Are independent optimal risks substitutes?

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Abstract

We examine the conditions under which independent risks are substitutes or complements. They are substitutes if the opportunity to invest in one risky project reduces the optimal investment in any other independent risky project. We obtain the following two results on the substitutability of independent risks. A necessary condition is that absolute risk aversion be convex, which is a sensible hypothesis on preferences. A sufficient condition is that absolute prudence be decreasing and less than twice the absolute risk aversion.

1 Introduction

How is the optimal exposure to one risk affected by the presence of other risks? This question is relevant in many circumstances e.g. for entrepreneurs who face different non-exclusive risky investment projects, portfolio managers who can invest in different currencies and sectors, or for policyholders who must select their level of insurance deductibles for their two cars.

The characterization of the optimal behavior of a risk-averse expected-utility (EU) maximizer toward a single risk is well documented in the literature, since Pratt (1964) and Arrow (1971). But progresses of the analysis for situations with multiple sources of risk have been extremely slow. Because of the nonlinearities of the EU objective function with respect to contingent payoffs, adding a second risky project to an initial one has a complex effect on the demand for it. This is the case even when these risks are independent, except when absolute risk aversion is constant since in this case the demands for the different risks are independent of each other.

Most people would argue that independent risks should be substitutes in the sense that the opportunity to invest in one risk should reduce the demand for other independent risks. For example, reducing the barriers to the ability of citizens of one country to invest in other countries should reduce the size of their investments in their own country.

The idea that independent risky projects are substitutes has not received the imprimatur of the theory, however. In fact, additional restrictions to the EU model are necessary to sustain this idea. For instances, to analyze the substitutability between independent take-it-or-leave-it risks, Pratt and Zeckhauser (1987) had to introduce the notion of proper risk aversion. Risk aversion is proper if an undesirable risk cannot become desirable if another undesirable risk is added to wealth. To simplify, this means that two independent risks cannot be complement. In the EU paradigm, not all increasing and concave utility functions u are proper. For example, a necessary condition for properness is that the fourth derivative of the utility function be negative, and that absolute risk aversion be decreasing. Ross (1999) found the terminology of Pratt and Zeckhauser (1987) improper. As Ross claims, another sensible property would be that a desirable risk can never be made undesirable by the addition of another desirable risk. In a sense, by suggesting that this would be a proper property of preferences under uncertainty, Ross rejects the idea that independent risks can be substitutes. Empirical researches are needed to check whether people have "proper" risk aversion.

In their papers, Pratt and Zeckhauser (1987) and Ross (1999) considered the case of fixed size lotteries in the face of which the decision maker can either accept or reject each lottery. In the present paper, we suppose alternatively that the decision-maker can decide the size of his exposure to each of the potential risks that he faces. This is the case for example when he can purchase more than one share of a given stock, or when he can select the level of insurance deductibility to cover a risk of damage. Then the question of the substitutability of independent risks is whether the opportunity to take an optimal exposure to one risk does reduce the optimal exposure to any other independent risk.

The bottom line of the controversy on risk substitutability is that there is no obvious argument for or against such a form of risk substitutability, either on the ground of common sense or on the theoretical ground. The absence of any such strong argument results from the existence of two contradictory effects of the option to invest in a risk. Indeed, introducing a new asset has a wealth effect and a pure risk effect. The wealth effect comes from the increase in the expected wealth generated by investing in this new risky asset. If investing in it would have a negative expected return, no risk-averse investor would participate. Under decreasing absolute risk aversion, this increase in expected wealth has a positive effect on the demand for other independent risks. If this would be the only effect, independent risky assets would be complements. But the opportunity to invest in a new risky asset has not the same comparative statics effect as just adding its certainty equivalent to the initial wealth of the investor. It also introduces a zero-mean, or pure, risk to background wealth. Gollier and Pratt (1996) examined the effect of a zero-mean background risk on the demand for stocks. They showed that it has a tempering effect if the agent is "risk vulnerable", a condition on preferences that is more general than properness. Thus, the pure risk effect is compatible with the substitutability of independent risks. It implies that risk vulnerability is a necessary condition for the substitutability of independent risks. Whether the effect of risk vulnerability dominates the wealth effect or not is the open question that we address in this paper.[1]

The ambition of this paper is to characterize the conditions under which

[1]It is important to stress the difference between our approach and that adopted in earlier papers on the subject (e.g. Sandmo (1969) and Dalal (1983)) where substitutability between assets is studied in the framework of Slutsky equation so that wealth effect is eliminated. In the present paper, we examine the total impact of introducing a new asset upon the demand for another asset.

the option to purchase one risk reduces or increases the optimal exposure to another independent risk. Relating these conditions to other known properties of the optimal risk-taking behavior will provide arguments for or against the hypothesis that independent risks are substitutes.

