

# EXISTENCE AND UNIQUENESS OF EQUILIBRIUM FOR A SPATIAL MODEL OF SOCIAL INTERACTIONS

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**Abstract** We extend Beckmann's spatial model of social interactions to the case of a two-dimensional spatial economy with a large class of utility functions, accessing costs, and space-dependent amenities. We show that spatial equilibria derive from a potential functional. By proving the existence of a minimiser of the functional, we obtain that of spatial equilibrium. Under mild conditions on the primitives of the economy, the functional is shown to satisfy displacement convexity, a concept used in the theory of optimal transportation. This provides a variational characterisation of spatial equilibria. Moreover, the strict displacement convexity of the functional ensures the uniqueness of equilibrium. Also, the spatial symmetry of equilibrium is derived from that of the primitives of the economy. Several examples illustrate the scope of our results. In particular, the emergence of multiple equilibria in the circular economy is interpreted as a lack of convexity of the problem.

**KEYWORDS:** social interaction, spatial equilibria, multiple cities, optimal transportation, displacement convexity.

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The authors acknowledge the support of the Agence Nationale de la Recherche through the project EVaMEF ANR-09-JCJC-0096-01, the Ramón y Cajal program at the Universidad de Alicante, and the COST Action IS1104. The authors wish to thank Masahisa Fujita, André Grimaud, Michel Le Breton, Yasasuda Murata, Daisuke Oyama, Jérôme Renault, François Salanié, as well as participants to the IAST LERNA - Eco/Biology Seminar and to the Tokyo workshop on Spatial Economics organized by RIETI and the University of Tokyo for helpful comments.

## 1. INTRODUCTION

Since [Marshall \[1920\]](#), it is known that both market and non-market forces play an important role in shaping the distribution of economic activities across space. The new economic geography literature has reemphasised the role of localised pecuniary externalities mediated by the market in a general equilibrium framework, see [Krugman \[1991\]](#). Social interactions through face-to-face contacts also contribute to the gathering of individuals in villages, agglomerations, or cities, see [Glaeser and Scheinkman \[2003\]](#). In [Beckmann \[1976\]](#), the urban structure results from the interplay between a spatial communication externality and the land market.

When studying the role of agglomeration forces on the urban structure, the existing literature traditionally relies on specific functional forms regarding utility functions or transportation costs. New economic geography models make a wide use of CES or quadratic preferences over manufacturing varieties and of iceberg transport costs, see [Fujita et al. \[1999\]](#) and [Ottaviano et al. \[2002\]](#). In Beckmann's spatial model of social interactions, the preference for land is logarithmic and the cost of accessing agents is linear, see [Fujita and Thisse \[2002\]](#).

More recently, some efforts have been made to build models allowing for more general preferences over goods, with internal or external increasing returns to scale. For instance, some works have extended the CES preferences traditionally used in general equilibrium models of monopolistic competition to the case of preferences with variable elasticity of substitution, see [Behrens and Murata \[2007\]](#), and more generally, [Zhelobodko et al. \[2012\]](#). Also, in a multi-district model with external increasing returns in the spirit of [Fujita and Ogawa \[1982\]](#), [Lucas and Rossi-Hansberg \[2002\]](#) have proved the existence of a symmetric spatial equilibrium from standard neoclassical assumptions on preferences and technology. Despite these various efforts in extending models addressing agglomeration forces mediated by the market mechanism, little progress has been made to extend further spatial models where agglomeration externalities are driven by non-market forces. The aim of this paper is to fill up this gap by addressing the existence and uniqueness of equilibrium for general spatial economies involving social interactions.

Our main results are the following. We generalise Beckmann's spatial model of social interactions to the case of a two-dimensional spatial economy with a large class of preferences for land, accessing costs, and space-dependent amenities. We prove the existence and the uniqueness of spatial equilibrium. So as to get our results, we start our analysis by providing conditions under which spatial equilibria derive from a potential. Stated differently, we build a functional of which the critical points correspond to the spatial equilibria of the economy. In this context, the conditions ensuring the existence of a minimiser of the functional also ensure the existence of a spatial equilibrium of the economy. As the functional is not

convex in the usual sense, we introduce another notion of convexity, referred to as *displacement convexity*, a concept widely used in the theory of optimal transportation. Under mild conditions on the primitives of the economy, the functional is shown to be displacement convex, and we obtain an equivalence between the minimisers of the functional and the spatial equilibria of the economy. This provides a variational characterisation of spatial equilibrium. Moreover, if the functional displays strict displacement convexity, we get the uniqueness of minimiser, and hence that of spatial equilibrium. Also, the spatial symmetry of equilibrium is derived from that of the primitives of the economy. We present several examples with the purpose of illustrating the scope of our existence and uniqueness results. In particular, one- and two-dimensional geographical spaces, linear and quadratic accessing costs, and linear and power residence costs are examined. Finally, the circular spatial economy is revisited so as to illustrate the role of non-convexities in explaining the emergence of multiple equilibria. A direct method allows us to derive all the spatial equilibria arising along the circle. The analysis completes the work initiated by [Mossay and Picard \[2011\]](#).

The remainder of the paper is organised as follows. Section 2 presents the economic environment and generalises Beckmann's spatial model of social interactions. In Section 3, we prove the existence of a spatial equilibrium. Section 4 is devoted to the variational characterisation and the uniqueness of equilibrium, as well as its spatial symmetry properties. In Section 5, we present several examples of spatial economies so as to illustrate the scope of our results. Section 6 is devoted to the analysis of the circular economy. Section 7 summarises the main results of the paper and concludes.

## 2. SPATIAL MODEL

In this Section we present the economic environment. We consider a closed spatial economy  $\mathcal{E}$  extending along a one- or two-dimensional geographical space  $\mathcal{K} \subset \mathbb{R}^d$ ,  $d = 1, 2$ , hosting a unit-mass of agents distributed according to the spatial density  $\lambda \in \mathcal{M}(\mathcal{K}) = \{\lambda \in L^1(\mathcal{K}) : \lambda \geq 0, \int_{\mathcal{K}} \lambda = 1\}$ , the set of non-negative, Lebesgue measurable and integrable functions with a unit norm. Agents meet each other so as to benefit from social contacts. The social utility  $S(x)$  that an agent in location  $x \in \mathcal{K}$  derives from interacting with other agents is given by

$$(2.1) \quad S(x) = B - W * \lambda(x)$$

where the constant  $B$  denotes the total benefit from interacting with other agents,  $W : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  the cost function of accessing to them, and  $W * \lambda(x)$  the convolution of  $W$  with  $\lambda$ ,  $\int_{\mathcal{K}} W(x-y)\lambda(y) dy$ , representing the accessing cost from location  $x$ . To ensure that social interactions are global,  $B$  is assumed to be large enough,  $B > \max_x W * \lambda(x)$ .

As agents in location  $x \in \mathcal{K}$  also consume a composite good  $z$  and some land space  $s$ , their utility  $\mathcal{U}$  is given by

$$\mathcal{U}(s, z, x) = z + u(s) + S(x) + A(x)$$

where  $S$  is the social utility defined in Expression (2.1),  $u : \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{-\infty\}$  the utility of land consumption, and  $A : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty\}$  the spatial distribution of amenities. The budget constraint faced by agents is

$$z + R(x) s = Y$$

where  $Y$  is the income of agents (e.g., the endowment of the composite good) and  $R(x)$  the land rent in location  $x$ .

As is usual in the urban economics literature, we assume the presence of an absentee landlord who collects the rent paid by agents. Also, we assume that land has no alternate use other than residence. The agent's bid rent function in location  $x$  is defined as the maximum rent he is willing to pay for residing in that location,

$$\psi(x, \bar{U}) = \max_{s, z} \frac{Y - z}{s} \quad \text{such that} \quad \mathcal{U}(s, z, x) = \bar{U}.$$

**ASSUMPTION 1** (Utility of land consumption) *The utility of land consumption  $u \in C^2(\mathbb{R}_+)$  is concave and increasing.*

**LEMMA 1** (Spatial indirect utility function) *Under Assumption 1, the spatial indirect utility function  $U_\lambda$  is given by*

$$(2.2) \quad U_\lambda(x) = Y - v(\lambda(x)) + S(x) + A(x)$$

where the residence cost  $v$  defined by

$$v(\lambda) = \frac{1}{\lambda} u' \left( \frac{1}{\lambda} \right) - u \left( \frac{1}{\lambda} \right)$$

is an increasing function of  $\lambda$ .

**PROOF:** Let  $\lambda$  be some fixed spatial distribution of agents. The bid-rent  $\psi(x, \bar{U})$  can be rewritten as  $\max_s (Y + u(s) + S(x) + A(x) - \bar{U})/s$ . Let  $\hat{s}(x, \bar{U})$  denote the bid-maximising consumption of land. The corresponding first-order condition can be written as  $\bar{U} = Y + u(\hat{s}) - u'(\hat{s})\hat{s} + S(x) + A(x)$ . By using the land market equilibrium condition ( $\lambda(x) = 1/\hat{s}$ ) and defining the residence cost  $v(\lambda) = (1/\lambda)u'(1/\lambda) - u(1/\lambda)$ , the utility obtained by agents in  $x$  is then  $Y - v(\lambda(x)) + S(x) + A(x)$  given the spatial density  $\lambda$ . This defines the spatial indirect utility  $U_\lambda(x)$  in location  $x$ . Finally, we have  $v'(\lambda) = -(1/\lambda^3)u''(\lambda) \geq 0$  as  $u$  is assumed to be concave. *Q.E.D.*

For a fixed spatial distribution  $\lambda$ , the spatial indirect utility  $U_\lambda(x)$  corresponds to the utility derived by agents located at  $x$  when the land market is in equilibrium. Its Expression (2.2) involves three non-constant terms: the residence cost  $v(\lambda)$ , the accessing cost  $W * \lambda$ , and the space-dependent amenities  $A$ .

In this context, we define a spatial equilibrium of the economy  $\mathcal{E}$  as follows.

**DEFINITION 1 (Spatial equilibrium)** A spatial distribution of agents  $\lambda \in \mathcal{M}(\mathcal{K})$  constitutes a *spatial equilibrium* of the economy  $\mathcal{E}$  if there exists  $\bar{U}$  such that

$$(2.3) \quad \begin{cases} U_\lambda(x) \leq \bar{U} & \text{for almost every } x \in \mathcal{K}, \\ U_\lambda(x) = \bar{U} & \text{for almost every } x \in \mathcal{K} \text{ such that } \lambda(x) > 0. \end{cases}$$

### 3. EXISTENCE OF EQUILIBRIUM

In this Section, we relate the search for spatial equilibrium to the optimisation of a functional. Consider some spatial distribution  $\lambda \in \mathcal{M}(\mathcal{K})$  and let  $V$  be defined by  $V(\lambda) = -\lambda u(1/\lambda) + \bar{V}$ , where  $\bar{V} = \lim_{s \rightarrow \infty} u(s)/s$  is finite and positive under Assumption 1.<sup>1</sup> We construct the functional  $\mathcal{F} : \mathcal{M}(\mathcal{K}) \rightarrow \mathbb{R} \cup \{+\infty\}$  as follows

$$(3.1) \quad \mathcal{F}[\lambda] = \mathcal{V}[\lambda] + \mathcal{W}[\lambda] + \mathcal{A}[\lambda]$$

where the terms  $\mathcal{V}$ ,  $\mathcal{A}$  and  $\mathcal{W}$  are defined respectively by<sup>2</sup>

$$\begin{aligned} \mathcal{V}[\lambda] &= \int_{\mathcal{K}} V[\lambda(x)] \, dx, & \mathcal{A}[\lambda] &= - \int_{\mathcal{K}} A(x)\lambda(x) \, dx \\ \text{and } \mathcal{W}[\lambda] &= \frac{1}{2} \iint_{\mathcal{K} \times \mathcal{K}} W(x-y)\lambda(x)\lambda(y) \, dx \, dy \end{aligned}$$

**ASSUMPTION 2 (Spatial symmetry)**

- (i) *The geographical space  $\mathcal{K}$  is symmetric: for all  $x \in \mathcal{K}$ ,  $-x \in \mathcal{K}$ ,*
- (ii) *The accessing cost  $W$  is even:  $W(z) = W(-z)$ , for all  $z$ .*

We now consider the minimisation of  $\mathcal{F}$  on  $\mathcal{M}(\mathcal{K})$ .

**LEMMA 2 (Sufficient condition for equilibrium)** *Under Assumptions 1 and 2, if the spatial distribution of agents  $\lambda$  minimises the potential functional  $\mathcal{F}$  in the set  $\mathcal{M}(\mathcal{K})$ , then it is a spatial equilibrium of the economy  $\mathcal{E}$ .*

<sup>1</sup>It can easily be checked that the derivative of  $V(\lambda)$  corresponds to the residence cost  $v(\lambda)$ .

<sup>2</sup>In the mathematics literature, these three integrals are referred to as the internal, the potential, and the interaction energies, see e.g., Villani [2003].

The proof argument relies on the fact that the spatial indirect utility function  $U_\lambda$  can be seen as a differential of  $\mathcal{F}$  in the following sense. For any admissible spatial densities  $(\lambda, \tilde{\lambda})$  in  $\mathcal{M}(\mathcal{K})$ , we have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\mathcal{F}[\lambda + \varepsilon(\tilde{\lambda} - \lambda)] - \mathcal{F}[\lambda]}{\varepsilon} = - \int_{\mathcal{K}} U_\lambda(x)(\tilde{\lambda}(x) - \lambda(x)) dx.$$

As  $\lambda$  is a minimiser of  $\mathcal{F}$ , the above limit is non negative

$$\int_{\mathcal{K}} U_\lambda(x)(\tilde{\lambda}(x) - \lambda(x)) dx \leq 0.$$

As this inequality holds for any arbitrary admissible density  $\tilde{\lambda}$ , the spatial indirect utility  $U_\lambda(x)$  achieves its maximum value  $\bar{U}$  on the support of  $\lambda$ . A detailed proof of Lemma 2 is provided in Appendix A.1.

Lemma 2 relates the concept of spatial equilibrium of the economy  $\mathcal{E}$  to the notion of minimiser of the potential functional  $\mathcal{F}$ . As in the theory of potential games (e.g., [Monderer and Shapley \[1996\]](#)), where maxima of the potential are Nash equilibria of the game, a minimizer of  $\mathcal{F}$  is a spatial equilibrium of  $\mathcal{E}$ . The spatial equilibrium problem described by 2.3 can actually be regarded as a *non-cooperative, non-atomic, anonymous game*  $\mathcal{G}$  with a continuum of players where each agent makes a location decision in  $\mathcal{K}$ . The payoff of choosing location  $x$  is given by the indirect utility  $U_\lambda(x)$ . Because of anonymity, the player's payoff depends on her action and the distribution of actions of other agents only. In the game theory literature on social interactions, the residence cost  $v(\lambda(x))$  is referred to as a *congestion* term and  $W * \lambda(x)$  to an *interaction* term, see [LeBreton and Weber \[2011\]](#). It is now apparent that the functional  $\mathcal{F}$  is a potential of the location game  $\mathcal{G}$  meaning that a minimizer of  $\mathcal{F}$  is a Nash equilibrium of  $\mathcal{G}$  which constitutes a spatial equilibrium of  $\mathcal{E}$ . Given this, it is no surprise that our proof strategy follows that adopted in the literature on potential games (see also [Dubey et al. \[2006\]](#) or [Jensen \[2010\]](#)) by first constructing an appropriate potential and then showing that the first-order condition of the minimization problem leads to a sufficient condition for equilibrium. As is the case in potential games, there might well exist spatial equilibria which do not minimize the potential. Moreover, should the potential admit several local minimizers, each of them would constitute a spatial equilibrium of the economy  $\mathcal{E}$ . Unlike in the literature on potential games cited earlier, the location game  $\mathcal{G}$  involves a continuum of players, which makes the potential a functional.

