NON-CAUSALITY IN CONTINUOUS TIME

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Abstract

In this paper, we define different concepts of non-causality for continuous-time processes, using conditional independence and decompostion of semi-martingales. These definitions extend the ones already given in the case of discrete-time processes. As in the discrete-time setup, continuous-time non-causality is a property concerned with the prediction horizon (global versus instantaneous non-causality) and the nature of the prediction (strong versus weak non-causality). Relations between the resulting continuous-time non-causality concepts are then studied for the class of decomposable semi-martingales, for which, in general, the weak instantaneous non-causality does not imply the strong global non-causality. The paper then characterizes these different concepts of non-causality in the cases of counting processes and Markov processes.

KEYWORDS :
Non-causality, continuous-time, semi-martingales, Doob-Meyer decomposition.

Résumé

On définit dans ce texte différents concepts de non-causalité en temps continu fondés sur la notion d'indépendance conditionnelle et sur les décompositions de semi-martingales. Ces définitions étendent au temps continu différents concepts déjà introduits en temps discret. Comme en temps discret la non-causalité est une propriété mettant en jeu un horizon de prévision (global contre instantané) et la nature de la prévision (non causalité faible ou forte). Les relations entre les différents définitions sont obtenues dans la classe des semi-martingales décomposables pour lesquelles par exemple la non-causalité faible instantanée n'implique pas la non causalité globale forte. Différents exemples sont enfin étudiés, notamment les processus de dénombrement et les processus de Markov.

MOTS-CLEFS :
Non-causalité, temps continu, semi-martingale, décomposition de Doob-Meyer.

1 Introduction

Following the seminal papers by Granger (1969) and Sims (1972), the non causality concept plays an increasing role in Econometrics and a mostly complete study of the relations between diverse forms of this notion has been yet performed. Non-causality expressed in terms of orthogonality in the Hilbert space of squared integrable random variables has been firstly studied by Hosoya (1977) and extensively treated by Florens and Mouchart (1985), while definitions in terms of conditional independence have been given, for example, by Florens and Mouchart (1982) and by Bouissou, Laffont and Vuong (1986). Non-causality is, in any case, a prediction property and the central question is : is it possible to reduce the available information in order to predict a given stochastic process? In these previous papers, two distinctions between various non-causality concepts appear, sometimes implicitely. One can first oppose a one-step ahead (or instantaneous) analysis (Granger's type definition) to a prediction property valid for any horizon (global non-causality or Sim's type definition). On the other hand, the definition may be focused on the prediction of the mean of the process (weak non-causality) or of any function of the process (strong non-causality). However, in any of these previous papers, the underlying processes are indexed by a discrete time set, which implies, in particular, that the notion of a one-step ahead forecast is defined unambigously.

Continuous-time models become more and more frequent in econometric practice. Let us mention two important fields of applications. In labor economics, duration models, markovian and, more generally, counting processes appear to be powerful tools describing individual mobilities between participation states (e.g. Flinn and Heckman, 1982, Heckman and Singer, 1984, Geweke, Marshall and Zarkin, 1986, Lancaster, 1990, Fougère and Kamionka, 1992a, 1992b) or to analyze cohort data in demographics (e.g. Heckman and Walker, 1990). In the same time, modern finance theory uses extensively diffusion processes (see, e.g., Merton, 1990, Melino, 1994).

The goal of our paper is to consider the different non-causality properties in continuous-time models and to analyze their relations. General definitions and results are provided, but a special attention is paid to counting and Markov processes. In the previous literature, Schweder (1970) obtained a first result for discrete state-space Markov processes, in which properties of transition rates and of transition probabilities are compared. Subsequently, Bremaud and Yor (1978) gave some very general results about changes of information sequences related to stochastic integration. Non-technical considerations about non-causality in continuous-time can be found in Aalen (1987). More recently, Comte and Renault (1996) analyze non-causality in continuous invertible moving average (CIMA) processes.

The paper is organized in three sections. First global non-causality is considered, i.e. non-causality involving prediction to any horizon. A subsection is devoted to the relations between martingale properties and non-causality. The second section introduces the concept of instantaneous non-causality and gives the main theorem concerning relations between instantaneous and global concepts. The last section presents three classes of examples : the counting processes, the CIMA processes and the Markov processes.