It is a tradition in economics to isolate each decision problem from the remaining environment of the decision maker. For example, when we study the level of optimal deductible of insurance, we do not take into account the policyholder's opportunity to invest in stocks. When we examine the socially efficient policy to reduce the risk of global warming, we do not consider the other risks surrounding the population. If we can confirm that independent risks are substitutes, this would tell us that such a procedure leads to an overestimation of the optimal risk exposure. An application of this idea is to the equity premium puzzle (Mehra and Prescott (1985), Kocherlakota (1996)). If the calibrator assumes that US citizens have no other opportunities than to invest in US stocks and bonds, he would overestimate the demand for US stocks, which would yield in turn an underestimation of the equity premium.

2 The effect of the introduction of an individually optimal risk

2.1 Definition

In this section, we consider a model with three assets: one is risk free with a zero return, whereas the other two assets are risky with independent returns $\widetilde{x}$ and $\widetilde{y}$, respectively. How does the opportunity to invest in $\widetilde{x}$ affect the optimal exposure to risk $\widetilde{y}$? We consider an expected-utility-maximizer with an increasing and concave utility function u. We also assume that u has three continuous derivatives. The decision problem for the agent with initial wealth z is written as

$$\max_{\alpha,\beta} \quad K(\alpha,\beta) = Eu(z+\alpha\widetilde{x}+\beta\widetilde{y}). \tag{1}$$

The domain of u is R^+, and we assume that $\lim_{z\to 0} u'(z) = +\infty$. This implies that the optimal solution is bounded.

We say that $\widetilde{x}$ and $\widetilde{y}$ are *uncompensated substitutes* (resp. uncompensated complements) if the introduction of an individually optimal risk $\widetilde{x}$ reduces (resp. increases) the optimal exposure to risk $\widetilde{y}$. Without loss of generality, let us suppose that it is optimal to invest 1 dollar in the asset with return $\widetilde{y}$ when this is the only available risky investment. This means that

$$\frac{\partial K}{\partial \beta}(0,1) = E\widetilde{y}u'(z+\widetilde{y}) = 0. \tag{2}$$

We now introduce an independent risk $\widetilde{x}$ whose size is optimal when taken in isolation, i.e.,

$$\frac{\partial K}{\partial \alpha}(1,0) = E\widetilde{x}u'(z+\widetilde{x}) = 0. \tag{3}$$

This means that $\alpha = 1$ is optimal when $\widetilde{x}$ is taken in isolation. Risks $\widetilde{x}$ and $\widetilde{y}$ are uncompensated substitutes if adding $\widetilde{x}$ to wealth reduces the optimal exposure to $\widetilde{y}$. This is true if[2]

$$\frac{\partial K}{\partial \beta}(1,1) = E\widetilde{y}u'(z+\widetilde{x}+\widetilde{y}) \leq 0. \tag{4}$$

To sum up, the utility function has the uncompensated risk substitutability property if and only if

$$\left.\begin{array}{l} E\widetilde{y}u'(z+\widetilde{y}) = 0 \\ E\widetilde{x}u'(z+\widetilde{x}) = 0 \end{array}\right\} \quad \Longrightarrow \quad E\widetilde{y}u'(z+\widetilde{x}+\widetilde{y}) \leq 0, \tag{5}$$

for all pairs of independent random variables $\widetilde{x}$ and $\widetilde{y}$, and at any wealth level z. In other words, adding an optimal risk exposure $\widetilde{x}$ reduces the demand for any other independent risk.

