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NOISY OBSERVATION IN ADVERSE SELECTION MODELS+

by B. Caillaud*
R. Guesnerie**
P. Rey***

* CEPREMAP, Paris
** DELTA, Paris
*** ENSAE, Paris

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ABSTRACT

NOISY OBSERVATION IN ADVERSE SELECTION MODELS

We consider a principal-agent contracting problem under incomplete information where some of the agent's actions are imperfectly observable. Contracts take the form of reward schedules based on the noisy observation of the agent's action. We first review situations where the principal can reach the same utility as in the absence of noise. Then we focus on the use of linear reward schedules, which allow universal implementation, i.e. implementation of a given mechanism for any unbiased noise of observation, and on quadratic reward schedules, which only require the knowledge of the variance of the noise. We exhibit sufficient conditions under which linear reward schedules implement a given mechanism. Finally, we characterize necessary conditions for a mechanism to be implementable under noisy observation by a linear schedule, and by quadratic schedules. We give the geometric intuition behind all results.

Journal of Economic Literature classification : 026

Key words : Principal-Agent, Contracts, Adverse Selection, Moral Hazard, Linear Reward Schedules.

I. INTRODUCTION

This paper is a contribution to principal-agent contracting theory. The starting point of the analysis is a standard adverse selection problem in which we introduce the possibility of errors (by the principal or by the enforcing party) in the observation of the actions of the agent. The model combines the "hidden knowledge" aspect of adverse selection problems and the "hidden action" aspect of pure moral hazard contexts. However, we restrict attention to risk neutral agents, which eliminates the insurance question that characterizes moral hazard problems. Thus the situation can be better described as a "noisy" adverse selection situation.

This type of situation is the focus of a recent strand of contract theory, pioneered by Laffont-Tirole [1986] and developed by Picard [1987], Melumad-Reichelstein [1989] and Rogerson [1988] , among others. Its applications and relevance to real world analysis are clear and illustrated in the following two examples. When the manager of a firm is controlled by a regulatory agency or a group of shareholders, the costs and profits achieved are merely a noisy estimator of the actual decisions made by the manager on the basis of his private information about the firm and his own talent. A worker's production depends upon his intrinsic productivity, but also on the random, unobserved quality of the materials he uses and can therefore only provide an imperfect assessment of the worker's quality.

In situations of noisy adverse selection, the pure adverse selection optimum, i.e. in the absence of noise, provides a desirable benchmark for the principal. The literature has focused on situations where this benchmark is achievable despite the noise of observation. In this paper, we study situations in which, even though some variables are subject to errors of observation, some others are perfectly observed. We show that in general there exist many ways of achieving the pure adverse selection benchmark. We then investigate the possibility of imposing an additional property on the mechanism that attains this benchmark, namely to be robust to the imperfect knowledge of the distribution of the noise of observation. We characterize cases where the benchmark can be achieved despite the fact that the principal does not know the distribution of the noise except that it is unbiased : we call this property "universal implementation". Again, our reflections on universal implementation are in line with Laffont-Tirole's initial attempt. However, by considering a general setting, the analysis casts the subject in an improved perspective. Previous positive results appear as special cases, the role of the different parameters of the contracting problem can be ascertained, and the analysis of universal implementation stresses the role and the limits of linear schemes in economic contracting.

The earlier literature on contracts has been faced with the contrast between the complexity of theoretically optimal schemes and the rough simplicity of many real world arrangements (See Hart-Holmstrom [1987]). The later literature has provided a number of explanations which mitigate the discrepancy between theory and practice. In particular Holmstrom-Milgrom [1987] have shown that simple linear schemes may be optimal in pure moral hazard problems where the action variable can be continuously corrected to respond to the accrual of information. The present paper (as well as the previous literature on which it relies) can be viewed as developing a different argument for the usefulness of linear schemes. The argument is that linear schemes provide a robust implementation of an optimum in a context of multidimensional adverse selection problems where "noise" significantly affects a subset of the contractual variables.

The paper is constructed as follows. Section II presents our simple framework where some actions are perfectly observed and some actions are observed with noise. Section III reviews the body of existing results that guarantee that the principal can achieve, under noisy observation, the same utility as in the pure adverse selection situation. This short survey encompasses more general settings as presented in Melumad-Reichelstein [1989] (hereafter MR) and Caillaud-Guesnerie-Rey [1988] (hereafter CGR). It shows that in our specific framework, there may be a large number of possibilities of implementation under noisy observation, some being very demanding on the knowledge of the noise distribution, some others being more robust to the imperfection of this knowledge.

Section IV constitutes the core of the paper and analyzes the conditions under which a principal can implement an adverse selection optimum under noisy observation, when ignoring the distribution of the noise of observation of the agent's actions (except the zero mean). We insist on geometric intuition. We identify the only reward schedule that can possibly yield universal implementation (IV A), we exhibit sufficient conditions for this schedule to be indeed an acceptable reward schedule, i.e. sufficient conditions for universal implementation (IV B), and we use the local incentive compatibility required of reward schedules to provide a necessary condition for universal implementation (IV C). Finally, in subsection IV D we relax our requirement of universality, to focus on reward schedules that require the knowledge of the variance (and of the zero mean) of the distribution of the noise of observation.

II. THE MODEL

We consider a standard principal-agent model under asymmetric information, with a multi-dimensional action space and a one-dimensional information space. The principal designs a contract with the agent who has private information on one characteristic denoted by θ; θ is assumed to belong to Θ , a compact interval of $\mathbb{R}$. The contract bears on a multidimensional action $\ell \in L$ that the agent can take and on the transfer $t \in \mathbb{R}$ that he can receive from the principal. L is a compact subset of $\mathbb{R}^n$. The agent's VNM utility function depends on action ℓ and characteristic θ and exhibits risk neutrality w.r.t. revenue; it is denoted by $t + U(\ell;\theta)$.

The <u>pure adverse selection</u> (or hidden knowledge) framework refers to the situation where action ℓ is observable and verifiable (therefore contractible). Using the Revelation Principle (Myerson [1979]), the principal can restrict attention to the set of Direct Incentive Compatible Mechanisms (DICM):

Definition 1 : A Direct Incentive Compatible Mechanism is a pair of functions $(\ell(.),t(.))$ mapping the set of characteristics Θ into $L \times \mathbb{R}$, such that for any $(\theta,\theta')\in \Theta^2$:

$$t(\theta) + U(\ell(\theta);\theta) \geq t(\theta') + U(\ell(\theta');\theta)$$

A continously differentiable DICM necessarily satisfies the following "first order" and "second order" conditions for truthful revelation (see for example Guesnerie-Laffont [1984] Theorem 1, p 336) :

$$(1) \qquad \frac{dt}{d\theta} + \sum_{i=1}^{n} \partial_{\ell_i} U(\ell(\theta);\theta) \frac{d\ell_i}{d\theta} = 0$$

$$(2) \qquad \sum_{i=1}^{n} \partial_{\ell_i\theta} U(\ell(\theta);\theta) \frac{d\ell_i}{d\theta} \geqslant 0$$

For a comprehensive analysis of this type of adverse selection problems we refer the reader to Guesnerie-Laffont [1984]. In particular, as a consequence of the "taxation principle" (Hammond [1979], Guesnerie [1981]), a DICM is equivalent to a non-linear tax schedule φ :

Proposition 1 : The pair $(\ell(.),t(.))$ is a DICM if and only if there exists a mapping φ from $\mathbb{R}^n$ to $\mathbb{R}$ such that:

$$\forall\theta\in\Theta,\ \ell(\theta) \in \underset{\ell\in L}{\text{Argmax}}\, [\varphi(\ell) + U(\ell;\theta)]$$

$$t(\theta) = \varphi(\ell(\theta))$$

Such a function φ will be called a <u>$(\ell(.),t(.))$-associated schedule</u>. (In the following when there is no ambiguity we will speak of an associated schedule without explicitly refering to the DICM $(\ell(.),t(.))$ with which it is associated).

In the present problem, the locus of points (ℓ,t) in $\mathbb{R}^{n+1}$ such that $(\ell,t) = (\ell(\theta),t(\theta))$ for some $\theta\in\Theta$ is generally a one-dimensional curve which we call the "contract curve". Given a DICM $(\ell(.),t(.))$, an associated