2 Global non-causality

A non-causality property requires the specification of the relevant stochastic process, the available information and the reduced information. Let us begin with the process z_t indexed by $t \in I \subset I\!R^+ = [0, \infty)$. We essentially consider the case where $I = I\!R^+$ but some concepts will also be presented with $t \in I\!N = \{0, 1, \cdots\}$ in order to relate our definitions to the previous ones. The case of a right bounded time set ($t \in [0, T]$) or the case of a both left and right unbounded time set ($t \in I\!R$) would not raise particular problems relative to the topic of this paper.

For any t, z_t is a real valued measurable function defined on a probability space $(\Omega, \mathcal{A}, P)$. Extensions of the definitions to vector processes are usually straightforward. We will analyze the non-causality as a property of a given probability P. In statistical applications, P is an element of a family of sampling probabilities and the usual statistical problem of non-causality is to test if the true sampling probability satisfies the non-causality condition.

The information available at time t is described by a sub-σ-field $\mathcal{F}_t$ of $\mathcal{A}$ and it is natural to assume :

i) That the family $(\mathcal{F}_t)_{t \in I}$, is a *filtration* , i.e. $t \leq t' \Rightarrow \mathcal{F}_t \subset \mathcal{F}_{t'}$,

ii) That z_t is *adapted* to $(\mathcal{F}_t)_{t \in I}$, i.e. z_t is $\mathcal{F}_t$-measurable for any t. Intuitively, $\mathcal{F}_t$ incorporates the knowlegde of the history of z_t up to t. Equivalently let us define $\mathcal{Z}_t$ as the sub-σ-field of $\mathcal{A}$ generated by the family of $z_s, 0 \leq s \leq t.(\mathcal{Z}_t)_{t \in I}$ is the canonical (or self exciting) filtration associated to the process z_t. Then z_t is adapted to $(\mathcal{F}_t)_{t \in I}$ if and only if $\mathcal{Z}_t \subset \mathcal{F}_t, \forall t \in I$.

Finally we introduce a subfiltration $(\mathcal{G}_t)_{t \in I}$ of $(\mathcal{F}_t)_{t \in I}$ representing the reduced information. We assume that z_t is still $\mathcal{G}_t$-measurable which is equivalent to :

$$\mathcal{Z}_t \subset \mathcal{G}_t \subset \mathcal{F}_t \quad , \quad \forall t \in I. \tag{2.1}$$

In applications, $\mathcal{F}_t$ is often the canonical filtration associated to a large multivariate stochastic process (z_t, y_t, w_t) where y_t and w_t are vector processes, $\mathcal{G}_t$ is the canonical filtration of (z_t, w_t) only and $\mathcal{Z}_t$ is still the canonical filtration of z_t. An important particular case, which will be analysed in the sequel of this paper, is the case where $\mathcal{G}_t = \mathcal{Z}_t$. In terms of stochastic processes, the process w_t disappears.

2.1 Definitions and elementary properties

Definition 2.1 . *(Weak global non-causality):*
$(\mathcal{F}_t)_t$ does not weakly globally cause z_t given $(\mathcal{G}_t)_t$ if :

$$E(z_t|\mathcal{F}_s) = E(z_t|\mathcal{G}_s) \qquad \forall s,t \in I.$$

■

The definition of strong non-causality uses the conditional independence notation which is :

$$\mathcal{M}_1 \perp\!\!\!\perp \mathcal{M}_2 | \mathcal{M}_3, \tag{2.2}$$

where $\mathcal{M}_i (i = 1,2,3)$ are sub-σ-fields of $\mathcal{A}$. This concept is in particular defined in Dellacherie and Meyer (1980a, chap. II) and the basic properties are collected e.g. in Florens and Mouchart (1982, appendix) and in Florens et al. (1990, chap.2).