2.2 A necessary condition

Defining $v(w) = Eu(w+\widetilde{x})$ for all w, with $E\widetilde{x}u'(z+\widetilde{x}) = 0$, condition (5) is equivalent to

$$E\widetilde{y}u'(z+\widetilde{y}) = 0 \quad \Longrightarrow \quad E\widetilde{y}v'(z+\widetilde{y}) \leq 0. \tag{6}$$

[2] Whereas $z+\widetilde{x}$ and $z+\widetilde{y}$ are in the domain of u, it is not necessarily true that the support of $z+\widetilde{x}+\widetilde{y}$ is in the same domain. This is a standard difficulty of doing comparative statics analysis with non marginal changes in the parameters. However, it is easy to check that when the support of $z+\widetilde{x}+\widetilde{y}$ is not in the domain of u, then the optimal β is less than 1 when $\widetilde{x}$ is introduced.

Condition (6) is necessary and sufficient for agent v to demand less risky assets than agent u who has the same wealth level z. As is well-known, a necessary condition for this is that $-v''(z)/v'(z)$ be larger than $-u''(z)/u'(z)$. Thus, using the definition of function v, a necessary condition for uncompensated substitutability is:

$$\forall z, \widetilde{x} : E\widetilde{x}u'(z+\widetilde{x}) = 0 \quad \Longrightarrow \quad \frac{-Eu''(z+\widetilde{x})}{Eu'(z+\widetilde{x})} \geq \frac{-u''(z)}{u'(z)}. \tag{7}$$

In other words, we must determine whether an optimal risk exposure increases local risk aversion. Let X denote the cumulative distribution function of $\widetilde{x}$. Denoting absolute risk aversion by $A(w) = -u''(w)/u'(w)$, the above condition is rewritten as

$$\int xu'(z+x)dX(x) = 0 \quad \Longrightarrow \quad \int \frac{u'(z+x)}{Eu'(z+\widetilde{x})}A(z+x)dX(x) \geq A(z). \tag{8}$$

The risk-neutral cumulative distribution $\widehat{X}$ is defined as

$$d\widehat{X}(x) = \frac{u'(z+x)}{Eu'(z+x)}dX(x). \tag{9}$$

Using $\widehat{E}h(\widetilde{x}) = \int h(x)d\widehat{X}(x)$ for the expectation operator under the risk neutral probability distribution, condition (7) is equivalent to

$$\forall z, \widetilde{x} : \widehat{E}\widetilde{x} = 0 \quad \Longrightarrow \quad \widehat{E}A(z+\widetilde{x}) \geq A(z). \tag{10}$$

Obviously, this is true if and only if the Arrow-Pratt absolute risk aversion is a convex function of wealth. This proves the following Proposition, which has originally been obtained by Gollier and Kimball (1997) who used a different technique of proof.

Proposition 1 *Adding an optimal risk exposure to wealth increases (reduces) local risk aversion if and only if absolute risk aversion is convex (concave) in wealth. Convex absolute risk aversion is a necessary condition for risk substitutability.*

All familiar utility functions as the power and logarithmic ones satisfy the condition that absolute risk aversion is convex in wealth.

2.3 The necessary and sufficient condition

Lemma 1, a standard hyperplane separation theorem, is useful for deriving the necessary and sufficient condition for (uncompensated) risk substitutability. This result is also in Gollier and Kimball (1997), but we provide a new proof of it.

Lemma 1 *Consider a set of functions f_i, $i = 0, 1, ..., n$, from R to R. The following two conditions are equivalent:*

- $Ef_i(\widetilde{x}) = 0 \quad i = 1, ..., n \Longrightarrow \quad Ef_0(\widetilde{x}) \leq 0 \quad$ *for all* $\widetilde{x}$*;*
- *There exists a vector* $(m_1, ..., m_n)$ *such that* $f_0(x) \leq \sum_{i=1}^{n} m_i f_i(x)$ *for all* x.

Moreover, if $f_i(0) = 0$ and $f_i'(0)$ exists, for all $i = 0, 1, ..., n$, vector $(m_1, ..., m_n)$ must satisfy the following condition:

$$f_0'(0) - \sum_{i=1}^{n} m_i f_i'(0) = 0. \tag{11}$$

Finally, if f_i is twice differentiable, $i = 0, 1, ..., n$, then a necessary condition is that

$$f_0''(0) - \sum_{i=1}^{n} m_i f_i''(0) \leq 0. \tag{12}$$

When $n = 1$, condition (11) provides the only potential candidate for m_1, which is equal to $f_0'(0)/f_1'(0)$.