Note that the global cost associated with  $\mathcal{F}$  does not correspond to the aggregate cost of the spatial economy  $\mathcal{E}$ . Though the integral  $\mathcal{A}$  is the spatial aggregate of amenities, the integral  $\mathcal{W}$  corresponds only to half the aggregate of the accessing cost  $W * \lambda$ . As a consequence, the spatial equilibria of the economy  $\mathcal{E}$  are not likely

to minimise the total aggregate cost. This is hardly surprising given the presence of the spatial communication externality.

Given Lemma 2, a preliminary step for proving the existence of a spatial equilibrium is to address the existence of a minimiser of  $\mathcal{F}$ .

ASSUMPTION 3

- (i) The utility of land consumption  $u$  satisfies  $\lim_{s \rightarrow 0^+} u(s) = -\infty$ ,
- (ii) The residence cost  $v$  satisfies  $\lim_{\lambda \rightarrow 0} v(\lambda) = 0$ ,
- (iii) The accessing cost  $W$  is continuous and bounded from below,
- (iv) The distribution of amenities  $A$  is continuous on  $\mathcal{K}$  and bounded from above,
- (v) If  $\mathcal{K}$  is unbounded, either  $\lim_{|x| \rightarrow \infty} A(x) = -\infty$  or  $A$  is constant and  $\lim_{|z| \rightarrow \infty} W(z) = +\infty$ .

The hyperbolic utility  $u(s) = -\beta/(2s)$ ,  $\beta > 0$ , used in Mossay and Picard [2011] is an example satisfying Assumption 3 (i-ii).

LEMMA 3 (Existence of a minimiser) *Under Assumptions 1 and 3, the potential functional  $\mathcal{F}$  admits a minimiser in  $\mathcal{M}(\mathcal{K})$ .*

PROOF: The proof consists in checking the lower semi-continuity of  $\mathcal{F}$  with respect to the weak convergence in  $\mathcal{M}(\mathcal{K})$  and showing that we can extract from a minimizing sequence of  $\mathcal{F}$  a subsequence which converges weakly to some element in  $\mathcal{M}(\mathcal{K})$ . This procedure is known as the direct method in the calculus of variations, see e.g., Dacorogna [2007]. See the details of the proof in Appendix B. *Q.E.D.*

The result on the equilibrium existence is summarised in the following Theorem.

THEOREM 1 (Existence of equilibrium) *Under Assumptions 1, 2 and 3, the spatial economy  $\mathcal{E}$  admits a spatial equilibrium.*

PROOF: This is an immediate consequence of Lemmas 2 and 3. *Q.E.D.*

As is standard in the theory of potential games, the convexity of the potential functional  $\mathcal{F}$  would ensure the critical points of  $\mathcal{F}$  to be minimisers of  $\mathcal{F}$ , and therefore spatial equilibria of  $\mathcal{E}$ . In addition, if the potential functional  $\mathcal{F}$  were strictly convex, it would not admit more than one minimiser. Unfortunately though, the potential functional  $\mathcal{F}$  fails to be convex because of the bi-linear form of the aggregate accessing cost  $\mathcal{W}$  which reflects the spatial externality associated with social interactions between agents located at different locations. The purpose of next section is to introduce another notion of convexity used in the theory of optimal transportation which will allow us to deal with this issue.

## 4. VARIATIONAL CHARACTERISATION AND UNIQUENESS OF EQUILIBRIUM

In this Section, in order to overcome the lack of standard convexity of the potential  $\mathcal{F}$ , we rely on a notion of convexity used for functionals defined over probability measures, referred to as *displacement convexity*. This concept has its origin in the theory of optimal transportation. We show that the functional  $\mathcal{F}$  is displacement convex under mild assumptions on the primitives of the spatial economy  $\mathcal{E}$  (*i.e.*, the geographical space  $\mathcal{K}$ , the utility function  $u$ , the accessing cost  $W$ , and the space-dependent amenities  $A$ ). This allows us to show an equivalence result between the critical points and the minimisers of  $\mathcal{F}$  for a large class of spatial economies, thereby providing a variational characterisation of the spatial equilibria of  $\mathcal{E}$ . Moreover, if  $\mathcal{F}$  is strictly displacement convex, the uniqueness of minimiser is ensured, and therefore that of equilibrium as well. Furthermore, the spatial symmetry of equilibrium is derived from the geometry of the spatial domain  $\mathcal{K}$ , and the spatial properties of the accessing cost  $W$  and of the space-dependent amenities  $A$ .

In the sequel, we assume that  $\mathcal{K} = \overline{\Omega}$  where  $\Omega$  is some open convex subset of  $\mathbb{R}^2$ .<sup>3</sup> We first introduce some basic concepts of the theory of optimal transportation. For a detailed exposition of this subject, we refer the interested reader to Villani [2003], Ambrosio et al. [2005], Villani [2009], or Rachev and Rüschendorf [1998]. Let  $\lambda_0$  and  $\lambda_1$  be two spatial densities in  $\mathcal{M}(\mathcal{K})$  and  $T$  a measurable map  $\mathcal{K} \rightarrow \mathcal{K}$ . The map  $T$  is said to transport the spatial density  $\lambda_0$  onto  $\lambda_1$  if, for any measurable set  $B \subset \mathcal{K}$ , we have

$$\int_B \lambda_1(x) \, dx = \int_{T^{-1}(B)} \lambda_0(x) \, dx.$$

This relation may also be expressed in terms of test functions in the following way

$$(4.1) \quad \int_{\mathcal{K}} \zeta(y) \lambda_1(y) \, dy = \int_{\mathcal{K}} \zeta[T(x)] \lambda_0(x) \, dx, \quad \forall \zeta : \mathcal{K} \rightarrow \mathbb{R} \text{ Lebesgue measurable}$$

The condition expressing that the map  $T$  transports  $\lambda_0$  onto  $\lambda_1$  is denoted by  $T\#\lambda_0 = \lambda_1$ , and  $T$  is referred to as the *transport map* between  $\lambda_0$  and  $\lambda_1$ .

Transport maps can be used to define distances between probability measures. As we focus our analysis on spatial densities in  $\mathcal{M}(\mathcal{K})$ , the Monge-Kantorovich distance  $w_2$  between  $\lambda_0$  and  $\lambda_1$  is defined by<sup>4</sup>

<sup>3</sup>When  $\Omega$  is unbounded, we restrict  $\mathcal{M}(\mathcal{K})$  to spatial densities  $\lambda$  with a finite second moment  $m_2 = \int_{\mathcal{K}} |y|^2 \lambda(y) \, dy < +\infty$ , so that the Monge-Kantorovich distance  $w_2$  defined later in this Section remains finite-valued.

<sup>4</sup>In the mathematics literature, the Monge-Kantorovich distance is often referred to as the Wasserstein distance. Note that the above expression can be extended to any possible probability measure on  $\mathcal{K}$ , though it needs to be modified in the case of atomic measures. Anyway, this latter case does not arise here as we are dealing with spatial densities, that is non-atomic and non-singular measures.



$$w_2(\lambda_0, \lambda_1) = \sqrt{\inf_{T: \lambda_1 = T\#\lambda_0} \int_{\mathcal{K}} |x - T(x)|^2 \lambda_0(x) dx}.$$

We will apply the result of [Brenier \[1991\]](#), stating that there is a unique optimal map  $T$  from  $\lambda_0$  onto  $\lambda_1$ , which can be written as  $T = \nabla\varphi$  for some convex function  $\varphi$ .

For any two spatial densities  $\lambda_0$  and  $\lambda_1$ , we consider the optimal transport map  $T$  transporting  $\lambda_0$  onto  $\lambda_1$  and consider the curve of spatial densities  $\{\lambda_t\}$

$$\lambda_t = [(1-t)Id + tT]\#\lambda_0 \quad \text{for } t \in [0, 1]$$

where  $Id$  is the identity operator. Note that  $\lambda_t|_{t=0} = \lambda_0$  and  $\lambda_t|_{t=1} = \lambda_1$ .

The curve  $\{\lambda_t\}_{t \in [0,1]}$  describes the gradual displacement of  $\lambda_0$  toward  $\lambda_1$ . It actually corresponds to the unique constant-speed geodesic connecting  $\lambda_0$  to  $\lambda_1$  under the metric  $w_2$ . For any  $(t, s) \in [0, 1]^2$ , we have

$$w_2(\lambda_t, \lambda_s) = |t - s| w_2(\lambda_0, \lambda_1).$$

Convexity in the space  $\mathcal{M}(\mathcal{K})$  endowed with the metric  $w_2$  has been studied first by McCann and is referred to as *displacement convexity* (or geodesic convexity), see [McCann \[1997\]](#).

**DEFINITION 2 (Displacement convexity)** The functional  $\mathcal{F}$  is said to be *displacement convex* (or *geodesically convex*) in  $\mathcal{M}(\mathcal{K})$ , if  $t \mapsto \mathcal{F}[\lambda_t]$  is convex, *i.e.* for all  $\lambda_0$  and  $\lambda_1$  in  $\mathcal{M}(\mathcal{K})$ ,

$$\mathcal{F}[\lambda_t] \leq (1-t)\mathcal{F}[\lambda_0] + t\mathcal{F}[\lambda_1] \quad \text{for } t \in [0, 1]$$

When the above inequality is strict for  $t \in (0, 1)$  and  $\lambda_0 \neq \lambda_1$ , the functional  $\mathcal{F}$  is said to be *strictly displacement convex*. When the inequality is strict for all  $t \in (0, 1)$  with  $\lambda_0$  and  $\lambda_1$  not being translates of each other, the functional  $\mathcal{F}$  is said to be *strictly displacement convex up to translation*.

[McCann \[1997\]](#) provided assumptions ensuring the displacement convexity of the functional  $\mathcal{F}$  defined in Expression (3.1). We now state those conditions in terms of the primitives of the economy.

**ASSUMPTION 4 (Displacement convexity)**

Let  $\mathcal{K} = \bar{\Omega}$  where  $\Omega$  is an open convex subset of  $\mathbb{R}^d$ ,  $d = 1, 2$ ,

- (i) The utility of land consumption  $u(s)$  is concave and increasing. Moreover, when  $d = 2$ ,  $u(s^2)$  is also concave,
- (ii) The accessing cost  $W$  is convex,
- (iii) The spatial distribution of amenities  $A$  is concave.

Both the logarithmic and the hyperbolic utility functions ( $u(s) = \beta(\log(s) + 1)$  and  $u(s) = -\beta/(2s)$ ,  $\beta > 0$ ) used in Beckmann [1976] and Mossay and Picard [2011] satisfy Assumption 4 (i).

The proof of the displacement convexity property of  $\mathcal{F}$  can be found in Ambrosio et al. [2005], McCann [1997], Villani [2003]. In order to provide an intuition of this convexity notion, we compare it to the usual convexity concept.

First, we provide a graphical illustration of the two notions. While the usual convexity requires  $\mathcal{F}[\lambda_t] \leq (1-t)\mathcal{F}[\lambda_0] + t\mathcal{F}[\lambda_1]$ , for  $\lambda_t = (1-t)\lambda_0 + t\lambda_1$ , with  $0 < t < 1$ , the displacement convexity requires  $\mathcal{F}[\lambda_t] \leq (1-t)\mathcal{F}[\lambda_0] + t\mathcal{F}[\lambda_1]$ , for  $\lambda_t = T_t\#\lambda_0 = [(1-t)Id + tT]\#\lambda_0$ , with  $0 < t < 1$ . This means that the usual convexity requires evaluating  $\mathcal{F}$  at some intermediary density  $\lambda_t$  which is a weighted average of  $\lambda_0$  and  $\lambda_1$ , while the displacement convexity requires evaluating  $\mathcal{F}$  at some intermediary density  $\lambda_t$  along the transport path from  $\lambda_0$  to  $\lambda_1$ . The weighted average  $\lambda_t^v$  (resp. the transported density  $\lambda_t^h$ ) corresponds to some vertical interpolation (resp. some horizontal interpolation) of  $\lambda_0$  and  $\lambda_1$ . See Figure 1 where the transport from  $\lambda_0$  to  $\lambda_1$  corresponds to a translation and Figure 2 where the transport involves some deformation of the density.

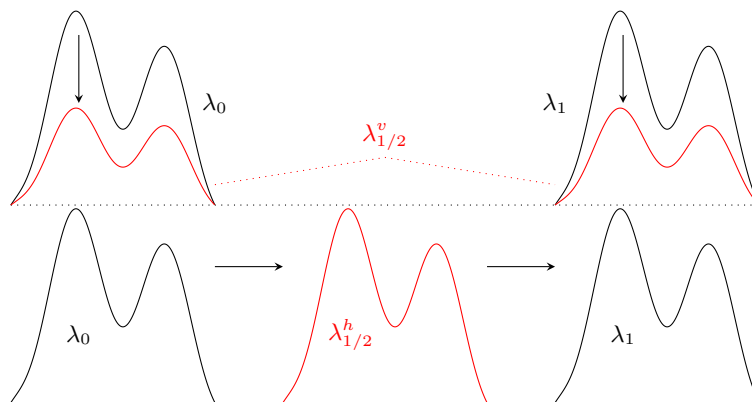


FIGURE 1.— While the upper panel illustrates the vertical interpolant  $\lambda_{1/2}^v$  of  $\lambda_0$  and  $\lambda_1$  which is used to evaluate the standard convexity of  $\mathcal{F}$ , the lower panel represents the horizontal interpolant  $\lambda_{1/2}^h$  which used to evaluate the displacement convexity of  $\mathcal{F}$ .

Second, we proceed formally and determine the convexity of each term of  $\mathcal{F}$ .

Because  $-\mathcal{A}(\lambda)$  is linear, it is convex in the usual sense. We now show that the concavity of  $A(x)$  ensures the displacement convexity of  $-\mathcal{A}$ . By applying the transport map  $T_t$  transporting  $\lambda_0$  onto  $\lambda_t$  (relation (4.1)), we get

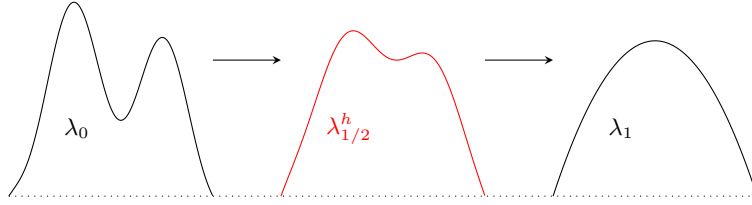


FIGURE 2.— The horizontal interpolant  $\lambda_{1/2}^h$  incurs some deformation during its transport from  $\lambda_0$  to  $\lambda_1$

$$\begin{aligned} \mathcal{A}(\lambda_t) &= \int_{\mathcal{K}} A(x)\lambda_t(x) \, dx = \int_{\mathcal{K}} A(T_t(x))\lambda_0(x) \, dx \\ &= \int_{\mathcal{K}} A((1-t)x + tT(x))\lambda_0(x) \, dx \end{aligned}$$

When  $A(x)$  is concave, we then get

$$\begin{aligned} \mathcal{A}(\lambda_t) &\geq \int_{\mathcal{K}} [(1-t)A(x) + tA(T(x))] \lambda_0(x) \, dx \\ &= (1-t) \int_{\mathcal{K}} A(x)\lambda_0(x) \, dx + t \int_{\mathcal{K}} A(T(x))\lambda_0(x) \, dx \\ &= (1-t) \int_{\mathcal{K}} A(x)\lambda_0(x) \, dx + t \int_{\mathcal{K}} A(x)\lambda_1(x) \, dx \\ &= \mathcal{A}(\lambda_0) + \mathcal{A}(\lambda_1) \end{aligned}$$

which means that  $-\mathcal{A}(\lambda)$  is displacement convex. Note that the strict displacement convexity of  $\mathcal{A}$  obtains if  $A(x)$  is strictly concave.

