Who should we believe? Collective risk-taking decisions with heterogeneous beliefs

Christian Gollier*
University of Toulouse (LEERNA-INRA and IDEI)

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Abstract

Suppose that a group of agents having divergent expectations can share risks efficiently. We examine how this group should behave collectively to manage these risks. We show that the beliefs of the representative agent is in general a function of the group's wealth level, or equivalently, that the representative agent has a state-dependent utility function. We also prove that the probability distribution used by the representative agent is biased in favor of the beliefs of the more risk tolerant agents in the group. From this central result, we show how does increasing disagreement on the state probability affect the state probability of the representative agent. When there are only two states of nature, we show that the representative agent has a bias towards certainty. Moreover, the divergence of opinions about the probability of occurence of a boom may help solving the equity premium puzzle.

Keywords: aggregation of beliefs, state-dependent utility, efficient risk sharing, disagreement, asset pricing, portfolio choices.

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1 Introduction

People have divergent opinions on a wide range of subjects, from the outcome of an election or of a war, the profitability of a new technology to the risk of global warming. Suppose that this heterogeneity of beliefs does not come from asymmetric information but rather from intrinsic differences in how to view the world. People agree to disagree, which implies that prices and observed behaviors of other market participants do not generate any Bayesian updating of individual beliefs. We examine how the group as a whole will behave towards risk. Aggregating beliefs when agents differ on their expectations is useful to solve various economic questions, from asset pricing to cost-benefit analyses of collective risk prevention.

The attitude towards risk of a group of agents depends upon how risk is allocated in the group. For example, if an agent is fully insured by other agents, it is intuitive that this agent's beliefs should not affect the social welfare function. Only those who bear a share of the risk should see their expectations be taken into account on the collective risk decision. In this paper, we assume that risks can be allocated in a Pareto-efficient way in the group. In such a situation, the willingness to take risk is increasing in the Arrow-Pratt index of absolute risk tolerance. It implies that the beliefs of agents with a larger risk tolerance should have a larger impact on how individual expectations are aggregated. At the limit, those with a zero risk tolerance do not influence the group's expectations.

The properties of the socially efficient probability distribution are derived from the characteristics of the efficient allocation of risk in the group, such as the one derived from the competitive allocation with complete Arrow-Debreu markets. Borch (1960,1962), Wilson (1968) and Rubinstein (1974) were the first to characterize the properties of Pareto-efficient risk sharings. Wilson (1968), and more recently Calvet, Grandmont and Lemaire (2001) showed that the standard methodology of the representative agent can still be used when agents have heterogeneous beliefs. Leland (1980) examined the competitive equilibrium asset portfolios when agents have different priors on the distribution of state probabilities.

The main comparative static exercise that we consider in this paper is to compare two states of nature for which the distribution of individual probabilities are different. Consider for example a situation where all agents believe that state s_2 has the same probability of occurrence than another state s_1, except agent θ. Suppose that this agent has a subjective probability for s_2

that is 1 percent larger than for s_1. By how much should we increase the probability of state s_2 with respect to s_1 in the collective decision making? The central technical result of the paper is to show that the collective probability should be increased by $x/100$ percents, where x is the share of the aggregate risk that is borne by agent θ, or the agent θ's tolerance to risk expressed as a share of the group's risk tolerance. More generally, the rate of change of the collective probability is a weighted mean of the rate of change of the individual probabilities. The weights are proportional to the individual risk tolerances. More risk tolerant agents see their beliefs better represented in the collective decision making under uncertainty. This intuitive result has several important consequences.

Observe first that, as initially observed by Hylland and Zeckhauser (1979),[1] the efficient aggregation of beliefs cannot be disentangled from the risk attitude of the group's members. Except in the case of constant absolute risk aversion, this individual risk attitudes depends upon the allocation of consumption in the group. It implies that the efficient collective probability distribution will be a function of the wealth per capita in the group. The representative agent has state-additive preferences as under the standard expected utility model, but the different terms of the sum cannot be written as a product of a probability that would depend only upon the state by a utility that would depend only upon consumption. Equivalently, this means that the representative agent has a state-dependent utility function, despite the fact that all members of the group have state-independent preferences. Drèze (2001) and Drèze and Rustichini (2001) examine the effect of the state dependency of the utility function for risk management and risk transfers. Another way to interpret this result is that the collective probability distribution depends upon the aggregate wealth level. Wilson (1968) showed that this wealth effect vanishes only when agents have an absolute risk tolerance that is linear with the same slope. We reexamine this wealth effect when this condition is not fulfilled.

The efficient aggregation result for beliefs states that the rate of change of the collective probability across states is a weighted mean of the rate of changes of individual probabilities. This result must be compared to the observation that the state probability used by the representative agent does not need to be in between the smallest and the largest state probabilities of

[1]See also Mongin (1995), and Gilboa, Samet and Schmeidler (2001). Our work differs much from this branch of the literature by allowing trade.

the agents. We will exhibit numerical counterexamples to this in the paper. Notice however that when deciding about transfers of wealth across states, what really matters are relative state prices per unit of probability. Thus the rate of change in the collective probabilities across states is the relevant information for determining the collective risk exposure, and our aggregation formula provides exactly that information.

The main objective of the paper is to determine how the divergence of opinions about the true probability distribution of the states of nature affects the optimal collective risk exposure, and the equilibrium asset prices. Let us compare two states of nature such that all individual probabilities for the second state are k percents larger than those of the first state. A direct consequence of our aggregation rule is that the collective probability will also be increased by k percents. Two classical aggregation rules satisfy this necessary condition. Under the geometric (arithmetic) mean approach, the collective state probability is assumed to be proportional to the geometric (arithmetic) Pareto-weighted mean of the individual subjective probabilities for that state.

The geometric mean approach for the aggregation of beliefs is socially efficient only if all members of the group have the same utility function exhibiting constant absolute risk aversion. When this condition is not fulfilled, the rate of change in the collective probability will dependent upon the relative degree of disagreement in the two states under consideration. We say that there is increasing disagreement if those agents with a larger subjective probability for the initial state also have a larger rate of increase of the likelihood of the second state relative to the first. We show that decreasing absolute risk aversion (DARA) implies that the geometric aggregation rule underestimates the effect of an increase in disagreement on the collective probability. To illustrate, suppose that Mrs Jones has a larger subjective probability for a flood to occur this year than Mr Jones. Compared her own beliefs about floods, Mrs Jones has a subjective probability for the risk of an earthquake that is k percents larger, whereas Mr Jones has a subjective earthquake probability that is k percents smaller than his estimate of the probability of a flood. Thus, the geometric mean probability in the couple is the same for the two risks, but there is more disagreement about the likelihood of an earthquake than for a flood. Under DARA, it implies that, when Mr and Mrs Jones decide about their collective prevention efforts and insurance, they should use a larger probability of occurrence for an earthquake than for a flood.

The alternative aggregation rule would be to take the arithmetic (Pareto-weighted) mean of individual probabilities for the collective beliefs. It is easily shown from the efficient aggregation rule that the arithmetic mean approach is efficient if and only if all agents have a logarithmic utility function. We show that this approach overestimates the effect of an increase in disagreement on the collective probability, if the sensitivity of absolute risk tolerance to changes in consumption is smaller than for the log utility function. Under constant relative risk aversion, this means that relative risk aversion is larger than unity, a plausible assumption. Suppose again that Mr and Mrs Jones have a subjective flood probability of respectively p_{Mr} and $p_{Mrs} > p_{Mr}$. Suppose also that for earthquake, Mr Jones has a probability $p_{Mr} - k$, and Mrs Jones has a probability $p_{Mrs} + k$. Here, the arithmetic means of individual probabilities are the same for the two events, and there is more disagreement for an earthquake than for a flood. Then, if relative risk aversion is larger than unity, the collective probability for an earthquake should be smaller than for a flood. This result has first been obtained by Varian (1985) and Ingersoll (1987). We extend it to the comparison of states where the means of individual probabilities are not equal.

These results describe how the heterogeneity of beliefs affects the difference in collective probabilities for any pair of states. Going from this partial analysis to a more global one, it is necessary to describe the structure of disagreements across states. This would be useful to determine whether the collective distribution function be stochastically dominated by the mean subjective distribution. Cecchetti, Lam and Mark (2000) and Abel (2002) examine the effect of a change in the beliefs of the representative agent on the equity premium. Contrary to us, they assume that all agents have the same beliefs that deviate from what could be inferred from the existing data. However, their work is useful to us because once the beliefs of the representative agent is obtained, our model becomes equivalent to an economy with homogeneous beliefs that can differ from a reference probability distribution. It is not true in general that a first-order-stochastically dominated shift in the subjective distribution of aggregate consumption raises the equity premium. Abel (2002) defines the notion of uniform pessimism by a leftward translation of the objective distribution of the aggregate consumption. He shows that uniform pessimism raises the equity premium. We provide another result which states that transferring probability mass from the wealthiest states uniformly to the other states also unambiguously raises the equity premium. Now, if most of the disagreement is concentrated on the likelihood of a boom, and if relative risk aversion is larger than unity, we know that this state should have a collective probability that is less than its mean individual probability. Thus, such concentration of disagreement on the boom state provides exactly the kind of transformation of the collective probability distribution for which we know that it raises the equity premium. We provide a numerical illustration that shows that a disagreement on the likelihood of a boom may have a sizeable positive effect on the equity premium. In a plausible simulation, it multiplies the equity premium by 4. The bad news is that the equity premium is reduced when most of the disagreement is about the likelihood of a krach. Calvet, Grandmont and Lemaire (2001) also examine the effect of heterogenous beliefs on the equity premium. They are able to sign this effect when the relative risk aversion of the representative agent is decreasing with average wealth.[2]

The structure of the paper is as follows. Section 2 is devoted to the description of the aggregation problem when agents have heterogeneous preferences and beliefs. In section 3, we solve the risk-taking decision problem of the representative agent, assuming that collective preferences are known. We show how to aggregate individual risk tolerances and individual beliefs in this framework in section 4. In section 5, we examine the problem of the multiplicative separability between utility and probability. In section 6, we define our concept of increasing disagreement, and we determine its effect on the collective degree of optimism. Section 7 provides a global analysis of the effect of the heterogeneity of beliefs on the collective probability distribution and the equity premium. An analysis specific to the two-state case is presented in Section 8. Finally, we present concluding remarks in section 9.

2 The aggregation problem

We consider an economy or a group of N heterogeneous agents indexed by $\theta = 1, ..., N$. Agents extract utility from consuming a single consumption good. The model is static with one decision date and one consumption date. At the decision date, there is some uncertainty about the state of nature

[2] Hara and Kuzmics (2001) obtained independently results about how to aggregate risk aversion when beliefs are homogeneous.

s that will prevail at the consumption date. The set of possible states of nature is denoted $S \subset R$. Agents are an expected-utility maximizer with a state-independent utility function $u(.,\theta) : R \to R$ where $u(c,\theta)$ is the utility of agent θ consuming c. We assume that $u_c = \partial u/\partial c$ is continuously differentiable and concave in c. As in Calvet, Grandmont and Lemaire (2001), we focus on interior solutions. To guarantee this, we assume that $\lim_{c\to 0} \partial u/\partial c = +\infty$ and that $\lim_{c\to +\infty} \partial u/\partial c = 0$.

We also assume that each agent θ has beliefs that can be represented by a density function $p(.,\theta) : S \to R^+$. Agents differ not only on their utility and beliefs, but also on their state-dependent wealth: $\omega(s,\theta)$ denotes the wealth of agent θ in state s. E denotes the expectation operator with respect to $\widetilde{\theta}$, i.e., $Ef(\widetilde{\theta}) = \sum_{\theta=1}^{N} f(\theta)/N$.

The group must take a decision towards a collective risk. This can be a portfolio choice, or a decision to invest in a prevention activity to reduce a global risk. In any case, the problem is to transfer wealth across states at an exogenously given exchange rate. The standard paradigm to analyze this problem is the Arrow-Debreu framework. We assume that there is a complete set of Arrow-Debreu securities in the economy. The equilibrium price of the Arrow-Debreu security associated to state s is denoted $\pi(s,\theta) > 0$. It means that agent θ must pay $\pi(s)$ ex-ante to receive one unit of the consumption good if and only if state s occurs. We normalize prices in such a way that $\int_S \pi(s,\theta) = 1$.

A consumption plan is described by a function $C(.,.) : S\times \Theta \to R$. The consumption per capita in state s is denoted $z(s)$:

$$EC(s,\widetilde{\theta}) = z(s) \tag{1}$$

for all $s \in S$. The mean initial endowment is denoted $\omega(s) = E\omega(s,\widetilde{\theta})$. When the group is active on contingent markets, z and ω need not to be equal.