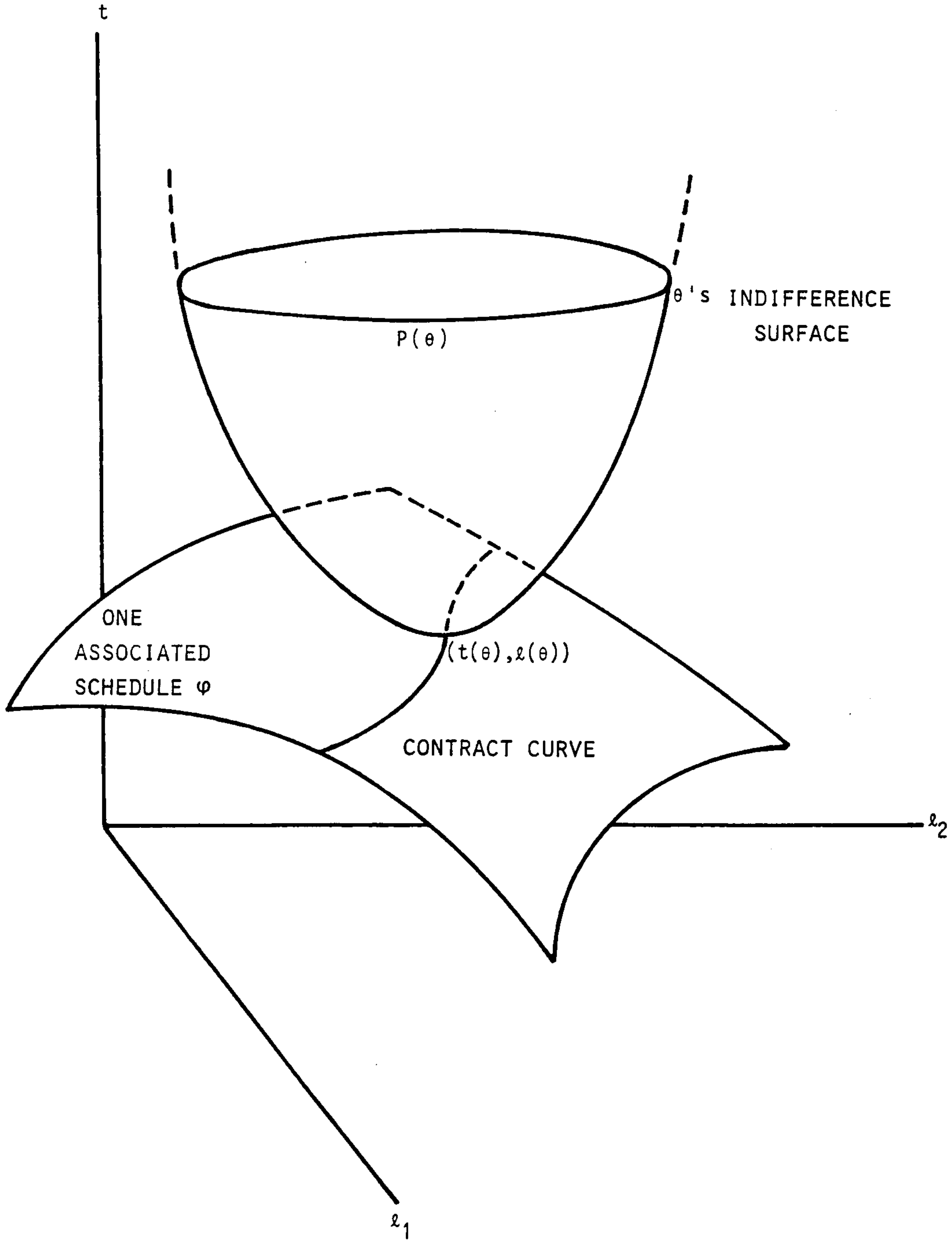


FIGURE 1 : AN ASSOCIATED SCHEDULE

schedule can be easily constructed : Take a "hypersurface" in $\mathbb{R}^{n+1}$ which contains the contract curve and such that for each type $\theta \in \Theta$, it has no intersection with the set $P(\theta)$ of actions ℓ strictly preferred by θ to the contractual action $\ell(\theta)$. (For all θ, these sets $P(\theta)$ must lie above (in the sense of increasing transfers) the hypersurface). Figure 1 visualizes such an associated schedule in the case n = 2.

Note that a $(\ell(.),t(.))$-associated schedule is uniquely defined only for $\ell \in \ell(\Theta)$, i.e. for the set of contractual actions. There is thus much freedom in the construction of an associated schedule outside $\ell(\Theta)$. In the paper, we will often make the assumption that the DICM $(\ell(.),t(.))$ under consideration satisfies :

(B) There exists a large enough penalty M^* such that:

$$\forall \theta \in \Theta, \qquad \max_{\ell \in L} U(\ell;\theta) - M^* \leq t(\theta) + U(\ell(\theta);\theta)$$

With (B), the penalty M^* will deter any agent from choosing ℓ rather than his preferred point on the contract curve. The assumption (B) allows us to build <u>"M-associated" schedules</u>, i.e. associated schedules that act as a penalty M against deviations far away from the contract curve.[1] In the limit, we can choose the associated schedule to be $\varphi(\ell) = -M$ with $M \geqslant M^*$ everywhere outside the support $\ell(\Theta)$; this would deter any agent of type θ, when facing the associated schedule $\varphi(.)$, from choosing an action ℓ outside $\ell(\Theta)$. This particular associated schedule is called a "knife-edge" associated schedule (In Figure 2, we present an associated schedule that is close to the knife-edge schedule).

Definition 2 : Given a DICM $(\ell(.),t(.))$ the M-knife edge associated schedule is defined by:

$\varphi(\ell) = t(\theta)$ if there exists $\theta \in \Theta$ such that $\ell = \ell(\theta)$,[2]

$= -M$ otherwise

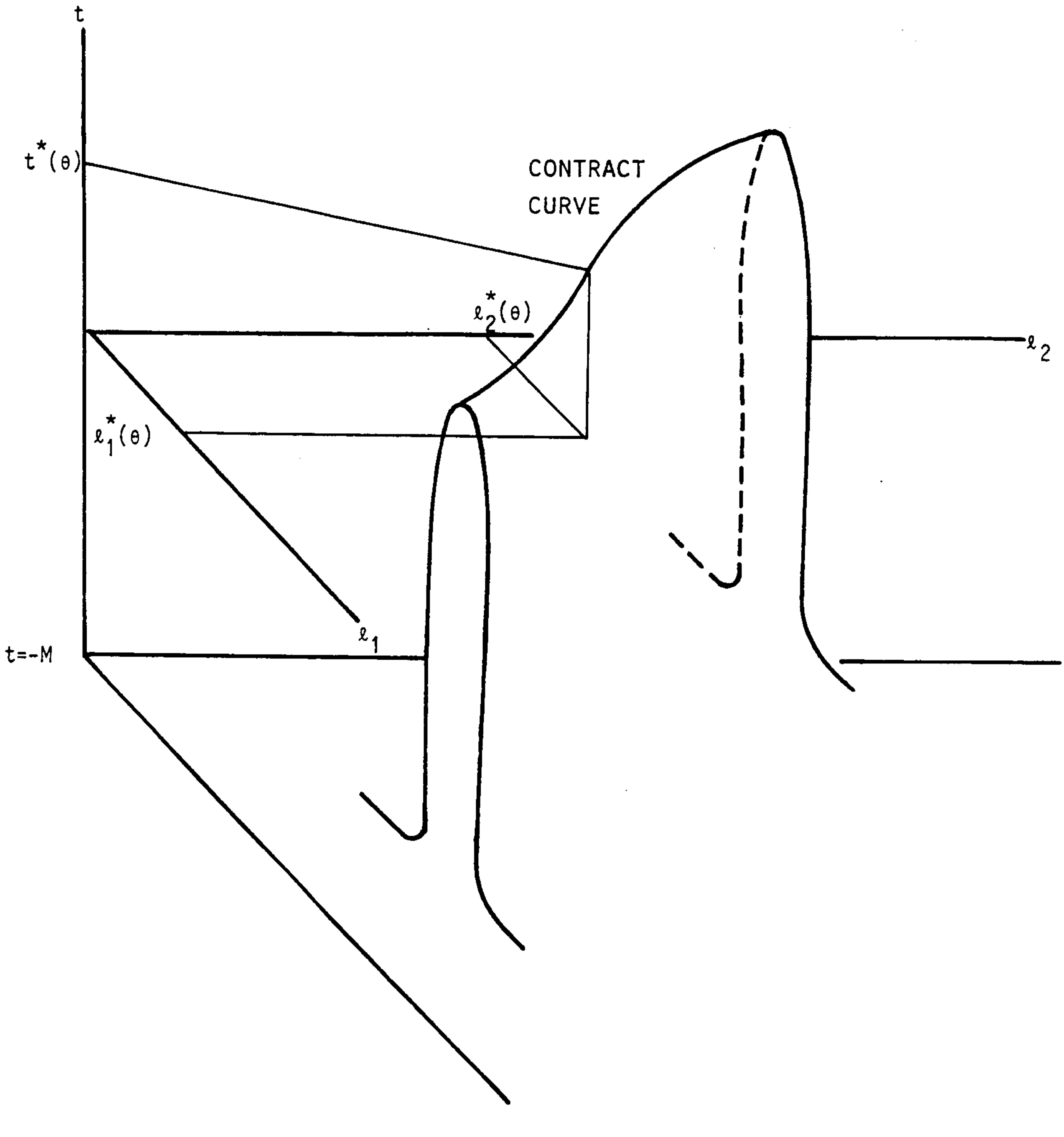


FIGURE 2 : A M-ASSOCIATED SCHEDULE
CLOSE TO THE KNIFE-EDGE SCHEDULE

<u>The (partially) noisy observation framework</u> relaxes the assumption of perfect observability of the agent's actions : we assume $\ell = (\ell_1, \ell_2) \in L = L_1 \times L_2$, where <u>$\ell_2$ is perfectly observed whereas ℓ_1 is not</u>. More precisely, L_i is a compact subset of $\mathbb{R}^{n_i}$ for $i \in \{1,2\}$ and instead of ℓ_1, the principal observes the signal $\ell_1' = \ell_1 - \varepsilon$, where the additive noise $\varepsilon \in \mathbb{R}^{n_1}$ is distributed independently of θ, according to a zero mean distribution conditional on ℓ. In Section III, we specialize to the case where ε is distributed independently of ℓ.[3]

In order to implement a given DICM $(\ell(.), t(.))$ in the noisy observation framework,[4] the principal can restrict attention, according to the revelation principle, to revelation mechanisms where an announcement $\theta \in \Theta$ determines an observable action $\ell_2(\theta)$ to be taken by the agent, and a payment $t = H(\ell_1', \theta)$ as a function of the observable signal ℓ_1'. This leads to the following definition of implementability :

Definition 3 : Under noisy observation of ℓ_1, a DICM $(\ell(.), t(.))$ <u>is implementable</u> if there is a function $H(\ell_1', \theta)$, from $L_1 \times \Theta$ to $\mathbb{R}$, such that for any θ in Θ :[5]

$$(\theta, \ell_1(\theta)) \in \underset{\tilde{\theta}, \tilde{\ell}_1}{\text{Argmax}}\ \mathbb{E}\left[H(\tilde{\ell}_1 - \varepsilon, \tilde{\theta}) + U(\tilde{\ell}_1, \ell_2(\tilde{\theta}); \theta)\right]$$

$$t(\theta) = \mathbb{E}[H(\ell_1(\theta) - \varepsilon, \theta)]$$

The mechanism discussed in the above definition consists of a menu of reward schedules $(H(.,\theta), \theta \in \Theta)$, indexed by the agent's announcement about θ. Alternatively, we could use the observable variable ℓ_2 as an index. This leads to the notion of implementability via a noisy reward schedule :

Definition 4 : A DIÇM $(\ell(.), t(.))$ is <u>implementable via a noisy reward schedule</u>, if there exists a mapping ψ from L to $\mathbb{R}$, such that:

$$\forall \theta \in \Theta,\ (\ell_1(\theta), \ell_2(\theta)) \in \underset{\ell \in L}{\text{Argmax}}\ \mathbb{E}[\psi(\ell_1 - \varepsilon, \ell_2) \mid \ell_1, \ell_2] + U(\ell_1, \ell_2; \theta)$$

$$t(\theta) = \mathbb{E}[\psi(\ell_1(\theta) - \varepsilon, \ell_2(\theta)) \mid \ell_1(\theta), \ell_2(\theta)]\ .$$

When $n_2 = 0$ or when ℓ_2 is not one-to-one, focusing on noisy reward schedules may entail some loss of generality, since it does not allow the principal to condition the agent's compensation schedule on the state of nature. A general revelation mechanism, however, allows this conditionning as seen above, provided incentive compatibility constraints are satisfied. MR investigates in detail the "value of communication", i.e. the value of using general revelation mechanisms compared with using only noisy reward schedules, and provide some examples in which this value is strictly positive. When there is a one-to-one relationship between the state of nature and the desired value of ℓ_2, we have the following adaptation of the above "taxation principle".[6]

Proposition 2 : Assume $n_2 \geqslant 1$ and consider a DICM $(\ell(.),t(.))$ such that $\ell_2(.)$ from Θ to L_2 is one-to-one. Then if this DICM is implementable, it is implementable via a noisy reward schedule ψ.