Though  $\mathcal{W}(\lambda)$  is quadratic in  $\lambda$ , it generally fails to be convex in the usual sense. For instance, when  $W(z) = |z|^2$ , we have

$$\begin{aligned} \mathcal{W}(\lambda) &= \iint_{\mathcal{K}^2} |x-y|^2 \lambda(x)\lambda(y) \, dx dy = 2 \int_{\mathcal{K}} |x|^2 \lambda(x) dx - 2 \iint_{\mathcal{K}^2} x \cdot y \lambda(x)\lambda(y) \, dx dy \\ &= 2 \int_{\mathcal{K}} |x|^2 \lambda(x) dx - 2 \left( \int_{\mathcal{K}} x \lambda(x) \, dx \right)^2 \end{aligned}$$

which shows that  $\mathcal{W}(\lambda)$  turns out to be concave in  $\lambda$ . Despite this lack of usual convexity, it turns out that  $\mathcal{W}$  is displacement convex. By applying the transport

map  $T_t$  transporting  $\lambda_0$  onto  $\lambda_t$ , we get

$$\begin{aligned} \mathcal{W}(\lambda) &= \iint_{\mathcal{K}^2} W(x-y)\lambda_t(x)\lambda_t(y) \, dx \, dy \\ &= \iint_{\mathcal{K}^2} W(T_t(x) - T_t(y))\lambda_0(x)\lambda_0(y) \, dx \, dy \\ &= \iint_{\mathcal{K}^2} W((1-t)x + tT(x) - (1-t)y - tT(y))\lambda_0(x)\lambda_0(y) \, dx \, dy \end{aligned}$$

Then by applying the convexity of  $W(x)$  to the pair of points  $x-y$  and  $T(x)-T(y)$ , we have

$$\begin{aligned} \mathcal{W}(\lambda) &\leq (1-t) \iint_{\mathcal{K}^2} W(x-y)\lambda_0(x)\lambda_0(y) \, dx \, dy + t \iint_{\mathcal{K}^2} W(T(x) - T(y))\lambda_0(x)\lambda_0(y) \, dx \, dy \\ &= (1-t) \iint_{\mathcal{K}^2} W(x-y)\lambda_0(x)\lambda_0(y) \, dx \, dy + t \iint_{\mathcal{K}^2} W(x-y)\lambda_1(x)\lambda_1(y) \, dx \, dy \\ &= (1-t)\mathcal{W}(\lambda_0) + t\mathcal{W}(\lambda_1) \end{aligned}$$

which shows that  $\mathcal{W}(\lambda)$  is displacement convex.

As  $V(\lambda)$  is convex,  $\mathcal{V}(\lambda)$  is convex in the usual sense. Under Assumption 4 (i), it turns out to be also displacement convex. For the sake of clarity, we first present the argument in the one-dimensional case  $d = 1$ . Still, it relies on optimal transport theory, which makes the exposition less transparent than for the two previous terms.

By the definition of  $V(\lambda)$ ,  $V(0) = \lim_{\lambda \rightarrow 0^+} V(\lambda) = \lim_{\lambda \rightarrow 0^+} -\lambda u(1/\lambda) + \bar{V} = 0$  as  $u(s)$  is concave and increasing. This condition is required to ensure the summability of  $V(\lambda)$  when  $\mathcal{K}$  is unbounded. Because  $T$  is the gradient of some convex function,  $T_t$  is non-decreasing. By the Monge-Ampère equation  $(\lambda_t(T_t(x))T'_t(x) = \lambda_0(x)$ , see (A.3)) and by a change of variables, we get

$$\int_{\mathcal{K}} V(\lambda_t(y)) \, dy = \int_{\mathcal{K}} -\lambda_t(y)u\left(\frac{1}{\lambda_t(y)}\right) \, dy = \int_{\mathcal{K}} -\frac{\lambda_0(x)}{T'_t(x)}u\left(\frac{T'_t(x)}{\lambda_0(x)}\right)T'_t(x) \, dx = \int_{\mathcal{K}} [-\lambda_0(x)u\left(\frac{T'_t(x)}{\lambda_0(x)}\right)] \, dx$$

As  $u$  is concave and increasing and  $T'_t = (1-t) + tT'$  is linear in  $t$ , the integrand is convex in  $t$ . So is  $t \mapsto \int_{\mathcal{K}} V(\lambda_t(y)) \, dy$ , which shows the displacement convexity of  $\mathcal{V}(\lambda)$ .

We now comment on the general condition ensuring the displacement convexity of  $\mathcal{V}(\lambda)$  provided by McCann [1997] which requires that the function  $\lambda \mapsto \lambda^d V(\lambda^{-d})$  is convex and non-increasing on  $(0, +\infty)$ . Here, this function is given by  $\lambda^d V(\lambda^{-d}) = -u(\lambda^d) + \bar{V}\lambda^d$ . The convexity of  $\lambda^d V(\lambda^{-d})$  is guaranteed as soon as  $u(\lambda^d)$  is concave, which is part of Assumption 4 (i). Regarding the monotonicity of  $\lambda^d V(\lambda^{-d})$ , it is enough to prove that  $s \mapsto -u(s) + \bar{V}s$  is decreasing. This is indeed the case as  $u'(s) \geq \bar{V}$  given that  $u$  is concave and  $\bar{V} = \lim_{s \rightarrow +\infty} u(s)/s =$

$\lim_{s \rightarrow +\infty} u'(s)$ . Notice that when  $d = 1$ , McCann [1997]'s condition coincides with  $u$  being concave and increasing. However, when  $d = 2$ , the concavity of  $u$  is no longer sufficient to have  $\lambda \mapsto \lambda^2 V(\lambda)^{-2} = -u(\lambda^2) + \bar{V} \lambda^2$  convex. This is the reason why Assumption 4 (i) is more restrictive in that case. Moreover, the behavior of the determinant of the Jacobian matrix associated with the change of variables has to be dealt with when checking the displacement convexity.

**THEOREM 2** (Variational characterisation) *Under Assumptions 2 and 4, the spatial distribution of agents  $\lambda$  is a spatial equilibrium of the economy  $\mathcal{E}$  if and only if it is a minimiser of  $\mathcal{F}$  in the set  $\mathcal{M}(\mathcal{K})$ .*

**PROOF:** The sufficient condition for spatial equilibrium was proved in Lemma 2 by using Assumption 1, which is now part of Assumption 4 (i), and Assumption 2. The necessary proof consists in studying the Euler-Lagrange equation associated with the minimisation of  $\mathcal{F}$ . Under Assumption 4, the functional  $\mathcal{F}$  is displacement convex. The proof makes use of displacement convexity and of optimal transportation arguments. See the details of the proof in Appendix A.2. *Q.E.D.*

To ensure the uniqueness of minimiser of functional  $\mathcal{F}$ , McCann [1997] also provided criteria so as to obtain the strict displacement convexity of  $\mathcal{F}$ . In particular, under Assumption 4, if  $W$  is strictly convex or if  $A$  is strictly concave, then the functional  $\mathcal{F}$  is strictly displacement convex.

**THEOREM 3** (Uniqueness of spatial equilibrium) *Under Assumption 4, if  $A$  is strictly concave (resp. if the accessing cost  $W$  is strictly convex or the function  $u(s^d)$  strictly concave), then any spatial equilibrium  $\lambda$  of the economy  $\mathcal{E}$  is unique (resp. unique up to translation).*

**PROOF:** By applying the criteria of McCann [1997], Assumption 4 and the strict concavity of  $A$  (resp. the strict convexity of  $W$  or the strict concavity of the function  $u(s^d)$ ) ensure the strict displacement convexity of functional  $\mathcal{F}$ . Let  $\lambda_0$  and  $\lambda_1$  be two distinct minimisers of  $\mathcal{F}$  and consider the optimal transport map  $T$  from  $\lambda_0$  onto  $\lambda_1$ . By applying the strict displacement convexity in  $\lambda_{1/2} = (\frac{1}{2}(Id + T))\# \lambda_0$ , we obtain  $\mathcal{F}[\lambda_{1/2}] < \frac{1}{2}(\mathcal{F}[\lambda_0] + \mathcal{F}[\lambda_1])$ , which is in contradiction with  $\lambda_0$  and  $\lambda_1$  being minimisers of  $\mathcal{F}$ . This proves the uniqueness of minimiser, and hence that of equilibrium by the variational characterisation provided in Theorem 2. As the accessing cost  $W$  is invariant under translation, the uniqueness holds up to translation if only  $W$  is strictly convex or the function  $u(s^d)$  is strictly concave. *Q.E.D.*

We now turn to the spatial properties of equilibrium.

**DEFINITION 3** A spatial density  $\lambda$  is said to be even (resp. radially symmetric) if  $\lambda(x) = \lambda(-x)$  for all  $x \in \mathcal{K}$  (resp.  $\lambda(x) = \lambda(y)$  for all  $x, y \in \mathcal{K}$  such that  $|x| = |y|$ ).

**ASSUMPTION 5** (Even symmetry) *Let the geographical space  $\mathcal{K}$  be symmetric: for all  $x \in \mathcal{K}$ ,  $-x \in \mathcal{K}$ . The accessing cost  $W$  and the spatial distribution of amenities  $A$  are even:  $W(x) = W(-x)$  and  $A(x) = A(-x)$ , for all  $x$ .*

**ASSUMPTION 6** (Radial symmetry) *Let  $\mathcal{K}$  be  $\mathbb{R}^2$  or a centred ball in  $\mathbb{R}^2$ . The accessing cost  $W$  and the spatial distribution of amenities  $A$  are radially symmetric:  $A(x) = A(y)$  and  $W(x) = W(y)$ , for all  $x, y$  such that  $|x| = |y|$ .*

**THEOREM 4** (Spatial symmetry of equilibria) *Suppose that the Assumptions of Theorem 3 hold (i.e., Assumption 4 holds and either  $A$  is strictly concave, or  $W$  is strictly convex, or the function  $u(s^d)$  is strictly concave). Then, under Assumption 5 (resp. Assumption 6), any spatial equilibrium  $\lambda$  is even (resp. radially symmetric).*

**PROOF:** The variational characterization provided in Theorem 2 allows us to focus our analysis on the symmetry of minimizers of  $\mathcal{F}$ . By Theorem 3, any minimizer  $\lambda$  of  $\mathcal{F}$  is unique, eventually up to translation. In this latter case, without loss of generality, we fix the barycenter of the spatial distribution  $\lambda$  as being the origin. When  $A$  and  $W$  are even, define  $\tilde{\lambda}$  by  $\tilde{\lambda}(x) = \lambda(-x)$ . Obviously,  $\mathcal{F}[\tilde{\lambda}] = \mathcal{F}[\lambda]$  meaning that  $\tilde{\lambda}$  is also a minimizer. The uniqueness of minimizer implies that  $\tilde{\lambda} = \lambda$  meaning that  $\lambda$  is even.

When  $A$  and  $W$  are radially symmetric, define  $\tilde{\lambda}$  by  $\tilde{\lambda}(x) = \lambda(Rx)$ , where  $R$  is any arbitrary rotation. Then  $\mathcal{F}[\tilde{\lambda}] = \mathcal{F}[\lambda]$  meaning again that  $\tilde{\lambda}$  is also a minimizer. By the uniqueness of equilibrium,  $\lambda(x) = \lambda(Rx)$  for any arbitrary rotation  $R$  meaning that the spatial equilibrium is radially symmetric. *Q.E.D.*

The existing literature on spatial equilibria systematically restricts the analysis to equilibria with certain symmetric properties (e.g., while Lucas [2001] and Lucas and Rossi-Hansberg [2002] focus their study on radially symmetric equilibria, Fujita and Ogawa [1982] and Berliant et al. [2002] restrict their work to even equilibria). In contrast, Theorem 4 derives the spatial symmetry properties of equilibria from that of the primitives of the economy, thereby providing an approach to characterize properties of equilibria. Under the stated assumptions, Theorem 4 shows that no asymmetric equilibrium exists.

## 5. EXAMPLES

In this Section, several examples illustrate the scope of the existence and uniqueness results obtained in the previous Sections. These examples extend existing models of the literature into many aspects: the dimension or the shape of the geographical space, and the class of utility functions or accessing costs.

In the sequel, we make use of the following Proposition which restates the spatial equilibrium condition (2.3) in a useful manner.

**PROPOSITION 1** *Suppose that the utility of land consumption  $u(s)$  is strictly concave and increasing and that the residence cost  $v$  is such that  $\lim_{\lambda \rightarrow 0} v(\lambda) = 0$ . Then the spatial distribution of agents  $\lambda$  is a spatial equilibrium of the economy  $\mathcal{E}$  if and only if the residence cost  $v$  corresponds to*

$$(5.1) \quad v(\lambda) = (Y - \bar{U} + S(x) + A(x))_+$$

where  $(\cdot)_+ = \text{Max}(\cdot, 0)$

**PROOF:** As  $u(s)$  is strictly concave, the proof of Lemma 1 ensures that the residence cost  $v$  is strictly increasing. Moreover, as  $\lim_{\lambda \rightarrow 0} v(\lambda) = 0$ , we have  $v(\lambda > 0) > 0$ . First, suppose the residence cost  $v$  satisfies  $v(\lambda) = (Y - \bar{U} + S(x) + A(x))_+$ . We have

$$\begin{cases} v(\lambda(x)) \geq Y - \bar{U} + S(x) + A(x) & \text{for almost every } x \in \mathcal{K}, \\ v(\lambda(x)) = Y - \bar{U} + S(x) + A(x) & \text{for almost every } x \in \mathcal{K} \text{ such that } \lambda(x) > 0, \end{cases}$$

and the spatial density  $\lambda$  constitutes a spatial equilibrium of the economy  $\mathcal{E}$ .

Conversely, by using the spatial equilibrium condition (2.3) along with the expression of the spatial indirect utility (2.2), we have  $v(\lambda(x)) = Y - \bar{U} + S(x) + A(x) > 0$  when  $\lambda(x) > 0$ , while  $v(0) = 0 \geq Y - \bar{U} + S(x) + A(x)$ . This can be summarised by  $v(\lambda) = (Y - \bar{U} + S(x) + A(x))_+$ . *Q.E.D.*

The above Assumption regarding the residence cost,  $\lim_{\lambda \rightarrow 0} v(\lambda) = 0$ , is satisfied by the hyperbolic utility function used in Mossay and Picard [2011], and by power residence costs given by  $v(\lambda) = \beta\lambda^\gamma$ , with  $\beta$  and  $\gamma > 0$ , see the Examples 5.2 and 5.6 for an illustration. More generally, when the residence cost  $v$  is bounded in  $\lambda = 0$  (i.e.,  $\lim_{\lambda \rightarrow 0} v(\lambda) = C \in \mathbb{R}$ ), Expression (5.1) should be replaced by  $v(\lambda) = \max(Y - \bar{U} + S(x) + A(x), C)$ . Last, in the case where  $v$  is unbounded from below in  $\lambda = 0$  (i.e.,  $\lim_{\lambda \rightarrow 0} v(\lambda) = -\infty$ ), Expression (5.1) becomes  $v(\lambda) = Y - \bar{U} + S(x) + A(x)$ . This last case corresponds to Beckmann's example,  $v(\lambda) = \beta \log \lambda$ ,  $\beta > 0$ , which is analyzed in Example 5.3.

A useful regularity result is provided by making use of Proposition 1.

**COROLLARY 1** *Suppose that Assumption 3 holds with the utility of land consumption  $u(s)$  being strictly concave and increasing. Then any spatial equilibrium  $\lambda$  is a continuous function of  $x$ .*

**PROOF:** As in the proof of Proposition 1, the residence cost  $v$  is continuous and strictly increasing. Relation (5.1) can then be inverted

$$(5.2) \quad \lambda = v^{-1}((Y - \bar{U} + A(x) + S(x))_+).$$

So as to show the continuity of  $\lambda$ , we have to prove that of  $A(x) + S(x)$ . As  $A$  is continuous by Assumption 3 (iv), we are left to prove the continuity of  $(W * \lambda)(x) = \int_{\mathcal{K}} W(x - y)\lambda(y)dy$  by using that of  $W$ . We consider several possible cases.