The crucial assumption of this paper is that the group can allocate risks efficiently among its members. An allocation $C(.,.) : S\times \Theta \to R^+$ is Pareto-efficient if it is feasible and if there is no other feasible allocation that raises the expected utility of at least one member without reducing the expected utility of the others. For a given Pareto-weight function $\lambda(.) : \Theta \to R^+$, normalized in such a way that $E\lambda(\widetilde{\theta}) = 1$, the group would select the portfolio of Arrow-Debreu securities and the allocation of the risk within the group that maximize the weighted sum of the members' expected utility under the feasibility constraint:

$$\max_{C} E\left[\lambda(\widetilde{\theta})\int_{S} p(s,\widetilde{\theta})u(C(s,\widetilde{\theta}),\widetilde{\theta})ds\right] \tag{2}$$

$$s.t. \quad \int_{S}\pi(s)E\left[C(s,\widetilde{\theta})-\omega(s,\widetilde{\theta})\right]ds=0. \tag{3}$$

It is useful to decompose this decision problem into two stages. Consider a specific state s and define

$$v(z,P)=\max_{c'(z,.,\theta)} E\left[\lambda(\widetilde{\theta})p(\widetilde{\theta})u(c(z,P,\widetilde{\theta}),\widetilde{\theta})\right] \quad s.t.\ Ec(z,P,\widetilde{\theta})=z, \tag{4}$$

where $P=(p(1),...,p(N))$ is a vector of N individual probabilities associated to the state under scrutiny. In this cake-sharing problem, z represents the consumption per capita, and $v(z,P)$ is the maximum sum of the members' utility weighted by the product of the Pareto weights $(\lambda(1),...,\lambda(N))$ and the subjective probabilities P. Our notation makes explicit that the efficient allocation c and the value function v depend upon the vector P of individual subjective probabilities associated to the state under scrutiny. Notice that by construction, v is homogenous of degree 1 with respect to P.

The second stage is a collective portfolio problem in which the group selects the state-dependent sizes of the cake that maximizes the sum of v across the states:

$$\max_{z(.)}\int_{S} v(z(s),P(s))ds \quad s.t. \quad \int_{S}\pi(s)\left[z(s)-\omega(s)\right]ds=0, \tag{5}$$

where $P(s)=((p(s,1),...,p(s,N))$ is the vector of subjective state probabilities across agents. Obviously, combining these two-stage problems generates the solution to program (2), with $C(s,\theta)=c(z(s),P(s),\theta)$.

Notice that the v function describes the risk attitude and beliefs of the representative agent in the sense of Constantinides (1982). It can be interpreted as the product of the collective state probability by the collective utility for consumption per capita z. However, it is not true in general that the representative agent has preferences and beliefs that are multiplicatively separable as in the standard expected utility model. In other words, the subjective probability distribution of the representative agent may be wealth-dependent, or equivalently, the utility function of the representative agent may be state-dependent. This is why we considered the non-separable case in the previous section. Being neither a probability nor a utility, but rather the contribution of the state to ex-ante expected utility, v will hereafter be referred to as the *contribution* function.[3]

3 Efficient collective risk exposure

In this section, we examine the determinants of the efficient collective risk exposure, assuming that the contribution function v is known. It solves problem (5) whose first-order condition is written as:

$$v_z(z(s), P(s)) = \xi\pi(s), \tag{6}$$

for all $s \in S$. This first-order condition defines the optimal risk exposure $z(s)$ as a function Z of $\pi(s)$ and $P(s) : z = Z(\pi, P)$. We consider the effect on Z of a marginal change of the state price π and of the vector P of individual probabilities. Fully differentiating condition $v_z(Z, P) = \xi\pi$ yields

$$v_{zz}dZ + \sum_{\theta=1}^{N} v_{zp(\theta)}dp(\theta) = \xi d\pi.$$

Eliminating the Lagrange multiplier from the above equation by using equation (6) implies that

$$dZ = T^v(Z, P)\left[\sum_{\theta=1}^{N} \frac{v_{zp(\theta)}(z, P)}{v_z(z, P)}dp(\theta) - \frac{d\pi}{\pi}\right], \tag{7}$$

where T^v is the absolute risk tolerance of the group, which is defined as

$$T^v(z, P) = -\frac{v_z(z, P)}{v_{zz}(z, P)}. \tag{8a}$$

Equation (7) describes how the optimal consumption per capita varies across states as a function of the distribution of individual beliefs and of state prices. Seen from ex-ante, it describes the efficient risk exposure of the group. The

[3] Wilson (1968) calls it the evaluation measure. Wilson takes it as a function of z and s. It is obvious however that it depends upon the state s only through the vector of individual probabilities $P(s)$. We make this explicit in our approach by making v dependent of P.

simplest case is when there is a single agent in the group with individual probability p, which implies that equation (7) simplifies to

$$dZ = T^u(Z,1)\left[\frac{dp}{p} - \frac{d\pi}{\pi}\right], \tag{9}$$

where $T^u(z,\theta) = -u_c(z,\theta)/u_{cc}(z,\theta)$ is the agent θ's risk tolerance. If higher states are better from the point of view of an external observer, he can interpret $dp(s)/p(s)$ as a local index of optimism of the agent. The variability of state consumption is proportional to the difference between the agent's degree of optimism and the price volatility $d\pi(s)/\pi(s)$. In the special case of actuarially fair prices, π/p is independent of the state, which implies that the bracketed term in (9) vanishes. As is well-known, a risk-free position would be optimal in that case. On the contrary, when there are two states s and s' such that $d\ln p$ is larger than $d\ln\pi$, equation (9) implies that the agent will consume more in state s' than in state s. This is because the agent perceives consumption in state s' as cheaper in actuarial terms than consumption in state s. The willingness to take advantage of the cheaper state consumption is limited ex-ante by risk aversion. As stated by equation (9), the optimal risk exposure is proportional to the agent's risk tolerance.

The extension of this standard optimal rules for individual portfolio choice to collective decision making is described by equation (7). Let us define function R as

$$R(z,P,\theta) = \frac{d\ln v_z(z,P)}{d\ln p(\theta)} = \frac{p(\theta)v_{zp(\theta)}(z,P)}{v_z(z,P)}. \tag{10}$$

$R(z,P,\theta)$ is the elasticity of the marginal contribution $v_z(z,P)$ to change in $p(\theta)$. By definition of the contribution function, it can also be interpreted as the elasticity of the collective state probability to a change in the subjective probability of agent θ. Because v_z is homogenous of degree 1 in P, it must be that

$$\sum_{\theta=1}^{N} R(z,P,\theta) = 1$$

for all (z,P). Property (7) can then be rewritten as

$$dZ = T^v(Z,P)\left[\sum_{\theta=1}^{N} R(z,P,\theta)\frac{dp(\theta)}{p(\theta)} - \frac{d\pi}{\pi}\right]. \tag{11}$$

Function R thus describes the efficient rule for the aggregation of beliefs. By analogy with the single agent case (9), if the subjective probability of agent θ is increased by k percents, this would have the same effect on the collective risk decision as an increase of the subjective probability of the representative agent by Rk percents. If $R = 1/N$, the representative agent has no bias in the aggregation of individual beliefs.

Keeping in mind equation (11), the remaining of the paper is devoted to this characterization of $T^v(z,P)$ and $R(z,P,\theta)$ of the collective contribution function v, which is defined by the cake-sharing program (4). Its first-order condition is written as

$$\lambda(\theta)p(\theta)u_c(c(z,P,\theta),\theta) = \psi(z,P) = v_z(z,P), \tag{12}$$

for all (z,P), and for all $\theta = 1,...,N$. The second inequality comes from the envelop theorem.

4 The aggregation rules

In this section, we characterize the group's degree of tolerance to risk on the wealth per capita z and the group's beliefs as functions of the primitives of the model, i.e., u and p.

The collective attitude towards risk depends upon how this collective risk is allocated to the members' risk on consumption. This is characterized by $\partial c/\partial z$. Fully differentiating first-order condition (12) with respect to z and using the feasibility constraint $Ec(z,P,\widetilde{\theta}) = z$ yields the following well-known Wilson (1968)'s result:

$$dc = \frac{T^u(c(z,P,\theta),\theta)}{ET^u(c(z,P,\widetilde{\theta}),\widetilde{\theta})}dz. \tag{13}$$

One can interpret this property of the efficient risk-sharing rule as follows: suppose that there are two states of nature that are perceived to be identical by all agents ($p(s,\widetilde{\theta}) = p(s',\widetilde{\theta})$), expect for the mean income z. Equation (13) shows how to allocate the collective wealth differential in the two states. Observe that the positiveness of the right-hand side of (13) means that individual consumption levels are all procyclical, ceteris paribus. More risk-tolerant agents should bear a larger fraction of the collective risk.

From this efficient collective risk-sharing rule, it is easy to derive the degree of risk tolerance of the group as a whole. Wilson (1968) obtains that

$$T^v(z,P) = ET^u(c(z,P,\widetilde{\theta}),\widetilde{\theta}). \tag{14}$$

The group's absolute risk tolerance is the mean of its members' tolerance. There is no bias in the aggregation of individual risk tolerances. We conclude that this rule already valid in the simpler Wilson's model is robust to the introduction of heterogeneous expectations.

In the classical case with homogeneous beliefs, an important property of any Pareto-efficient allocation of risk is the so-called mutuality principle. It states that efficient individual consumption levels depend upon the state only through the wealth per capita z. Its economic interpretation is that all diversifiable risks are eliminated through sharing. In this classical case, the wealth level per capita z is a sufficient statistic for efficient individual consumption levels. The mutuality principle is obviously not robust to the introduction of heterogeneous beliefs because efficient allocation plans $c(z,P,\theta)$ depend also upon the distribution of individual subjective probabilities associated to the state. For example, agents will find mutually advantageous exchanges of zero-sum lotteries in order to gamble on states that they believe to be more likely than their counterpart. In the following, we examine the effect of a change in the distribution of individual probabilities P on the allocation of wealth and on the marginal contribution v_z.

The aggregation of beliefs cannot be disentangled from how the heterogeneity of beliefs affects the allocation of risk in the group. In the following Proposition we derive altogether the aggregation rule of beliefs and the allocation of diversifiable risks. The comparative exercise there and in the remaining of the paper consists in comparing two states of nature s and s' with $P(s') = P(s) + \Delta P$. It does *not* consist in increasing the subjective probability of state s by agent θ. We do not modify the structure of the beliefs in the economy.

Proposition 1 *The elasticity of the collective state probability to the subjective state probability of agent θ is proportional to agent's θ risk tolerance:*

$$R(z,P,\theta) = \frac{d\ln v_z(z,P)}{d\ln p(\theta)} = \frac{T^u(c(z,P,\theta),\theta)}{N\, ET^u(c(z,P,\widetilde{\theta}),\widetilde{\theta})}. \tag{15}$$

The efficient allocation of consumption satisfies the following condition:

$$\frac{dc(z,P,\theta)}{d\ln p(\theta')} = \begin{cases} T^u(c(z,P,\theta),\theta)\left[1 - \dfrac{T^u(c(z,P,\theta),\theta)}{N\, ET^u(c(z,P,\widetilde{\theta}),\widetilde{\theta})}\right] & if \quad \theta = \theta' \\ -\dfrac{T^u(c(z,P,\theta),\theta)T^u(c(z,P,\theta'),\theta')}{N\, ET^u(c(z,P,\widetilde{\theta}),\widetilde{\theta})} & if \quad \theta \neq \theta'. \end{cases} \tag{16}$$

Proof: Fully differentiating the first-order condition (12) with respect to $p(\theta')$ and dividing both side of the equality by $\lambda p u_c = \psi$ yields

$$dc(z,P,\theta) = -T^u(c(z,P,\theta),\theta)\frac{d\psi}{\psi} \tag{17}$$

for all $\theta \neq \theta'$, and

$$dc(z,P,\theta') = T^u(c(z,P,\theta'),\theta')\left[\frac{dp(\theta')}{p(\theta')} - \frac{d\psi}{\psi}\right]. \tag{18}$$

By the feasibility constraint, it must be that $Edc(z,P,\widetilde{\theta}) = 0$. Replacing $dc(z,P,\theta)$ by its expressions given above implies that

$$\frac{d\psi}{\psi} = \frac{T^u(c(z,P,\theta),\theta)}{N\, ET^u(c(z,P,\widetilde{\theta}),\widetilde{\theta})}\frac{dp(\theta')}{p(\theta')}. \tag{19}$$

Combining (17), (18) and (19) yields (16). By the envelop theorem, we also know that $v_z(z,P) = \psi(z,P)$. It implies that

$$d\ln v_z(z,P) = \frac{d\psi}{\psi}. \tag{20}$$

Combining equations (19) and (20) yields property (15).■

Let us first focus on property (16). Ceteris paribus, an increase in the state probability by agent θ increases his efficient consumption and it reduces the consumption by all other members of the group. Ex-ante, this means that the members take risk on their consumption even when there is no social risk, i.e., when z is state independent. Agents take a long position on states that they perceive to have a relatively larger probability of occurrence relative to the other members of the group. This illustrates the violation of the mutuality principle. Notice that the size of these side bets among the members of the group is proportional to the members' risk tolerance. At the limit, if an agent θ has a zero tolerance to risk, it is not efficient for him to gamble with others in spite of the divergence of opinions in the group.

Condition (15) provides a nice characterization of the aggregation of individual beliefs in groups that can share risk efficiently. The elasticity of the collective probability to a change in an agent's subjective probability is proportional to that agent's degree of absolute risk tolerance. Thus, the aggregation of individual beliefs is biased in favor of those agents who are more risk tolerant. Combining properties (15) and (13), we obtain that

$$R(z, P, \theta) = \frac{1}{N}\frac{dc(z, P, \theta)}{dz}. \tag{21}$$

The collective probability distribution is biased towards those who actually bear the collective risk in the group.

In the remaing of the paper, we use the aggregation rule (15) to derive properties of the collective probability distribution.