The following Proposition is obtained by using the $n = 1$ version of Lemma 1 twice.

Proposition 2 *Independent risky assets are uncompensated substitutes (resp. complements) if and only if function $H(z, x, y)$ defined by equation (13) is nonpositive (resp. nonnegative) for every (z, x, y) in the feasible domain.*

$$H(z, x, y) = y \left\{ u'(z + x + y)u'(z) - u'(z + x)u'(z + y) \right\} \tag{13}$$

$$+xyu'(z + x)u'(z + y) \left[A(z + y) - A(z) \right].$$

It is important to observe that the convexity of absolute risk aversion guarantees that H is convex in y locally around $(z, x, 0)$. Indeed, we have that

$$\left.\frac{\partial^2 H}{\partial y^2}\right|_{y=0} = 2u'(z)u'(z+x)\left[A(z) + xA'(z) - A(z+x)\right]. \tag{14}$$

Because H and its derivative with respect to y at $(z, x, 0)$ are zero, it implies that adding endogenous risk $\widetilde{x}$ reduces the demand for any other endogenous risk $\widetilde{y}$ whose optimal exposure is *small*, i.e. for risk $\widetilde{y}$ whose expectation is small with respect to its standard deviation.

One might believe that imposing the local convexity of A at any wealth level, i.e., global convexity, would imply that the local property obtained in the corollary would be global. Let us make a parallel with the notions of local and global risk aversion. Local risk aversion means that the agent rejects any small fair risk, whereas global risk aversion means that the agent rejects any fair risk. We know since Pratt (1964) that if an agent is locally risk-averse at any wealth level, then he is globally risk-averse. To show that this kind of property does not hold here, we propose the following counterexample. The investor has a power utility function

$$u(z) = \frac{z^{1-\gamma}}{1-\gamma}, \tag{15}$$

where $\gamma > 0$ is the constant relative risk aversion (CRRA). The degree of absolute risk aversion equals $A(z) = \gamma/z$, which is a convex function of z. Function H is then written as

$$H(z,x,y) = y\left\{(z+y+x)^{-\gamma}z^{-\gamma} - (z+x)^{-\gamma}(z+y)^{-\gamma} - \frac{\gamma y x(z+x)^{-\gamma}(z+y)^{-\gamma}}{z(z+y)}\right\} \tag{16}$$

By Proposition 2, independent risks are uncompensated substitutes if H is nonpositive for all (z, x, y) in the feasible domain $\{z + x > 0; z + y > 0 \text{ and } z + x + y > 0\}$. Fix x and y positive, and let z approach 0. Then, the first and the third terms in the right-hand side of (16) tend to infinity. If γ is larger than 1, this is the first which dominates, and H tends to $+\infty$. Function H is depicted in Figure 1 for $\gamma = 5$, $z = 1$, $x = 1$. We see that H and $\partial H/\partial y$ are both zero at $(z, x, 0)$. The convexity of A implies that $\partial^2 H/\partial y^2$ is negative at $(z, x, 0)$,

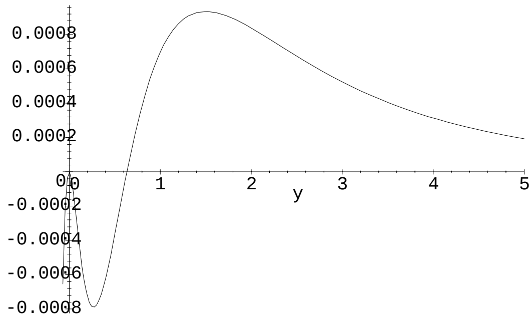

Figure 1: Evaluation of $H(1,1,y)$ for the CRRA utility function, with $\gamma = 5$.

which implies that H is nonpositive in the neighborhood of $y = 0$. But H is not nonpositive everywhere.

To be more explicit, take the CRRA utility function with $\gamma = 5$, $z = 1$ and $\widetilde{x}$ and $\widetilde{y}$ being i.i.d. and distributed as $(-0.1, 4/10; 10, 6/10)$. After some computations, we get that the optimal investment when only one risky asset is offered is 0.1678. When the second risky asset is also offered, the investor increases his demand for the first asset up to 0.1707. Our conclusions from this example are twofold: first, the convexity of absolute risk aversion is not sufficient for uncompensated substitutability. Second, CRRA utility functions with a relative risk aversion larger than unity may not yield uncompensated substitutability for all pairs of independent risks.