When  $\mathcal{K}$  is bounded, it may be contained in some ball  $B_R$ . We need to control the behavior of  $W(x - y)$ , where  $x, y \in \mathcal{K}$ . This requires to look at the behavior of  $W$  on  $B_{2R}$ , and indeed  $W$  is uniformly continuous on  $B_{2R}$ . Let  $\varepsilon > 0$  and  $\delta > 0$  such that  $|z - z'| < \delta$ , the uniform continuity of  $W$  means that  $|W(z) - W(z')| \leq \varepsilon$ . Then consider  $|x - x'| \leq \delta$ , we have  $|W * \lambda(x) - W * \lambda(x')| = |\int_{\mathcal{K}} (W(x - y) - W(x' - y))\lambda(y)dy| \leq \varepsilon$ , which ensures the continuity of  $W * \lambda$ , and therefore that of  $\lambda$ .

When  $\mathcal{K}$  is unbounded, we show that  $\lambda$  has compact support under Assumption 3 (v). First, if  $\lim_{|x| \rightarrow \infty} A(x) = -\infty$ , we have  $\lim_{|x| \rightarrow \infty} A(x) - (W * \lambda)(x) = -\infty$  as  $W$  is bounded from below. Relation (5.2) then implies that  $\lambda$  has a compact support, which may be contained in some ball  $B_R$ . As above, the uniform continuity of  $W$  on  $B_{2R}$  implies the continuity of  $W * \lambda$ . Second, if  $\lim_{|z| \rightarrow \infty} W(z) = +\infty$ , a similar argument applies. Consider a sequence  $\{x_n\}$  such that  $|x_n| \rightarrow \infty$ . The sequence of functions  $y \mapsto W(x_n - y)$  converges pointwise to  $+\infty$  and is bounded from below. By applying Fatou's lemma, we get  $\liminf_{n \rightarrow \infty} \int_{\mathcal{K}} W(x_n - y)\lambda(y)dy \geq \int_{\mathcal{K}} \lim_{n \rightarrow \infty} W(x_n - y)\lambda(y)dy = +\infty$ . This shows that  $\lim_{|x| \rightarrow \infty} A(x) - (W * \lambda)(x) = -\infty$  as  $A$  is bounded from above. Again,  $\lambda$  has compact support which implies the continuity of  $W * \lambda$ .

*Q.E.D.*

### 5.1. Linear accessing and residence costs

This case has been studied by Mossay and Picard [2011]. The economy extends along the real line  $\mathcal{K} \equiv \mathbb{R}$ . The utility of land consumption is hyperbolic, the accessing cost linear and there are no amenities. Their corresponding expressions are respectively given by  $u(s) = -\beta/(2s)$ ,  $W(x) = \tau|x|$ , with  $\beta, \tau > 0$ . The residence cost is then linear  $v(\lambda) = \beta\lambda$ .

As Assumption 3 is satisfied, a spatial equilibrium exists by Theorem 1. Also, as  $u(s)$  is strictly concave and increasing, the strict displacement convexity of functional  $\mathcal{F}$  is ensured. As a consequence, Theorem 3 provides the uniqueness of



equilibrium up to translation. By Proposition 1, we get the following equilibrium spatial distribution

$$\lambda(x) = \frac{1}{\beta} (Y + B - \bar{U} - W * \lambda(x))_+ .$$

By inspection of the above expression, as both  $W$  and  $x \mapsto x_+$  are Lipschitz continuous, so is the spatial density  $\lambda$ . Moreover, given that  $(W * \lambda)'' = W'' * \lambda$ , the convexity of the accessing cost  $W$  implies that of function  $W * \lambda$ . In particular, the lower level set of  $W * \lambda$ ,  $\{x : W * \lambda < c\}$ , is an interval meaning that the spatial density  $\lambda$  will be positive on some interval and vanish everywhere else. Along this interval, the equilibrium spatial distribution is *unimodal* and *concave* as it corresponds to the positive part of a concave function.

For an analytical expression of the spatial equilibrium, see [Mossay and Picard \[2011\]](#).

## 5.2. A two-dimensional model

We extend the previous example into several aspects by considering a two-dimensional geographical space, a residence cost given by a power function, and a general accessing cost. The economy  $\mathcal{E}$  extends along  $\mathcal{K} = \mathbb{R}^2$ . The residence cost is given by  $v(\lambda) = \beta\lambda^\gamma$ , with  $\beta, \gamma > 0$ . The accessing cost  $W$  is Lipschitz continuous, strictly convex, and radially symmetric with  $\lim_{|x| \rightarrow \infty} W(x) = +\infty$  (e.g.,  $W(x) = \tau|x|^2$  with  $\tau > 0$ ).

Assumption 3 is satisfied so that a spatial equilibrium exists. As the accessing cost  $W$  is strictly convex, the strict displacement convexity of  $\mathcal{F}$  ensures that *the equilibrium is unique up to translation* by Theorem 3.

Moreover, Theorem 4 proves that *the equilibrium is radially symmetric around its barycentre*. By Proposition 1, we have

$$(5.3) \quad \lambda(x)^\gamma = \frac{1}{\beta} (Y + B - \bar{U} - W * \lambda(x))_+ .$$

From Corollary 1, the equilibrium  $\lambda$  is a continuous function with compact support. Moreover, as in the previous example, the convexity of the accessing cost  $W$  implies that of function  $W * \lambda$ . As the support of  $\lambda$  is bounded, it corresponds to a ball given the radial symmetry of  $\lambda$ .

We now derive some further regularity of the spatial equilibrium. Since the accessing cost  $W$  is convex, it is Lipschitz continuous on bounded sets. From the proof of Corollary 1, the Lipschitz continuity is inherited by  $\lambda * W$  and by the spatial density  $\lambda$ . As a consequence, the term  $\nabla(\lambda * W) = \nabla\lambda * W$  corresponds to the convolution of a bounded function with a Lipschitz continuous function. Hence, it is Lipschitz continuous as well. This means that the function  $\lambda * W \in C^{1,1}$  (i.e.,

it is differentiable at every point and its gradient is Lipschitz continuous), so that the spatial density  $\lambda$  is globally Lipschitz continuous on  $\mathcal{K}$  and  $C^{1,1}$  on the ball  $\{x : \lambda(x) > 0\}$ .

Finally, when the accessing cost is quadratic,  $W(x) = \tau|x|^2/2$ , the equilibrium spatial density can be written as

$$\lambda(x) = \frac{1}{\beta^{1/\gamma}} \left( C - \frac{\tau}{2}|x - x_0|^2 \right)_+^{1/\gamma}$$

with  $C = Y + B - \bar{U} + \tau(x_0^2/2 - m_2/2)$  where the barycentre  $x_0$  and the second moment  $m_2$  of the spatial distribution  $\lambda$  are given by  $x_0 = \int_{\mathcal{K}} y\lambda(y) dy$  and  $m_2 = \int_{\mathcal{K}} |y|^2\lambda(y) dy$  respectively. The result is obtained by plugging the expression of function  $W * \lambda$

$$\begin{aligned} \tau \int_{\mathcal{K}} |x - y|^2 \lambda(y) dy &= \tau[|x|^2 \int_{\mathcal{K}} \lambda(y) dy - 2x \cdot \int_{\mathcal{K}} y\lambda(y) dy + \int_{\mathcal{K}} |y|^2 \lambda(y) dy] \\ &= \tau[|x - x_0|^2 - x_0^2 + m_2]. \end{aligned}$$

into Relation (5.3).

### 5.3. A two-dimensional Beckmann model

The model of Beckmann [1976], revisited by Fujita and Thisse [2002], is extended to the case of a two-dimensional spatial domain  $\mathcal{K} = \mathbb{R}^2$ . The logarithmic utility of land consumption is  $u(s) = \beta(\log(s) + 1)$ ,  $\beta > 0$  and the accessing cost is quadratic,  $W(x) = \tau|x|^2/2$ ,  $\tau > 0$ . The resulting residence cost is given by  $v(\lambda) = \beta \log \lambda$  and the corresponding function  $V$  is given by

$$V(\lambda) = \begin{cases} \beta(\lambda \log \lambda - \lambda) & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda = 0. \end{cases}$$

As  $u(s^2)$  is strictly concave and increasing and the accessing cost  $W$  is strictly convex and radially symmetric, the strict displacement convexity of  $\mathcal{F}$  ensures the uniqueness of equilibrium up to translation as well as its radial symmetry by Theorems 3 and 4. Contrary to the other examples presented in this Section, Assumption 3 does not hold here. This because as  $\lim_{\lambda \rightarrow 0} V'(\lambda) = -\infty$ , Assumption 3 (ii) is violated, which means that  $\mathcal{V}[\lambda]$  is not bounded from below anymore. Anyway, as we are able to determine the analytical expression of the spatial equilibrium, existence is shown by construction.

The spatial equilibrium condition (2.3) can be written as

$$\begin{cases} -\beta \log(\lambda(x)) - W * \lambda(x) \leq \bar{U} - Y - B & \text{for almost every } x, \\ -\beta \log(\lambda(x)) - W * \lambda(x) = \bar{U} - Y - B & \text{for almost every } x \text{ such that } \lambda(x) > 0. \end{cases}$$

Note that here, the equilibrium condition cannot be written as Relation (5.1) given that  $v(0) \neq 0$ . By inspection of the above equilibrium conditions, there is no point  $x$  for which  $\lambda(x) = 0$ . Otherwise, the first equilibrium condition would imply  $\bar{U} = +\infty$  and the second one would not be satisfied. Hence, we can write

$$(5.4) \quad \beta \log(\lambda(x)) = Y + B - \bar{U} - W * \lambda(x)$$

which leads to

$$\lambda(x) = e^{[Y+B-\bar{U}-W*\lambda(x)]/\beta} > 0.$$

By determining the function  $W * \lambda$  as done in Example 5.2, we get the following expression

$$\lambda(x) = C e^{-\frac{\tau}{2\beta}|x-x_0|^2}$$

where  $C = e^{[Y+B-\bar{U}+\tau(x_0^2-m_2)/2]/\beta}$ , and the barycenter  $x_0$  and the second moment  $m_2$  of the spatial density  $\lambda$  are given respectively by  $x_0 = \int_{\mathcal{K}} y \lambda(y) dy$  and  $m_2 = \int_{\mathcal{K}} |y|^2 \lambda(y) dy$ . The solution is translation-invariant and the constant  $C$  is determined by using the total population constraint ( $\int \lambda(x) dx = 1$ ) so that

$$\lambda(x) = \frac{\tau}{2\pi\beta} e^{-\frac{\tau}{2\beta}|x-x_0|^2}$$

#### 5.4. A city centre model

The economy extends along  $\mathcal{K} = \mathbb{R}^2$  and amenities decrease with distance from the city centre  $x = 0$ ,  $A(x) = -\alpha|x|^2/2$ ,  $\alpha > 0$ . The residence cost is given by  $v(\lambda) = \beta\lambda$ ,  $\beta > 0$ , and the accessing cost is quadratic,  $W(x) = \tau|x|^2/2$ ,  $\tau > 0$ .

Assumption 3 holds so that a *spatial equilibrium exists*. As the spatial distribution of amenities  $A$  is strictly concave, the strict displacement convexity of  $\mathcal{F}$  ensures the *uniqueness of equilibrium*. Moreover, the radial symmetry of  $A$  and  $W$  ensures that of equilibrium by Theorem 4. By Proposition 1 and the expression of function  $W * \lambda$ , the equilibrium spatial distribution corresponds to the following truncated regular paraboloid centred in the city centre  $x = 0$

$$\lambda(x) = \frac{1}{\beta} \left( Y + B - \bar{U} - \tau \frac{m_2}{2} - (\tau + \alpha) \frac{|x|^2}{2} \right)_+.$$

where the second moment  $m_2$  of the spatial density  $\lambda$  is given by  $m_2 = \int_{\mathcal{K}} |y|^2 \lambda(y) dy$ .

### 5.5. A linear city model

The economy extends along  $\mathcal{K} = \mathbb{R}^2$ . We consider a linear city where amenities are distributed along a road and decrease with distance from the road. The residence cost is given by  $v(\lambda) = \beta\lambda$ ,  $\beta > 0$ , and the accessing cost is quadratic,  $W(x) = \tau|x|^2/2$ ,  $\tau > 0$ . Amenities are given by  $A(x) = -\alpha|x \cdot e|^2/2$ , with  $\alpha > 0$  and  $e = (1, 0)$ . The larger the distance to the road  $e^\perp = \{x \in \mathbb{R}^2 : x \cdot e = 0\}$ , the lower the amenities.

Assumption 3 holds, so that a spatial equilibrium exists. As the accessing cost  $W$  is strictly convex, *the uniqueness of equilibrium is ensured up to translation parallel to  $e^\perp$ .*

By Proposition 1 and the computation of  $W * \lambda$ , we get the following spatial equilibrium density

$$\lambda(x) = \frac{1}{\beta} \left( Y + B - \bar{U} + \tau \frac{x_0^2}{2} - \tau \frac{m_2}{2} - \tau \frac{(x - x_0)^2}{2} - \alpha \frac{|x \cdot e|^2}{2} \right)_+.$$

where the barycenter  $x_0$  and the second moment  $m_2$  of the spatial distribution  $\lambda$  are given respectively by  $x_0 = \int_{\mathcal{K}} y \lambda(y) dy$  and  $m_2 = \int_{\mathcal{K}} |y|^2 \lambda(y) dy$ .

As the support of the equilibrium is an ellipse with a transverse axis corresponding to the road  $e^\perp$  and a conjugate axis orthogonal to the road  $e^\perp$ , the equilibrium distribution corresponds to a truncated elliptic paraboloid.

### 5.6. A seashore model

We consider half a space in  $\mathbb{R}^2$  representing a seashore. The economy extends along the convex domain  $\mathcal{K} = \{x \in \mathbb{R}^2 : x \cdot e \geq 0\}$ , with  $(0, 0) \neq e \in \mathbb{R}^2$ . The residence cost is  $v(\lambda) = \beta\lambda^\gamma$ , with  $\beta, \gamma > 0$  and the accessing cost is quadratic  $W(x) = \tau|x|^2/2$ . The spatial distribution of amenities is given by  $A : x \mapsto -x \cdot e$  so that  $-A$  stands for the distance from the boundary of  $\mathcal{K}$ , that is the hyperplane  $e^\perp = \{x : x \cdot e = 0\}$ .

Assumption 3 is satisfied, so that a spatial equilibrium exists. The strict convexity of the accessing cost  $W$  ensures *the uniqueness of equilibrium up to translation.*

By Proposition 1 and the computation of  $W * \lambda$ , the equilibrium spatial distribution corresponds to the following truncated paraboloid centred in  $y_0 = x_0 - e$

$$\begin{aligned} \lambda(x)^\gamma &= \frac{1}{\beta} \left( Y + B - \bar{U} - \tau \frac{|x - x_0|^2}{2} - \tau x \cdot e + \tau \frac{x_0^2}{2} - \tau \frac{m_2}{2} \right)_+ \\ &= \frac{1}{\beta} \left( C - \tau \frac{|x - (x_0 - e)|^2}{2} \right)_+, \end{aligned}$$

where  $C = Y + B - \bar{U} + \tau[e^2/2 + x_0 \cdot e + x_0^2/2 - m_2/2]$ , and the barycenter  $x_0$  and the second moment  $m_2$  of the spatial density  $\lambda$  are given respectively by  $x_0 = \int_{\mathcal{K}} y \lambda(y) dy$  and  $m_2 = \int_{\mathcal{K}} |y|^2 \lambda(y) dy$ .