5 Wealth effect on the aggregation of beliefs

The fact that all members of the group have a multiplicatively separable contribution function $p(s, \theta)u(c, \theta)$ does not imply that the contribution function of the representative agent inherits this property from them. In other words, it is not necessarily true that $v(z, P) = p^v(P)h(z)$, where $p^v(P)$ could be interpreted as the collective probability of a state whose distribution of subjective state probabilities across agents is $P = (p(1), ..., p(N))$, and $h(z)$ would be the utility of mean wealth z. This non-separability implies that the collective tolerance to risk T^v is a function of P, and that the aggregation rule R for beliefs depends upon wealth z. The equivalence between these two non-separability properties of the contribution function is expressed by the following equality:

$$\frac{\partial(-1/T^v(z, P))}{\partial p(\theta)} = \frac{\partial R(z, P, \theta)}{\partial z}.$$

When the contribution is not multiplicatively separable, one can say that the representative agent has a state-dependent utility function, or equivalently, that its subjective probability distribution is sensitive to changes in aggregate wealth.

We start with a rephrasing of another Wilson's result where such a wealth effect does not exist. It corresponds to situations where the derivative of individual risk tolerances $\partial T^u/\partial c$ are all identical and consumption independent. The corresponding set of utility functions is referred to as ISHARA. A utility function has an Harmonic Absolute Risk Aversion (HARA) if its absolute risk tolerance is linear in consumption: $\partial T^u/\partial c(c,\theta) = 1/\gamma(\theta)$ for all c. A set of utility functions satisfies the Identically Sloped HARA (ISHARA) property if their absolute risk tolerances are linear in consumption with the same slope: $\gamma(\theta) = \gamma$ for all θ. The set of utility functions that satisfies these conditions must be parametrized as follows:

$$u(c,\theta) = \kappa \left(\frac{c - a(\theta)}{\gamma} \right)^{1-\gamma} \tag{22}$$

These utility functions are defined over the consumption domain such that $\gamma^{-1}(\ c - a(\theta)) > 0$. When $\gamma > 0$, parameter $a(\theta)$ is often referred to as the minimum level of subsistence. This preference set includes preferences with heterogeneous exponential utility functions $u(c,t,\theta) = -\exp(-A(\theta)c)$ when γ tends to $+\infty$, and $a(\theta)/\gamma$ tends to $-1/A(\theta)$. Taking $a(\theta) = 0$ for all θ, it also includes the set of power (and logarithmic) utility functions with the same relative risk aversion γ for all θ.

Proposition 2 *The aggregation rule R for beliefs is independent of the wealth per capita in the group if and only if the members of the group have ISHARA preferences (22):*

$$\frac{\partial R(z,P,\theta)}{\partial z} = 0 \quad \forall (z,P,\theta) \quad \Longleftrightarrow \quad \frac{\partial T^u(c,\theta)}{\partial c} \text{ is independent of } \theta, \text{ for all } c.$$

Proof: Fully differentiating equation (15) with respect to z and using property (13) yields

$$\frac{N(ET^u(\widetilde{c},\widetilde{\theta}))^2}{T^u(c(z,P,\theta),\theta)} \frac{\partial R}{\partial z}(z,P,\theta) = \frac{\partial T^u}{\partial c}(c(z,P,\theta),\theta) - \widehat{E}_{zP} \frac{\partial T^u}{\partial c}(\widetilde{c},\widetilde{\theta}), \tag{23}$$

where $\widetilde{c} = c(z,P,\widetilde{\theta})$, and where $\widehat{E}_{zP}$ is a "risk-neutral" expectation operator defined as

$$\widehat{E}_{zP} f(\widetilde{\theta}) = \frac{Ef(\widetilde{\theta})T^u(c(z,P,\widetilde{\theta}),\widetilde{\theta})}{ET^u(c(z,P,\widetilde{\theta}),\widetilde{\theta})}$$

For ISHARA preferences, $\partial T^u / \partial c$ is a constant, which implies that the right-hand side of equation (23) vanishes, and R is independent of the per capita wealth in the group. Reciprocally, R independent of z implies that

$$\frac{\partial T^u}{\partial c}(c(z, P, \theta), \theta) = \widehat{E}_{zP} \frac{\partial T^u}{\partial c}(c(z, P, \widetilde{\theta}), \widetilde{\theta})$$

for all θ and P. This can be possible only if $\partial T^u / \partial c$ is independent of c and θ, which means that the group has ISHARA preferences.■

The ISHARA condition guarantees that R remains constant when the wealth level changes in the group. This result is equivalent to the property that efficient sharing rules are linear in z in ISHARA groups with homogenous beliefs. In Appendix A, we derive an analytical solution to the aggregation problem when the ISHARA condition is satisfied.

In Proposition 2, we assumed that the derivative of individual absolute risk tolerances be identical across agents. In the next Proposition, we show that agents with a large sensitivity of risk tolerance to changes in consumption have a share R in the aggregation of beliefs that is increasing with wealth.

Proposition 3 *To each vector* (z, P), *there exists a scalar* m *belonging to* $[\min_\theta \partial T^u(c(z, P, \theta), \theta)/\partial c, \max_\theta \partial T^u(c(z, P, \theta), \theta)/\partial c]$ *such that*

$$\frac{\partial R(z, P, \theta)}{\partial z} \geq 0 \text{ if and only if } \frac{\partial T^u(c(z, P, \theta), \theta)}{\partial c} \geq m(z, P).$$

Proof: This is a direct consequence of equation (23) with

$$m(z, P) = \widehat{E}_{zP} \frac{\partial T^u}{\partial c}(c(z, P, \widetilde{\theta}), \widetilde{\theta}).■$$

Agents with a large sensitivity of risk tolerance to changes in consumption are those who increase their bearing of the collective risk when the group's wealth increase. The result follows from the fact that the share R of agent $\theta's$ beliefs in the aggregation of beliefs is proportional agent θ's share in the group's risk.

In the special case of utility functions exhibiting constant relative risk aversion (CRRA), viz. $u(c, \theta) = c^{1-\gamma(\theta)}/(1 - \gamma(\theta))$, there is a negative relationship between relative risk aversion $\gamma(\theta)$ and $\partial T^u(c, \theta)/\partial c = 1/\gamma(\theta)$. Thus, the above Proposition applied in the case of CRRA utility functions

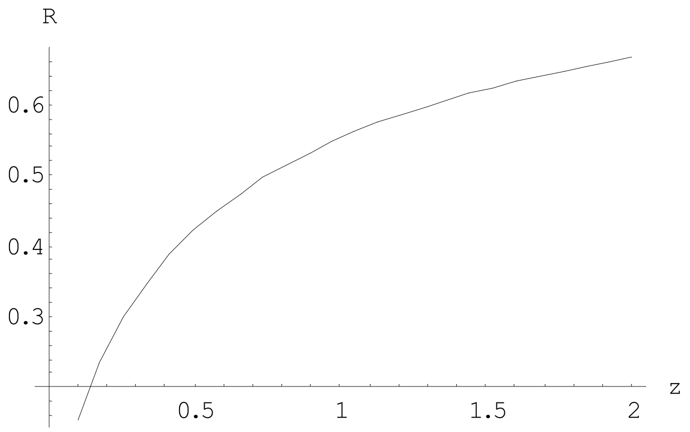

Figure 1: The share R of the beliefs of a low risk-averse agent in the collective beliefs as a function of the group's wealth per capita.

means that *less risk-averse agents have a share R in the aggregation of beliefs that is increasing with wealth.* To illustrate, suppose that there are two agents in the group, respectively with constant relative risk aversion $\gamma(\theta_1) = 1$ and $\gamma(\theta_2) = 2$. In Figure 1, we have drawn the share R of agent θ_1's beliefs in the collective beliefs has a function of z, for P such that $\lambda(\theta_1)p(\theta_1) = \lambda(\theta_2)p(\theta_2)$. Because agent 1 is relatively less risk-averse than agent 2, this curve is upward sloping.

6 The effect of increasing disagreement on the collective state probability

In this section, we want to determine the effect of the divergence of opinions on the collective probability distribution. The efficient collective risk exposure is governed by how v_z fluctuates with the wealth per capita and with the distribution of individual probabilities P. In section 4, we already determined the relationship between v_z and z by characterizing the collective tolerance T^v to an aggregate risk. In the ISHARA case, this relationship is not affected by the divergence of opinions in the population. We now turn to the effect of the heterogeneity of beliefs on the beliefs of the representative agent. We first consider marginal changes from P to $P + dP$. Latter in this section, we extend our results to comparing two distributions of individual state probabilities P and P'. We want to determine the sign of $v_{zP}(z,P)$ in the small, or to compare $v_z(z,P')$ to $v_z(z,P)$ in the large. If $v_z(z,P')$ is larger than $v_z(z,P)$, the demand for the contingent claim is increased by the shift in the distribution of beliefs from P to P'. Ceteris paribus, the optimal consumption per capita in the group would be increased by this shift. Equation (11) describes the link between the optimal state consumption per capita and the distribution of individual beliefs in a more precise way.

The effect of a shift in distribution on the collective probability depends upon the dispersion of individual beliefs in a complex way. As a benchmark, consider the proportional shift in distribution with $P' = kP$ for some scalar $k \neq 1$. Each member of the group believes that state s' is k time more likely than state s. The decision problem (4) is unchanged by this multiplicative change in the parameters of the problem. The efficient allocation of z will be the same in the two states, and $v_z(z,kP) = kv_z(z,P)$, for all z and P. Thus, as stated before, v and its partial derivatives with respect to z are homogeneous of degree 1 in the vector of individual probabilities P. In the following, we define a family of shifts in P that are not proportional.

In order to focus on the heterogeneity of beliefs, we assume in this section that risk preferences are homogenous: $\partial T^u/\partial\theta \equiv 0$. We first define the notion of increasing disagreement.

6.1 Increasing disagreement in the small and in the large

With proportional shifts in the distribution of individual probabilities, the rate of change $d\ln p(\theta)$ is the same for all θ: $d\ln p(\theta) = k \quad d\ln p$. We say that the degree of disagreement on the probability is unaffected. As said above, such a shift raises the collective probability by the same factor k. We hereafter define a concept of increasing disagreement that is based on the Monotone Likelihood Ratio (MLR) order. However, the main ingredient in this section is not the individual subjective probabilities $p(\theta)$, but rather the Pareto-weighted ones $q(\theta) = \lambda(\theta)p(\theta)$. We say that a marginal shift dP yields increasing disagreement if those agents with a larger $q(\theta)$ also have a larger rate of increase $d\ln q(\theta) = d\ln p(\theta)$. Compared to a proportional increase, the distribution of individual probabilities becomes more dispersed.

Definition 1 *Consider a specific distribution of individual probabilities* $P =$

$(p(1), ..., p(N))$ *and a specific Pareto-weight vector* $(\lambda(1), ..., \lambda(N))$*. We say that a marginal shift* dP *yields increasing disagreement if* $q(\widetilde{\theta}) = \lambda(\widetilde{\theta})p(\widetilde{\theta})$ *and* $d\ln q(\widetilde{\theta})$ *are comonotone: for all* (θ, θ') :

$$[q(\theta') - q(\theta)]\,[d\ln q(\theta') - d\ln q(\theta)] \geq 0. \tag{24}$$

Those with a larger subjective probability also have a larger rate of increase of their probability. If we assume without loss of generality that q is increasing in θ, this is equivalent to require that $p(\theta')/p(\theta)$ be increased by the shift whenever $\theta' > \theta$. This is a MLR property. Notice that our definition of increasing disagreement does not constrain in any way how the mean (log) (Pareto-weighted) probability is affected by the shift in distribution. In the following Proposition, we show that increasing disagreement generates a Rothschild-Stiglitz (1970) spread in the distribution of Pareto-weighted individual log probabilities.

Proposition 4 *Any marginal shift* dP *that preserves the mean* $E\ln q(\widetilde{\theta})$ *is an increase in disagreement if and only if it generates a Rothschild-Stiglitz increase in risk of* $\ln q(\widetilde{\theta})$*. Any marginal shift* dP *that increases disagreement and preserves the mean* $Eq(\widetilde{\theta})$ *generates a Rothschild-Stiglitz increase in risk of* $q(\widetilde{\theta})$*.*

Proof: We check that for any concave function h, $Eh(\ln q(\widetilde{\theta}))$ is decreased by the marginal shift dP. Suppose without loss of generality that q is increasing in θ. Because dP is an increase in disagreement, we have that $d\ln q(\theta)$ is increasing in θ. By the covariance rule, it implies that

$$dEh(\ln q(\widetilde{\theta})) = E\left[h'(q(\widetilde{\theta}))d\ln q(\widetilde{\theta})\right] \leq Eh'(q(\widetilde{\theta}))Ed\ln q(\widetilde{\theta}) = 0. \tag{25}$$

The last equality comes from the assumption that dP preserves the mean of $\ln q(\widetilde{\theta})$. This proofs the sufficiency part of the first statement. The proof of necessity is by contradiction. Suppose that $d\ln q(\theta)$ is not increasing in the neighborhood K of some θ_0. Then, inequality (25) is reversed for any function h that is linear outside K and concave in K, thereby contradicting the condition that $d\ln q(\widetilde{\theta})$ yields a Rothschild-Stiglitz increase in risk of $\ln q(\widetilde{\theta})$. The proof of the second statement of the Proposition follows the some line as the sufficiency proof above by replacing function h by function $\widehat{h}(.) = h(\ln(.))$, which is concave if h is concave.■

Our assumption of increasing disagreement is more restrictive than the Rothschild-Stiglitz notion of an increase in risk in the distribution of individual probabilities. The latter just requires that $q(\theta)$ and $dq(\theta)$ be comonotone, whereas the former requires that $q(\theta)$ and $d\ln q(\theta)$ be comonotone, which is more demanding.