2.4 A sufficient condition for uncompensated substitutability

Remember that uncompensated risk substitutability means that adding an optimal risk exposure $\widetilde{x}$ reduces the demand for any other independent risk. Kimball (1993) examined the same question, but replacing the restriction that $\widetilde{x}$ be optimal by the restriction that $\widetilde{x}$ be expected-marginal-utility increasing: $Eu'(z + \widetilde{x}) \geq u'(z)$. Kimball proved that decreasing absolute prudence and decreasing absolute risk aversion are necessary and sufficient for this property. Absolute prudence is defined as $P(w) = -u'''(w)/u''(w)$. It measures the willingness to save in the face of uncertain future incomes. Positive prudence is necessary for decreasing absolute risk aversion. Kimball (1990) provided arguments to justify decreasing absolute prudence. In particular, decreasing absolute prudence implies that the level of saving by wealthier people is less sensitive to an increase in risk on future incomes.

From these observations, decreasing absolute risk aversion and decreasing absolute prudence would be sufficient for uncompensated substitutability if an optimal risk would raise expected marginal utility, i.e., if

$$E\widetilde{x}u'(z+\widetilde{x})=0 \quad\Longrightarrow\quad Eu'(z+\widetilde{x})\geq u'(z). \tag{17}$$

Using risk-neutral probabilities as defined in equation (9), we can rewrite property (17) as[3]

$$\widehat{E}\widetilde{x}=0 \quad\Longrightarrow\quad \widehat{E}\frac{1}{u'(z+\widetilde{x})}\leq\frac{1}{u'(z)}. \tag{18}$$

This is true if and only if $1/u'$ is concave. The reader can easily check that this is equivalent to $P\geq 2A$.

Lemma 2 *An optimal risk exposure raises the expected marginal utility if and only if absolute prudence is larger than twice absolute risk aversion: $P(z)\geq 2A(z)$ for all z.*

The intuition to explain that prudence must be sufficiently larger than risk aversion is quite simple. An optimal risk must have a positive expectation. Because marginal utility is decreasing, this increase in expected consumption has a negative impact on expected marginal utility. The intensity of this effect depends upon the sensitivity of u' to change in consumption, which is measured by A. In addition to this wealth effect, there is a risk effect: the uncertainty affecting the return around its means increases expected marginal utility if marginal utility is convex. The size of this effect is proportional to the degree of convexity of u', which is measured by P. The risk effect must be sufficiently larger than the wealth effect for an optimal risk to raise expected marginal utility.

[3] The risk-neutral expectation of $1/u'(z+\widetilde{x})$ exists because it must be that the support of $z+\widetilde{x}$ be in the domain of u. Otherwise $\alpha=1$ would not be optimal when $\widetilde{x}$ is considered in isolation.

Notice that condition $P \geq 2A$ is stronger than decreasing absolute risk aversion, which is itself equivalent to condition $P \geq A$. Indeed, we have that $A'(w) = A(w)\left[A(w) - P(w)\right]$. This equation also tells us that

$$A''(w) = A(w)\left[A(w) - P(w)\right]\left[2A(w) - P(w)\right] - A(w)P'(w). \tag{19}$$

In consequence, decreasing absolute prudence and $P \geq 2A$ imply that A is convex.

We can now combine the following results:

- (Kimball (1993)) An expected-marginal-utility-increasing risk reduces the demand for other independent risk if and only if absolute risk aversion and absolute prudence are decreasing;
- (Lemma 2) All optimal risk exposures are expected-marginal-utility-increasing if P is larger than $2A$;

This yields the following Proposition.

Proposition 3 *Independent risks are uncompensated substitutes if absolute prudence is decreasing and larger than twice the absolute risk aversion.*

Another question is whether this sufficient condition is far from the necessary and sufficient condition presented in Proposition 2. To answer this question, let us consider again the case of utility functions with constant relative risk aversion, for which we know that absolute risk aversion is convex in wealth. If we assume that u satisfies condition (15), condition $P \geq 2A$ becomes equivalent to relative risk aversion γ being less than 1, since $A(w) = \gamma/w$ and $P(w) = (\gamma + 1)/w$ in that case. Our results in the CRRA case are gathered in the following Corollary.