In the following, we will focus on implementation via noisy reward schedules. Section III reviews some general results on this type of implementation (irrespectively of any condition on n_2, or on the one-to-one property of $\ell_2(.)$), and addresses the following questions. When is a DICM implementable via a noisy reward schedule in the sense of Definition 4 ? How can a noisy reward schedule ψ be computed for a given specification of the noise ? Are there many noisy reward schedules ?

Section IV considers situations in which Proposition 2 holds, and focuses on "universal" implementation : Can we find reward schedules when we have only partial information on the noise distribution (e.g. its variance) ? Could there exist, and under what conditions, "universal" reward schedules, i.e. schedules that implement a DICM whatever the distribution of the observation error ?

III. IMPLEMENTATION VIA NOISY REWARD SCHEDULES.

In this section, we start from a given DICM $(\ell^*(.),t^*(.))$, and we analyze its implementability via noisy reward schedules in the sense of Definition 4. This section is a short summary of results that appeared independently in MR or in CGR.

The next proposition (for example proposition 2 in CGR) shows that the problem is equivalent to solving a functional equation that guarantees that, in expectation over ε, the noisy reward schedule gives the same incentives to the agent as a $(\ell^*(.),t^*(.))$ - associated schedule.

Proposition 3 : A DICM $(\ell^*(.),t^*(.))$ is implementable via a noisy reward schedule ψ if and only if there exists φ, a $(\ell^*(.),t^*(.))$ -associated schedule, such that for all $(\ell_1,\ell_2)\in L$

(3) $$\varphi(\ell_1,\ell_2) = \mathbb{E}[\psi(\ell_1-\varepsilon,\ell_2)|\ell_1,\ell_2]$$

We provide the (straightforward) proof for the sake of completeness. Suppose $(\ell^*(\theta),t^*(\theta))$ is implementable via ψ, then :

$$\begin{cases} \ell^*(\theta) \in \underset{(\ell_1,\ell_2)\in L}{\text{Argmax}} \ \mathbb{E}[U(\ell_1,\ell_2,\theta) + \psi(\ell_1-\varepsilon,\ell_2)|\ell_1,\ell_2] \\ \\ t^*(\theta) = \mathbb{E}\left[\psi(\ell_1^*(\theta)-\varepsilon,\ \ell_2^*(\theta)|\ell_1^*(\theta),\ell_2^*(\theta)\right] \end{cases}$$

Then $\varphi(\ell_1,\ell_2) \equiv \mathbb{E}[\psi(\ell_1-\varepsilon,\ell_2)|\ell_1,\ell_2]$ satisfies (3) and is an associated schedule. Since the agent's choice of action is subject to the same incentives with the noisy reward schedule and with the associated schedule the sufficiency part is immediate.

The associated schedule φ only reflects pure adverse selection considerations. On the contrary, ψ also takes into account the observation error. Given a DICM $(\ell^*(.),t^*(.))$, one finds the noisy reward schedules that implement $(\ell^*(.),t^*(.))$ by first choosing <u>some</u> $(\ell^*(.),t^*(.))$ - associated schedule and then solving the resulting functional equation (3). Proposition 3 holds also for multidimensional Θ ; it can take the form of a more general functional equation when ℓ_1' and ℓ_1 are supposed to be of different dimensions (see MR).

Depending on the properties of the possible associated schedules and of the noise distribution, equation (3) may or may not admit a solution, i.e. the DICM may or may not be implementable under noisy observation ; a noisy reward schedule may even be explicitly computable. In the following, we mention a number of cases in which a DICM $(\ell^*(.),t^*(.))$ is implementable under noisy observation when $n_1 = 1$ and the noise is distributed <u>independently of ℓ</u>, according to a density f(.). Some of these results could be extended in more general frameworks, e.g. when Θ is multidimensional, or $n_1 > 1$ or $n_2 = 0$ (See CGR) ; MR also provides technical results when ε is not independent of ℓ or θ.

- If the density f(.) is uniform on a compact [-a,a] and there exists a continuous M-associated schedule φ that admits a partial derivative with respect to ℓ_1, then (3) is solvable explicitly by :

$$\psi(\ell_1',\ell_2) = 2a\left[\sum_{k=0}^{\infty} \frac{\partial\varphi}{\partial\ell_1}(\ell_1'-(2k+1)a,\ell_2)\right] + G(\ell_1',\ell_2) - M$$

where G is a function periodical in ℓ_1' (period 2a) such that $\int_{-a}^{+a} G(x,\ell_2)dx = 0$ for any ℓ_2. (See CGR).

- More generally, if f is continuously differentiable on a compact support [a,b] and there exists a continuously differentiable associated schedule φ, then (3) is solvable (under a mild assumption on the relative size of [a,b] and L, see MR).

- Assume that f has a Fourier transform $\mathcal{F}(f)$ (e.g. f normal), and that there exists a $(\ell^*(.),t^*(.))$-associated schedule φ with Fourier transform $\mathcal{F}(\varphi)$ taken with respect to ℓ_1 for each ℓ_2. Whenever $\mathcal{F}(\varphi)/\mathcal{F}(f)$ has an inverse Fourier transform $\mathcal{F}^{-1}(\mathcal{F}(\varphi)/\mathcal{F}(f))$, (3) is solvable and a solution is $\psi = \mathcal{F}^{-1}(\mathcal{F}(\varphi)/\mathcal{F}(f))$ (See (CGR)).

- When there exists an associated schedule that is a polynomial in ℓ_1, then Proposition 4 below indicates the conditions under which (3) is solvable and the way to compute explicitly the solution.

Note that we could depart from the exact resolution of equation

(3) and adopt a concept of almost implementation in the spirit of MR. Basically, this approach consists in solving (3) for a function $\tilde{\varphi}$ that is well behaved and close to an associated schedule under the topology of the uniform convergence. But it is possible to adopt other topologies, in particular if we allow for unbounded, or even infinite-valued functions. As an example, when f has compact support [a,b] , ℓ_2^* is one-to-one (for simplicity), and φ is the M-knife edge schedule, the solution of (3) converges, when M goes to infinity, towards a "Mirrlees scheme" : $\psi(\ell_1', \ell_2) = t^*(\theta)$ if $\ell_2 = \ell_2^*(\theta)$ and $\ell_1' \in \left[\ell_1^*(\theta)-b, \ell_1^*(\theta)-a\right]$, $= -\infty$ otherwise. This limit noisy reward schedule implements the DICM under the noise ε. (As is known, this construction extends for example to normal distributions).

The main conclusion of these results is that there is much leeway in the choice of noisy reward schedules that implement a DICM for two reasons. First there is much freedom in solving the convolution equation (3) for a given $(\ell^*(.), t^*(.))$-associated schedule. Second, one can choose any associated schedule provided it satisfies some regularity and integrability properties. This last requirement is intuitively weak, and we will give a more precise assessment of this fact later.

The question is now : can we exploit the freedom in the choice of reward schedules to impose on them some additional desirable properties. An interesting property [7] would be that the noisy reward schedules do not heavily depend on the specification of the noise density f(.). Indeed, the information on f(.) required to compute a noisy reward schedule, may crucially depend on the associated schedule we are starting from. This fact is clearly illustrated by the next proposition.

Proposition 4 : Assume that $n_1 = 1$ and the distribution of the noise ε is independent of ℓ. Let us also suppose that for every ℓ_2, the associated schedule is a polynomial in ℓ_1 of degree smaller or equal to m. Then there exists a noisy reward schedule ψ which, for a given ℓ_2 , is a polynomial of degree smaller or equal to m (in ℓ_1'), the coefficients of which only depend on the moments of the noise distribution of order smaller or equal to m.

Proof : Suppose $\varphi(\ell_1,\ell_2) = \sum_{p\leqslant m} a_p(\ell_2)\ell_1^p$ and let us look for a solution of the form $\psi(\ell_1',\ell_2) = \sum_{p\leqslant m} b_p(\ell_2)\ell_1'^p$. Denoting C_q^p the binomial coefficient $\binom{p}{q}$, the convolution equation can be written : $\forall(\ell_1,\ell_2)\in L$,

$$\sum_{p\leqslant m} a_p(\ell_2)\ell_1^p = \int \sum_{p\leqslant m} b_p(\ell_2)(\ell_1-\varepsilon)^p f(\varepsilon)d\varepsilon$$

$$= \sum_{p\leqslant m} \int b_p(\ell_2)\left[\sum_{q=0}^{p} C_p^q \ell_1^q (-\varepsilon)^{p-q}\right] f(\varepsilon)d\varepsilon$$

$$= \sum_{r\leqslant m} \ell_1^r \left\{\sum_{s=r}^{m} C_s^r\ b_s(\ell_2)\int (-\varepsilon)^{s-r} f(\varepsilon)d\varepsilon\right\}$$

Identifying each coefficient in this triangular, non-degenerate, linear system only requires the knowledge of all moments of ε of order at most m.

Q.E.D.

Proposition 4 shows that when there exists a $(\ell^*(.),t^*(.))$-associated schedule which is a surface such that all sections by an hyperplane $\ell_2 = C$ (where C is a constant) are polynomial of order smaller than m, then one only needs to know the m first moments of the distribution of ε to compute a noisy reward schedule that implements the DICM. This is much less demanding than knowing the whole distribution f(.) of ε, as would be required to compute a reward schedule using e.g. a Fourier transform. Note that the independence assumption could be relaxed. What is only needed is that the m first moments of ε do not depend on ℓ.