We can also examine non marginal changes in distribution by comparing two distribution P_0 and P_1. Our definition of an increase in disagreement "in the large" is as follows.

Definition 2 *Consider a specific Pareto-weight vector $(\lambda(1),...,\lambda(N))$. We say that P_1 yields more disagreement than P_0 if $q_0(\widetilde{\theta}) = \lambda(\widetilde{\theta})p_0(\widetilde{\theta})$ and $\ln q_1(\widetilde{\theta}) - \ln q_0(\widetilde{\theta})$ are comonotone: for all (θ, θ') :*

$$[q_0(\theta') - q_0(\theta)]\left[\ln\frac{q_1(\theta')}{q_0(\theta')} - \ln\frac{q_1(\theta)}{q_0(\theta)}\right] \geq 0.$$

If q_0 is increasing in θ, this is equivalent to require that (P_0, P_1) satisfies the Monotone Likelihood Ratio (MLR) property that $p_1(\theta)/p_0(\theta)$ be increasing in θ. Because agents with a larger lnq under P_0 get a larger increase in log probabilities under P_1, it implies that the distribution of log probabilities under P_1 is a spread of the individual log probabilities under P_0. Thereby, it amplifies the dispersion of $q(\widetilde{\theta})$.

It is useful to decompose any shift in distribution from P_0 to P_1 as a sequence of infinitesimal changes in probabilities $dP(\tau) = (dp(\tau,1),...,dp(\tau,N))$ indexed by τ going from 0 to 1 with

$$P(t) = P_0 + \int_0^t dP(\tau) \geq 0 \text{ and } P(1) = P_1.$$

Among the various ways to do this, we are interested in the paths $P(.)$ that preserve the property of increasing disagreement for each infinitesimal change $dP(\tau)$ in the vector of individual probabilities. The following Lemma proves that such paths exist.

Lemma 1 *If P_1 exhibits more disagreement than P_0, there exists a path $P(.)$ linking P_0 to P_1 in which each increment $dP(\tau)$ yields an increase in disagreement.*

Proof: We check that $P(t) = P_1^t P_0^{1-t} = P_0 \exp[t \ln P_1/P_0]$ satisfies this property. Define $q(t,\theta) = \lambda(\theta)p(t,\theta) = \lambda(\theta)p_1(\theta)^t p_0(\theta)^{1-t}$. It implies that

$$d \ln q(t,\theta) = \ln \frac{p_1(\theta)}{p_0(\theta)} dt, \tag{26}$$

which is independent of t. Without loss of generality, suppose that P_0 is such that $q_0(1) \leq q_0(2) \leq ... \leq q_0(N)$. Because P_1 exhibits more disagreement than P_0, it must be that $p_1(\theta)/p_0(\theta) = q_1(\theta)/q_0(\theta)$ be increasing in θ. Combining this with equation (26) implies that the right bracketed term in (24) is positive if $\theta' > \theta$. It remains to prove that $\lambda(\theta)p_1(\theta)^t p_0(\theta)^{1-t}$ is increasing in θ. This is immediate from the observation that

$$\lambda(\theta)p_1(\theta)^t p_0(\theta)^{1-t} = q_1(\theta)^t q_0(\theta)^{1-t} = q_0(\theta)\left[\frac{q_1(\theta)}{q_0(\theta)}\right]^t$$

is the product of two positive increasing functions of θ. Notice that this implies that $\lambda(\theta)p(t,\theta)$ increases with θ at a rate that increases with t,or that $\lambda(\theta)p(t,\theta)$ is logsupermodular. ■

This Lemma is useful because it allows us to focus on marginal changes in distribution. Any result holding for increasing disagreement in the small can be extended to increases in disagreement in the large. For example, because a sequence of increases in risk is an increase in risk, Proposition 4 implies that P_1 is riskier than P_0 in the sense of Rothschild-Stiglitz if P_1 exhibits more disagreement than P_0 and $Eq_1(\widetilde{\theta}) = Eq_0(\widetilde{\theta})$.

Consider an initial distribution $P = (p(1), ..., p(N))$ of individual probabilities, and a shift $dP = (dp(1), ..., dp(N))$ in this distribution. We hereafter examine the effect of this shift on the collective probability. From definition (10) of the aggregation function $R(z,P,\theta)$, we have that

$$d \ln v_z(z,P) = \sum_{\theta=1}^{N} R(z,P,\theta)\frac{dp(\theta)}{p(\theta)}.$$

Using Proposition 1 together with the assumption that all agents have the same utility function, this can be rewritten as

$$d \ln v_z(z,P) = \frac{E\left[T^u(c(z,P,\widetilde{\theta}))d\ln p(\widetilde{\theta})\right]}{ET^u(z,P,\widetilde{\theta})}. \tag{27}$$

The left-hand side of this equality can be interpreted as the rate of increase in the collective probability. Equation (27) states that it is a weighted mean of the rate of increase in the individual probabilities. The weights are proportional to the individual absolute risk tolerance.

6.2 The geometric mean approach

In the special case of constant absolute risk aversion (CARA), T^u is constant. It implies that equation (27) can be rewritten in this special case as

$$d \ln v_z(z, P) = E d \ln p(\widetilde{\theta}). \tag{28}$$

This means that the contribution function $v(z, P)$ can be written as a product of a collective utility $h(z)$ by a collective probability $p^v(P)$ that takes the form of

$$p^v(P) = a \left[\prod_{\theta=1}^{N} p(\theta) \right]^{1/N}, \tag{29}$$

where a is a normalizing constant that guarantees that $\int_S p^v(P(s))ds = 1$. In the CARA case, the efficient collective state probabilities are proportional to the geometric average of the individual state probabilities. This is called the geometric aggregation rule for beliefs. In this section, we suppose that the social planner takes this geometric aggregation rule of individual beliefs into consideration, and we determine the error that it generates when the CARA condition on individual preferences is not satisfied.

Under the geometric aggregation rule, a shift in the distribution dP that preserves the geometric mean has no effect on the collective probability. Suppose that, comparing to a reference state, we examine an alternative state where the probability of Mr Jones is increased by $k + \varepsilon\%$ and the probability of Mrs Smith is reduced by $k - \varepsilon\%$. The couple using the geometric mean approach would increase the probability of this alternative state by $k\%$ without taking into account of their increased divergence of opinions. However, the socially efficient rule would increase the collective probability by ηk, where η is defined by

$$\eta(z, P, dP) = \frac{d \ln v_z(z, P)}{\frac{1}{N} \sum_{\theta=1}^{N} d \ln p(\theta)}.$$

When η is larger (smaller) than unity, using the geometric aggregation rule would underestimate (overestimate) the rate of increase of the collective probability. Thus $\eta - 1$ measures the error of using the geometric rule.

We see that $\eta(z, P, dP)$ is larger than unity if and only if

$$E\left[T^u(c(z,P,\widetilde{\theta}))d\ln p(\widetilde{\theta})\right] \geq E\left[T^u(c(z,P,\widetilde{\theta}))\right]\ E\left[d\ln p(\widetilde{\theta})\right]. \qquad (30)$$

When it is T^u is not constant, the allocation of consumption in the group will affect the weights in the aggregation formula (27). Suppose without loss of generality that $q(\theta) = \lambda(\theta)p(\theta)$ is increasing in θ. Combining the first-order condition (12) with risk aversion implies that $c(z, P, \theta)$ is increasing in θ. Under decreasing absolute risk aversion (DARA), it implies in turn that $T^u(c(z, P, \theta))$ is also increasing in θ. In short, the definition of increasing disagreement just guarantees that T^u and $d\ln p$ be comonotone under DARA. Applying the covariance rule to $E[T^u d\ln p]$ directly implies (30), or $\eta \geq 1$. Of course, switching to either increasing absolute risk aversion or decreasing disagreement would yield $\eta \leq 1$.

Proposition 5 *Suppose that the individual utility functions are identical. The following two conditions are equivalent:*

1. *For any wealth z, any initial distribution of individual probabilities P and any shift dP yielding increasing disagreement, the geometric mean approach underestimates the rate of increase of the collective probability: $\eta(z, P, dP) \geq 1$;*
2. *Absolute risk aversion is decreasing: $\partial T^u/\partial c \geq 0$.*

Proof: The sufficiency of DARA has been proved above. Suppose now by contradiction that T^u is locally decreasing in the neighborhood B of c_0. Then, take $z = c_0$ and an initial distribution $P(\varepsilon)$ such that $\lambda(\theta)p(\theta) = k + \varepsilon\theta$ for all θ. When $\varepsilon = 0$, $c(z, P(0), \theta) = c_0$ for all θ. Take a small ε such that $c(z, P(\varepsilon), \theta)$ remains in B for all θ. By assumption, the shift dP exhibits increasing disagreement, which means that $c(z, P(\varepsilon), \theta)$ and $d\ln p(\theta)$ are comonotone. This implies that $T^u(c(z, P(\varepsilon), \theta))$ and $d\ln p(\theta)$ are anti-comonotone, thereby reversing the inequality in (30). This implies that DARA is necessary for property 1.■

Under DARA, a mean-preserving spread in log probabilities, in the small or in the large, always raises the collective probability. The intuition of this result is easy to derive from the central property (15) of the aggregation of heterogeneous beliefs. Under DARA, this property states that those who consume more see their beliefs better represented in the aggregation. But by definition of an increase in disagreement, those who consume more are also those who have a larger rate of increase in their subjective probability. We conclude that, because of the bias in favor of those who consume more, an increase in disagreement raises the collective probability even when the mean rate of increase in individual probabilities is zero.

When the geometric mean of individual probabilities is independent of the state, the social planner should consider that the different states are equally likely only when absolute risk aversion is constant. Under DARA, doing so would underestimate the efficient collective probability of the state with larger disagreement. Let us illustrate this result by the following example. There are two agents, $\theta = 1$ and $\theta = 2$, both with a constant relative risk aversion $\gamma = 0.1$. This implies that their absolute risk aversion is decreasing. There is a continuum of states of nature $s \in S = [0, 1]$. The beliefs of agent θ is represented by an exponential density function $p(s, \theta) = \delta_\theta \exp[\delta_\theta s]/(\exp[\delta_\theta] - 1)$. This means that agent θ has a constant rate of increase $\delta_\theta = d \ln p(s, \theta)/ds$ of his state probabilities . We assume that $\delta_1 = -\delta_2 = \delta = 5$, which implies that any change in s preserves the mean of the log probabilities, or that the geometric mean of individual probabilities is independent of s. When increasing state s from 0 to 1, the subjective state probability of agent $\theta = 1$ increases at rate δ, whereas the one of agent $\theta = 2$ decreases at rate δ. In Figure 2, we draw these density functions. In Figure 3, we describe the efficient allocation of risk in the group when there is no aggregate risk ($z = 1$) and when $\lambda(1) = \lambda(2)$. From Figure 2, we see that increasing s at the margin everywhere between 0 and 0.5 decreases disagreement in the group, whereas marginally increasing s everywhere between 0.5 and 1 increases disagreement.

In Figure 4, we draw the collective probability density which is characterized by

$$p^v(s) = p(s, 1) \left[\left(\frac{p(s, 2)}{p(s, 1)} \right)^{1/\gamma} + 1 \right]^\gamma .$$

Because when $s > 0.5$ a marginal increase in s reduces disagreement, DARA implies that the collective probability is increasing in s in this region. The slope of the collective density function is very similar to the slope of the density function of agent $\theta = 1$. This is because, as seen from Figure 3, most of the aggregate wealth is consumed by that agent in these states. This implies that the social planner who considers transferring wealth across states in this region will mostly be concerned by the beliefs of that agent.

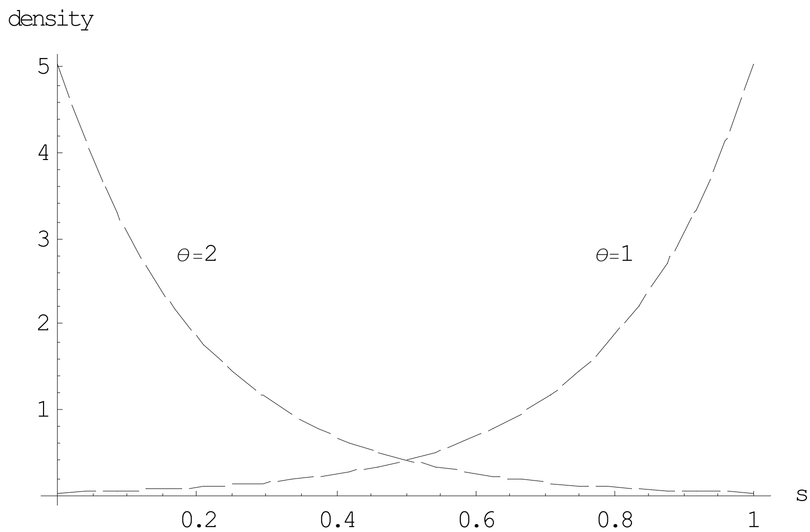

Figure 2: A set of beliefs for which the average of individual log probabilities is independent of the state.