Corollary 1 *Suppose that relative risk aversion is constant. Then, small independent risks are uncompensated substitutes. Moreover, the following properties hold:*

1. *independent risks are uncompensated substitutes if relative risk aversion is less than or equal to unity.*
2. *it is always possible to find a pair of independent risks that are uncompensated complements if relative risk aversion is larger than unity.*

Proof: The fact that small independent risk are uncompensated substitutes comes from the fact that absolute risk aversion is convex for CRRA utility functions. Property 2 in this corollary comes from the fact that H is not nonpositive when γ is larger than unity. As observed earlier, if γ is larger than 1, $H(z,x,y)$ tends to $+\infty$ when z tends to zero and x and y are positive. ■

3 The compensated substitutability of independent risky assets

In the previous section, we examined the condition under which the introduction of an individually optimal risk $\widetilde{x}$ reduces the demand for any independent risk $\widetilde{y}$. This condition, which we called uncompensated substitutability, does not necessarily imply substitutability. Risks are substitutes if the introduction of the *opportunity* to invest in any risk $\widetilde{x}$ reduces the demand for any other independent risk $\widetilde{y}$.

Characterizing the necessary and sufficient condition for risk substitutability is more complex when risks are not identically distributed. Let (α^*,β^*) be the pair which maximizes $K(\alpha,\beta)=Eu(z+\alpha\widetilde{x}+\beta\widetilde{y})$. Assuming that $E\widetilde{y}u'(z+\widetilde{y})=0$, risks $\widetilde{x}$ and $\widetilde{y}$ will be substitutes if β^* is less than 1. Because K is concave, the standard method to determine whether β^* is less than unity consists in first obtaining the α that maximizes $K(\alpha,1)$. Let us denote it $\widehat{\alpha}$. Then, β^* is less than 1 if $\frac{\partial K}{\partial \beta}$ is negative when evaluated at $(\widehat{\alpha},1)$.

We normalize $\widehat{\alpha}$ to unity, which means that

$$\frac{\partial K}{\partial \alpha}(1,1)=E\widetilde{x}u'(z+\widetilde{x}+\widetilde{y})=0 \tag{20}$$

Risks $\widetilde{x}$ and $\widetilde{y}$ are substitutes if $\dfrac{\partial K}{\partial \beta}(1,1)=E\widetilde{y}u'(z+\widetilde{x}+\widetilde{y})\leq 0$, otherwise they are complements. To sum up, independent risky assets are substitutes if and only if

$$\left.\begin{array}{l} E\widetilde{y}u'(z+\widetilde{y})=0 \\ E\widetilde{x}u'(z+\widetilde{x}+\widetilde{y})=0 \end{array}\right\} \implies E\widetilde{y}u'(z+\widetilde{x}+\widetilde{y})\leq 0, \tag{21}$$

Observe that this definition of risk substitutability differs from the definition of uncompensated substitutability appearing in equation (5) only by the replacement of condition $E\widetilde{x}u'(z+\widetilde{x})=0$ by condition $E\widetilde{x}u'(z+\widetilde{x}+\widetilde{y})=0$. Risk $\widetilde{x}$ may not be individually optimal. Rather, its size must be optimal when added to the individually optimal risk $\widetilde{y}$. Risks are substitutes if adding a new risk whose size is optimal *given the existence of the other risk* in the portfolio induces the agent to reduce the size of this other risk.

We now prove that the sufficient condition for uncompensated substitutability that we presented in Proposition 3 is also sufficient for the substitutability of independent risks. The proof of this result is based on the property that the indirect utility function inherits the property that $P\geq 2A$ from the original utility function, as stated in the following Lemma.

Lemma 3 *Consider any random variable $\widetilde{y}$ and define the associated indirect utility function $\widehat{u}$ by $\widehat{u}(w)=Eu(w+\widetilde{y})$ for all w. Suppose that $-u'''(w)/u''(w)$ is smaller than $-2u''(w)/u'(w)$ for all w. Then, -$\widehat{u}'''(w)/\widehat{u}''(w)$ is smaller than $-2\widehat{u}''(w)/\widehat{u}'(w)$.*

Proof: See the Appendix. The proof is based on Lemma 1.