Definition 2.2 *(Strong global non-causality) :*
$(\mathcal{F}_t)_t$ does not strongly cause z_t given $(\mathcal{G}_t)_t$ if :

$$\mathcal{Z}_t \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_s \qquad \forall s,t \in I.$$

■

Note that this independence is trivially satisfied if $t \leq s$. This property means in particular that :

$$\forall f : \Omega \to I\!R \quad , \quad \mathcal{Z}_t\text{-measurable and } P\text{-integrable,}$$
$$E(f|\mathcal{F}_s) = E(f|\mathcal{G}_s) \quad P\text{-a.s..} \tag{2.3}$$

In this paper we identify the forecast of f given $\mathcal{F}_s$ with $E(f|\mathcal{F}_s)$ and the property (2.5) means that $\mathcal{G}_s$ is sufficient to forecast f given $\mathcal{F}_s$. If $\mathcal{Z}_t = \mathcal{G}_t$, we say that $\mathcal{F}_t$ does not cause z_t (or $\mathcal{Z}_t$).

Obviously the strong global property implies the weak one : property (2.5) is in particular true when f is the identity function. The weak global non-causality only involves the conditional expectation of z_t while the strong concept is a property of the whole conditional distribution of the process.

We will give now alternative characterizations of strong global non-causality, using in particular σ-fields associated to stopping times. Let us recall that a stopping time relative to a filtration $(\mathcal{F}_t)_t$ is a real valued measurable function σ such that $\sigma^{-1}([0,t]) \in \mathcal{F}_t, \forall t \in I$. For a given stopping time σ, the associated sub-σ-field of $\mathcal{A}, \mathcal{F}_\sigma$, is defined by :

$$\mathcal{F}_\sigma = \{X \in \mathcal{A} / X \cap \sigma^{-1}([0,t]) \in \mathcal{F}_t\} \tag{2.4}$$

(for details, see Dellacherie and Meyer (1980b, chap. IV)).

Theorem 2.1 *$(\mathcal{F}_t)_t$ does not strongly globally cause $(\mathcal{Z}_t)_t$ given $(\mathcal{G}_t)_t$ if and only if one of the following properties is satisfied :*

i) $\mathcal{Z}_\infty \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_s \qquad \forall s \in I$

where $\mathcal{Z}_\infty = \bigvee_{t \in I} \mathcal{Z}_t$ *i.e.* $\mathcal{Z}_\infty$ *is generated by* $\bigcup_{t \in I} \mathcal{Z}_t$.

ii) $\forall p \in \mathbb{N}$, $\forall t_1, \cdots, t_p \in I$, $\forall \varphi : \mathbb{R}^p \to \mathbb{R}$ *measurable and bounded :*

$$E(\varphi(z_{t_1}, \cdots, z_{t_p}) | \mathcal{F}_s) = E(\varphi(z_{t_1}, \cdots, z_{t_p}) | \mathcal{G}_s) \qquad P\text{-a.s.}.$$

iii) For any stopping time σ relative to the filtration $(\mathcal{G}_t)_t$,

$$\mathcal{Z}_\infty \perp\!\!\!\perp \mathcal{F}_\sigma | \mathcal{G}_\sigma.$$

iv) For any stopping time τ relative to $(\mathcal{Z}_t)_t$ and any stopping time σ relative to $(\mathcal{G}_t)_t$:

$$\mathcal{Z}_\tau \perp\!\!\!\perp \mathcal{F}_\sigma | \mathcal{G}_\sigma.$$

■

All proofs are given in the appendix.

The property (i) means that $\mathcal{G}_s$ is sufficient to forecast any function of the whole trajectory of z_t given $\mathcal{F}_s$.

Property (ii) shows that it is sufficient to check the equalities between conditional expectations (given $\mathcal{G}_s$ and $\mathcal{F}_s$) for any function depending on a finite set of realizations of the z_t-process only. Property (iii) extends property (i) from fixed times to stopping times. For example, the analysis of non-causality in counting processes crucially relies on (iii). Property (iv) extends the definition to stopping times.