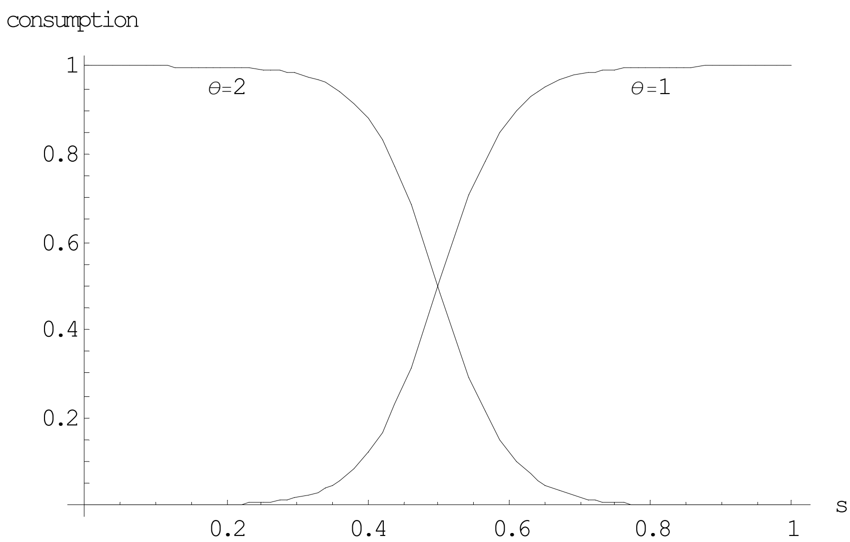

Figure 3: Optimal allocation of consumption $c(1, P(s), \theta)$.

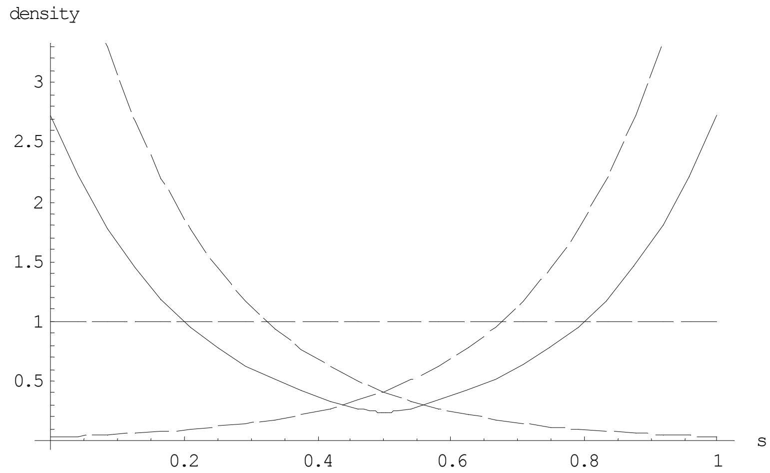

Figure 4: The collective density function for $u(c,\theta) = c^{1-\gamma}/(1-\gamma)$, $p(s,1) = 5\exp[5s]/(\exp[5]-1)$, and $p(s,2) = -5\exp[-5s]/(\exp[-5]-1)$.

In region $s < 0.5$ on the contrary, the collective probability is decreasing in s because a larger s yields less disagreement. Most of the aggregate wealth is consumed by agent $\theta = 2$ in these states, which implies that the social planner who consider transferring wealth across these states will use beliefs whose sensitivity to changes in s is close to the one of the subjective density function of that agent.

This is an example for which the geometric mean of individual probabilities is constant. The geometric aggregation rule (29) applied in this case would generate a uniform collective probability density represented by the horizontal line in Figure 4. In terms of collective risk management, a social planner using the geometric rule would not purchase enough insurance for the states with the largest divergence of opinions, i.e., extreme states in this example.

Figure 4 illustrate another important feature of the aggregation of beliefs. Contrary to the intuition, the collective probability of any state s needs not to belong to the interval bounded by $\min_{\theta\in\Theta} p(s,\theta)$ and $\max_{\theta\in\Theta} p(s,\theta)$. This is in sharp contrast with the rate of increase in the collective probability, which is a weighted mean of the rate of increase in the individual probabilities, as stated by equation (27).

6.3 The arithmetic mean approach

In the previous section, we justified the geometric aggregation rule by considering the CARA case. Consider alternatively that all agents have a logarithmic utility function. Using the first-order condition (12), the optimal consumption plan is such that

$$c(z,P,\theta)=T^u(c(z,P,\theta))=b(z,P)q(\theta)$$

for all θ. In this case, property (27) can be rewritten as

$$d\ln v_z(z,P)=\frac{Eq(\widetilde{\theta})d\ln q(\widetilde{\theta})}{Eq(\widetilde{\theta})}=\frac{Edq(\widetilde{\theta})}{Eq(\widetilde{\theta})}=d\ln Eq(\widetilde{\theta}). \tag{31}$$

This means that the contribution function $v(z,P)$ can be written as a product of a collective utility $h(z)$ by a collective probability $p^v(P)$ that takes the form of

$$p^v(P)=a\frac{1}{N}\sum_{\theta=1}^{N}\lambda(\theta)p(\theta). \tag{32}$$

This describes the arithmetic aggregation rule. In a group with agents having a logarithmic utility function, the efficient rule to aggregate heterogeneous beliefs consists in computing for each state the Pareto-weighted mean of the individual subjective probabilities. If two states have the same weighted mean, they should have the same collective probability. If agents do not have a logarithmic utility, this arithmetic aggregation rule is inefficient. The associated error can be estimated by

$$\phi(z,P,dP)=\frac{d\ln v_z(z,P)}{d\ln Eq(\widetilde{\theta})}. \tag{33}$$

When ϕ is larger than unity, using the arithmetic aggregation rule would underestimate the rate of increase in the collective probability generated by the marginal shift dP. It is useful to observe that, for any increasing disagreement dP, ϕ is smaller than η, as states in the following Proposition.

Proposition 6 *Consider a marginal shift in distribution dP that increases disagreement. It implies that $d\ln Eq(\widetilde{\theta})$ is larger than $dE\ln q(\widetilde{\theta})$, and that*

$$\phi(z,P,dP)\leq\eta(z,P,dP). \tag{34}$$

Proof: If dP yields increasing disagreement, it implies that

$$Eq(\widetilde{\theta})\ \frac{dq(\widetilde{\theta})}{q(\widetilde{\theta})} \geq Eq(\widetilde{\theta})\ E\frac{dq(\widetilde{\theta})}{q(\widetilde{\theta})},$$

or that $d\ln Eq(\widetilde{\theta}) \geq dE\ln q(\widetilde{\theta})$. The definition of η and ϕ directly implies (34). ■

When disagreement is increasing, the rate of increase of the expected individual probabilities is larger than the expected rate of increase of individual probabilities. It implies that ϕ is smaller than η, or that $\phi > 1 \Longrightarrow \eta > 1$. In words, for an increase in disagreement, requiring that the rate of increase in the collective probability be larger than the rate of increase in the mean individual probabilities *is more demanding than* it be larger than the mean rate of increase of individual probabilities. Thus DARA is necessary but not sufficient for $\phi \geq 1$.

Using equation (27), ϕ is larger than unity if and only if for all α

$$Edq(\widetilde{\theta}) = \alpha Eq(\widetilde{\theta}) \Longrightarrow E\left[\frac{T^u(c(z,P,\widetilde{\theta}))}{q(\widetilde{\theta})}dq(\widetilde{\theta})\right] \geq \alpha E\left[T^u(c(z,P,\widetilde{\theta}))\right]. \quad (35)$$

The following Lemma is useful to solve this determine the condition under which this property holds.

Lemma 2 *Consider two functions f_1 and f_2 from $X \subset R$ to R^{++}, and $g: X \to R$. Then, the following two conditions are equivalent:*

1. *For any random variable $\widetilde{\theta}$ with support in X and for any constant α, $Ef_1(\widetilde{\theta})g(\widetilde{\theta}) = \alpha Ef_1(\widetilde{\theta})$ implies that $Ef_2(\widetilde{\theta})g(\widetilde{\theta}) \leq \alpha Ef_2(\widetilde{\theta})$.*
2. *$f_1(\theta)/f_2(\theta)$ and $g(\theta)$ are comonotone in X.*

Proof: It is a direct consequence of the "diffidence theorem" in Gollier and Kimball (1996) (Proposition 11 in Gollier (2001)). ■

This Lemma generalizes the covariance rule that we used to prove Proposition 5.

Proposition 7 *Suppose that the individual utility functions are identical. The following two conditions are equivalent:*

1. *For any wealth z, any initial distribution of individual probabilities P and any shift dP yielding increasing disagreement, the arithmetic mean approach underestimates the rate of increase of the collective probability: $\phi(z, P, dP) \geq 1$;*

2. *The derivative of absolute risk tolerance with respect to consumption is larger than unity: $T_c^u(c) \geq 1$ for all c.*

Proof: Because we consider the case of increasing disagreement, we can assume without loss of generality that q and $d\ln q(\theta)$ are increasing in θ. By the first-order condition (12), c must be increasing in θ. We use Lemma 2 with $f_1(\theta) = q(\theta)$, $f_2(\theta) = T^u(c(z, P, \theta))$ and $g(\theta) = d\ln q(\theta)$. By the first-order condition (12), we have that $f_1(\theta) = \psi(z, P)/u_c(c(z, P, \theta))$. Lemma 2 requires that $f_1(\theta)/f_2(\theta) = \psi(z, P)/[u_c(c(z, P, \theta))T^u(c(z, P, \theta))]$ be increasing in θ. Because c is increasing in θ, this is true if and only if $u'(c)T^u(c)$ is increasing in c. This is the case if

$$u''(c)T^u(c) + u'(c)T_c^u(c) = u''(c)T^u(c)\left[1 - T_c^u(c)\right] \geq 0$$

for all c. This is equivalent to require that $T_c^u(c) \geq 1$.■

Again, an equivalent result can be obtained for non marginal changes in distribution. Considering the particular case in which $E\lambda(\widetilde{\theta})p(\widetilde{\theta})$ is preserved across states, we obtain the following Corollary which is due to Varian (1985) and Ingersoll (1987).

Corollary 1 *Suppose that the individual utility functions are identical. Consider two distribution P_0 and P_1 of individual probabilities, where the Pareto-weighted mean of individual probabilities is unchanged: $E\lambda(\widetilde{\theta})p_1(\widetilde{\theta}) = E\lambda(\widetilde{\theta})p_0(\widetilde{\theta})$. Suppose moreover that P_1 exhibits more disagreement than P_0. If $T_c^u(c) \geq 1$ for all c, then the collective subjective probability is larger under P_1 than under $P_0 : v_z(z, P_1) \geq v_z(z, P_0)$.*

Notice that

$$T_c^u(c) = \frac{P^u(c)}{A^u(c)} - 1 \text{ with } A^u(c) = -\frac{u''(c)}{u'(c)} \text{ and } P^u(c) = -\frac{u'''(c)}{u''(c)}. \tag{36}$$

A^u and P^u are respectively the degree of absolute risk aversion and absolute prudence. Kimball (1990) shows that absolute prudence is useful to measure the impact of risk on the marginal value of wealth. Namely, he shows that the effect of risk on the marginal value of wealth is equivalent to a sure reduction of wealth that is approximately proportional to the product of the variance of the risk by P^u. Using equation (36), the derivative of absolute risk tolerance is larger than unity if and only if absolute prudence is larger than twice the absolute risk aversion:

$$T_c^u(c) \geq 1 \Leftrightarrow P^u(c) \geq 2A^u(c). \tag{37}$$

There is a simple intuition to Corollary 1. It states that, everything else unchanged, the group should devote more effort to finance aggregate consumption in states with more disagreement if P^u is larger than $2A^u$. The paradigm of the veil of ignorance is useful for this intuition, using Proposition 4. Under the veil of ignorance, the cake sharing problem (4) is equivalent to an Arrow-Debreu portfolio problem. More disagreement in the cake sharing problem can be reinterpreted as more risk in the portfolio problem, which has two conflicting effects on the marginal value of aggregate wealth v_z. The first effect is a precautionary effect. The increase in risk has an effect on v_z that is equivalent to a sure reduction of aggregate wealth that is approximately proportional to absolute prudence. But this does not take into account of the fact that the group does rebalance consumption towards those who have a larger probability. This endogenous negative correlation between the weighted probability $q(\theta)$ and individual consumption is favorable to the expected consumption $E\lambda(\widetilde{\theta})p(s,\widetilde{\theta})c(z,s,\widetilde{\theta})$. Under the veil of ignorance, this makes the representative agent implicitly wealthier, thereby reducing the marginal value of wealth. This wealth effect is proportional to the rate at which marginal utility decreases with consumption. It is thus proportional to A^u. Globally, more disagreement raises the marginal value of wealth if the precautionary effect dominates the wealth effect, or if absolute prudence is sufficiently larger than absolute risk aversion. This provides an intuition to condition $P^u \geq 2A^u$, or $T_c^u \geq 1$.[4]

The assumption that agents have decreasing absolute risk aversion is a widely accepted hypothesis in our profession. The plausibility of condition $T_c^u \geq 1$ is much more questionable. In fact, most specialists in the field believe that T_c^u is *smaller* than unity. The argument goes as follows. In the absence of any direct estimate of the sensitivity of absolute risk tolerance to changes

[4] In the Arrow-Debreu portfolio context, Gollier (2002) shows that condition $P^u \geq 2A^u$ is necessary and sufficient for a mean-preseving spread in the distribution of state price per unit of probability to raise the marginal value of wealth.

in wealth, we usually consider the CRRA specification $u(c) = c^{1-\gamma}/(1-\gamma)$ for which $T^u(c) = c/\gamma$. It implies that $T^u_c \geq 1$ if and only if γ is smaller than unity. Relying on asset pricing data and the equity premium puzzle, one must conclude that relative risk aversion must be much larger than unity. Therefore, Proposition 7 should be interpreted by considering their contraposition with $T^u_c(c) \leq 1$! Notice that the limit case is the logarithmic utility function for which $T^u_c(c) \equiv 1$. This is the only case in which using the alternative aggregation rule based on $p^v(s) = a\ E\lambda(\widetilde{\theta})p(s,\widetilde{\theta})$ is socially efficient. When $T^u_c(c) \leq 1$, using this rule will induce the planner to demand too many Arrow-Debreu securities in states with a low level of disagreement, and not enough in states with a high level of disagreement. Observe that this conclusion is exactly opposite to the one presented in the previous subsection when the planner use the geometric aggregation rule for individual beliefs. In fact, when $0 \leq T^u \leq 1$, we get the nice property that the rate of increase of the collective probability is in between the mean rate of increase of the individual probabilities and the rate of increase of the mean probabilities:

$$Ed\ln q(\widetilde{\theta}) \leq d\ln v_z(z,P) \leq d\ln Eq(\widetilde{\theta})$$

for an increase in disagreement.