Proposition 4 *Independent risks are substitutes if absolute prudence is decreasing and larger than twice the absolute risk aversion.*

Proof: Kimball (1993) proved that decreasing absolute risk aversion and decreasing absolute prudence are necessary and sufficient for the following property to hold:

$$\left.\begin{array}{l} E\widetilde{y}u'(z+\widetilde{y})=0 \\ Eu'(z+\widetilde{x}+\widetilde{y})\geq Eu'(z+\widetilde{y}) \end{array}\right\} \quad\Longrightarrow\quad E\widetilde{y}u'(z+\widetilde{x}+\widetilde{y})\leq 0. \tag{22}$$

As before, we would be done if we could prove that

$$E\widetilde{x}u'(z+\widetilde{x}+\widetilde{y})=0 \quad\Longrightarrow\quad Eu'(z+\widetilde{x}+\widetilde{y})\geq Eu'(z+\widetilde{y}), \tag{23}$$

or, equivalently, if

$$E\widetilde{x}\widehat{u}'(z+\widetilde{x})=0 \quad\Longrightarrow\quad E\widehat{u}'(z+\widetilde{x})\geq \widehat{u}'(z,) \tag{24}$$

where $\widehat{u}(w)=Eu(w+\widetilde{y})$. By Proposition 2, we know that property (24) holds if and only if the absolute prudence of $\widehat{u}$ is larger than twice the absolute risk aversion of $\widehat{u}$. From Lemma 3, we know that this is true when the absolute prudence of u is larger than twice the absolute risk aversion of u. ■

Thus, the sufficient condition for uncompensated risk substitutability is also sufficient for the more demanding condition of risk substitutability.

4 Conclusion

Pratt and Zeckhauser (1987) suggested that independent risks should be substitutes in the sense that two individually undesirable risks should never be jointly desirable. They showed that this is true only under some specific restrictions on preferences that they called "properness". In this paper, we considered another concept of substitutability. Independent risks are substitutes if the opportunity to invest in one risk always reduces the investment in other risks. Our conclusions are twofold. First, the hypothesis that independent risks are substitutes seems plausible if we consider small risks. Indeed, the only condition on preferences that is required to get this result is that absolute risk aversion be convex. Under DARA, this means that the risk premium associated to a small risk is decreasing with wealth at a decreasing rate. Second, this hypothesis is questionable for larger risks. In particular, we can always find a pair of independent risks that are complements when the utility function exhibits constant relative risk aversion larger than unity. However, a sufficient condition for the substitutability of independent risks is that absolute prudence be decreasing and larger than twice the absolute risk aversion. Whether actual preferences exhibit this property is an empirical question that is left for future research.

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Proof of Lemma 1

We first prove that condition $f_0(x) \leq \sum_i m_i f_i(x)$ is necessary and sufficient for $Ef_0(\widetilde{x}) \leq 0$ whenever $Ef_i(\widetilde{x}) = 0$ for all $i = 1, ..., n$. A counter-example would be found by characterizing a random variable $\widetilde{x}$ such that $Ef_i(\widetilde{x}) = 0$ for all i, with $Ef_0(\widetilde{x})$ positive. Our best chance to get this is obtained by finding the cumulative distribution function G of $\widetilde{x}$ that satisfies the constraint and which maximizes the expectation of f_0. Thus, we should first solve the following problem:

$$\begin{array}{lll} \max_{dG \geq 0} & \int_a^b f_0(x)dG(x) & \\ s.t. & \int_a^b f_i(x)dG(x) = 0 \quad i = 1, ..., n & \\ & \int_a^b dG(x) = 1. & \end{array} \tag{25}$$