We still need to determine the admissible translations. The support of the spatial density  $\lambda$  corresponds to the intersection of a ball centred in  $y_0$  and the spatial domain  $\mathcal{K}$ . Since the spatial density  $\lambda$  is unique up to translation, the shape of the support of any possible spatial equilibrium must be unique. In particular, this shape depends on the distance from  $y_0$  to the boundary  $e^\perp$  (see balls  $B_1$  and  $B_2$  in Figure 3), unless  $y_0$  would be so far from that boundary that the ball would not intersect it. In this latter case, the support would be an entire ball (such as ball  $B_3$  in Figure 3). However, this latter scenario cannot arise because if the support of  $\lambda$  were an entire ball, then  $x_0$  would correspond to  $y_0$ , which is not possible. This means that the support of all possible spatial equilibria intersects the boundary  $e^\perp$  and that the distance from  $y_0$  to that boundary is constant, that is, the same for all spatial equilibria.

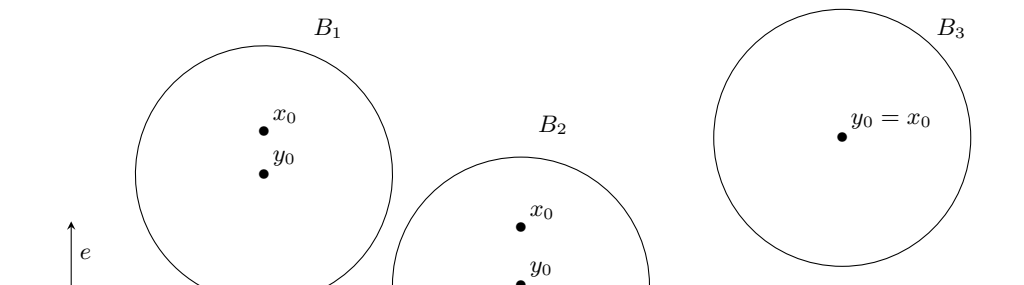


FIGURE 3.— Examples of equilibrium supports for the seashore model. Ball  $B_1$  and  $B_2$  are located at different distances from the boundary  $e^\perp$ . Ball  $B_3$  is not admissible as  $y_0 = x_0$ .

## 6. A CIRCULAR ECONOMY: A NON-CONVEX EXAMPLE

In this Section, we revisit the model by [Mossay and Picard \[2011\]](#) along the unit circle  $\mathcal{K} = \mathcal{C} = [0, 2\pi]$ . In the light of Assumption 4 and Theorem 3, the emergence of multiple spatial equilibria can be explained by a lack of convexity of the spatial domain. As the problem along the circle is not convex, Theorem 3 does not apply. This is the reason why the model exhibits multiple equilibria along the circle while it admits a unique spatial equilibrium along the real line, see Example 5.1.

Studying spatial economies extending along a circle has a long tradition in economics, ranging from the circular Hotelling model in the industrial economics literature to the more recent racetrack economy in the New Economic Geography literature. Here, we show that the circular model of spatial interactions cannot be interpreted as a simple variant of the corresponding model along the real line. As the spatial equilibria arising along the circle may involve disconnected cities, we

find it useful to introduce the following Definition.

**DEFINITION 4** (City, city-centre and multiple cities) Let  $\lambda$  be a spatial density of agents. A *city* is defined as a connected component of the support of  $\lambda$ , and a *city-centre* (or centre) of a city as any point  $x$  which is a strict local maximum of  $\lambda$ . The spatial economy is said to be a *multiple-city* economy if it consists of several disjoint cities.

Following Mossay and Picard [2011], we consider an hyperbolic utility function,  $u(s) = -\beta/(2s)$  where  $\beta$  denotes the preference for land, and a linear accessing cost  $W(x)$  equal to  $\tau x$ , for  $x \in [0, \pi]$ , and to  $\tau(2\pi - x)$ , for  $x \in [\pi, 2\pi]$ , where  $\tau$  is the accessing cost.

Mossay and Picard used a constructive method to solve the model, making conjectures about candidates for equilibrium and, only then, determining which of these candidates do actually satisfy the equilibrium condition (2.3). In contrast to their approach, we propose a direct method which allows to determine all the spatial equilibria of the economy as solutions to a differential equation.

By spatial periodicity, we impose that  $\lambda(x+2\pi) = \lambda(x)$ . Also, the point opposite to  $x$  along  $\mathcal{C}$  is denoted by  $\bar{x}$ . By Proposition 1, as the residence cost is given by  $v(\lambda) = \beta\lambda$ , any spatial equilibrium  $\lambda$  satisfies

$$\lambda(x) = \frac{1}{\beta} (Y + B - \bar{U} - W * \lambda(x))_+.$$

We make the following change of functions by defining the auxiliary function  $\phi$

$$(6.1) \quad \phi(x) = \frac{1}{\tau} W * \lambda(x) - \frac{\pi}{2}.$$

This allows to rewrite the spatial distribution  $\lambda$  as

$$(6.2) \quad \lambda(x) = \frac{1}{2} (C - \delta^2 \phi(x))_+$$

where  $\delta^2 = 2\tau/\beta$  and  $C = 2[Y + B - \bar{U} - \tau\pi/2]/\beta$ .

We now derive an equation for function  $\phi$ .

**PROPOSITION 2** (Differential equation for  $\phi$ ) *If  $\lambda$  is a spatial equilibrium along the spatial domain  $\mathcal{C}$ , then the function  $\phi$  defined in Expression (6.1) belongs to  $C^2(\mathcal{C})$  and satisfies the following ordinary differential equation*

$$(6.3) \quad \phi'' = (C - \delta^2 \phi)_+ - (C + \delta^2 \phi)_+$$

with the periodic condition

$$(6.4) \quad \phi(x) = -\phi(x \pm \pi), \quad \forall x \in [0, \pi)$$

PROOF: By using relation (6.1), function  $\phi$  can be rewritten as

$$\phi(x) = \int_{x-\pi}^x (x-y)\lambda(y) dy + \int_x^{x+\pi} (2\pi-x+y)\lambda(y) dy - \pi.$$

By inspection of this expression,  $\phi$  is differentiable. Its derivative is given by

$$\phi'(x) = \int_{x-\pi}^x \lambda(y) dy - \int_x^{x+\pi} \lambda(y) dy.$$

As  $\phi$  is differentiable and thus continuous,  $\lambda$  is also continuous given Relation (6.2). The fundamental theorem of calculus allows to differentiate  $\phi'$

$$\phi''(x) = \lambda(x) - \lambda(x-\pi) - \lambda(x+\pi) + \lambda(x) = 2[\lambda(x) - \lambda(\bar{x})].$$

This implies that  $\phi \in C^2(\mathcal{C})$ . By using Relation (6.2), we get  $\phi''(x) = (C - \delta^2\phi(x))_+ - (C - \delta^2\phi(\bar{x}))_+$ . We also have

$$\phi(x) + \phi(\bar{x}) = [W * \lambda(x) + 1/\tau W * \lambda(\bar{x})]/\tau - \pi = 0$$

given the relation  $W * \lambda(x) + W * \lambda(\bar{x}) = \tau\pi$  and the total population constraint  $\int_{\mathcal{C}} \lambda(y) dy = 1$ . Finally, we get  $\phi''(x) = (C - \delta^2\phi(x))_+ - (C + \delta^2\phi(x))_+$ . *Q.E.D.*

Our resolution method consists in determining the solutions  $\phi$  to Equation (6.3) with the periodic condition (6.4). Only then, the spatial equilibria  $\lambda$  will be obtained by Relation (6.2). Mossay and Picard identified spatial equilibria involving cities distributed according to a cosine function given by  $\cos(\delta x)$ . In what follows, these equilibria are referred to as one-frequency ( $\delta$ ) equilibria, as opposed to other solutions derived in this paper involving two frequencies ( $\delta$  and  $\sqrt{2}\delta$ ). All the details of the resolution are provided in Appendix C. We summarize them in the following Proposition.

**PROPOSITION 3** (Spatial equilibria along the circle) *The spatial equilibria arising in the circular economy  $\mathcal{C}$  can be described as follows. Of course, the uniform spatial distribution is always an equilibrium. If  $\sqrt{2}\delta$  happens to be an odd number, there exists a spatial equilibrium with full support exhibiting  $\sqrt{2}\delta$  centres, see the illustration in Figure 4. When  $\sqrt{2}\delta$  is not an odd number, for any odd number  $J$  such that  $J \leq \delta$  (resp. such that  $\delta < J \leq \sqrt{2}\delta$ ), there is a one-frequency (resp. two-frequency) spatial equilibrium with  $J$  identical and evenly spaced cities, see the illustration in Figure 5 (resp. Figure 6).*

Our direct resolution method has allowed us to determine all the spatial equilibria of the circular economy. This completes the analysis initiated by Mossay and Picard and reemphasizes the emergence of multiple equilibria, which has been interpreted here as a lack of convexity arising in the circular model.

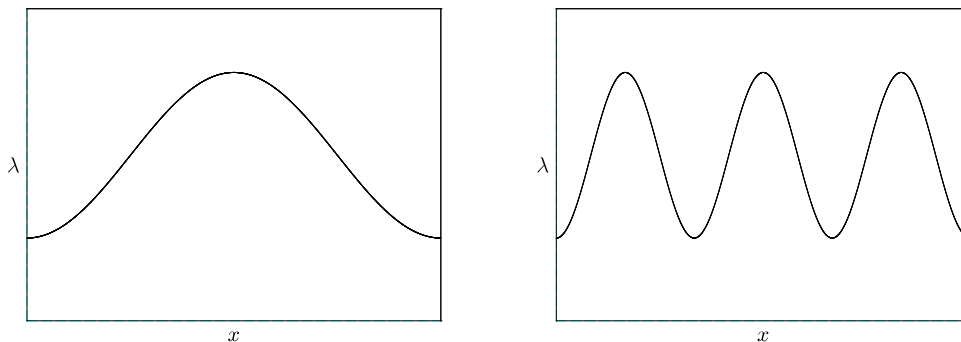


FIGURE 4.— Spatial equilibria with full support involving an odd number of centres. In the left panel, the spatial economy displays one centre for  $\delta = \sqrt{2}/2$ . In the right panel, the spatial economy displays three centres for  $\delta = 3\sqrt{2}/2$ .

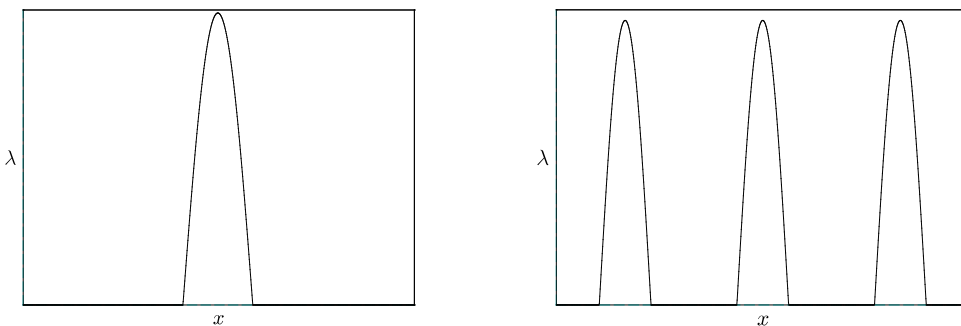


FIGURE 5.— One-frequency spatial equilibria involving an odd number of cities. In the left panel, the spatial economy displays  $J = 1$  city for  $\delta = 3$ . In the right panel, the spatial economy displays  $J = 3$  cities for  $\delta = 4$ .

## 7. CONCLUSION

We have studied a spatial model of social interactions for a large class of preferences for land, accessing costs and space-dependent amenities in a one- or two-dimensional spatial domain. By showing that spatial equilibria derive from a potential and by providing their variational characterisation, we have proved their existence and uniqueness under mild conditions on the primitives of the economy. Various examples drawn from the existing literature as well as some new ones have



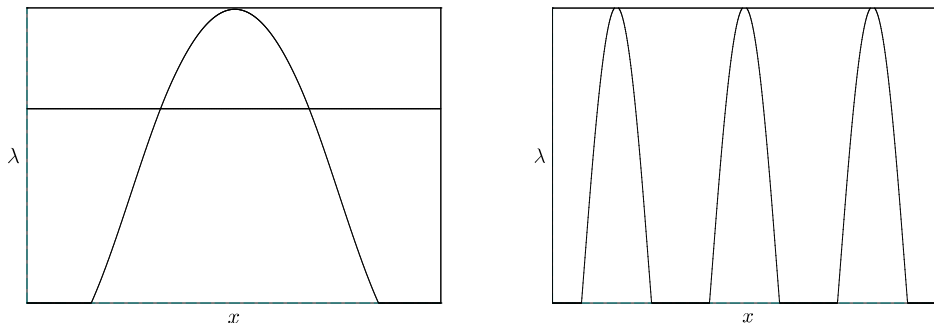


FIGURE 6.— Two-frequency spatial equilibria involving an odd number of cities. In the left panel, for  $\delta = 3/4$ , the equilibrium displays  $J = 1$  city where the frequency is  $\sqrt{2}\delta$  for the portion of the curve above the line and  $\delta$  for the portion of the curve below that line. In the right panel, for  $\delta = 2.8$ , the equilibrium displays  $J = 3$  cities.

been used to illustrate the scope of our results. In particular, the role of strict displacement convexity has been shown to be crucial for the uniqueness of equilibrium. Moreover, the emergence of multiple equilibria arising along the circular economy has been explained by a lack of convexity of the problem.

Several extensions are of interest for future research. Here are some suggestions. First, considering heterogeneous populations of agents should allow to study intra- and inter-group social interactions, and therefore to tackle spatial segregation and integration issues. Second, the extension of the model along a sphere seems natural. However, dealing with spatial symmetries in our economic environment is far from obvious. Third, a further analysis of the multiple equilibria arising along a circle could study whether some dynamics induced by the spatial mobility of agents could be used as a device to select equilibria.

## APPENDIX A: VARIATIONAL CHARACTERISATION

### A.1. Proof of Lemma 2

Let  $\lambda$  be a minimizer  $\mathcal{F}$  on  $\mathcal{M}(\mathcal{K})$ . We consider some admissible spatial density  $\tilde{\lambda} \in \mathcal{M}(\mathcal{K})$  and a family of perturbations  $\lambda_\varepsilon = (1 - \varepsilon)\lambda + \varepsilon\tilde{\lambda}$ , indexed by  $0 \leq \varepsilon \leq 1$ .