We illustrate these results by the following example. There is a continuum of states represented by $S = [0,1]$. There are two agents in the group. The beliefs of agent $\theta = 1$ are represented by the density function $p(s,1) = 2s$, whereas agent $\theta = 2$ has a density function $p(s,2) = 2-2s$. They are represented by the dashed lines in Figure 5. We assume that agents have the same state-independent utility function $u(c) = c^{1-\gamma}/(1-\gamma)$, where γ is constant relative risk aversion. We consider the Pareto efficient allocation associated to $\lambda(1) = \lambda(2)$. Increasing s from 0 first tends to reduce disagreement. Above $s = 1/2$, increasing s increases disagreement. Because $\overline{p}(s) = E\lambda(\widetilde{\theta})p(s,\widetilde{\theta}) = 1$ for all $s \in S$, these changes in disagreement preserves the mean of $q(s,\widetilde{\theta})$. We draw Figure 5 where the two plain curves are for the efficient collective densities, respectively for $\gamma = 10$ and $\gamma = 0.1$. The efficient collective density function is uniform when agents are logarithmic ($\gamma = 1$). When $\gamma = 10$, the efficient density function is hump-shaped, whereas it is U-shaped in the case of $\gamma = 0.1$. These are direct consequences of Proposition 7.

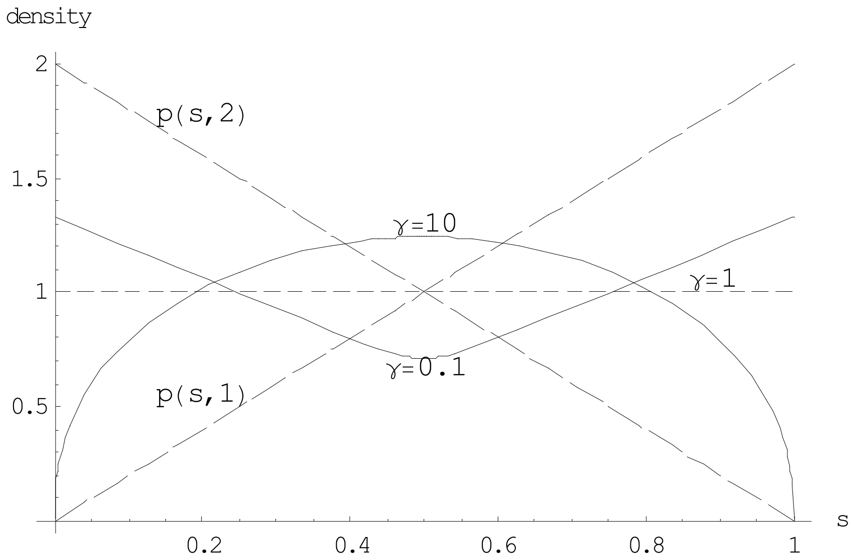

Figure 5: The collective density function for $u(c,\theta) = c^{1-\gamma}/(1-\gamma)$, $p(s,1) = 2s$, and $p(s,2) = 2-2s$.

7 The equity premium and doubts on the occurrence of a boom

What are the implications of these results on the equity premium? The equity premium is the expected excess return that is obtained at equilibrium when accepting to bear the collective risk rather than investing in the risk free asset. There is a specific difficulty to use this concept when there is no agreed-upon probability distribution for the excess return. We consider the position of the econometrician who can use a long time series of observations. The sample average state probability $\overline{q}(s)$ equals the objective probability of state s. For some reasons, investors disagree on these state probabilities, but we assume that their expectations are unbiased on average: $Eq(s,\widetilde{\theta}) = \overline{q}(s)$ for all $s \in S$.[5]

The group is here reinterpreted as the set of all consumers in an exchange economy à la Lucas (1978), which implies that the equilibrium condition

[5] This is in sharp contrast with Abel (2002) who assumes alternatively that investors share their beliefs $p(s,\theta) = p(s)\ \forall\theta$, which are systematically biased with respect to the objective probability distribution p^*.

is $\omega(s) = z(s)$ for all s, where $\omega(s)$ and $z(s)$ are respectively the mean endowment and the mean consumption in state s. The first-order condition of program (5) can be rewritten as an equilibrium condition as follows:

$$v_z(\omega(s), P(s)) = \pi(s),$$

where $\pi(s)$ is the price of the Arrow-Debreu security associated to state s. Suppose that investors have state-independent ISHARA utility functions. We know from Proposition 2 that v_z is multiplicatively separable in that case: $v_z(\omega, P) = p^v(P)h'(\omega)$. The risk free rate equals

$$R^f = \left[\int_S \pi(s)ds\right]^{-1} = \left[\int_S p^v(P(s))h'(\omega(s))ds\right]^{-1}$$

The price of equity equals

$$P^e = \int_S \omega(s)\pi(s)ds = \int_S p^v(P(s))\omega(s)h'(\omega(s))ds.$$

The objective expected payoff of equity is equal to $\overline{\mu} = \int_S \overline{q}(s)\omega(s)ds$. As in Abel (2002), the objective equity premium is thus equal to

$$\phi = \frac{\overline{\mu}}{P^e R^f} = \frac{\left[\int_S \overline{q}(s)\omega(s)ds\right]\left[\int_S p^v(P(s))h'(\omega(s))ds\right]}{\int_S p^v(P(s))\omega(s)h'(\omega(s))ds}. \quad (38)$$

When beliefs are homogeneous, $p^v(.)$ and $\overline{q}(.)$ coincide, and the calibration of the parameters in equation (38) using historical data yields an equity premium that is much smaller than the average equity premium observed on financial markets during the last century. In this section, we examine whether the heterogeneity of beliefs could explain this puzzle.

It is intuitive that the equity premium would be increased if the representative agent would perceive equity as riskier than what can be inferred by the historical data. A larger equity premium would then be necessary to compensate for the larger subjective risk. Consider the example summarized in Figure 5. Suppose that aggregate state wealth $\omega(s)$ equals s for all s, so that the horizontal axis in this figure measures wealth. The objective distribution of wealth is therefore uniform on [0,1] in this example. When relative risk aversion is larger than unity, the divergence of opinions tends to reduce the probability of the extreme states, and to raise the probability of medium-wealth states. This means that the representative agent perceives the collective risk as *smaller* than what is given by the objective distribution. Following Proposition 7, this is due to the fact that the divergence of opinions is strongest for the probability of the extreme events. Under the assumption that the derivative of absolute risk aversion is less than unity, this reduces the probability of these events, thereby reducing the risk perceived by the representative agent. Solving the equity premium puzzle would rather require concentrating the divergence of opinions to the medium-wealth states. This is not the most plausible assumption that one can do about the distribution of beliefs on financial markets. We will therefore not pursue this line of research.[6]

Let us alternatively assume that most of the divergence of opinions is concentrated on the probability of occurrence of a boom. The structure of individual beliefs are similar to those described in Figure 6. Suppose that $\omega(s) = s$, and assume that $S =]a, b]$. There is a boom if $s \in [b - \Delta, b]$. Individual θ believes that the probability of a boom state s is $\theta\, \overline{q}(s)$, where $\overline{q}(s)$. For all $s < b - \Delta - \varepsilon$, the subjective probability of state s for agent θ equals $(1-\theta k)\, \overline{q}(s)/(1-k)$, where $k = \int_{b-\Delta}^{b} \overline{q}(s)ds$ is the objective probability of a boom. In the small interval $I = [b - \Delta - \varepsilon, b - \Delta[$, the individual beliefs are selected in order to guarantee that $p(s, \theta)$ be differentiable with respect to s. By construction, we have that $d\ln q(s, \theta) = d\ln \overline{q}(s)$ for all θ and for all s except in interval I. Using equation (27), this implies that

$$d\ln p^v(P(s)) = d\ln \overline{q}(s)$$

for all s except in interval I. We assume that k and Δ are small, so that most of the disagreement is concentrated in the boom probability. It implies that the degree of disagreement is decreasing in interval I. Under the assumption that $T_c^u \leq 1$, Proposition 7 implies that

$$d\ln p^v(P(s)) \leq d\ln \overline{q}(s)$$

in this interval. To sum up, the rate of increase in the collective probability is everywhere equal to the rate of increase in the mean probability, except in interval I where it is smaller. It implies that the representative agent has

[6] An additional reason for this is that it is in general not true that an increase in the equity risk reduces the demand for equity by all risk-averse investors, thereby reducing the equilibrium price of equity. See Rothschild and Stiglitz (1971), Gollier (1995) and Abel (2002).

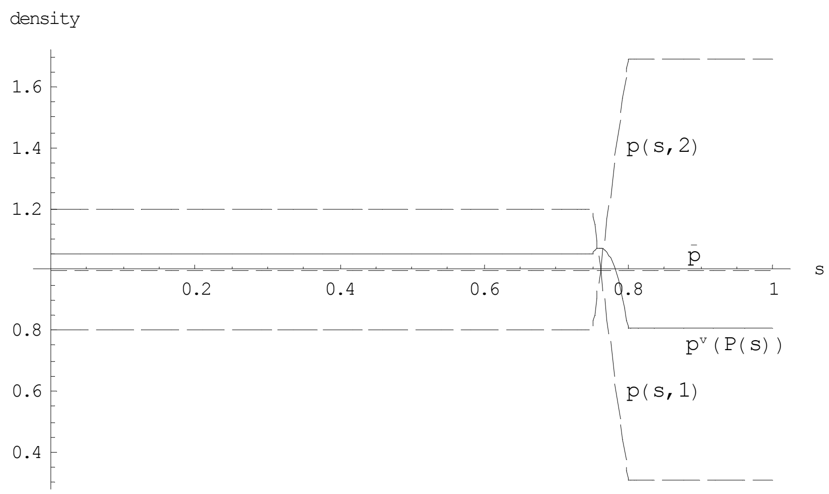

Figure 6: Most of the divergence of opinions is concentrated on the probability of occurence of a boom. Relative risk aversion γ equals 10. The beliefs p^v of the representative agent is first-order stochastically dominated by the mean beliefs $\overline{p}$.

beliefs that are first-order stochastically dominated by the objective probability distribution. More specifically, the representative agent uses a density function p^v such that

$$p^v(P(s)) = \begin{cases} k_1\ \overline{q}(s) \geq\ \overline{q}(s) & if \quad s \leq b - \Delta - \varepsilon \\ k_2\ \overline{q}(s) \leq\ \overline{q}(s) & if \quad s \geq b - \Delta. \end{cases} \tag{39}$$

with $k_1 = (1 - k_2 k)/(1 - k) \geq 1$. Perceiving equity returns as less favorable, the representative agent would reduce his demand for equity, thereby increasing the equilibrium equity premium. Figure 6 illustrates such a phenomenon.

Proposition 8 *Suppose that most of the divergence in opinions is concentrated in the probability of occurrence of a boom. Suppose that agents have CRRA preferences $u(c) = c^{1-\gamma}/(1-\gamma)$ with $\gamma > 1$. It implies that the heterogeneity of beliefs raises the equity premium.*

Proof: It remains to prove that a transfer of probability mass from the wealthiest state to the other states raises the equity premium ϕ. In Appendix B, we prove that $P^e R^f$ is reduced by a change in the probability density

p^v that transfers probability mass from any state with wealth per capita $\omega \geq \overline{\omega}$ uniformly the other states, as in (39). This critical wealth level $\overline{\omega}$ is strictly smaller than the upper bound of the support of the wealth distribution. It implies that the equity premium $\phi = \mu / P^e R^f$ is increased by the heterogeneity of beliefs when the divergence of opinions is concentrated in the probability of occurrence of the wealthiest state.■

In the rest of this section, we consider a credible calibration exercise whose objective is to show that the effect of divergent opinions may have a strong impact on the equity premium. We assume that all agents have the same utility function with a constant relative risk aversion γ equaling 4. Suppose that agents form expectations for the future growth rate of the economy from observing the realized growth rates of real GDP per capita in the U.S.A. over the 70's and 80's (source Penn-World data). The mean growth rate has been 1.72% per year, and the standard deviation equalled 2.48%. This is not far from the historical mean (1.8%) and standard deviation (3.56%) that have been reported by Kocherlakota (1996) for the period 1889-1978. In a situation where agents agree on the probability distribution that would attach a probability 1/20 to each of the 20 observations, the equilibrium equity premium would equal 0.25% per year, far below the average equity premium of 6% observed during the last century.