The second constraint simply states that G is indeed a cumulative distribution function. Let G^* be the distribution function of the best candidate random variable to violate The first condition in Lemma 1. The Kuhn-Tucker conditions for this maximization problem are written as:

$$f_2(x) - \sum_{i=1}^{n} m_i f_i(x) + k \begin{cases} = 0 & \text{if} \quad dG^*(x) > 0 \\ \leq 0 & \text{if} \quad dG^*(x) = 0 \end{cases} \tag{26}$$

for all x, where m_i and k are the Lagrangian multipliers associated respectively to conditions $\int f_i(x)dG(x) = 0$ and $\int dG(x) = 1$. Let $I \subset [a, b]$ be the subset of points with a positive density ($dG^*(x) > 0$). From condition (26), we obtain that

$$\int_a^b f_0(x)dG^*(x) = \int_I f_0(x)dG^*(x) = \sum_{i=1}^{n} m_i \int_I f_i(x)dG^*(x) + k \int_I dG^*(x). \tag{27}$$

Because G^* must satisfy the constraints of program (25), we obtain from the above equality that the best candidate to violate property (25) is such that $Ef_0(\widetilde{x}) = k$. Thus a necessary and sufficient condition for the first property in Lemma 1 to hold is that k be negative. From conditions (26), this means that we need that

$$h(x, m_1, ..., m_n) = f_2(x) - \sum_{i=1}^{n} m_i f_i(x) \leq 0 \tag{28}$$

for all x in $[a, b]$.

Suppose now that $f_i(0) = 0$ and f_i is differentiable at zero, for all $i = 0, 1, ..., n$. Then, we have that $h(0, m) = 0$, and h is differentiable with respect to x at $x = 0$. A necessary condition for $h(x, m)$ to be nonpositive for all x is that $\partial h/\partial x$ evaluated at $x = 0$ be zero. This implies condition (11). Another necessary condition is that $\partial^2 h/\partial x^2$ evaluated at $x = 0$ be nonpositive. This implies condition (12).■

Proof of Proposition 2

We use Lemma 1 twice. Define indirect utility function v as $v(w) = Eu(w + \widetilde{x})$ with $E\widetilde{x}u'(z + \widetilde{x}) = 0$. Then, condition (5) is rewritten as, $\forall \widetilde{y}$:

$$E\widetilde{y}u'(z + \widetilde{y}) = 0 \quad \Longrightarrow \quad E\widetilde{y}v'(z + \widetilde{y}) \leq 0. \tag{29}$$

Using Lemma 1, this is equivalent to

$$y\frac{v'(z + y)}{u'(z + y)} \leq y\frac{v'(z)}{u'(z)} \tag{30}$$

for all y. Now, we remember the way we constructed function v. Condition (30) is thus equivalent to

$$E\widetilde{x}u'(z + \widetilde{x}) = 0 \quad \Longrightarrow \quad yE\left[u'(z + \widetilde{x} + y)u'(z) - u'(z + \widetilde{x})u'(z + y)\right] \leq 0, \tag{31}$$

Using Lemma 1 again yields the result.■

Proof of Lemma 3

Condition $\frac{-\widehat{u}'''(z)}{\widehat{u}''(z)} \geq 2\frac{-\widehat{u}''(z)}{\widehat{u}'(z)}$ may be rewritten as follows:

$$E\left[u''(z + \widetilde{x}) + \lambda u'(z + \widetilde{x})\right] = 0 \quad \Longrightarrow \quad E\left[u'''(z + \widetilde{x}) + 2\lambda u''(z + \widetilde{x})\right] \geq 0$$

Using, Lemma 1, we know that this condition holds for any z and $\widetilde{x}$ if and only if there exists $m = m(\lambda, z)$ such that

$$u'''(z + x) + 2\lambda u''(z + x) \geq m\left[u''(z + x) + \lambda u'(z + x)\right] \tag{32}$$

for all x.

By risk aversion, condition $P \geq 2A$ implies that

$$\begin{aligned} u'''(z+x) + 2\lambda u''(z+x) &= -A(z+x)u'(z+x)\left[2\lambda - P(z+x)\right] \\ &\geq -2A(z+x)u'(z+x)\left[\lambda - A(z+x)\right]. \end{aligned} \tag{33}$$

We are looking for a m such that

$$-2A(z+x)u'(z+x)\left[\lambda - A(z+x)\right] \geq m\left[u''(z+x) + \lambda u'(z+x)\right] \tag{34}$$

Combining conditions (33) and (34) would yield the necessary and sufficient condition (32). Taking $m = -2\lambda$ is a good candidate, since condition (34) is then equivalent to

$$2u'(z+x)\left[\lambda - A(z+x)\right]^2 \geq 0,$$

which is true. ■