IV. NECESSARY AND SUFFICIENT CONDITIONS FOR UNIVERSAL IMPLEMENTATION

IV.A/Ruled Schedules and Universal Implementation

The less exhaustive the information on f(.) required to compute a reward schedule, the more robust the reward schedule to mispecifications in the observation disturbance. Robustness is typically a major concern in contract theory, since applications to the real world cannot reasonably assume a complete knowledge of the distribution of noises in the economy. There are then strong motivations for finding simple associated schedules, and thereby simple noisy reward schedules, so that implementation under noisy observation be possible despite a limited knowledge of the distribution f(.). In particular, from the proof of Proposition 4, a noisy reward schedule derived from a linear-in-section (affine in ℓ_1 for each ℓ_2) associated schedule is itself linear-in-section and is obtained by moving upwards the associated schedule by an amount equal to the expectation of ε. This implies that a linear-in-section associated schedule is itself a noisy reward schedule for any unbiased noise. The schedule has then some "universal" validity as defined below :

Definition 6 : A noisy reward schedule is universal if it implements the DICM for any unbiased noise of observation.[8]

We saw that linear-in-section associated schedules are universal noisy reward schedules. We now show that they are in fact the only possible ones. For simplicity we will assume $n_1 = n_2 = 1$, but now again ε and ℓ may be correlated. We will also assume that the agent's preferences are smooth :

(D) U(.) is twice continuously differentiable.

We will also focus on a DICM $(\ell^*(.), t^*(.))$ that satisfies the following monotonicity property :

(MO) $\ell_2^*(.)$ is one-to-one from Θ to $\ell_2^*(\Theta) \subset L_2$.

Assumption (MO) is made partly for ease of presentation : we shall later discuss its precise role in the analysis.

In the present setting, there is essentially a unique candidate for being a linear-in-section associated schedule. We call it the "ruled schedule" and define it as follows.

Definition 5 : Under (D), (MO) and (B), the "<u>M-ruled schedule</u>" is defined for $M \geqslant M^*$ as the surface

$$t = \varphi(\ell_1,\ell_2) \equiv t^*(\theta) - \partial_{\ell_1} U(\ell^*(\theta);\theta)\left[\ell_1 - \ell_1^*(\theta)\right] \quad \text{if } \ell_2 = \ell_2^*(\theta) \text{ for some } \theta\epsilon\Theta$$

$$\equiv - M \quad \text{otherwise}$$

The ruled schedule is built as follows. For every ℓ_2 that can be attained by the DICM, i.e. that is a contractual action for some type θ (unique by (MO)), consider the intersection between the tangent hyperplane to this type θ's indifference surface passing through $(\ell^*(\theta),t^*(\theta))$, and the hyperplane of coordinate ℓ_2. The intersection defines a line, and as ℓ_2 varies in $\ell_2^*(\Theta)$, these lines generate a <u>ruled surface</u>. This ruled surface is then extended using a penalty as described in the discussion of (B). In the following, we will omit the details of the extension outside $\ell_2^*(\Theta)$ and we will refer to <u>the</u> ruled surface. This construction is visualized in Figure 3.

The next proposition shows that the ruled schedule is the only possible schedule leading to universal implementation.

Proposition 5 : Under (B), (D) and (MO), there exists an universal noisy reward schedule if and only if the ruled schedule is an associated schedule. In this case, the ruled schedule is itself an universal noisy reward schedule.

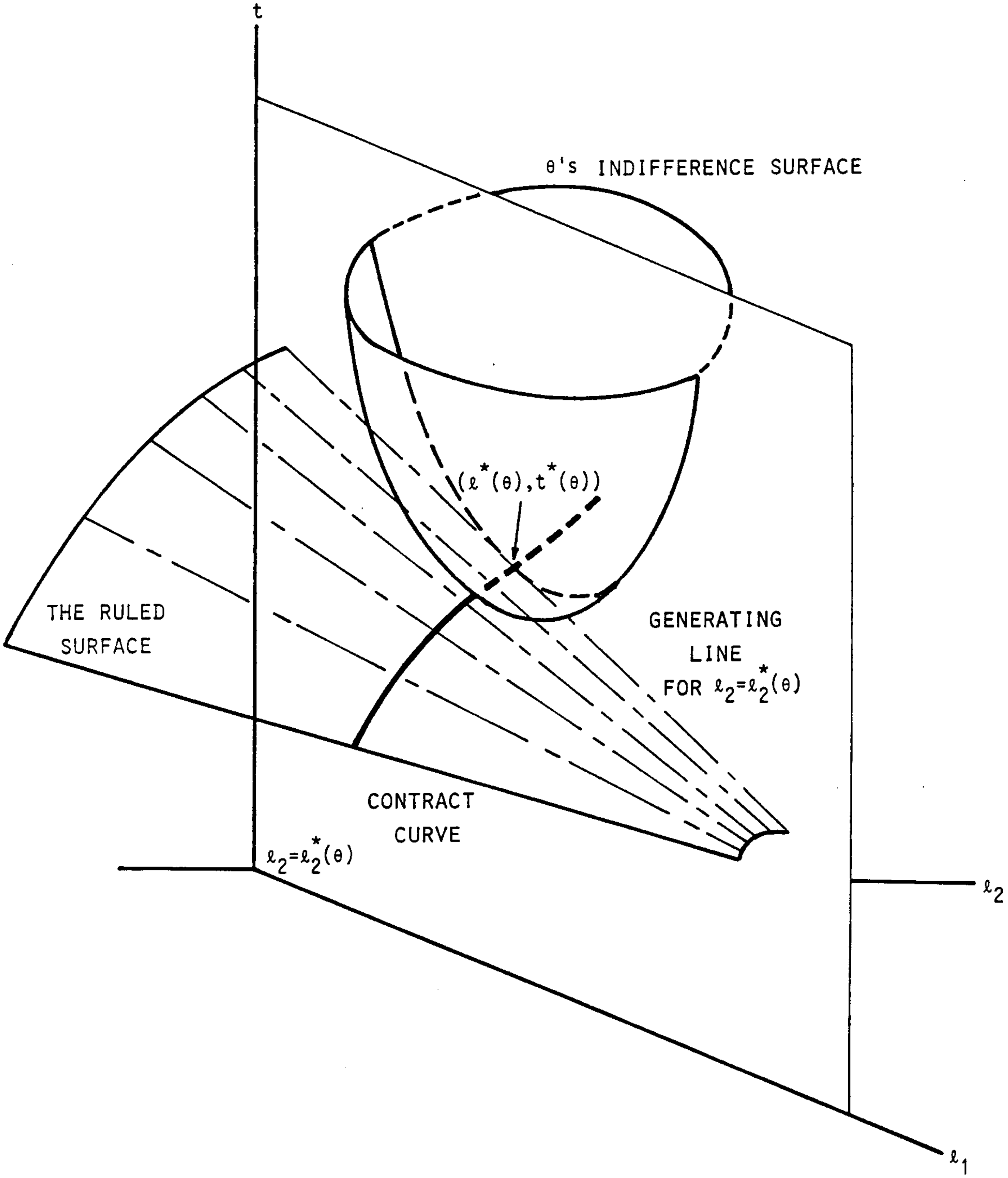


FIGURE 3 : CONSTRUCTION OF THE RULED SURFACE

Proof : The "if" part has been already proved. We therefore focus on the "only if" part.

For any unbiased distribution of ε, a universal schedule ψ must satisfy

(4) $\quad \forall\theta\in\Theta,\ \mathbb{E}\left[\psi(\ell_1^*(\theta)-\varepsilon,\ell_2^*(\theta))\,|\,\ell^*(\theta)\right] = t^*(\theta)$

By choosing the Dirac distribution at $\varepsilon = 0$, the previous equality implies :

(5) $\quad \forall\theta\in\Theta,\ \psi(\ell_1^*(\theta),\ell_2^*(\theta)) = t^*(\theta)$

Next, it is easy to show that if ψ is universal, then it must be affine in ℓ_1', i.e. for any $(\varepsilon_1,\varepsilon_2)\in\mathbb{R}^2_{+*}$, the points $\left\{\ell_1^*(\theta),\psi(\ell_1^*(\theta),\ell_2^*(\theta))\right\}$, $\left\{\ell_1^*(\theta)-\varepsilon_1,\psi(\ell_1^*(\theta)-\varepsilon_1,\ell_2^*(\theta))\right\}$ and $\left\{\ell_1^*(\theta)+\varepsilon_2,\psi(\ell_1^*(\theta)+\varepsilon_2,\ell_2^*(\theta))\right\}$ are on the same line. If this were not the case, then one could find a probability distribution putting almost all the weight on neighborhoods of ε_1 and ε_2, of zero mean, and such that (4) is wrong.

So ψ is universal only if : $\psi(\ell_1',\ell_2^*(\theta)) = t^*(\theta)+k(\theta)(\ell_1'-\ell_1^*(\theta))$. Finally the condition that a θ-agent chooses $\ell^*(\theta)$ (i.e. that ψ be a noisy reward schedule) implies that $k(\theta) = -\partial_{\ell_1}U(\ell^*(\theta);\theta)$.

Q.E.D.

IV.B/Sufficient Conditions for Universal Implementation

We can now come to the heart of the analysis : For a given DICM when is the ruled schedule an associated schedule ? We first give sufficient conditions for having this property.

Proposition 6 : Assume that (B), (D), (MO) hold. If one of the following set of properties holds, then the ruled schedule is a $(\ell^*(.),t^*(.))$ -associated schedule (and hence is a universal noisy reward schedule) :

<u>either</u> : i1) Preferences are independent of ℓ_2.

i2) The DICM (ℓ^*,t^*) is continuously differentiable.

i3) $\exists$ an associated schedule $\bar{\varphi}$ independent of ℓ_2 and <u>convex in</u> ℓ_1 on L_1.

<u>or</u> : ii1)The restriction of the ruled surface to $\ell_2^*(\Theta)$ is a subset of an hyperplane.

ii2)The agent's preferences are convex.