Remark :

Let us consider a process f_t which is $\mathcal{Z}_t$-adapted. We have implicitely taken $(E(f_t|\mathcal{F}_s))_s$ as the natural sequence of the predictions of f_t given the sequence of information $(\mathcal{F}_s)_s$. However it is well known that $E(f_t|\mathcal{F}_s)$ is only almost surely defined and the sequence of the predictions obtained by selecting a version of $E(f_t|\mathcal{F}_s)$ for any s could have non regular sample paths. The concept of optional projection (see Meyer (1968) and Dellacherie and

Meyer (1980b, chap. VI-2)) avoids this selection problem. In our case, the optional projection of f_t on $\mathcal{F}_s$ is the unique process $\hat{f}_s$ (up to an evanescent process) such that : for any stopping time τ adapted to $(\mathcal{F}_s)_s$,

$$E(f_\tau \cdot \mathbb{I}\{\tau < \infty\}|\mathcal{F}_\tau) = \hat{f}_\tau \cdot \mathbb{I}\{\tau < \infty\} \tag{2.5}$$

Recall that an evanescent process has almost all its sample paths equal to zero at any time.

It follows from proposition 2.1 iii) that the non-causality assumption implies that $\hat{f}_s$ and f_s^* (the optional projection of f_t on the filtration $(\mathcal{G}_s)_s$) are equal up to an evanescent process. An identical property occurs with the predictable projection in which the stopping time is constrained to be predictable. ■

2.2 Martingale property and non-causality

Let us recall that a stochastic process ξ_t is a $(\mathcal{F}_t)_t$ martingale if ξ_t is $(\mathcal{F}_t)_t$-adapted and if :

$$E(\xi_t|\mathcal{F}_s) = \xi_s \qquad \text{P-a.s..} \tag{2.6}$$

If ξ_t is $(\mathcal{G}_t)_t$-adapted and if $\mathcal{G}_t \subset \mathcal{F}_t, \xi_t$ is still a $(\mathcal{G}_t)_t$-martingale : the martingale property remains valid if the filtration decreases. However this property is in general not preserved if the filtration increases. When the filtration is getting larger, the preservation of the martingale property is strongly connected to the non-causality concept. The intuitive reason of the interest of the martingale property in a non-causality analysis is the following. The variation of a martingale relative to a given filtration is "unpredictable" given the information provided by this filtration (the best prediction in the L^2 sense is zero). It is then natural to study the processes which remains unpredictable even if the information σ-fields increase.

This paragraph is concerned by the analysis of the connection between the preservation of the martingale property and the non-causality concept. Moreover, the following results will be the corner stone of the relations between global and instantaneous non-causality definitions.

Theorem 2.2

i) If $(\mathcal{F}_t)_t$ does not strongly globally causes z_t given $(\mathcal{G}_t)_t$, any $(\mathcal{Z}_t)_t$-adapted $(\mathcal{G}_t)_t$-martingale process is a $(\mathcal{F}_t)_t$-martingale.,

ii) If any $(\mathcal{Z}_t)_t$-martingale is a $(\mathcal{F}_t)_t$-martingale, $(\mathcal{F}_t)_t$ does not strongly globally cause z_t. ■

In the case $\mathcal{Z}_t = \mathcal{G}_t$, strong global non-causality is then equivalent to the preservation of the martingale property. Without references to non-causality considerations, a first proof of this result is given by Bremaud and Yor (1978). A proof in our framework is provided in the appendix.

The next theorem is more original and is especially interesting in relation with the Sims (1972) non-causality definition . This theorem will be stated in the case where $\mathcal{G}_t = \mathcal{Z}_t$ and involves in the reciprocal theorem an assumption of initial non-causality.

Theorem 2.3

i) If $(\mathcal{F}_t)_t$ does not strongly globally cause z_t, for any $t_0 \in I$, any stochastic process $\eta_t, t \in [0, t_0]$, which is a $(\mathcal{Z}_{t_0} \vee \mathcal{F}_t)_t$-martingale, is a $(\mathcal{Z}_\infty \vee \mathcal{F}_t)_t$-martingale.

ii) Under the assumption : $\mathcal{Z}_\infty \perp\!\!\!\perp \mathcal{F}_0 | \mathcal{Z}_t$, if any $\eta_t, t \in [0, t_0]$, $(\mathcal{Z}_{t_0} \vee \mathcal{F}_t)_t$-martingale is a $(\mathcal{Z}_\infty \vee \mathcal{F}_t)_t$-martingale, then $(\mathcal{F}_t)_t$ does not strongly globally cause z_t. ■