Suppose alternatively that agents disagree on the probability distribution of the growth rate of the economy. We gathered the five best years of these two decades, for which the growth rate exceeded 3.7%. These states are referred to as boom states, whereas the other states are referred to as "normal" states . From our data, the objective probability of a boom equals 1/4. The objective probability of the occurrence of a normal state is 3/4. But, for some reason, agent of type θ attaches a probability θ not necessarily equal to 3/4 to the occurrence of a normal state. The probability of occurrence of a boom is the only source of divergence of opinions. This means in particular that all agents agree on the probability of 1/5 (1/15) of each of the observed growth rates conditional to a (no-)boom. We stress the fact that our simulations preserve the mean expectations which are equal to the objective ones. More precisely, we assume that the mean $\widetilde{\theta}$ is 3/4. There are two groups of equal size and with the same Pareto-weights. The members of the optimistic group believe that the probability of occurrence of a normal state is θ_m that is smaller than 3/4 , whereas the members of the optimistic group believes that the probability of occurrence of a normal state is $1.5 - \theta_m > 3/4$. Consider in particular case $\theta_m = 1/2$, in which case the optimistic group believes

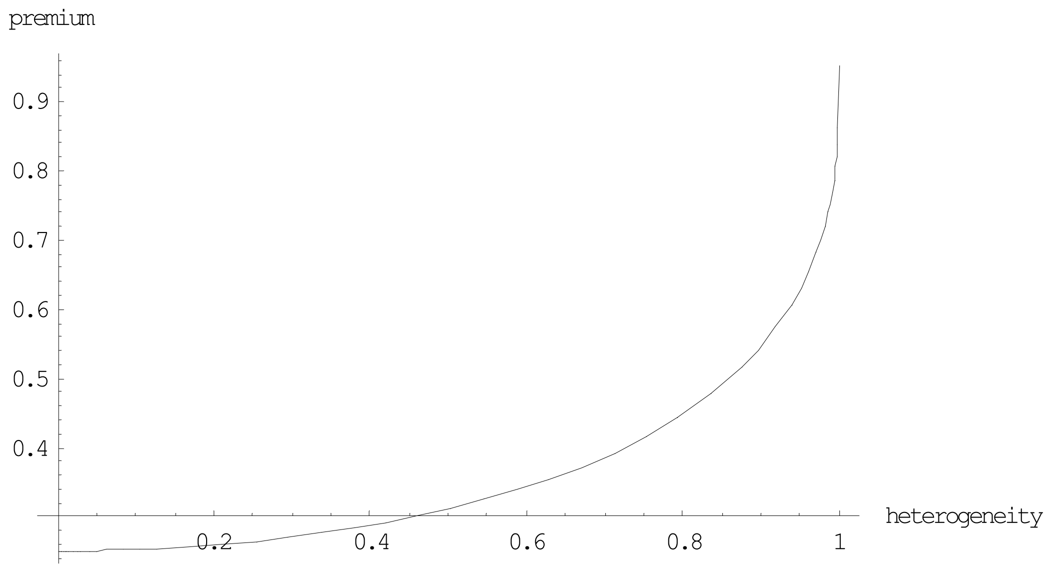

Figure 7: The equity premium in percent per year as a function of the degree of heterogeneity. The degree of heterogeneity is unity when $\widetilde{\theta}$ is distributed as $(0.5, 1/2; 1, 1/2)$. It is zero for $\widetilde{\theta} \sim (0, 75, 1/2; 0.75, 1/2)$. More generally, the degree of heterogeneity is measured by $3 - 4\theta_m$.

that a boom will occur with certainty, whereas the optimistic group believes that this event will occur only with probability 1/2. With such a strong divergence of opinions on the occurrence of a boom, the equilibrium equity premium goes up to 0.96%. It is far to explain the equity premium puzzle, but it multiplies almost by a factor 4 the equity premium that would have been observed without heterogeneous beliefs. In Figure 7, we describe the equity premium as a function of the degree of divergence of opinions. We see that the equity premium increases with the degree of heterogeneity at an increasing rate. The details of our computations are presented at the end of Appendix C.

We could have alternatively assumed that most of the divergence of opinions is concentrated on the probability of occurrence of a krach. Under the assumption that the derivative of absolute tolerance is less than unity, this would tend to reduce the equity premium. Why shouldn't we take into account of the extremely optimistic agents who believe that the likelihood of a krach is high? Just because these agents will purchase insurance against the occurrence of the krach state. At the limit, they will stay out of the equity market. Only the optimistic agents will participate. That will have a negative impact on the equity premium.

8 The two-state case

In this section, we consider a very specific situation in which all agents agree on the fact that there are only two states of nature, s_1 and s_2, but they disagree on the probability of these two states. This would be a natural modeling to examine earthquake insurance markets for example. None of the results presented earlier in this paper is helpful to describe the binary case. In particular, the notion of increasing disagreement, which has been instrumental to characterize the effect of the heterogeneity of beliefs on their aggregation, is useless in the binary case. This is because $p(s_1,\widetilde{\theta})$ and $p(s_2,\widetilde{\theta})$ must have the same degree of disagreement, since $p(s_2,\widetilde{\theta}) = 1 - p(s_1,\widetilde{\theta})$.

Suppose that the agents have an identical state-independent utility function. Returning to the sources of the original problem, we can use the first-order condition (12) of the cake-sharing problem (4) together with the feasibility constraint to characterize the marginal value of wealth by the following condition:

$$z = Ef\left(\frac{q(s_i,\widetilde{\theta})}{v_z(z,P(s_i))}\right),$$

where function f is defined as $u'(f(1/t)) = t$. Notice that f is increasing and that it is concave (convex) if and only if T_c^u is uniformly smaller (larger) than unity. One can define the collective probability distribution as follows:

$$p_i^v = \frac{v_z(z,P(s_i))}{v_z(z,P(s_1)) + v_z(z,P(s_2))}.$$

The collective state probabilities are nonnegative and they sum up to unity. In this section, we are interested in determining whether the collective state probability be larger than the corresponding Pareto-weighted mean of individual state probabilities, i.e., whether

$$p_i^v \geq \frac{Eq(s_i, \widetilde{\theta})}{Eq(s_1, \widetilde{\theta}) + Eq(s_2, \widetilde{\theta})}.$$

Proposition 9 *Suppose that there are only two states of nature. Suppose also that individual Pareto-weighted state probabilities are symmetrically distributed around their mean, which implies that $q(s_i, \widetilde{\theta}) = \overline{q}_i + \widetilde{\varepsilon}$, with $E\widetilde{\varepsilon} = 0$. If all agents have the same utility function u with $T_c^u(c) \geq 1$ for all c, then we have that*

$$p_1^v \begin{cases} \geq \dfrac{Eq(s_1, \widetilde{\theta})}{Eq(s_1, \widetilde{\theta}) + Eq(s_2, \widetilde{\theta})} \leq 0.5 & if \quad Eq(s_1, \widetilde{\theta}) < Eq(s_2, \widetilde{\theta}) \\ = \dfrac{Eq(s_1, \widetilde{\theta})}{Eq(s_1, \widetilde{\theta}) + Eq(s_2, \widetilde{\theta})} = 0.5 & if \quad Eq(s_1, \widetilde{\theta}) = Eq(s_2, \widetilde{\theta}) \\ \leq \dfrac{Eq(s_1, \widetilde{\theta})}{Eq(s_1, \widetilde{\theta}) + Eq(s_2, \widetilde{\theta})} \geq 0.5 & if \quad Eq(s_1, \widetilde{\theta}) > Eq(s_2, \widetilde{\theta}). \end{cases}$$

These inequalities are reversed if $T_c^u(c)$ is smaller than unity for all c.

Proof: Suppose that $Eq(s_1, \widetilde{\theta}) < Eq(s_2, \widetilde{\theta})$, or $\overline{q}_1 < \overline{q}_2$. Define function ξ in such a way that

$$z = Ef\left(\frac{x + \widetilde{\varepsilon}}{\xi(x)}\right)$$

for all x. By definition, we have that $v_z(z, P(s_i)) = \xi(\ \overline{q}_i)$. Observe that

$$p_1^v = \frac{\xi(\ \overline{q}_1)}{\xi(\ \overline{q}_1) + \xi(\ \overline{q}_2)} \geq \frac{\overline{q}_1}{\overline{q}_1 +\ \overline{q}_2}$$

if and only if

$$\frac{\xi(\ \overline{q}_1)}{\overline{q}_1} \geq \frac{\xi(\ \overline{q}_2)}{\overline{q}_2}.$$

A sufficient condition for this condition to hold is that $\xi(x)/x$ be decreasing in x, or $x\xi'(x) - \xi(x) \leq 0$. By definition of function ξ, this is true if

$$x \frac{Ef'\left(\dfrac{x + \widetilde{\varepsilon}}{\xi(x)}\right)}{E\dfrac{x + \widetilde{\varepsilon}}{\xi(x)} f'\left(\dfrac{x + \widetilde{\varepsilon}}{\xi(x)}\right)} - \xi(x) \leq 0,$$

or, equivalently, if

$$E\widetilde{\varepsilon}f'\left(\frac{x+\widetilde{\varepsilon}}{\xi(x)}\right) \geq 0.$$

Because $E\widetilde{\varepsilon} = 0$ and $\xi(x) \geq 0$, this is true if and only if f is convex, or $T_c^u \geq 1$.■

Because by construction the degrees of disagreement are the same in the two states, these results are different in nature from what we obtained in section 6.3 where we compared two states that differ only on their degrees of disagreement. This is in spite of the fact that the technical conditions on the utility function are the same for the two sets of results.

When the two states are equally likely on the basis of the Pareto-weighted mean individual beliefs, i.e., when $Eq(s_1, \widetilde{\theta})$ equals $Eq(s_2, \widetilde{\theta})$, the social planner should assume that the two states are equally likely. This is a situation where the disagreement on the probability of state s_1 exactly counterbalances the disagreement on the probability of state s_2. In fact, this neutrality result that is specific to that distribution of beliefs does not rely on any condition on the utility function. Notice that the assumption that $p(s_i, \widetilde{\theta})$ is symmetrically distributed around its mean is essential here, since it implies that $p(s_2, \widetilde{\theta})$ is obtained from $p(s_1, \widetilde{\theta})$ by a translation. The structure of disagreement must be exactly the same for the two states. When this symmetry assumption is relaxed, the first and third moments of these two random variables will differ, yielding a third-order effect.

Suppose alternatively that, on the basis of the Pareto-weighted mean of individual beliefs, state s_1 is less likely than state s_2: $Eq(s_1, \widetilde{\theta}) \leq Eq(s_2, \widetilde{\theta})$. Under the realistic assumption that $T_c^u < 1$, it implies that the collective subjective probability of state s_1 should be smaller than the mean one. Reciprocally, when state s_1 is more likely than state s_2 on average, the collective probability of state s_1 must be larger than the mean one. In a word, when $T_c^u < 1$, the social planner should be an extremist. Or, he should bias his beliefs towards certainty. He should raise the larger probability and, symmetrically, he should reduce the smaller one. On the contrary, when T_c^u is larger than unity, the representative agent biases probabilities towards probability 0.5.

9 Conclusion

Our aim in this paper was to characterize the probability distribution that should be used for collective decision making when individuals differ about their expectations. To examine this question, we assumed that agents can share risk efficiently, thereby relying on techniques borrowed from the theory of finance. The basic ingredient behind our results is that, in aggregating individual beliefs, one should favor the beliefs of agents that bear a larger share of the risk. However, the allocation of risk in the economy is endogenous and it depends upon the individual beliefs. Therefore, efficient risk allocations are more difficult to characterize under expectations disagreement. For example, it is not necessarily efficient to wash out diversifiable risks in that case. It may be efficient for agents to gamble against each others in spite of their risk aversion. Horsetrack betting is Pareto-improving when agents have different beliefs about the chances of the competing horses.

In an Arrow-Debreu framework, the risk exposure of an individual is a local concept that is measured by local differences in state consumption levels across states. As is well-known, the socially efficient local risk exposure for an agent is proportional to his local degree of absolute risk tolerance which measures the rate at which marginal utility decreases with consumption. We showed that this result remains true with heterogeneous beliefs. The basic property of the aggregation of beliefs is that an increase in the subjective probability of agent θ should raise the collective probability also proportionally to agent θ's degree of absolute risk tolerance. If an agent bears a share x of the collective risk, a one percent increase in his subjective probability should raise the collective probability by $x/100$ percents. This result has several important consequences.

First, it implies that the socially efficient collective probability distribution depends upon the aggregate wealth level of the group. This is because the aggregate wealth level affects the way risks should be allocated in the group. However, when agents have the same HARA utility function, changes in aggregate wealth has no effect on the allocation of risks. This implies that the collective probability distribution is independent of wealth in that case. We showed that the identically-sloped HARA case is the *only* case in which such separability property between beliefs and utility holds. In all other cases, the representative agent has a state-dependent utility function.

Second, we derived various results that are useful to understand the effect of the divergence of opinions on the shape of the collective probability distribution. To do this, we defined the concept of increasing disagreement. In short, there is more disagreement about the probability of state s' than about the probability of state s if the individual subjective probabilities are more dispersed in state s' than in state s. We showed that, with such a shift in the distribution of individual probabilities, the rate of increase of the collective probability is larger than the mean rate of increase of individual probabilities if and only if absolute risk aversion is decreasing. We also showed that it is smaller than the rate of increase of the mean probability if and only if the derivative of absolute risk tolerance is smaller than unity, a plausible hypothesis on preferences. It must be stressed that these results are purely local. They do not provide a global view about how the beliefs of the representative agent are affected by the heterogeneity of beliefs.