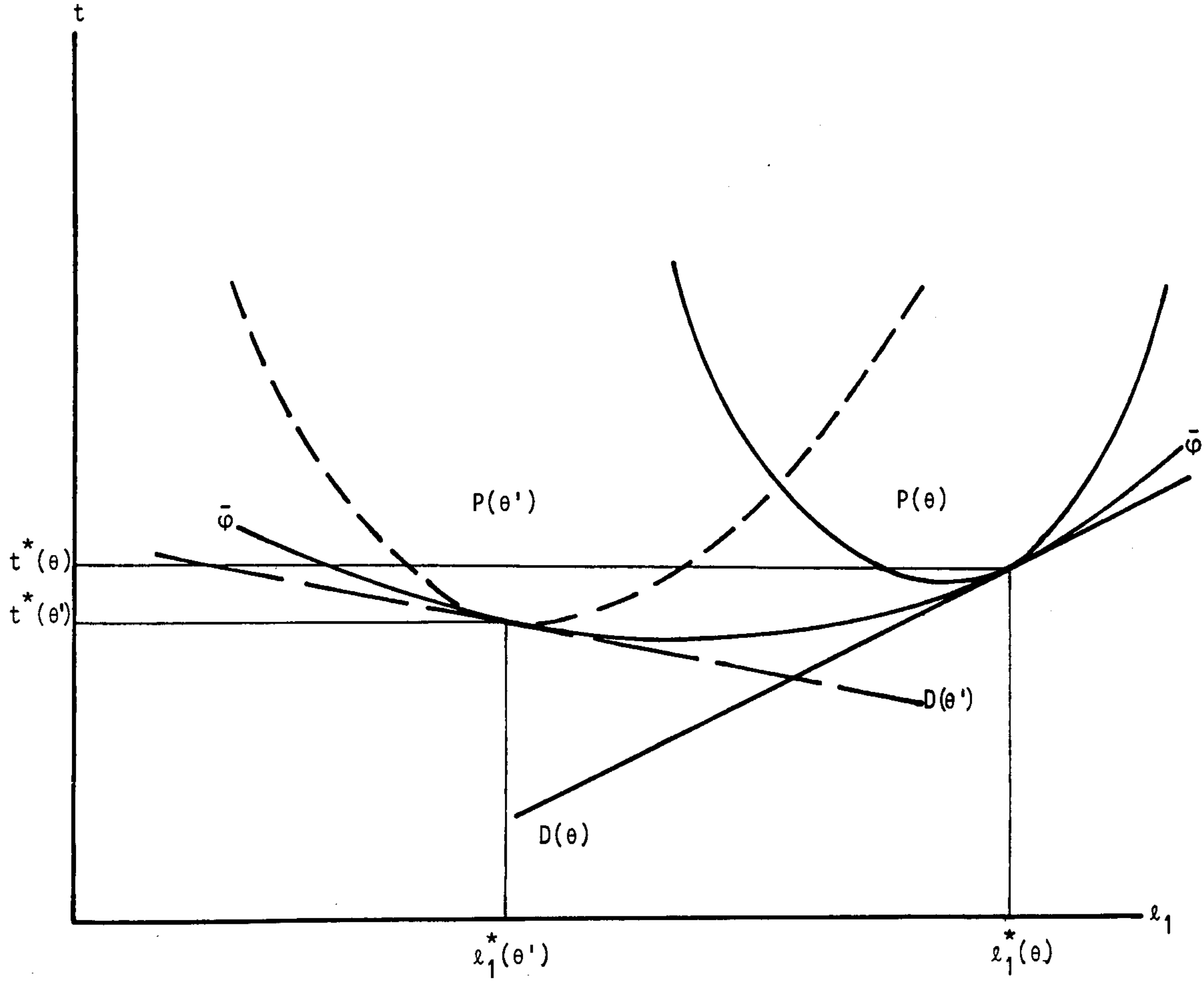


FIGURE 4 : ILLUSTRATION OF PROPOSITION 6I)

Proof : i)As the agent's utility does not depend on ℓ_2, we simply write it as $U(\ell_1,\theta)$. Let us define

$$D(\theta) = \left\{(\ell_1,t)\in\mathbb{R}^2 \mid t = t^*(\theta) - \partial_{\ell_1}U(\ell_1^*(\theta);\theta)\left[\ell_1 - \ell_1^*(\theta)\right]\right\}$$

$$P(\theta) = \left\{(\ell_1,t)\in\mathbb{R}^2 \mid t + U(\ell_1;\theta) > t^*(\theta) + U(\ell_1^*(\theta);\theta)\right\}$$

$P(\theta)$ is the set of points (ℓ_1,t) that are strictly preferred to $(\ell_1^*(\theta),t^*(\theta))$ by a θ-agent. $D(\theta)$ is the projection in the plane (ℓ_1,t) of the generating line of the ruled surface passing through $(\ell^*(\theta),t^*(\theta))$. Since U is independent of ℓ_2, the agent's indifference surfaces are cylinders parallel to the ℓ_2-axis ; then, the ruled schedule is an associated schedule if and only if : $D \cap P = \varnothing$, where $D = \bigcup_{\theta\in\Theta} D(\theta)$ and $P = \bigcup_{\theta\in\Theta} P(\theta)$. (See Figure 4).

Consider the associated schedule $\bar{\varphi}$, and let $\bar{Z}$ be its epigraph on L_1, (i.e. $\left\{(\ell_1,t)\in L_1\times \mathbb{R} \mid t \geqslant \bar{\varphi}(\ell_1)\right\}$). Using the definition of an associated schedule, it can be shown by contradiction that $P(\theta)\subset \text{int}\bar{Z}$.

Moreover from (D) and the regularity of the DICM, $D(\theta)$ is <u>the</u> tangent line to the lower boundary of $\bar{Z}$ at point $(\ell_1^*(\theta),t^*(\theta))$. Then from the convexity of $\bar{Z}$, $D(\theta) \cap \text{Int}\,\bar{Z}$ is empty. It follows that for any θ $D(\theta) \cap P = \varnothing$ and then $D \cap P = \varnothing$.

ii) If an agent chooses $(\ell^*(\theta),t^*(\theta))$, where $\ell_2^*(\theta)\in \text{Int}\,(\ell_2^*(\Theta))$, the corresponding indifference surface is tangent to the hyperplane. As $P(\theta)$ is convex for all θ, it never intersects the hyperplane.

In the case where $(\ell^*(\theta),t^*(\theta))$ is such that $\ell_2^*(\theta)$ is on the boundary of $\ell_2^*(\Theta)$, the conclusion remains if M is large enough.

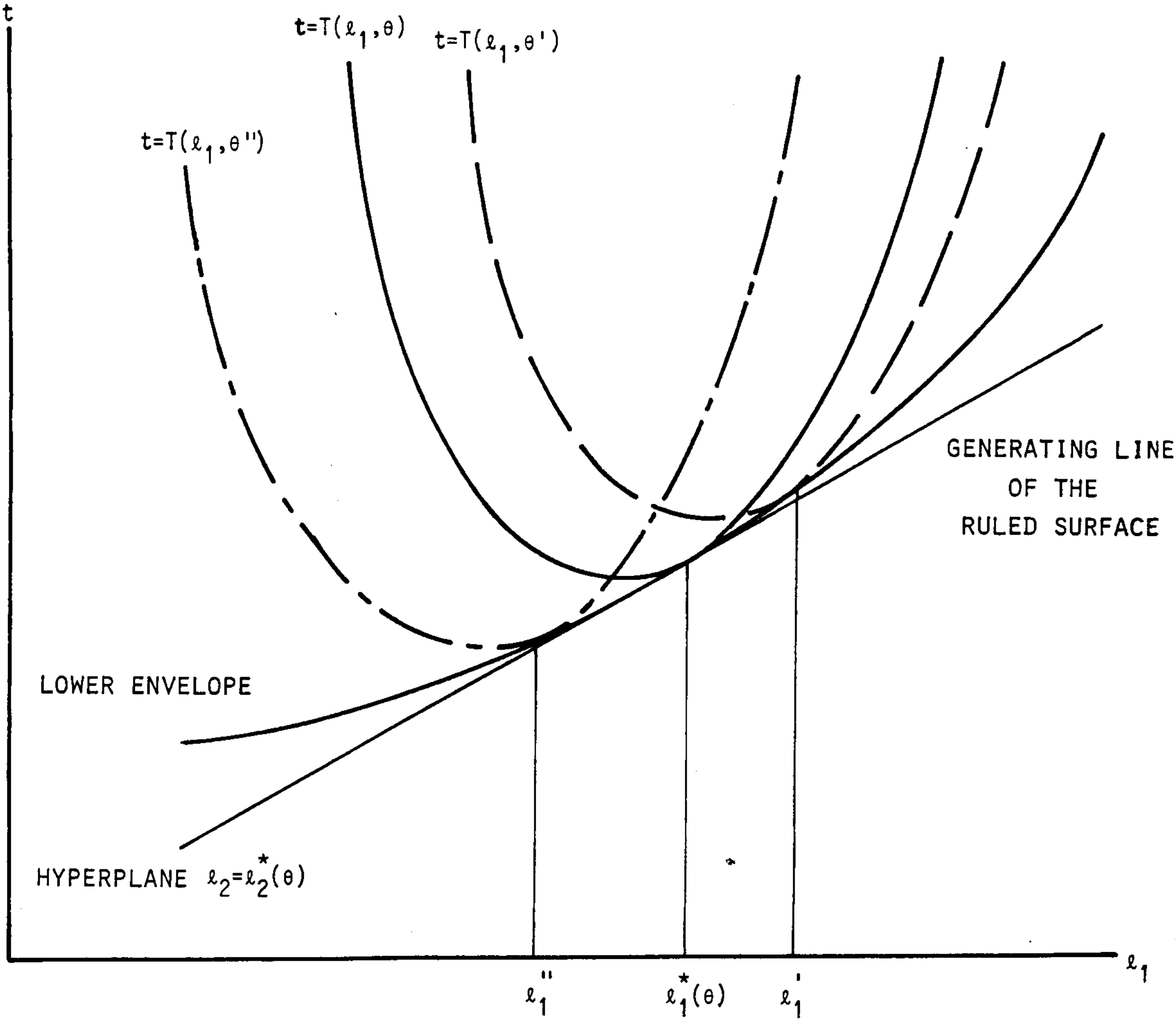
Q.E.D.