Given that  $\lambda$  minimises  $\mathcal{F}$ , we have

$$(A.1) \quad 0 \leq \frac{d}{d\varepsilon} \mathcal{F}[\lambda_\varepsilon]_{\varepsilon=0} = \frac{d}{d\varepsilon} \mathcal{V}[\lambda_\varepsilon]_{\varepsilon=0} + \frac{d}{d\varepsilon} \mathcal{A}[\lambda_\varepsilon]_{\varepsilon=0} + \frac{d}{d\varepsilon} \mathcal{W}[\lambda_\varepsilon]_{\varepsilon=0}$$

As  $V' = v$ , the first derivative in Relation (A.1) is given by

$$\frac{d}{d\varepsilon} \mathcal{V}[\lambda_\varepsilon]_{|\varepsilon=0} = \int_{\mathcal{K}} V'(\lambda(x)) \frac{d}{d\varepsilon} \lambda_\varepsilon(x) dx_{|\varepsilon=0} = \int_{\mathcal{K}} v(\lambda(x)) [\tilde{\lambda}(x) - \lambda(x)] dx.$$

The second derivative in Relation (A.1) can be written as

$$\frac{d}{d\varepsilon} \mathcal{A}[\lambda_\varepsilon]_{|\varepsilon=0} = - \int_{\mathcal{K}} A(x) (\tilde{\lambda}(x) - \lambda(x)) dx.$$

Under Assumption 2, the accessing cost  $W$  is even, so that the third derivative in Relation (A.1) leads to

$$\begin{aligned} \frac{d}{d\varepsilon} \mathcal{W}[\lambda_\varepsilon]_{|\varepsilon=0} &= \frac{1}{2} \iint_{\mathcal{K} \times \mathcal{K}} W(x-y) \left( \lambda(x) [\tilde{\lambda}(y) - \lambda(y)] + [\tilde{\lambda}(x) - \lambda(x)] \lambda(y) \right) dx dy \\ &= \iint_{\mathcal{K} \times \mathcal{K}} W(x-y) \lambda(y) [\tilde{\lambda}(x) - \lambda(x)] dx dy \\ &= \int_{\mathcal{K}} W * \lambda(x) (\tilde{\lambda}(x) - \lambda(x)) dx. \end{aligned}$$

By plugging the expressions of these three derivatives into Relation (A.1), we obtain

$$\int_{\mathcal{K}} [A(x) - v(\lambda(x)) - W * \lambda(x)] \tilde{\lambda}(x) dx \leq \int_{\mathcal{K}} [A(x) - v(\lambda(x)) - W * \lambda(x)] \lambda(x) dx.$$

Given the expression of the indirect utility function  $U_\lambda(x)$  as given by Lemma 1, we get

$$\int_{\mathcal{K}} U_\lambda(x) (\tilde{\lambda}(x) - \lambda(x)) dx \leq 0$$

As this inequality holds for any admissible density  $\tilde{\lambda}$ , the spatial density  $\lambda$  is concentrated on the set where the indirect utility function  $U_\lambda(x)$  reaches its maximum value  $\bar{U}$ . Stated differently,  $U_\lambda(x)$  achieves its maximum value  $\bar{U}$  on the support of  $\lambda$ . Hence,  $\lambda$  is a spatial equilibrium of the economy  $\mathcal{E}$ .

### A.2. Necessity Proof of Theorem 2

Under Assumption 4, the functional  $\mathcal{F}$  is displacement convex. Let  $\lambda$  be a spatial equilibrium of the economy  $\mathcal{E}$ ,  $\tilde{\lambda}$  some admissible spatial density, and  $T$  the optimal transport map from  $\lambda$  onto  $\tilde{\lambda}$ . For the moment, we assume that  $T$  is  $C^1$ . The changes corresponding to the case  $T \notin C^1$  will be discussed at the end of the proof. We define the maps  $T_\varepsilon := (1 - \varepsilon)Id + \varepsilon T$  and consider the family of perturbations  $\lambda_\varepsilon = T_\varepsilon \# \lambda$ , indexed by  $0 \leq \varepsilon \leq 1$ .<sup>5</sup>

As the curves  $\{\varepsilon \mapsto \lambda_\varepsilon\}_{\varepsilon \in [0,1]}$  are geodesics in  $\mathcal{M}(\mathcal{K})$  and  $\mathcal{F}$  is displacement convex (i.e. geodesically convex), the function  $\varepsilon \mapsto \mathcal{F}[\lambda_\varepsilon]$  is convex. In what follows, we show that the derivative of that function is positive in  $\varepsilon = 0$ . This will prove that  $\mathcal{F}[\tilde{\lambda}] \geq \mathcal{F}[\lambda]$ .

First we derive the equation for the perturbation  $\lambda_\varepsilon$ . By Expression (4.1), as the map  $T_\varepsilon$  transports  $\lambda$  onto  $\lambda_\varepsilon$ , we have

$$\int_{\mathcal{K}} \zeta(y) \lambda_\varepsilon(y) dy = \int_{\mathcal{K}} \zeta[T_\varepsilon(x)] \lambda(x) dx \quad \forall \zeta : \mathcal{K} \rightarrow \mathcal{K}.$$

<sup>5</sup>Note that we rely on a family of perturbations which is distinct from that of an additive type  $(1 - \varepsilon)\lambda + \varepsilon\tilde{\lambda}$  used in the sufficiency part (Lemma 2), see Appendix A.1. These two different types of perturbations are equally used in the theory of optimal transportation, see for instance Santambrogio [2012].

By performing the change of variable  $y = T_\varepsilon(x)$  in the left-hand side term, we obtain

$$\int_{\mathcal{K}} \zeta(T_\varepsilon(x)) \lambda_\varepsilon(T_\varepsilon(x)) |J_{T_\varepsilon}(x)| dx = \int_{\mathcal{K}} \zeta[T_\varepsilon(x)] \lambda(x) dx \quad \forall \zeta : \mathcal{K} \rightarrow \mathcal{K},$$

where  $|J_{T_\varepsilon}|$  denotes the determinant of the Jacobian matrix related to the map  $T_\varepsilon$

$$(A.2) \quad |J_{T_\varepsilon}| = \det((1 - \varepsilon)I + \varepsilon J_T) = \det(I + \varepsilon(J_T - I))$$

and  $J_T$  denotes the Jacobian matrix of map  $T$ . By equating the expressions of the two integrands appearing in the above relation, we obtain the following equation<sup>6</sup>

$$(A.3) \quad \lambda_\varepsilon(T_\varepsilon(x)) = \frac{\lambda(x)}{|J_{T_\varepsilon}(x)|} \quad \text{or equivalently} \quad \lambda_\varepsilon(y) = \frac{\lambda(T_\varepsilon^{-1}(y))}{|J_{T_\varepsilon}(T_\varepsilon^{-1}(y))|}.$$

Let us now evaluate the derivative of  $\mathcal{F}$  in  $\varepsilon = 0$

$$(A.4) \quad \frac{d}{d\varepsilon} \mathcal{F}[\lambda_\varepsilon]_{|\varepsilon=0} = \frac{d}{d\varepsilon} \mathcal{V}[\lambda_\varepsilon]_{|\varepsilon=0} + \frac{d}{d\varepsilon} \mathcal{A}[\lambda_\varepsilon]_{|\varepsilon=0} + \frac{d}{d\varepsilon} \mathcal{W}[\lambda_\varepsilon]_{|\varepsilon=0}.$$

By Equation (A.3), the first derivative in Relation (A.4) can be rewritten as

$$\int_{\mathcal{K}} V(\lambda_\varepsilon(x)) dx = \int_{\mathcal{K}} V\left(\frac{\lambda(T_\varepsilon^{-1}(x))}{|J_{T_\varepsilon}(T_\varepsilon^{-1}(x))|}\right) dx.$$

By performing the change of variable  $y = T_\varepsilon^{-1}(x)$ , we obtain

$$\int_{\mathcal{K}} V\left(\frac{\lambda(T_\varepsilon^{-1}(x))}{|J_{T_\varepsilon}(T_\varepsilon^{-1}(x))|}\right) dx = \int_{\mathcal{K}} V\left(\frac{\lambda(y)}{|J_{T_\varepsilon}(y)|}\right) |J_{T_\varepsilon}(y)| dy.$$

So as to differentiate this expression, we need to compute the derivative of the Jacobian term  $J_{T_\varepsilon}$ . As  $\det(I + H) = 1 + \text{tr}(H) + o(\|H\|)$ , using the Jacobian determinant (A.2) leads to

$$|J_{T_\varepsilon}| = 1 + \varepsilon \text{tr}(J_T - I) + o(\varepsilon) = 1 + \varepsilon((\text{div } T) - d) + o(\varepsilon),$$

where  $\text{div } T$  denotes the divergence of  $T$ , that is the trace of the Jacobian matrix  $J_T$ . As a consequence,

$$\frac{d}{d\varepsilon} |J_{T_\varepsilon}|_{|\varepsilon=0} = (\text{div } T) - d \quad \text{and} \quad \frac{d}{d\varepsilon} \frac{1}{|J_{T_\varepsilon}|_{|\varepsilon=0}} = -\frac{1}{|J_{T_\varepsilon}|^2} \frac{d}{d\varepsilon} |J_{T_\varepsilon}|_{|\varepsilon=0} = -(\text{div } T) + d.$$

Hence, by integration by parts, the first derivative in Relation (A.4) can be written as

$$\begin{aligned} \frac{d}{d\varepsilon} \int_{\mathcal{K}} V(\lambda_\varepsilon(x)) dx_{|\varepsilon=0} &= \frac{d}{d\varepsilon} \int_{\mathcal{K}} V\left(\frac{\lambda(y)}{|J_{T_\varepsilon}(y)|}\right) |J_{T_\varepsilon}(y)| dy_{|\varepsilon=0} \\ &= - \int_{\mathcal{K}} \lambda(y) ((\text{div } T)(y) - d) V'(\lambda(y)) dy \\ &\quad + \int_{\mathcal{K}} V(\lambda(y)) ((\text{div } T)(y) - d) dy \\ &= \int_{\mathcal{K}} [V(\lambda(y)) - \lambda V'(\lambda(y))] ((\text{div } T)(y) - d) dy \\ &= - \int_{\mathcal{K}} \nabla [V(\lambda(y)) - \lambda V'(\lambda(y))] \cdot (T(y) - y) dy \\ &\quad + \int_{\partial \mathcal{K}} [V(\lambda(y)) - \lambda(y) V'(\lambda(y))] (T(y) - y) \cdot n d\sigma, \end{aligned}$$

---

<sup>6</sup>In the mathematics literature, the condition relating the density of the transported density to the Jacobian determinant of the transport map is referred to as the Monge-Ampère equation.

where  $n$  is the normal outward vector. By convexity of  $\mathcal{K}$ ,  $(T(y) - y) \cdot n \leq 0$ . Also, by convexity of  $V$  and  $V(0) = 0$ ,  $V(\lambda(x)) - \lambda V'(\lambda(x))$  is negative. Thus, the boundary integral is positive. Moreover,  $\nabla [V(\lambda) - \lambda V'(\lambda)] = V'(\lambda)\nabla\lambda - V''(\lambda)\lambda\nabla\lambda = -\lambda\nabla(V'(\lambda)) = \lambda\nabla(v(\lambda))$ . This allows to write

$$\frac{d}{d\varepsilon} \int_{\mathcal{K}} V(\lambda_\varepsilon(x)) dx|_{\varepsilon=0} \geq \int_{\mathcal{K}} \lambda(y)\nabla[v(\lambda(y))] \cdot (T(y) - y) dy$$

By the push-forward Definition (4.1), the second derivative in Relation (A.4) can be written as

$$\begin{aligned} -\frac{d}{d\varepsilon} \int_{\mathcal{K}} A(x)\lambda_\varepsilon(x) dx|_{\varepsilon=0} &= -\int_{\mathcal{K}} A(T_\varepsilon(x))\lambda(x) dx|_{\varepsilon=0} \\ &= -\int_{\mathcal{K}} \nabla A(x) \cdot (T(x) - x)\lambda(x) dx . \end{aligned}$$

Similarly, the last derivative in Relation (A.4) is given by

$$\begin{aligned} \frac{d}{d\varepsilon} \mathcal{W}[\lambda_\varepsilon]|_{\varepsilon=0} &= \frac{d}{d\varepsilon} \frac{1}{2} \iint_{\mathcal{K}^2} W(T_\varepsilon(x) - T_\varepsilon(y))\lambda(x)\lambda(y) dx dy|_{\varepsilon=0} \\ &= \frac{1}{2} \iint_{\mathcal{K}^2} \nabla W(x - y) \cdot [(T(x) - x) - (T(y) - y)] \lambda(x)\lambda(y) dx dy \\ &= \iint_{\mathcal{K}^2} \nabla W(x - y) \cdot (T(x) - x) \lambda(x)\lambda(y) dx dy \\ &= \int_{\mathcal{K}} \nabla W * \lambda(x) \cdot (T(x) - x) \lambda(x) dx . \end{aligned}$$

Thus, by summing up the expressions of the three derivatives in Relation (A.4), we obtain

$$\frac{d}{d\varepsilon} \mathcal{F}[\lambda_\varepsilon]|_{\varepsilon=0} \geq -\int_{\mathcal{K}} \nabla [v(\lambda(x)) - A(x) + W * \lambda(x)] \cdot \lambda(x)(T(x) - x) dx = 0$$

This integral vanishes because  $v(\lambda) - A + W * \lambda$  is constant on the set where  $\lambda(x) > 0$  for almost every  $x \in \mathcal{K}$ . Thus, the derivative of  $\mathcal{F}$  at  $\varepsilon = 0$  is positive and the spatial equilibrium  $\lambda$  is a minimiser of  $\mathcal{F}$ .

We now comment on the case where the optimal transport map  $T$  is not  $C^1$ . This may often arise depending on the spatial density  $\lambda$ . The main issue lies in the distinction between the divergence  $\operatorname{div}(T - Id)$ , which appears when computing the first derivative in Relation (A.4) (it is computed pointwise), and the divergence that we need to perform the integration by parts, which corresponds to the divergence in the distributional sense. For non-regular maps, these two notions may differ. However here, the formal computations to be made in the case  $T \notin C^1$  can be rigorously justified in the framework of non-smooth analysis, see [Villani, 2003, Theorem 5.30]. As  $T$  is the gradient of a convex function  $\varphi$ , we have  $(\operatorname{div} T) = \Delta_A \varphi$  almost everywhere, where  $\Delta_A \varphi$  denotes the Alexandroff Laplacian of  $\varphi$ , which is also the absolutely continuous part of the distributional Laplacian  $\Delta \varphi$ . By convexity,  $\Delta \varphi$  is a positive measure and  $\Delta_A \varphi \leq \Delta \varphi$ . This shows that the pointwise divergence  $(\operatorname{div} T)$  is smaller than the distributional divergence  $\operatorname{div}_{\text{dist}} T$ . This implies that the first derivative in Relation (A.4) is smaller than  $\int_{\mathcal{K}} [V(\lambda) - \lambda V'(\lambda)] \nabla_{\text{dist}} \cdot (T - Id) dx$ . This then leads to the same result as that obtained when assuming  $T \in C^1$ .

APPENDIX B: EXISTENCE OF A MINIMIZER

*Proof of Lemma 3*

First,  $\mathcal{F}$  is shown to be bounded from below by checking that each term of  $\mathcal{F}$  is so. By Assumption 3 (iii)-(iv),  $A$  is bounded from above and  $W$  from below, so that we have  $\mathcal{A}[\lambda] \geq \int_{\mathcal{K}} (-\sup A)\lambda(x) dx = -\sup A$  and  $\mathcal{W}[\lambda] \geq (\iint_{\mathcal{K} \times \mathcal{K}} \inf W \lambda(x)\lambda(y) dx dy)/2 = (\inf W)/2$ . Regarding the term  $\mathcal{V}[\lambda]$ , we show that  $V(\lambda) \geq 0$ . By the definition of  $V(\lambda)$ ,  $V(0) = \lim_{\lambda \rightarrow 0^+} V(\lambda) = \lim_{\lambda \rightarrow 0^+} -\lambda u(1/\lambda) + \bar{V} = 0$  as  $u(s)$  is concave and increasing. Also, by Assumption 3 (ii),  $v(0) = 0$ , which gives  $V'(0) = v(0) = 0$ . Moreover,  $V''(\lambda) = v'(\lambda) > 0$  by Lemma 1. As we have shown that  $V$  is convex with  $V(0) = V'(0) = 0$ , it is increasing and positive, and therefore  $\mathcal{V}(\lambda) \geq 0$ .

This means that  $\inf \mathcal{F}$  is finite and there exists a minimizing sequence in  $\mathcal{M}(\mathcal{K})$ ,  $\{\lambda_n\}$ ,  $n = 1, 2, \dots$ , such that  $\mathcal{F}(\lambda_n) \rightarrow \inf_{\lambda \in \mathcal{M}(\mathcal{K})} \mathcal{F}$ .