The last step is to link the structure of disagreement at the global level to the global properties of the collective probability distribution. When most disagreements are concentrated in the wealthier states, the collective distribution function is dominated by the average individual probability distribution in the sense of first-order stochastic dominance. This tends to raise the equity premium. We showed in a simple numerical example that the heterogeneity of individual beliefs may have a sizeable effect on the equity premium.

The critical assumption of this model is that the group can allocate risk efficiently. This assumption is difficult to test. For example, the efficient coverage of earthquake coverage in various regions can be interpreted in two ways. The optimistic view is that homeowners are more pessimistic than insurers about the risk, which implies that the low insurance coverage is socially efficient. But alternatively, it could be interpreted as a proof that markets are incomplete. A similar problem arises to explain the insurance crisis after 9/11/01, or about the difficulty to share the risk related to global warming on an international basis. A possible extension of this work would be to consider an economy with incomplete markets.

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Appendix A: The case of ISHARA preferences

In this appendix, we examine the special case of ISHARA preferences (22) for which we know from proposition 2 that R is independent of z. It implies that $v_z(z, P)$ is separable into a product $p^v(P)h'(z)$. Moreover, ISHARA preferences (22) yield an analytical solution for the aggregation problem. Indeed, in this particular case, the first-order condition to state-dependent the Pareto program (4) implies that

$$c(z, P, \theta) - a(\theta) = k\left[\lambda(\theta)p(\theta)\right]^{1/\gamma}.$$

Since $T^u(c, \theta) = (c - a(\theta))/\gamma$, property (15) can be rewritten in the ISHARA case as

$$R(z, P, \theta) = \frac{\left[\lambda(\theta)p(\theta)\right]^{1/\gamma}}{NE\left[\lambda(\widetilde{\theta})p(\widetilde{\theta})\right]^{1/\gamma}}. \tag{40}$$

The definition of R applied to the ISHARA case implies that

$$R(z, P, \theta) = \frac{p(\theta)p^v_\theta(P)}{p^v(P)}, \tag{41}$$

where $p^v_\theta = \partial p^v / \partial p(\theta)$. Combining (40) and (41) yields

$$\frac{p^v_\theta(P)}{p^v(P)} = \frac{\lambda(\theta)^{1/\gamma}p(\theta)^{-1+1/\gamma}}{NE\left[\lambda(\widetilde{\theta})p(\widetilde{\theta})\right]^{1/\gamma}} \tag{42}$$

for $\theta = 1, ..., N$. The solution to this system of partial differential equations has the following form:

$$p^v(P) = C \quad \left[E\left[\lambda(\widetilde{\theta})p(\widetilde{\theta})\right]^{1/\gamma}\right]^\gamma, \tag{43}$$

where C is a constant. In order for p^v to be a probability distribution, we need to select the particular solution with

$$p^v(P) = \frac{\left[E_{\widetilde{\theta}}\left[\lambda(\widetilde{\theta})p(s, \widetilde{\theta})\right]^{1/\gamma}\right]^\gamma}{\int_S \left[E_{\widetilde{\theta}}\left[\lambda(\widetilde{\theta})p(t, \widetilde{\theta})\right]^{1/\gamma}\right]^\gamma dt}. \tag{44}$$

Calvet, Grandmont and Lemaire (2001) obtained the same solution. Rubinstein (1974) derives it in the special cases $\gamma = 1$ and $\gamma = +\infty$.[7] Thus, in the ISHARA case, we can directly compute the socially efficient probability distribution of risk as a function of individual beliefs p, the Pareto weights λ, and the concavity coefficient γ. Two special cases are worthy to examine. Consider first the case with γ tending to zero. This corresponds to risk-neutral preferences above a minimum level of subsistence. Under this specification, condition (44) is rewritten as

$$p^v(P(s)) = p^n(P(s)) =_{def} \frac{\max_{\theta\in\Theta}\lambda(\theta)p(s,\theta)}{\int_S \max_{\theta\in\Theta}\lambda(\theta)p(t,\theta)dt} \quad \text{for all } s. \text{ (risk-neutral case)}$$

With risk-neutral preferences, the efficient allocation produces a flip-flop strategy where the cake in state s is entirely consumed by the agent with the largest Pareto-weighted probability associated to that state. It implies that the group will use a state probability p^n proportional to it to determine its attitude toward risk ex ante.

In the case of logarithmic preferences ($\gamma = 1$), the denominator in (44) equals $E\lambda(\widetilde{\theta})$ since

$$\int_S E_{\widetilde{\theta}}\lambda(\widetilde{\theta})p(t,\widetilde{\theta})dt = E_{\widetilde{\theta}}\left[\lambda(\widetilde{\theta})\int_S p(t,\widetilde{\theta})dt\right] = E\lambda(\widetilde{\theta}) = 1.$$

It implies that

$$p^v(P(s)) = p^{\ln}(P(s)) =_{def} E\lambda(\widetilde{\theta})p(s,\widetilde{\theta}) \quad \text{for all } s. \text{ (logarithmic case)}$$

With these Bernoullian preferences, the efficient probability that should be associated to any state s is just the weighted mean $p^{\ln}(s)$ of the individual subjective probabilities of that state s. This is the limit case $T_c^u \equiv 1$ of the result presented in Proposition 7.

Appendix B: Change in beliefs and the equity premium

[7] When γ tends to infinity, we obtain the following aggregation formula:

$$p^v(s) = \frac{\exp E\ln p(s,\widetilde{\theta})^{a(\widetilde{\theta})/Ea(\widetilde{\theta})}}{\int_S \exp E\ln p(t,\widetilde{\theta})^{a(\widetilde{\theta})/Ea(\widetilde{\theta})}dt}.$$

In this Appendix, we show that a uniform transfer of probability mass from a high wealth state reduces the future price of equity $P^e R^f$. It implies that the equity premium $\phi = \mu / P^e R^f$ is increased.

Proposition 10 *Consider a random variable $\widetilde{y}(p)$ which is distributed as $(\widetilde{x}, p; x_0, 1-p)$, with $x_0 \in R$ and $p \in [0,1]$. Suppose that the real-valued utility function h is increasing and concave. The future price of equity*

$$FP^e(p) = P^e(p) R^f(p) = \frac{E\widetilde{y}(p) h'(\widetilde{y}(p))}{Eh'(\widetilde{y}(p))}$$

is decreasing in p if and only if x_0 is larger than $\overline{x} = E\widetilde{x}h'(\widetilde{x})/Eh'(\widetilde{x})$, which is the risk-neutral mean of $\widetilde{x}$.

Proof: $FP^e(p)$ is decreasing in p if and only if

$$\left[E\widetilde{x}h'(\widetilde{x}) - x_0 h'(x_0)\right] Eh'(\widetilde{y}) - \left[Eh'(\widetilde{x}) - h'(x_0)\right] E\widetilde{y}h'(\widetilde{y}) \leq 0,$$

where $\widetilde{y}$ is distributed as $(\widetilde{x}, p; x_0, 1-p)$. This condition can be rewritten as

$$p\left[h'(x_0)E\widetilde{x}h'(\widetilde{x}) - x_0 h'(x_0)Eh'(\widetilde{x})\right] + (1-p)\left[h'(x_0)E\widetilde{x}h'(\widetilde{x}) - x_0 h'(x_0)Eh'(\widetilde{x})\right] \leq 0,$$

or, equivalently, as

$$h'(x_0)\left[E\widetilde{x}h'(\widetilde{x}) - x_0 Eh'(\widetilde{x})\right] \leq 0.$$

This is true if and only if x_0 is larger than $\overline{x} = E\widetilde{x}h'(\widetilde{x})/Eh'(\widetilde{x})$. ■

Appendix C: Numerical illustration

In this appendix, we provide the details of the numerical illustration presented at the end of section 7. We assume that all agents have the same state-independent utility function $u(c) = -c^{-3}/3$, which yields a constant relative risk aversion equaling $\gamma = 4$. We consider the Pareto-efficient solution with equal weights. The percentage yearly growth rate of real GDP per capita for period 1970-1990 in the United States is

$$\begin{aligned} D \log GDP \;\; = \;\; & (2.04, 3.74, 4.4, -2.14, -2.89, 4.13, 3.78, 3.94, 1.26, -0.02, \\ & 1.34, -4.06, 2.57, 5.94, 1.90, 1.65, 1.96, 2.96, 2.13, -0.23) \end{aligned}$$

(source Penn-World data). Let $\widetilde{\omega}_1$ be the uniformly distributed and discrete random variable whose support is given by the 15 smallest elements of $D\log GDP$. Similarly, let $\widetilde{\omega}_2$ be the uniformly distributed and discrete random variable whose support is given by the 5 largest elements of $D\log GDP$. The "objective" risk on the growth rate is the random variable $\widetilde{\omega}$ which is distributed as $(\widetilde{\omega}_1, 3/4; \widetilde{\omega}_2, 1/4)$. Its mean equals $E\widetilde{\omega} = \overrightarrow{\mu} = 1.0172$, which implies a mean growth rate of real GDP per capita of 1.72% per year during the period. When all agents have the same beliefs that is represented by $\widetilde{\omega}$, the equity premium that sustains the corresponding Pareto-efficient allocation equals

$$\phi = \frac{E\widetilde{\omega}Eu'(\widetilde{\omega})}{E\widetilde{\omega}u'(\widetilde{\omega})} = 1.0025,$$

or 0.25% per year.

We now consider an economy with heterogenous beliefs. Agent θ has beliefs on the growth rate per capita that are distributed as $\widetilde{y}(\theta) \sim (\widetilde{y}_1, \theta; \widetilde{y}_2, 1-\theta)$. Let types be distributed as $\widetilde{\theta} \sim (\theta_m, 1/2; 1.5-\theta_m, 1/2)$, so that the mean type is $E\widetilde{\theta} = 3/4$. It means that individual expectations are equal to $\widetilde{y}$ on average. It implies that $p(\omega, \theta) = \theta/15$ if ω is in the support of $\widetilde{\omega}_1$, and $p(\omega, \theta) = (1-\theta)/5$ if ω is in the support of $\widetilde{\omega}_2$. From Appendix B, we know that the beliefs $p^v(.)$ of the representative agent is such that

$$p^v(P(\omega)) = \frac{\left[(\theta_m/15)^{1/\gamma} + ((1.5-\theta_m)/15)^{1/\gamma}\right]^\gamma}{15\left[(\theta_m/15)^{1/\gamma} + ((1.5-\theta_m)/15)^{1/\gamma}\right]^\gamma + 5\left[((1-\theta_m)/5)^{1/\gamma} + ((\theta_m-0.5)/5)^{1/\gamma}\right]^\gamma}$$

if ω is in the support of $\widetilde{\omega}_1$, and

$$p^v(P(\omega)) = \frac{\left[((1-\theta_m)/5)^{1/\gamma} + ((\theta_m-0.5)/5)^{1/\gamma}\right]^\gamma}{15\left[(\theta_m/15)^{1/\gamma} + ((1.5-\theta_m)/15)^{1/\gamma}\right]^\gamma + 5\left[((1-\theta_m)/5)^{1/\gamma} + ((\theta_m-0.5)/5)^{1/\gamma}\right]^\gamma}$$

if ω is in the support of $\widetilde{\omega}_2$. It simplifies to

$$p^v(P(\omega)) = \begin{cases} k_1/15 & \text{if } \ \omega \text{ is in the support of } \ \widetilde{\omega}_1; \\ (1-k_1)/5 & \text{if } \ \omega \text{ is in the support of } \ \widetilde{\omega}_2. \end{cases},$$

with

$$k_1 = \frac{\left[{\theta_m}^{1/\gamma} + (1.5-\theta_m)^{1/\gamma}\right]^\gamma}{\left[{\theta_m}^{1/\gamma} + (1.5-\theta_m)^{1/\gamma}\right]^\gamma + \left[(1-\theta_m)^{1/\gamma} + (\theta_m-0.5)^{1/\gamma}\right]^\gamma}$$

It yields the following pricing formula:

$$\phi = \frac{E\widetilde{\omega}.\left[k_1 E\widetilde{\omega}_1^{-\gamma} + (1-k_1)E\widetilde{\omega}_2^{-\gamma}\right]}{\left[k_1 E\widetilde{\omega}_1^{1-\gamma} + (1-k_1)E\widetilde{\omega}_2^{1-\gamma}\right]}$$

For example, when $\widetilde{\theta}$ is distributed as $(0.5, 1/2; 1, 1/2)$, i.e., when $\theta_m = 0.5$, we obtain that $k_1 \simeq 0.96$. This is to be compared to $k_1 = 0.75$ in the homogenous case with $\theta_m = 0.75$. This represents a massive transfer of probability mass to the lower wealth states compared to the objective distribution. Because

$$E\widetilde{\omega}_1^{-\gamma} = 0.973, \quad E\widetilde{\omega}_2^{-\gamma} = 0.894, \quad E\widetilde{\omega}_1^{1-\gamma} = 0.978, \quad E\widetilde{\omega}_2^{1-\gamma} = 0.878,$$

we conclude that $\phi = 1.0096$, yielding an equity premium equaling 0.96%. In Figure 7, we evaluate the equity premium for other values of θ_m.