The hypothesis $\mathcal{Z}_\infty \perp\!\!\!\perp \mathcal{F}_0 | \mathcal{Z}_t$ was introduced in Florens and Mouchart (1982) and analysed in the linear case in their (1985) paper. Let us only remark here that this conditional independence is implied by the strong global non-causality but remains a very weak assumption. It is in particular satisfied if the initial conditions are deterministic ($\mathcal{F}_0$ then becomes the trivial σ-field $\mathcal{F}_0 = \{\phi, \Omega\}$) or if $\mathcal{F}_0 = \mathcal{Z}_0$. If the time index is replaced by $\mathbb{R}, \mathcal{F}_0$ becomes $\bigcap_{t \in \mathbb{R}} \mathcal{F}_t$ and this σ-field is trivial for the purely stochastic processes. However this extension is useful for discrete time processes but is not common for continuous-time analysis.

The application of theorems 2.2 and 2.3 to instantaneous non-causality definitions require an extension of these theorems to local martingales. As usual in the stochastic process litterature (see Dellacherie and Meyer (1980b) or Protter (1990) for examples), local martingales are introduced with some regularity conditions.

We assume that the filtrations satisfy "les conditions habituelles", i.e. :

i) The probability space $(\Omega, \mathcal{A}, P)$ is completed in the Lebesgue sense and all the σ-fields contain all the null sets.

ii) The filtrations are right continuous, i.e. for a filtration $(\mathcal{F}_t)$,

$$\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s. \tag{2.7}$$

A stochastic process ξ_t, adapted to $(\mathcal{F}_t)_t$ with almost surely right continuous left limit trajectories (cadlag process) is a local martingale if there exists an increasing sequence of stopping times $(\sigma_n)_{n \geq 0}$ such that : $\sigma_n \to +\infty$ and that :

$$\xi_{t \wedge \sigma_n} = \xi_t \mathbb{I}(t < \sigma_n) + \xi_{\sigma_n} \mathbb{I}(t \geq \sigma_n) \tag{2.8}$$

is a uniformly integrable martingale for any $n \geq 0$. However, in this section, regularity conditions (conditions habituelles, cadlag process and uniform integrability) are not essential.

Using this definition, we obtain the following corollary :

Corollary 2.1 *In theorems 2.2 and 2.3, martingales may be replaced by local martingales.*

3 Granger instantaneous non-causality

To introduce concepts of instantaneous non-causality, we begin by considering discrete-time processes.

In discrete time, a weak concept of instantaneous non-causality may be defined as :

$$E(z_t|\mathcal{F}_{t-1}) = E(z_t|\mathcal{G}_{t-1}) \quad \text{a.s. } \forall t \in I\!N, \tag{3.1}$$

with $\mathcal{F}_{-1} = \mathcal{F}_0$ and $\mathcal{G}_{-1} = \mathcal{G}_0$. A strong extension of this definition would be :

$$\mathcal{Z}_t \perp\!\!\!\perp \mathcal{F}_{t-1}|\mathcal{G}_{t-1} \quad \forall t \in I\!N. \tag{3.2}$$

Properties (3.1) and (3.2) can be viewed as restatements of the seminal Granger (1969) definition. These properties are respectively restrictions of definitions (2.1) and (2.2) to the case $s = t-1$. However, it has been proved that, if $\mathcal{G}_t = \mathcal{F}_t$, definition (2.2) and property (3.2) are equivalent (see Florens and Mouchart, 1982, theorem 1).

We first extend the weak instantaneous non-causality to continuous-time. Note that (3.1) is equivalent to

$$\forall t \in I\!N \quad , \quad E(\Delta z_t|\mathcal{F}_{t-1}) = E(\Delta z_t|\mathcal{G}_{t-1}) \quad \text{a.s. }, \tag{3.3}$$

where $\Delta z_t = z_t - z_{t-1}$ and $\Delta z_0 = z_0$.