In 6i), the requirement that preferences be independent of ℓ_2 is restrictive, but the assumption that $\bar{\varphi}$ does not depend on ℓ_2 is innocuous. One could take the projection of the contract curve on an hyperplane of coordinate $\ell_2 = C$ (where C denotes a constant), and construct a cylinder orthogonal to that hyperplane with identical section ; this cylinder would be an associated schedule independent of ℓ_2. A crucial assumption however, is that $\bar{\varphi}$ be convex (in ℓ_1). Given that any generating line of the ruled surface, tangent to the section of θ's indifference surface in the hyperplane $\ell_2 = C$, is also tangent to $\bar{\varphi}$ in $(\ell_1^*(\theta), t^*(\theta))$, the convexity of $\bar{\varphi}$ guarantees that no type θ' has a preferred point below (or on) the generating line, therefore below (or on) the ruled surface. The independence of preferences on ℓ_2 makes our model equivalent to Picard [1987]'s unidimensional problem of implementation via a family of linear schedules, and our condition of convexity is similar to his condition. It should be noted that the case is more general than it might appear. Indeed if $U(\ell;\theta) = U_1(\ell_1,\ell_2) + U_2(\ell_1;\theta)$, a similar argument would apply : what matters is that the cross derivative of U with respect to ℓ_2 and θ be zero, for in this case, the problem is essentially a one-dimensional incentive problem with action ℓ_1 and transfers $t + U_1(\ell_1,\ell_2)$.

Proposition 6ii) allows the agent's preferences to depend on ℓ_2 but at the cost of strong conditions on the ruled surface, namely the fact that it is an hyperplane.

IV.C/Necessary Condition for Universal Implementation

Thus we have identified a small subset of problems for which universal implementation holds. We now characterize a broader class of problems where this property might hold. These problems are those for which a necessary condition for implementation is satisfied. The intuition behind this necessary condition is first explained.



$$\theta' = \hat{\theta}(\ell_1')$$
$$\theta'' = \hat{\theta}(\ell_1'')$$

FIGURE 5 : NECESSARY CONDITION FOR UNIVERSAL IMPLEMENTATION

THE ENVELOPE CURVE

Consider the hyperplane $\ell_2 = \ell_2^*(\theta)$ and the sections in this hyperplane of all θ'-indifference surfaces going through $(\ell^*(\theta'), t^*(\theta'))$, for θ' close to θ. Three of these sections are depicted on Figure 5, one for θ and two for θ' and θ'' close to θ. Note that these sections are (implicitly) defined by equation (6) :

(6) $$t \equiv T(\ell_1,\theta') \equiv t^*(\theta') + U(\ell^*(\theta');\theta') - U(\ell_1,\ell_2^*(\theta)\ ;\ \theta')$$

The diagram suggests that the family of curves so defined has a lower envelope. The necessary condition which we will stress is that the lower envelope be locally above the section of the ruled schedule, i.e. above the tangent in $(\ell_1^*(\theta),t^*(\theta))$ to the section of θ's indifference surface in $(\ell^*(\theta),t^*(\theta))$: Figure 5 shows that if this condition were not satisfied, one agent θ' would prefer some point on the generating line of the ruled surface going through $(\ell^*(\theta),t^*(\theta))$ to his own bundle $(\ell^*(\theta'),t^*(\theta'))$. Therefore this condition is needed for universal implementation. The necessary condition will express the requirement that the epigraph of the lower envelope be convex i.e. that the second derivative of the function defining the envelope be positive.

Assuming enough regularity, we know that the equation of the lower envelope is $t = T(\ell_1,\hat\theta(\ell_1))$, where $\hat\theta(\ell_1)$ is implicitly defined by (7) :

(7) $$\partial_{\theta'}T(\ell_1,\hat\theta(\ell_1)) = 0$$

We can now compute the second derivative of the equation of the lower envelope, which we denote $\frac{d^2t}{d\ell_1^2}$. With straightforward notation, we have :

$$\frac{d^2t}{d\ell_1^2} = \frac{d}{d\ell_1}\left[\partial_{\ell_1}T + \partial_{\theta'}T\,\frac{d\hat\theta}{d\ell_1}\right]_{(\ell_1,\hat\theta(\ell_1))}$$

$$= \frac{d}{d\ell_1}\left[\partial_{\ell_1}T\right]_{(\ell_1,\hat\theta(\ell_1))}$$

$$= \partial_{\ell_1\ell_1}T(\ell_1,\hat\theta(\ell_1)) + \partial_{\ell_1\theta'}T(\ell_1,\hat\theta(\ell_1))\,\frac{d\hat\theta}{d\ell_1}$$

Moreover, differentiating (7) yields :

$$\partial_{\theta'\ell_1}T(\ell_1,\hat\theta(\ell_1)) + \partial_{\theta'\theta'}T(\ell_1,\hat\theta(\ell_1))\,\frac{d\hat\theta}{d\ell_1} = 0$$

So that :

(8) $$\frac{dt^2}{d\ell_1^2} = \partial_{\ell_1\ell_1}T(\ell_1,\hat{\theta}(\ell_1)) - \frac{\left[\partial_{\ell_1\theta'}T(\ell_1,\hat{\theta}(\ell_1))\right]^2}{\partial_{\theta'\theta'}T(\ell_1,\hat{\theta}(\ell_1))}$$

an expression that only depends on the derivatives of T and therefore of U. Differentiating repeatedly (6) one finds

$$\partial_{\ell_1\ell_1}T(\ell_1,\theta') = -\partial_{\ell_1\ell_1}U(\ell_1,\ell_2^*(\theta),\theta')$$
$$\partial_{\ell_1\theta'}T(\ell_1,\theta') = -\partial_{\ell_1\theta}U(\ell_1,\ell_2^*(\theta),\theta')$$

Differentiating (6) and using the incentive compatibility condition on the contract curve (that implies that the derivative of the agent's objective function with respect to his announcement is <u>identically</u> equal to zero, i.e. (1)) one gets :

$$\partial_{\theta'}T(\ell_1,\theta') = \partial_\theta U(\ell^*(\theta'),\theta') - \partial_\theta U(\ell_1,\ell_2^*(\theta),\theta')$$

So that

$$\partial_{\theta'\theta'}T(\ell_1^*(\theta),\theta) = \Gamma(\theta)$$

where $\Gamma(\theta) = \partial_{\ell_1\theta}U(\ell^*(\theta),\theta)\,\frac{d\ell_1^*}{d\theta} + \partial_{\ell_2\theta}U(\ell^*(\theta),\theta)\,\frac{d\ell_2^*}{d\theta}$

Finally taking all derivatives in $\ell_1 = \ell_1^*(\theta)$, $\theta' = \hat{\theta}(\ell_1^*(\theta)) = \theta$, one obtains :

$$\frac{d^2t}{d\ell_1^2} = -\partial_{\ell_1\ell_1}U(\ell^*(\theta),\theta) - \frac{\left[\partial_{\ell_1\theta}U(\ell^*(\theta),\theta)\right]^2}{\Gamma(\theta)}$$

where as recalled after definition 1, $\Gamma(\theta)$ has to be positive, for incentive compatibility reasons.

The necessary condition we are looking for, is obtained by requiring that $\frac{d^2t}{d\ell_1^2}$ is positive. The proof sketched above is made fully rigorous in appendix and justifies our main theorem :

Theorem : Assume that (B), (D) and (MO) hold. Consider a DICM $(\ell^*(.),t^*(.))$ which is continuously differentiable and a.e. twice differentiable. The ruled schedule is an associated schedule (allowing universal implementation) <u>only</u> if the following condition holds :

(9) $$\forall\,\theta\in\Theta,\ \left[\partial_{\ell_1\theta}U(\ell^*(\theta),\theta)\right]^2 \leqslant -\,\partial_{\ell_1\ell_1}U(\ell^*(\theta),\theta)\,\Gamma(\theta)$$

where $\Gamma(\theta) = \partial_{\ell_1\theta}U(\ell^*(\theta),\theta)\,\frac{d\ell_1^*}{d\theta} + \partial_{\ell_2\theta}U(\ell^*(\theta),\theta)\,\frac{d\ell_2^*}{d\theta}$.

Before going further, we should convince ourselves that the above formula fits the intuition of the phenomenon, as developed throughout the proof. For that let us consider the three terms in (9) and check that they indeed play in the direction one should expect.

First the LHS of (9) is non negative, and if $(\ell^*(.),t^*(.))$ is a DICM, then necessarily $\Gamma(\theta)$ is non negative. So the term $\left[-\partial_{\ell_1\ell_1}U(\ell^*(\theta);\theta)\right]$ must be non negative for the ruled schedule to be a $(\ell^*(.),t^*(.))$-associated schedule. Indeed, if the indifference surface of a θ-agent had a section in the hyperplane $\ell_2 = \ell_2^*(\theta)$ which were not locally concave around the contractual point $(\ell^*(\theta),t^*(\theta))$, then the ruled surface would not be an associated schedule since it would require the agent to choose an action ℓ_1 that minimizes (locally) his utility. Now, given that $\left[-\partial_{\ell_1\ell_1}U(\ell^*(\theta);\theta)\right]$ must be positive, it increases with the curvature of the indifference surface of the agent in ℓ_1. Everything else being given, the higher the curvature of the family of curves, the better the chance that the lower envelope of this family be itself convex. When the term $\left[-\partial_{\ell_1\ell_1}U(\ell^*(\theta');\theta')\right]$ increases, a θ'-agent will require a higher compensation to deviate to the point $\ell^*(\theta)$ for θ' close to θ, and, for highly convex preferences in ℓ_1, local implementation by the ruled schedule will be possible.

Second, it has already been said that the term $\Gamma(\theta)$ is non-negative. Consider now the transfer needed to make a θ'-agent choose $(\ell^*(\theta),t^*(\theta))$ instead of his contractual point : it is given by

$$\{t^*(\theta') + U(\ell^*(\theta') \; ; \; \theta')\} - \{t^*(\theta) + U(\ell^*(\theta);\theta')\}$$

For θ' close to θ, this expression is equal to $\Gamma(\theta)(d\theta)^2$ to the second order. Thus $\Gamma(.)$ "measures" the distance in terms of transfers between the point chosen by θ-agents and the indifference curve of neighbor agents. Increasing this distance is a favorable factor for the convexity of the epigraph of the lower envelope of all these curves, i.e. for the ruled schedule not to allow this amount of transfers that would trigger local deviations. This phenomenon conforms to expression (9) in the theorem.

Finally $|\partial_{\ell_1\theta}U(\ell^*(\theta);\theta)|$ must be as small as possible for the ruled schedule to be an associated schedule. The intuition here can be understood as follows. $\partial_{\ell_1}U(.)$ measures the slope of the ruled surface or/and of the indifference surface of θ-agents in the hyperplane $\ell_2^*(\theta)$, so that $|\partial_{\ell_1\theta}U|$ measures the speed of rotation of the generating lines of the ruled schedule when θ varies. Given the (local) convexity of θ's indifference surface, it is clear that a high speed of rotation will imply that close to $(\ell^*(\theta),t^*(\theta))$, the generating lines of the ruled surface will intersect θ's indifference surface, thereby ruling out the ruled surface as an associated schedule.

