surface flow, it will never be optimal to replenish depleted stocks of

[5]All the possible path configurations which may occur under such circumstances are analyzed in detail in Amigues, Gaudet and Moreaux (1992). It is done there to show that if we allow for aquifers that are replenishable and some that are not, then, under an increasing population, the order of depletion is not always irrelevant and there are also cases where the groundwater should be valued differently according to which type of aquifer it is being drawn from.

groundwater and will always be optimal to eventually exhaust these stocks if population is decreasing or constant. The situation is quite different when population is increasing. We have shown that it may then well become optimal to, at times, replenish the depleted stocks of groundwater. When this is the case, then the aquifer itself, as distinct from the stock of water it holds, may have value as a reservoir for storing water for future consumption. This will occur when the size of the aquifer becomes a binding constraint, thus restricting the cumulative restocking of groundwater to less than would otherwise be desirable. The marginal valuation of water during a phase where it is being drawn from the groundwater stock will then exceed that of water during a phase when it is being drawn strictly from surface water. The difference reflects the implicit rent that must then be imputed to the aquifer in its function as a reservoir, permitting the required intertemporal transfers of surface water. This must also be reflected in the pricing of surface and groundwater in order to assure a proper allocation of water resources over time.

Appendix

A Proof of Proposition 1

Suppose $c(t') \leq \alpha/N(t')$. Two cases are possible:

1. if $c(t') < \alpha/N(t')$, then $\alpha/N(t) - c(t')$ is wasted, since $S(t') = \bar{S}$ by assumption. Since $U'(c(t)) > 0$ for all $t < \tau$, the value of $\mathcal{U}$ can therefore be increased by, for example, setting $c(t') = \alpha/N(t')$.

2. if $c(t') = \alpha/N(t')$, then some transfer of consumption between t' and any $t'' \in (t', \tau)$ can increase $\mathcal{U}$ unless

$$e^{-rt''}N(t'')U'(c(t'')) = e^{-rt'}N(t')U'(\alpha/N(t')).$$

Since $N(t'') \leq N(t')$, this implies

$$c(t'') < \alpha/N(t')$$

which itself implies

$$c(t'') < \alpha/N(t'') \quad \text{and} \quad S(t'') = \bar{S}.$$

But this means that the previous case holds at t'', a possibility which was just ruled out along an optimal consumption path.

Q.E.D.

B Proof of Proposition 2

Simply consuming less than $c^*(t)$ over some interval beginning at $t > T$ clearly reduces the value of $\mathcal{U}$, since $U'(c) > 0$ for $c < \bar{c}$. Consuming more over an interval beginning at some date $t \geq \tau$ cannot increase the utility, by definition of τ and $\bar{c}$.

There remains the possibility of consuming more over an interval beginning at some date $t < \tau$. But this requires that the stock be built up by consuming less over some previous interval. Consider then a deviation from $c^*(t)$ which consists in reducing consumption by some amount $dc > 0$ on the interval $(t', t' + dt)$, $T < t' < \tau$, and increasing it by an amount γ/dc, $\gamma > 0$, on the interval $(t'', t'' + dt/\gamma)$, $t' + dt < t'' < \tau$. The net effect on $\mathcal{U}$ of this deviation is

$$\Delta\mathcal{U} = e^{-rt''}N(t'')U'(\alpha/N(t''))dcdt - e^{-rt'}N(t')U'(\alpha/N(t'))dcdt,$$

which is negative since $N(t'') \leq N(t')$ and $U''(c(t)) < 0$ for all $t < \tau$. Q.E.D.

C Proof of Proposition 3

Suppose there does not exist a $T \in [0, \tau]$ such that $S(T) = 0$. Then, since $S(0) > 0$, it must be the case that $S(t) > 0$ for all $t \in [0, \tau]$. We also know that for all $t \geq \tau$, $c^*(t) = \bar{c} \leq \alpha/N(t)$ is optimal. It is therefore possible to increase the consumption of water over some interval $(t', t' + dt)$, $t' + dt < \tau$, without having to reduce it over some other interval, thereby generating an increase in $\mathcal{U}$ since $U'(c(t)) > 0$ for $t < \tau$. Q.E.D.

D Proof of Proposition 4

Suppose $c(t') < \alpha/N(t)$ for some $t' \in [0, T]$. Therefore, by (2), $\dot{S}(t') \geq 0$. But since $S(T) = 0$, it must be that for some t'', $t' < t'' < T$, $\dot{S}(t'') < 0$ and hence $c(t'') > \alpha/N(t'')$. Therefore, since $N(t'') \leq N(t')$,

$$c(t'') > \alpha/N(t'') \geq \alpha/N(t') > c(t'),$$

which, since $U''(c(t)) < 0$ for all $t < T$, implies that

$$e^{-rt''}N(t'')U'(c(t'')) < e^{-rt'}N(t')U'(c(t')).$$

But when this is the case, a net increase in $\mathcal{U}$ can be generated by reducing consumption at t'' and increasing it at t'. Q.E.D.

References

Amigues, Jean-Pierre, Gérard Gaudet and Michel Moreaux (1992) "Principes de valorisation d'une ressource semi-renouvelable: Elements pour une tarification rationnelle de l'usage de l'eau," Cahier no 92.23.270, GREMAQ, Université de Toulouse.

Burt, Oscar R. (1964a) "The Economics of Conjunctive Use of Ground and Surface Water," *Hilgardia*, **36**, 31–111.

Burt, Oscar R. (1964b) "Optimal Resource Use Over Time with an Application to Groundwater," *Management Science*, **11**, 80–93.

Tsur, Yacov and Theodore Graham-Tomasi (1991) "The Buffer Value of Groundwater with Stochastic Surface Water Supplies," *Journal of Environmental Economics and Management*, **21**, 201–224.