Second, we apply the Theorems of Dunford-Pettis and Prokhorov<sup>7</sup> to show that we can extract from the minimizing sequence  $\{\lambda_n\}$  a subsequence, still denoted by  $\{\lambda_n\}$  for notational convenience, converging weakly to some element  $\bar{\lambda} \in \mathcal{M}(\mathcal{K})$ , that is  $\lim_{n \rightarrow \infty} \int_{\mathcal{K}} \phi(x)[\lambda_n(x) - \bar{\lambda}(x)] dx = 0, \forall \phi \in \mathcal{B}(\mathcal{K})$ , the set of bounded functions  $\phi$  over  $\mathcal{K}$ .

We now check the boundeness and the equi-integrability of  $\lambda_n$  so as to apply the Theorem of Dunford-Pettis as well as the tightness condition of the sequence to use Prokhorov's Theorem. The sequence  $\lambda_n$  is bounded in  $L^1(\mathcal{K})$  given its unit norm. Given that there exists some constant  $C$  with  $\int V(\lambda(x)) dx \leq C$  and that  $V$  is superlinear ( $\lim_{\lambda \rightarrow \infty} V(\lambda)/\lambda = -\lim_{s \rightarrow 0^+} u(s) = +\infty$  by Assumption 3 (i)),  $\lambda_n$  is also *equi-integrable* (i.e., for every  $\varepsilon > 0$ , there is  $\delta > 0$  such that for all sets  $S \subset \mathcal{K}$  with a measure  $\mu(S)$  smaller than  $\delta$ , we have  $\int_S \lambda_n(x) dx < \varepsilon$ , for every  $n$ ). On the other hand, the tightness of  $\lambda_n$  requires that for every  $\varepsilon > 0$  there is a compact set  $K \subset \mathcal{K}$  such that  $\int_{\mathcal{K} \setminus K} \lambda_n(x) dx < \varepsilon$  for every  $n$ . This last condition is satisfied under Assumption 3 (v) on  $A$  and  $W$ . When  $\lim_{|x| \rightarrow \infty} -A(x) = +\infty$ , given that there is some constant  $C$  such that  $\int_{\mathcal{K}} -A(x)\lambda_n(x) dx \leq C$ ,  $\lambda_n$  cannot concentrate too much on points outside some large compact set  $K = \overline{B(0, R)}$ . Otherwise, by choosing the radius  $R$  such that  $-A(x) > C'/\varepsilon = (C - \sup_K A)/\varepsilon$ , for any  $|x| \geq R$ , the integral  $\int_{\mathcal{K} \setminus K} -A(x)\lambda_n(x) dx$  could be made larger than  $C'$ , and we would have  $\int_{\mathcal{K}} -A(x)\lambda_n(x) dx = \int_K -A(x)\lambda_n(x) dx + \int_{\mathcal{K} \setminus K} -A(x)\lambda_n(x) dx \geq \sup_K A + C' = C$ . On the other hand, when  $A$  is constant and  $\lim_{|z| \rightarrow \infty} W(z) = +\infty$ , the functional  $\mathcal{F}$  is translation-invariant, i.e replacing some density  $\lambda(x)$  by  $\lambda(x-h)$  leaves the value of each term of  $\mathcal{F}$  unchanged. Therefore, one can choose a minimizing sequence  $\lambda_n$  such that the barycenter of  $\lambda_n$  corresponds to the origin, e.g.  $\int x\lambda_n(x) dx = 0$ . Then an argument similar to that applied earlier can be used. Given that there is some constant  $C$  such that  $\iint_{\mathcal{K} \times \mathcal{K}} W(x-y)\lambda(x)\lambda(y) dx dy \leq C$ , it is not possible for the density  $\lambda$  to concentrate too much on points outside some large compact set  $K = \overline{B(0, R)}$ . Otherwise by choosing  $R$  large enough, the integral  $\iint_{\mathcal{K} \times \mathcal{K}} W(x-y)\lambda(x)\lambda(y) dx dy$  could exceed  $C$ .

Third,  $\mathcal{F}$  is shown to be lower-semicontinuous (lsc) with respect to the weak convergence. For this, we construct functionals  $\mathcal{F}_k = \mathcal{A}_k + \mathcal{W}_k + \mathcal{V}_k$ ,  $k \in \mathbb{N}$ , such that  $\sup_k \mathcal{A}_k[\lambda] = \mathcal{A}[\lambda]$ ,  $\sup_k \mathcal{W}_k[\lambda] = \mathcal{W}[\lambda]$ , and  $\sup_k \mathcal{V}_k[\lambda] = \mathcal{V}[\lambda]$ . Then showing that  $\mathcal{A}_k$ ,  $\mathcal{W}_k$ , and  $\mathcal{V}_k$  are lsc for any  $k$ , will ensure that  $\mathcal{F} = \sup_k \mathcal{F}_k$  is also lsc.

We construct  $\mathcal{A}_k[\lambda] = \int A_k(x)\lambda(x) dx$ , with  $A_k(x) = \min\{-A(x), k\}$  being an increasing sequence. For fixed  $\lambda$ , we have  $\lim_{k \rightarrow \infty} \mathcal{A}_k[\lambda] = \int -A(x)\lambda(x) dx = \mathcal{A}[\lambda]$  by Lebesgue monotone convergence Theorem, and therefore  $\sup_k \mathcal{A}_k[\lambda] = \mathcal{A}[\lambda]$ . Indeed,  $\mathcal{A}_k$  is continuous since  $\int A_k(x)\lambda_n(x) dx \rightarrow \int A_k(x)\bar{\lambda}(x) dx$  as  $\lambda_n \rightharpoonup \bar{\lambda}$  given that  $A_k$  is bounded (from below as  $-A$  is so by Assumption 3 (iv), and from above by  $k$  by construction).

<sup>7</sup>For a general exposition of the Theorems of Dunford-Pettis and Prokhorov, see for instance [Attouch et al. \[2005\]](#)

Similarly, we construct  $\mathcal{W}_k = \iint_{\mathcal{K} \times \mathcal{K}} W_k(x-y)\lambda(x)\lambda(y) dx dy$ , with  $W_k(z) = \min\{W(z), k\}$  so that  $\sup_k \mathcal{W}_k = \lim_{k \rightarrow \infty} \mathcal{W}_k[\lambda] = \mathcal{W}[\lambda]$ . Notice that  $\lambda_n \rightarrow \bar{\lambda}$  implies that the sequence  $\lambda_n(x)\lambda_n(y) \in \mathcal{M}(\mathcal{K} \times \mathcal{K}) \rightarrow \bar{\lambda}(x)\bar{\lambda}(y)$  in  $\mathcal{M}(\mathcal{K} \times \mathcal{K})$  in duality with bounded functions over  $\mathcal{K} \times \mathcal{K}$ . Then  $\mathcal{W}_k$  is continuous since  $\iint_{\mathcal{K} \times \mathcal{K}} W_k(x-y)\lambda_n(x)\lambda_n(y) dx dy \rightarrow \iint_{\mathcal{K} \times \mathcal{K}} W_k(x-y)\lambda(x)\lambda(x) dx dy$  as  $\lambda_n(x)\lambda_n(y) \rightarrow \bar{\lambda}(x)\bar{\lambda}(y)$  given that  $W_k$  is bounded (from below as  $W$  is so by Assumption 3 (iii), and from above by  $k$  by construction).

Regarding  $\mathcal{V}(\lambda)$ , we construct  $\mathcal{V}_k[\lambda] = \int_{\mathcal{K}} V_k(\lambda(x)) dx$ ,  $k \in \mathbb{N}$ , in the following way.  $V_k(t)$  is a Lipschitz convex function corresponding to the graph of  $V(t)$  until  $t = k$  and to the line tangent to  $V$  at  $k$  for  $t > k$ ,

$$V_k(t) = \begin{cases} V(k) + V'(k)(t - k) & \text{for } t > k, \\ V(t) & \text{for } t \leq k. \end{cases}$$

Given the convexity of  $V_k$ , we have  $V_k(\lambda_n(x)) \geq V_k(\bar{\lambda}) + (\lambda_n(x) - \bar{\lambda}(x))(V_k)'(\bar{\lambda}(x))$ . Hence,  $\mathcal{V}_k[\lambda_n] \geq \mathcal{V}_k[\bar{\lambda}] + \int (\lambda_n(x) - \bar{\lambda}(x))(V_k)'(\bar{\lambda}(x)) dx$ . As  $(V_k)'$  is a bounded function,  $\lim_{n \rightarrow \infty} \int (\lambda_n(x) - \bar{\lambda}(x))(V_k)'(\bar{\lambda}(x)) dx = 0$  as  $\lambda_n \rightarrow \bar{\lambda}$ . Therefore  $\liminf_{n \rightarrow \infty} \mathcal{V}_k[\lambda_n] \geq \mathcal{V}_k[\bar{\lambda}]$ .

Finally, collecting the above results leads to  $\inf \mathcal{F} \leq \mathcal{F}(\bar{\lambda}) \leq \liminf_{n \rightarrow \infty} \mathcal{F}(\lambda_n) \leq \inf \mathcal{F}$ , which means that  $\bar{\lambda}$  is a minimizer of  $\mathcal{F}$  in  $\mathcal{M}(\mathcal{K})$ .

### APPENDIX C: SPATIAL EQUILIBRIA OF THE CIRCULAR SPATIAL ECONOMY

In this Appendix, the explicit solutions  $\phi$  to the differential equation (6.3) with the periodic condition (6.4) are determined. Then, the spatial equilibria  $\lambda$  are obtained by using Relation (6.2). For notational convenience, we will denote the maximum value of  $\phi$  along  $\mathcal{C}$  by  $\Phi$ . Without loss of generality, we will assume that this maximum value  $\Phi$  is attained in  $x = 0$ . It is convenient to rewrite the problem (6.3)- (6.4) as  $\phi'' = f(\phi)$ , with  $\phi(0) = \Phi$  and  $\phi'(0) = 0$ , where the function  $f$  is defined by

$$(C.1) \quad f(t) = (C - \delta^2 t)_+ - (C + \delta^2 t)_+.$$

We distinguish three families of solutions: one-frequency equilibria ( $C \leq 0$ ), two-frequency equilibria ( $C > 0$  and  $\Phi > C/\delta^2$ ), and equilibria with full support ( $C > 0$  and  $\Phi \leq C/\delta^2$ ). Note that unlike parameters  $\beta$  and  $\tau$ , the values of  $C$  and  $\Phi$  have to be determined in equilibrium.

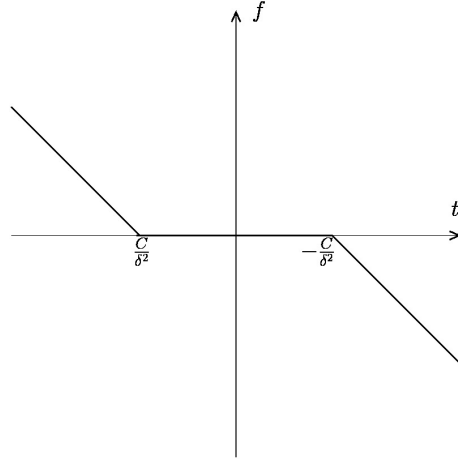
#### C.1. Case 1: $C \leq 0$ (One-frequency spatial equilibria)

When  $C \leq 0$ , the function  $f$  defined in Expression (C.1) can be rewritten as

$$f(t) = \begin{cases} C - \delta^2 t & \text{if } t < \frac{C}{\delta^2}, \\ 0 & \text{if } \frac{C}{\delta^2} \leq t \leq -\frac{C}{\delta^2}, \\ -C - \delta^2 t & \text{if } t > -\frac{C}{\delta^2}. \end{cases}$$

The graph of  $f$  is illustrated in Figure 7.

First of all, the case  $\Phi \leq -C/\delta^2$  can be discarded for the following reason. As the function  $f$  vanishes in  $[C/\delta^2, -C/\delta^2]$ , the solution to Equation (6.3) is linear. Hence, no linear periodic function  $\phi$  with  $\phi(x) + \phi(\bar{x}) = 0$  can be expected, except  $\phi = 0$ . However, in this latter case,  $\lambda = C_+/2 = 0$  since  $C < 0$ , which is not an equilibrium as the total population constraint cannot be satisfied.

FIGURE 7.— Graph of  $f$  in the case  $C \leq 0$ .

We now consider the case  $\Phi > |C|/\delta^2$ . In the neighbourhood of  $x = 0$ , we have to solve the Cauchy problem associated to the following second order linear differential equation  $\phi'' = -C - \delta^2\phi$ , with  $\phi(0) = \Phi$  and  $\phi'(0) = 0$ . This equation has the following unique solution

$$\phi_1 : x \mapsto \left( \Phi + \frac{C}{\delta^2} \right) \cos(\delta x) - \frac{C}{\delta^2}.$$

This expression is valid as long as  $\phi_1(x) > -C/\delta^2$ . Let  $a$  be the first value of  $x$  for which  $\phi_1(x) = -C/\delta^2$ , so that  $\phi_1(x) > -C/\delta^2$  in the interval  $(-a, a)$ . Note that this interval is symmetric as  $\phi_1$  is even. It follows that  $a = \pi/(2\delta)$ . In the neighbourhood at the right of  $x = a$ , we have to solve the equation  $\phi'' = 0$  with  $\phi(a) = -C/\delta^2$ . By Proposition 2,  $\phi$  is regular so that  $\phi'(a) = \phi_1'(a) = -\delta(\Phi + C/\delta^2)$ . The solution to the differential equation at the right of  $x = a$  is given by

$$\phi_2 : x \mapsto -\delta \left( \Phi + \frac{C}{\delta^2} \right) x + \frac{\pi}{2} \left( \Phi + \frac{C}{\delta^2} \right) - \frac{C}{\delta^2}.$$

This expression is valid in  $(a, a + 2b)$  where  $a + b$  denotes the first zero of  $\phi_2$ , *i.e.*

$$(C.2) \quad a + b = \frac{|C|}{\delta(\delta^2\Phi + C)} + \frac{\pi}{2\delta}.$$

The construction of solution  $\phi$  can be extended to obtain a solution of period  $T = (4a + 4b)$

$$(C.3) \quad \phi(x) = \begin{cases} \phi_1(x) = \left( \Phi + \frac{C}{\delta^2} \right) \cos(\delta x) - \frac{C}{\delta^2} & \text{if } -a \leq x \leq a, \\ \phi_2(x) = -\delta \left( \Phi + \frac{C}{\delta^2} \right) x + \frac{\pi}{2} \left( \Phi + \frac{C}{\delta^2} \right) - \frac{C}{\delta^2} & \text{if } a \leq x \leq a + 2b, \\ -\phi_1(x - 2a - 2b) & \text{if } a + 2b \leq x \leq 3a + 2b, \\ -\phi_2(x - 2a - 2b) & \text{if } 3a + 2b \leq x \leq 3a + 4b. \end{cases}$$

The period  $T$  of function  $\phi$  has to satisfy the periodic condition (6.4), which can be written as  $(2j+1)(4a+4b) = 2\pi$ , for  $j \in \mathbb{N}$ . We still need to determine the admissible values of period  $T$

$$T = 4(a+b) = \frac{2\pi}{\delta} + \frac{4|C|}{\delta(\delta^2\Phi + C)}.$$

By inspection of the above expression, when  $C \neq 0$ , the period  $T$  is a monotone function of the ratio  $\Phi/C$ . As  $\Phi/|C| > 1/\delta^2$ , the admissible values of period  $T$  are the interval  $(2\pi/\delta, +\infty)$ . Hence, for a given value of  $\delta$  and for any  $j \in \mathbb{N}$  such that  $2\pi/(2j+1) > 2\pi/\delta \Leftrightarrow 2j+1 < \delta$ , a unique value of  $\Phi/C$  can be determined so that the above solution  $\phi$  is of period  $T = 2\pi/(2j+1)$ . The values of  $C$  and  $\Phi$  are determined by imposing the total population constraint

$$\begin{aligned} 1 &= (2j+1) \int_{a+2b}^{3a+2b} (\delta^2\Phi + C) \cos(\delta(x-2a-2b)) dx \\ &= (2j+1) (\delta^2\Phi + C) \int_{-a}^a \cos(\delta x) dx = 2(2j+1) \left( \delta\Phi + \frac{C}{\delta} \right). \end{aligned}$$

which leads to  $\delta^2\Phi + C = \delta/(2(2j+1))$ . The positive part of the equilibrium spatial distribution  $\lambda$  in the interval  $(-a, 3a+4b)$  is obtained by using Relation (6.2)

$$\lambda(x) = \frac{\delta}{2(2j+1)} \cos(\delta(x-2a-2b)) \quad \text{if } a+2b \leq x \leq 3a+2b$$

Note that by using Expression (C.2), the values of  $\Phi$  and  $\bar{U}$  can also be obtained. The solution  $\phi$  is illustrated in Figure 8. The corresponding spatial equilibrium  $\lambda$  is represented in Figure 5.