To obtain a better understanding of the theorem, it is useful to specialize it to the case where preferences are independent of ℓ_2. We know from Proposition (6i) that if the DICM is continuously differentiable, the convexity in ℓ_1 of an associated schedule is a sufficient condition for the ruled schedule to implement the DICM as an associated schedule. The next corollary shows that it is equivalent to condition (9) so that (9) (or the convexity of an associated schedule) is a necessary and sufficient condition for universal implementation.

Corollary 1. Under the conditions of the Theorem, if preferences are independent of ℓ_2, then condition (9) is a necessary and sufficient condition for universal implementation.

Proof : (9) is a necessary condition for universal implementation by the previous theorem. Next we prove that (9) implies the existence of an associated schedule satisfying 6i3) (convex in ℓ_1, independent of ℓ_2).

For that consider the projection of the contract curve in the plane (ℓ_1,t) : it is a C^1-manifold. Moreover as $\ell_1^*(\theta_1)=\ell_1^*(\theta_2) \Longrightarrow t^*(\theta_1)=t^*(\theta_2)$, it can be written as a function $t = \mathfrak{T}(\ell_1)$. From incentive compatibility

$\frac{dt^*}{d\theta} + \partial_{\ell_1} U\big(\ell_1^*(\theta);\theta\big) \frac{d\ell_1^*}{d\theta} = 0$. First, since $\partial_1 U$ is bounded (on the compact $L_1 \times \Theta$), $\mathfrak{T}$ is C^1. Second, for almost all ℓ_1, the local inverse $\theta^*(\ell_1)$ of $\ell_1^*(\theta)$ is defined, and $\mathfrak{T}'(\ell_1) = -\partial_{\ell_1} U\big(\ell_1, \theta^*(\ell_1)\big)$ and $\mathfrak{T}''(\ell_1) = \left[- \partial_{\ell_1 \ell_1} U(\ell_1, \theta^*(\ell_1)) \frac{d\ell_1^*}{d\theta} - \partial_{\ell_1 \theta} U(\ell_1, \theta^*(\ell_1))\right] / \frac{d\ell_1^*}{d\theta}$ exists.

Condition (9) is equivalent to $\mathfrak{T}''(\ell_1) \geqslant 0$ for almost every ℓ_1, given that $\partial_{\ell_1 \theta} U(\ell_1, \theta^*(\ell_1)) \frac{d\ell_1^*}{d\theta} \geqslant 0$ by incentive compatibility.

Now $\mathfrak{T}$ being C^1 and with a positive second derivative almost every where is globally convex.

It is now easy to take the cylinder generated by the section $t = \mathfrak{T}(\ell_1)$, i.e. the cylinder embedding the contract curve. It obviously is an associated schedule, and we showed that it is convex. Hence it satisfies 6i3).

Now applying proposition 6i), we know that the ruled surface is an associated schedule, hence universal implementation holds. So (9) is also sufficient.

Q.E.D.

The necessary requirement for universal implementation in the case where preferences are independent of ℓ_2 coincides with the sufficient condition of Proposition (6i) and can be satisfied only by special DICM (those with associated schedules which are convex in the sense of Proposition 6). This stringent restriction relates to the special form of preferences under consideration. (In fact, for any a priori given DICM, one can find preferences for which the DICM can be universally implemented.)

Let us examine how our Theorem and Corollary 1 relate to the existing literature on linear implementation. In Laffont-Tirole's [1986] regulation model, the manager's objectives only depend upon his personal effort (e) and on the regulator's transfer (t) : t-H(e). The effort e reduces the unit cost of production of the firm, and is moreover assumed to be a perfect substitute for the cost parameter θ : the unit cost is equal to $c = \theta - e$.

Laffont and Tirole assume that the level of production q is perfectly observable whereas the unit cost is observable with some noise. Setting $\ell_1 = \theta - e$ and $\ell_2 = q$, the agent's utility function does not depend upon ℓ_2 and can be writen in our framework with : $U(\ell_1, \ell_2, \theta) = H(\theta - \ell_1)$. Since the agent's preferences do not depend upon ℓ_2, condition (9) is not only necessary, but also sufficient for universal implementation of "smooth" DICM's (in particular, it implies that there exists an associated schedule which is convex in ℓ_1). Because of the substitutability between θ and ℓ_1, condition (9) boils down to : $\frac{d\ell_1^*}{d\theta} \geqslant 1$.

Laffont and Tirole also consider a particular type of objective function for the principal, which is linear in ℓ_1 and independent of θ (the regulator's utility function is given by : $S(\ell_1, \ell_2, \theta) = s(\ell_2) - (1+\lambda)\ell_1\ell_2$). Altogether, these assumptions imply that the optimal DICM satisfies $\frac{d\ell_1^*}{d\theta} \geqslant 1$. Note however that modifying the principal's objective (e.g., having $\partial_\theta S > 0$ or $\partial_{\ell_1\ell_2} S > 0$ would not affect the necessary and sufficient condition (9), but could lead the optimal DICM to violate this condition. In this case, linear implementation would fail.

Apart from Laffont-Tirole [1986] , most other examples in the literature dealing with linear schemes, focus on $n_1 = 1$ and $n_2 = 0$, and study reward schedules that depend on an announcement : $\varphi(\ell_1', \theta')$, and that lead to truthful revelation. Given (MO), our results still hold in such a framework, and condition (9) is necessary and sufficient for universal implementation and takes the same form except that $\Gamma(\theta) = \partial_{\ell_1\theta} U(\ell^*(\theta), \theta) \frac{d\ell_1^*}{d\theta}$. In Picard [1987] , $U(\ell_1, \theta) = -H(\ell_1 + \theta)$. So that : $\partial_{\ell_1\theta} U = -H''(\ell_1^*(\theta) + \theta) \leqslant 0$ and therefore $\frac{d\ell_1^*}{d\theta} \leqslant 0$ for implementability.(9) is equivalent to $\frac{d\ell_1^*}{d\theta} \leqslant -1$ i.e. $\ell_1^*(\theta) + \theta$ non increasing, which is precisely the condition stated in Picard for linear implementation (Proposition 2, p 309). Similarly, in MacAfee-MacMillan [1987] , $\partial_{\ell_1\ell_1} U < 0$, $\partial_{\ell_1\theta} U > 0$, so that $\frac{d\ell_1^*}{d\theta} \geqslant 0$ for implementation so (9) is equivalent to $\partial_{\ell_1\theta} U \leqslant - \partial_{\ell_1\ell_1} U \frac{d\ell_1^*}{d\theta}$, i.e. to

$d_\theta(\partial_{\ell_1} U(\ell_1^*(\theta),\theta)) \leq 0$ which McAfee and MacMillan (Theorem 2, p 301 and condition (16)) prove to be necessary and sufficient for linear implementation.

We now show that a "reasonable" dependence on ℓ_2 is a favorable factor for the possibility of universal implementation.

Consider the family of agent's preferences, indexed by $\alpha \in \mathbb{R}^+$:

(10) $$U_\alpha(\ell_1,\ell_2;\theta) = V(\ell_1;\theta) + \alpha\, W(\ell_2;\theta)$$

where V and W are fixed functions such that

(P) $$\partial_{\ell_1\theta}V > 0 \quad , \quad \partial_{\ell_2\theta}W > 0 \quad , \quad \partial_{\ell_1\ell_1}V < 0$$

In this setting it is known (cf Guesnerie-Laffont [1984]) that the piecewise continuously differentiable functions (ℓ_1,ℓ_2) which satisfy

$$\frac{d\ell_1}{d\theta} > 0 \ , \quad \frac{d\ell_2}{d\theta} > 0$$

are implementable, for any positive α . Let us focus attention on this set $\mathcal{L}$ of implementable functions which are common to all preference parameters α.

Condition (9) takes the following form

(9') $$\left[\partial_{\ell_1\theta}V\right]^2 \leq \left[-\partial_{\ell_1\ell_1}V\right] \left[\left(\partial_{\ell_1\theta}V\right)\frac{d\ell_1}{d\theta} + \alpha\left(\partial_{\ell_2\theta}W\right)\frac{d\ell_2}{d\theta}\right]$$

It is easy to check that if for a given (ℓ_1,ℓ_2), condition (9') holds for α, it also holds for any $\alpha' > \alpha$. Thus :

Corollary 2 : Consider the class of functions indexed by α defined by (10) and assume in addition that they satisfy (B) (D) and (P).Then the set $\mathcal{L}_\alpha = \{(\ell_1,\ell_2) \in \mathcal{L} \mid \text{condition(9') holds for } \alpha\}$ is increasing in α.

Then Corollary 2 provides a clear illustration of the fact that the dependence of preferences on ℓ_2 is a favorable factor for universal implementation. [9] Loosely speaking, among preferences which rationalize a given DICM, the more likely to pass the <u>necessary</u> test for universal implementation are those which depend strongly (and correctly) on ℓ_2. Our test which is very demanding for preferences independent of ℓ_2 can be less stringent when preferences depend upon ℓ_2. In counterpart, the criterion which we have stated and which is (almost) sufficient for preferences independent of ℓ_2 is no longer sufficient when preferences depend upon ℓ_2.

Let us summarize our findings. For a given DICM, condition (9) provides a description, which we have shown to be intuitively plausible, of the border between preferences for which universal implementation is possible and those for which it cannot work. In some sense, the addition of the dimension ℓ_2 to our problem, when it is effective, increases the power of the ruled schedule for universal implementation.

As final comments, it should be noted first that the condition we have exhibited, although being only necessary for universal implementation, is necessary and sufficient when one considers the implementation problem under small noises of observation (small variance and small support). In CGR, we developed a theory based on the consideration of truncated ruled schedules for problems with small disturbances.

Second, the fact that ℓ_2 is one-dimensional does not seem crucial to our analysis. The important point is the fact that the dimension n_2 of the space of observable variables ℓ_2 is at least equal to the dimension of characteristics Θ.

Finally, the assumption that ℓ_2^* is one-to-one has an ambiguous effect on the result of our analysis. The local considerations leading to our necessary condition hold independently of this assumption, but in the absence of such a condition of strong monotonicity, universal implementation is likely to be hopeless. The condition remains relevant however, for the analysis of implementation in the presence of small noises.

IV.D/Implementation via Quadratic-in-Section Schedules.

Let us consider now the problem of implementation via noisy reward schedules which are quadratic-in-section. As shown by proposition 4, such an implementation is possible whenever there exist associated schedules that are themselves quadratic-in-section. In fact, as shown by Picard

[1987] the noisy reward schedule that implements a given quadratic associated schedule, is deduced, by translation, from the associated schedule : in particular it has the same "curvature" as the associated schedule.

Implementation via noisy reward schedules which are quadratic-in-section is less appealing, although easier, than universal implementation. The design of a quadratic-in-section schedule requires the knowledge of the variance of the noise. May be more importantly, it has the inconvenience of leading to a rather high variance of the agents remuneration, a fact which the recent theory of contracts has shown undesirable for self enforcement of contracts (See for example the debate on penalties enforcement).

Keeping this problem in mind, it is useful to look for the associated schedules with the smallest possible curvature (as measured by the inverse of the radius of curvature of the associated schedule at the point of the DICM on this schedule - which is also the radius of curvature at the corresponding point of the noisy reward schedule). Indeed, our analysis provides a lower bound to such a curvature.

Proposition 7 : Assume (B), (D) and (MO) hold, and that $(\ell^*(.),t^*(.))$ is C^1 a.e. twice differentiable DICM. A necessary condition for a M quadratic-in-section schedule to be an associated schedule, is that its curvature in section in the hyperplane $\ell_2^*(\theta)$, is

at least larger than : $\frac{1}{2}\left\{\frac{\left[\partial_{\ell_1\theta}U(\ell^*(\theta);\theta)\right]^2}{\Gamma(\theta)} + \partial_{\ell_1\ell_1}U(\ell^*(\theta);\theta)\right\}$

The expression is reminiscent of condition (9) and the proof of Proposition 7 given in Appendix is a by-product of the proof of the Theorem. The differentiability assumptions guarantee that the lower envelope presented in the previous subsection is smooth around $(\ell^*(\theta);t^*(\theta))$, so that its curvature is bounded from below. Any parabola passing through $(\ell^{*}(\theta), t^{*}(\theta))$, tangent to the envelope at this point, and with high enough curvature will lie below this envelope curve : the surface generated by all these parabolas fulfills all local incentive compatibility constraints. Proposition 7 gives a precise bound for the minimal curvature.

Proposition 7 has two consequences. Since a formal statement would be somewhat heavy, we only provide the intuition of the straightforward arguments underlying them.

First, for small noises, (with ε independent of ℓ) , there always exists a quadratic-in-section associated schedule : any one generated by a family of parabolas which satisfy the curvature condition of the proposition (See CGR).

Second, for general noises, Proposition 6 only gives the right local curvature, but global incentive compatibility constraints (i.e. between θ and θ' not close) might not hold. However, the reader will convince himself that "generically" one can increase the curvature of the parabola beyond the lower bound exhibited above, up to a point where the corresponding quadratic-in-section schedule is globally incentive compatible and therefore an associated schedule (compactness, uniform continuity).

This remark as well as the above proposition generalizes the results obtained by Picard [1987] for one-dimensional action variable, principal-agent problems.

V. CONCLUSION

As stressed on the introduction the present paper can be viewed as developing a possible argument for the usefulness of simple and linear schemes, namely that such schemes allow a robust implementation of optimal pure adverse selection schemes in the context of a multidimensional problem where "noise" signicantly affects a subset of the contractual variables. It is hoped that the analysis has informed the reader on both the significance and the limits of this argument.

APPENDIX : PROOF OF THE THEOREM AND OF PROPOSITION 7.

Consider the quadratic-in-section surface defined by :

$$\text{(A1)}\quad \begin{cases} \ell_1 = u \\ \ell_2 = \ell_2^*(v) \\ t = t^*(v) - \partial_1 U(\ell^*(v);v)\left[u-\ell_1^*(v)\right] - q(v)\left[u-\ell_1^*(v)\right]^2 \end{cases}$$

which is parametrized by $(u,v) \in \ell_1^*(\Theta) \times \Theta$ and quadratic in u. Let us consider some agent θ in Θ and call $F_{q,\theta}(u,v)$ the value of his utility over the surface : $F_{q,\theta}(u,v) = t+U(\ell_1,\ell_2;\theta)$, where t, ℓ_1 and ℓ_2 are derived from u and v by (A1).

It is first obvious that $(\ell_1^*(\theta),\theta)$ is a stationary point of $F_{q,\theta}$. Computing the Hessian quadratic form of the function F and using the incentive compatibility constraints fulfilled on the contract curve, one gets :

$$\partial^2_{u\,u}F_{q,\theta}(\ell_1^*(\theta);\theta) = \partial^2_{11}U(\ell^*(\theta),\theta) - 2q(\theta),$$

$$\partial^2_{u\,v}F_{q,\theta}(\ell_1^*(\theta),\theta) = -\left[\partial^2_{11}U(\ell^*(\theta),\theta) - 2q(\theta)\right]\frac{d\ell_1^*}{d\theta}(\theta) - \partial^2_{1\theta}U(\ell^*(\theta),\theta)$$

$$\partial^2_{v\,v}F_{q,\theta}(\ell_1^*(\theta),\theta) = \left[\partial^2_{11}U(\ell^*(\theta),\theta) - 2q(\theta)\right]\left[\frac{d\ell_1^*}{d\theta}\right]^2(\theta) + \frac{d\ell_1^*}{d\theta}(\theta)\ \partial^2_{1\theta}U(\ell^*(\theta),\theta) - \frac{d\ell_2^*}{d\theta}(\theta)\ \partial^2_{2\theta}U(\ell^*(\theta),\theta)$$

A necessary condition for the surface defined by (A1) to implement the DICM $(\ell^*(.),\ t^*(.))$ is that the Hessian quadratic form be negative semi definite :

$$\partial^2_{11}U(\ell^*(\theta),\theta) - 2q(\theta) \leq 0 \qquad \text{(A2)}$$

$$-\left[\partial^2_{11}U(\ell^*(\theta),\theta) - 2q(\theta)\right]\left[\frac{d\ell_1^*}{d\theta}(\theta)\ \partial^2_{1\theta}U(\ell^*(\theta),\theta) + \frac{d\ell_2^*}{d\theta}(\theta)\ \partial^2_{2\theta}U(\ell^*(\theta),\theta)\right]$$
$$\geqslant \left[\partial_{1\theta}U(\ell^*(\theta),\theta)\right]^2 \qquad \text{(A3)}$$

From second-order incentive compatibility condition of the DICM : $\forall\theta\in\Theta$, $\frac{d\ell_1^*}{d\theta}(\theta)\ \partial^2_{1\theta}U(\ell^*(\theta),\theta) + \frac{d\ell_2^*}{d\theta}(\theta)\ \partial^2_{2\theta}U(\ell^*(\theta),\theta) \geqslant 0$, so that (A3) implies (A2).

Take $q(.) \equiv 0$. (A3) is a necessary condition for the ruled schedule to implement the DICM, and is identical to (9). Hence the theorem.

Take $q(.)$ non null, (A3) is a necessary condition for the quadratic-in-section schedule to be an associated schedule, and it gives a lower bound for the value of the curvature $q(.)$. Hence Proposition 7.

FOOTNOTES

1. For this, consider any $(\ell(.),t(.))$ - associated schedule φ, and take two compact sets K_1 and K_2, such that the interior of K_1 contains K_2 and K_2 contains $\ell(\Theta)$. Consider then the function $\tilde{\varphi}$ defined by :

$$\tilde{\varphi}(\ell) = \varphi(\ell), \ \forall \ \ell \in K_2$$

$$\tilde{\varphi}(\ell) = -M, \ \forall \ \ell \in \mathbb{R}^n/K_1$$

and by piecing together these parts on K_1/K_2. It is obvious that $\tilde{\varphi}$ is an $(\ell(.),t(.))$ - associated schedule, and it can be constructed to be as regular as desired on $\mathbb{R}^n/K_2$.

2. This is well defined because incentive compatibility on the DICM implies :

$$\ell(\theta) = \ell(\theta') \Longrightarrow t(\theta) = t(\theta')$$

3. Given that the signal has the same dimension as ℓ_1, the additive form involves no loss of generality.

4. Although we focus on situations where the observation noise does not induce any efficiency loss, it should be stressed that this need not be always the case. In particular, implicit here is the assumption that the noise satisfies some kind of "spanning" condition, such as $\mathbb{E}(\ell_1' | \ell_1) = \ell_1$. Otherwise, the noise reduces drastically the available information and is thus likely to generate some loss of efficiency. (For example, in the extreme case where $(\mathbb{E}(\ell_1' | \ell_1) = K$ (i.e., $\mathbb{E}(\varepsilon | \ell_1) = K-\ell_1$), then clearly observing ℓ_1' has no value).

5. $\mathbb{E}$ denotes the expectation operator with respect to the distribution of ε.

6. When $n_2 \geqslant 1$, the one-to-one assumption stated in proposition 2 is stronger than required. What is needed is that whenever there is "pooling" in ℓ_2 for the desired DICM $(\ell(.),t(.))$, there is also pooling in the reward function (that is, $\ell_2(\theta) = \ell_2(\theta')$ implies $H(.,\theta) = H(.,\theta')$ (for (one of) the revelation mechanism implementing $(\ell(.),t(.))$).

7. Another desirable property would be that the variance of transfers actually paid to the agent be limited, or that transfers be stochastically bounded. Unlimited transfers (as in a Mirrlees Scheme) or highly variable transfers may limit the enforceability of contracts, and will cease to be optimal when a small amount of risk aversion is introduced, so that it is reasonable to try to implement an allocation without using them. This requirement also leads us to favor linear rather than quadratic schedules, as witnessed in section IV.

8. This definition applies even if ε is correlated with θ provided $\mathbb{E}(\varepsilon \mid \theta) = 0$.

9. Some technical comments are in order : first condition (P) and the monotonicity of the DICM could be generalized to a constant sign assumption (see Guesnerie-Laffont [1984]). Second, the set $\mathcal{L}$ is only a subset of all implementable functions. It is the only one, in the present state of knowledge, for which one can be sure that it belongs to the intersection over α, of the sets of implementable functions.

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