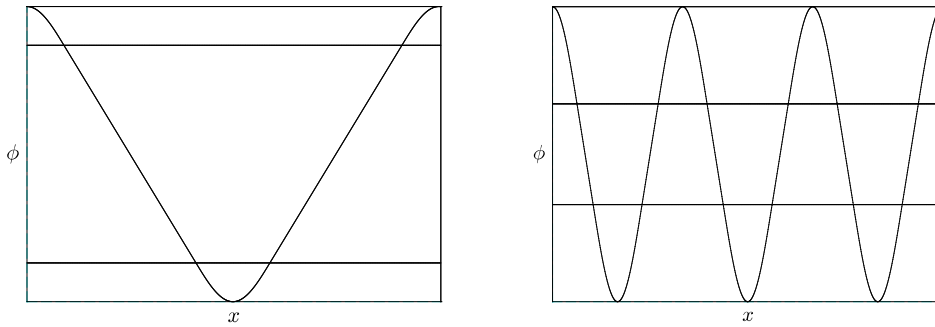


FIGURE 8.— Solution  $\phi$  for  $C = -1$  and  $\delta = 3$  in the left panel and for  $C = -1$  and  $\delta = 4$  in the right panel. The horizontal lines correspond to the values  $\pm C/\delta^2$ .



C.2. Case 2:  $C > 0$ 

When  $C > 0$ , the function  $f$  defined in Expression (C.1) can be rewritten as

$$f(t) = \begin{cases} C - \delta^2 t & \text{if } t < -\frac{C}{\delta^2}, \\ -2\delta^2 t & \text{if } -\frac{C}{\delta^2} \leq t \leq \frac{C}{\delta^2}, \\ -C - \delta^2 t & \text{if } t > \frac{C}{\delta^2}. \end{cases}$$

The graph of  $f$  is illustrated in Figure 9.

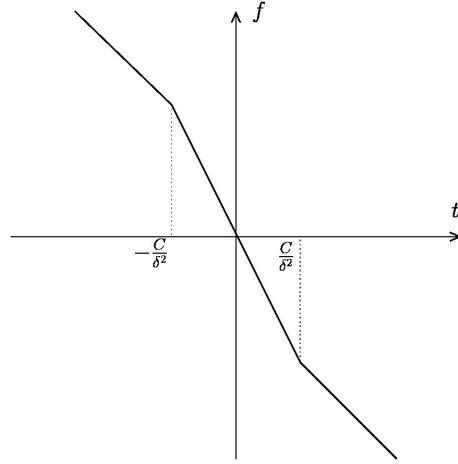


FIGURE 9.— Graph of  $f$  in the case  $C > 0$ .

C.2.1. Case 2.1:  $\Phi \leq C/\delta^2$  (Spatial equilibria with full support)

In this case, we have to solve the following differential equation  $\phi''(x) = -2\delta^2\phi$ , with  $\phi(0) = \Phi$  and  $\phi'(0) = 0$ . The unique solution to this equation is given by  $\phi(x) = \Phi \cos(\sqrt{2}\delta x)$ . The periodic condition  $\phi(x) = -\phi(\bar{x})$  leads to two cases, either  $\Phi = 0$  or  $\cos(\sqrt{2}\delta x) = -\cos(\sqrt{2}\delta(x + \pi))$ . When  $\Phi = 0$ ,  $\phi = 0$ , and  $\lambda = C/2$ . By using the total population constraint, we get the uniform spatial equilibrium  $\lambda = 1/(2\pi)$ . The other case corresponds to  $\sqrt{2}\delta$  being an odd number  $J$ , that is  $\sqrt{2}\delta = J = (2j + 1)$  for some  $j \in \mathbb{N}$ . By making use of Relation (6.2) and of the total population constraint, the spatial distribution  $\lambda$  is then given by  $\lambda(x) = 1/(2\pi) - m \cos(\sqrt{2}\delta x)$ ,  $\forall m \in [-1/(2\pi), 1/(2\pi)]$ . Examples of such equilibria are drawn in Figure 4.

C.2.2. Case 2.2:  $\Phi > C/\delta^2$  (Two-frequency spatial equilibria)

In the neighbourhood of  $x = 0$ , we have to solve the following second order linear differential equation  $\phi'' = -C - \delta^2\phi$  with  $\phi(0) = \Phi$  and  $\phi'(0) = 0$ . The unique solution to the equation is given by  $\phi_1(x) = (\Phi + C/\delta^2) \cos(\delta x) - C/\delta^2$ . This expression is valid for any  $x \in (-a, a)$  where  $a$  is the first value of  $x$  for which  $\phi_1(a) = C/\delta^2$ , that is  $a = (1/\delta) \arccos[2C/(\delta^2\Phi + C)]$ . In  $x = a$ , the function  $\phi_1$  satisfies

$$\phi_1(a) = \frac{C}{\delta^2} \quad \text{and} \quad \phi_1'(a) = -\delta \left( \Phi + \frac{C}{\delta^2} \right) \sin(\delta a) = -\frac{1}{\delta} \sqrt{(\delta^2\Phi + C)^2 - 4C^2}.$$

Since the solution  $\phi$  is  $C^1$ , at the right of  $x = a$ , we have to solve the following second order linear differential equation  $\phi'' = -2\delta^2\phi$  with  $\phi(a) = \phi_1(a)$  and  $\phi'(a) = \phi_1'(a) < 0$ . There is a unique solution  $\phi_2$  to this equation in the interval  $(a, a+b)$  where  $a+b$  is the first root of  $\phi_2$ . The solution  $\phi_2$  is given by

$$\phi_2(x) = \frac{C}{\delta^2} \cos(\sqrt{2}\delta(x-a)) - \frac{1}{\delta^2\sqrt{2}} \sqrt{(\delta^2\Phi + C)^2 - 4C^2} \sin(\sqrt{2}\delta(x-a)).$$

This expression of  $\phi_2$  remains valid in the interval  $(a+b, a+2b)$ . So, we have constructed a solution  $\phi$  of period  $T = (4a+4b)$  with  $(2j+1)(2a+2b) = \pi$

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } -a \leq x \leq a, \\ \phi_2(x) & \text{if } a \leq x \leq a+2b, \\ -\phi_1(x-2a-2b) & \text{if } a+2b \leq x \leq 3a+2b, \\ -\phi_2(x-2a-2b) & \text{if } 3a+2b \leq x \leq 3a+4b. \end{cases}$$

We now need to determine the value of  $b$  by imposing that  $\phi_2(a+b) = 0$ , that is

$$\frac{C}{\delta^2} \cos(\sqrt{2}\delta b) - \frac{1}{\delta^2\sqrt{2}} \sqrt{(\delta^2\Phi + C)^2 - 4C^2} \sin(\sqrt{2}\delta b) = 0,$$

which leads to

$$b = \frac{1}{\delta\sqrt{2}} \arctan\left(\frac{C\sqrt{2}}{\sqrt{(\delta^2\Phi + C)^2 - 4C^2}}\right).$$

Thus, the period  $T$  of solution  $\phi$  can be written as

$$T = 4(a+b) = 4\frac{1}{\delta} \arccos\left(\frac{2\alpha}{\delta^2\Phi + C}\right) + 4\frac{1}{\delta\sqrt{2}} \arctan\left(\frac{C\sqrt{2}}{\sqrt{(\delta^2\Phi + C)^2 - 4C^2}}\right).$$

We still need to determine  $C$  and  $\Phi$  in equilibrium. Let us define  $r = (\delta^2\Phi + C)/C = 1 + \delta^2\Phi/C$ . For  $\Phi \geq C/\delta^2$ , the value of  $r$  ranges from 2 to  $+\infty$ . We now study the monotonicity of the following function

$$r \mapsto \arccos\left(\frac{2}{r}\right) + \frac{1}{\sqrt{2}} \arctan\left(\frac{\sqrt{2}}{\sqrt{r^2 - 4}}\right), \quad r \in [2, +\infty[.$$

By computing the derivative of the above function, it can readily be checked that the above function is strictly decreasing. The image of this function on  $[2, +\infty)$  is given by  $[\pi/(2\sqrt{2}), \pi/2[$ . This means that, for a given value of  $\delta$ , any period  $T \in [\sqrt{2}\pi/\delta, 2\pi/\delta[$  may be obtained for a unique value of the ratio  $\Phi/C \geq 1/\delta^2$ . In particular, given the value of  $\delta$  and any  $j \in \mathbb{N}$  satisfying

$$\frac{2\pi}{2j+1} \in \left[\frac{\sqrt{2}\pi}{\delta}, \frac{2\pi}{\delta}\right[ \quad \Leftrightarrow \quad \delta < 2j+1 \leq \sqrt{2}\delta,$$

we can determine a unique value of  $\Phi/C$  such that the solution  $\phi$  that we have constructed above is of period  $T = 2\pi/(2j+1)$ . Note that the limit case  $2j+1 = \sqrt{2}\delta$  actually corresponds to the case (C.2.1) where  $\Phi = C/\delta^2$ .

There is still one degree of freedom left as only the ratio  $\Phi/C$  has been determined. The values of  $C$  and  $\Phi$  can be determined by using the total population constraint. The spatial equilibrium  $\lambda$  is obtained from the solution  $\phi$  by using Relation (6.2). If both  $C$  and  $\Phi$  are multiplied by some constant  $K$ , so will be the spatial density  $\lambda$ . This allows to tune the values of  $C$  and  $\Phi$  so as to get the total population of agents equal to 1. The solution  $\phi$  is illustrated in Figure 10. The corresponding spatial equilibrium  $\lambda$  is represented in Figure 6.

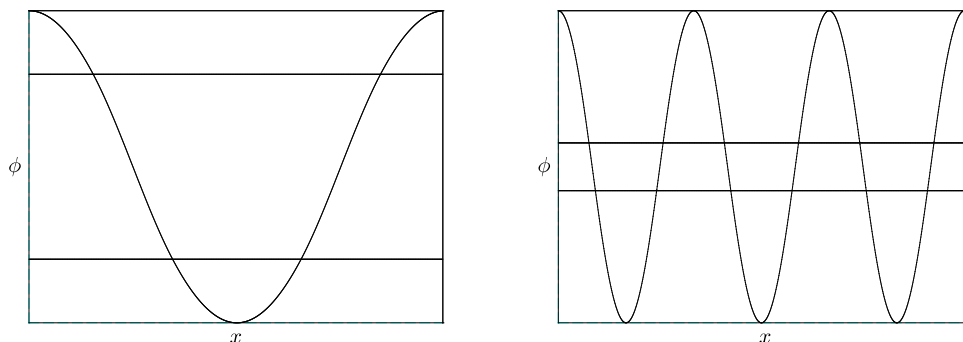


FIGURE 10.— Solution  $\phi$  for  $C = 1$ ,  $\delta = 3/4$  and  $\Phi = 3$  in the left panel, and for  $C = 1$ ,  $\delta = 2.8$  and  $\Phi = 1$  in the right panel. The horizontal lines represent the values  $\pm C/\delta^2$  so that the frequency changes when crossing these lines.

## REFERENCES

- L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics. Birkhäuser, 2005.
- H. Attouch, G. Buttazzo, and G. Michaille. *Variational analysis in Sobolev and BV spaces : applications to PDEs and optimization*. MPS-SIAM series on optimization. Society for Industrial and Applied Mathematics, Philadelphia, 2005.
- M.J. Beckmann. *Spatial equilibrium in the dispersed city*, pages 117–125. Y.Y. Papageorgiou (ed.), *Mathematical Land Use Theory*, Lexington, MA:Lexington Books, 1976.
- K. Behrens and Y. Murata. General equilibrium models of monopolistic competition: A new approach. *Journal of Economic Theory*, 136(1):776–787, September 2007.
- M. Berliant, P. Shin-Kun, and P. Wang. Production externalities and urban configuration. *Journal of Economic Theory*, 104(2):275–303, June 2002.
- Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44(4):375–417, 1991.
- B. Dacorogna. *Direct Methods in the Calculus of Variations*. Applied Mathematical Sciences. Springer, 2007.
- P. Dubey, O. Haimanko, and A. Zapechelnyuk. Strategic complements and substitutes, and potential games. *Games and Economic Behavior*, 54:77–94, 2006.
- M. Fujita and H. Ogawa. Multiple equilibria and structural transition of non-monocentric urban configurations. *Regional Science and Urban Economics*, 12(2):161–196, May 1982.
- M. Fujita and J. F. Thisse. *Economics of Agglomeration: Cities, Industrial Location, and Regional Growth*. Cambridge University Press, 2002.
- M. Fujita, P. Krugman, and A. J. Venables. *The Spatial Economy: Cities, Regions, and International Trade*. MIT Press, 1999.
- E. Glaeser and J. Scheinkman. *Non-market interactions*, pages 339–369. M. Dewatripont, L.P. Hansen, and S. Turnovsky (Eds.), *Advances in Economics and Econometrics: Theory and Applications*, Eight World Congress, Cambridge University Press, 2003.
- M. Jensen. Strategic complements and substitutes, and potential games. *Economic Theory*, 43:

- 45–66, 2010.
- P. Krugman. Increasing returns and economic geography. *Journal of Political Economy*, 99(3): 483–499, 1991.
- M. LeBreton and S. Weber. Games of social interactions with local and global externalities. *Econ. Letters*, 111:88–90, 2011.
- R. E. Lucas. Externalities and cities. *Review of Economic Dynamics*, 4(2):245–274, April 2001.
- R. E. Lucas and E. Rossi-Hansberg. On the internal structure of cities. *Econometrica*, 70(4): 1445–1476, July 2002.
- A. Marshall. *Principles of Economics*. (Revised Edition ed.). London: Macmillan; reprinted by Prometheus Books, 1920.
- R. J. McCann. A convexity principle for interacting cases. *Adv. Math.*, 128(1):153–179, 1997.
- D. Monderer and L.S. Shapley. Potential games. *Games and Economic Behavior*, 14:124–143, 1996.
- P. Mossay and P. Picard. A spatial model of social interactions. *Journal of Economic Theory*, 146(6):2455–2477, 2011.
- G. Ottaviano, T. Tabuchi, and J.-F. Thisse. Agglomeration and trade revisited. *International Economic Review*, 43(2):409–435, 2002.
- S.T. Rachev and L. Rüschendorf. *Mass Transportation problems. Vol. I: Theory; Vol. II : Applications*. Springer-Verlag,, 1998.
- F. Santambrogio. Gradient flows in Wasserstein spaces and applications to crowd movement. To appear in the proceedings of Séminaire X-EDP, 2012.
- C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.
- C. Villani. *Optimal transport: old and new*. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 2009.
- E. Zhelobodko, S. Kokovin, M. Parenti, and J.-F. Thisse. Monopolistic competition: Beyond the constant elasticity of substitution. *Econometrica*, 80(6):2765–2784, 2012.