If $h_t = E(\Delta z_t|\mathcal{F}_{t-1}), h_0 = 0$ and $m_t = \Delta z_t - h_t, m_t$ is a martingale difference and the process can be reconstructed by :

$$z_t = H_t + M_t \tag{3.4}$$

where $M_t = \sum_{i=0}^{t} m_i$ is a martingale with respect to the filtration $(\mathcal{F}_t)_t$ and $H_t = \sum_{i=0}^{t} h_i$ is $\mathcal{F}_{t-1}$-measurable.

An analogous decomposition as (3.4) can be reconstructed with respect to the filtration $(\mathcal{G}_t)_t$:

$$z_t = H_t^* + M_t^* \tag{3.5}$$

where M_t^* is a martingale with respect to $(\mathcal{G}_t)_t$ and H_t^* is $\mathcal{G}_{t-1}$-measurable. Using these decompositions the property (3.3) may be restated by : the two decompositions (3.4) and (3.5) are identical ($H_t = H_t^*$ or $M_t = M_t^*$ a.s.).

Heuristically (3.3) may be generalized in continuous-time models by :

$$\forall t \quad E(dz_t|\mathcal{F}_{t-}) = E(dz_t|\mathcal{G}_{t-}) \quad \text{a.s.} \tag{3.6}$$

or by the equality between H_t and H_t^* now defined by

$$H_t = \int_0^t E(dz_s|\mathcal{F}_{s-}) \text{ and } H_t^* = \int_0^t E(dz_s|\mathcal{G}_{s-}) \tag{3.7}$$

Here $\mathcal{F}_{t-} = \bigvee_{s<t} \mathcal{F}_s$ (the σ-field generated by all the $\mathcal{F}_s, s < t$) replaces $\mathcal{F}_{t-1}$ and dz_t replaces Δz_t. This heuristic approach can be formalized using the notion of special semi-martingale.

We still assume the validity of the "conditions habituelles". The process z_t is a special semi-martingale with respect to the filtration $\mathcal{F}_t$ if z_t may be decomposed into :

$$z_t = z_0 + H_t + M_t, \tag{3.8}$$

where H_t is a $(\mathcal{F}_t)_t$-predictable process and M_t a zero mean $(\mathcal{F}_t)_t$-local martingale. Let us recall that a predictable process is measurable (as a function of (t,ω)) with respect to the σ-field on $I \times \Omega$ generated by all the left continuous processes with right limits. Intuitively, if H_t is predictable, the knowledge of H_s for any $s < t$ determines the knowledge of H_t. It is usually assumed that H_t has bounded integrable variations , i.e. $E(\int_0^\infty |dH_t|) < \infty$. The decomposition (3.8) is unique up to an evanescent process.

Roughly speaking the decomposition (3.8) is obtained by integrating the decomposition $dz_t = E(dz_t|\mathcal{F}_{t-}) + dM_t$, or in the case of derivable processes, one has :

$$dH_t = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} E(z_{t+\Delta t} - z_t|\mathcal{F}_{t-})dt. \tag{3.9}$$

In this expression processes are assumed to be square integrable and the limits taken in quadratic norm. Important special cases of semi-martingales are cadlag super–martingales or sub–martingales. A cadlag process is a super– martingale (resp., a sub–martingale) with respect to $\mathcal{F}_t$ if $E(z_t|\mathcal{F}_s) \leq z_s$, $\forall s \leq t$ (resp. $E(z_t|\mathcal{F}_s) \geq z_s$). For such processes, the Doob-Meyer theorem (see Ito and Watanabe (1965) , Dellacherie and Meyer (1980b, chap. VII) or Karr (1986, Appendix B)) guaranties the special semi-martingale decomposition. Moreover H_t is decreasing for a supermartingale and increasing for a submartingale.

We can now define instantaneous non-causality in the Granger sense.

Definition 3.1 *Let us assume that z_t is a special semi-martingale with respect to $(\mathcal{G}_t)_t$, characterized by the decomposition $z_t = z_0 + H_t^* + M_t^*$. Then*

$(\mathcal{F}_t)_t$ does not weakly instantaneously cause z_t given $(\mathcal{G}_t)_t$ in the Granger sense if z_t remains a semi-martingale with respect to $(\mathcal{F}_t)_t$ with the same decomposition.

The Stricker theorem (see Dellacherie and Meyer, 1980b, VII.60) states that the semimartingale decomposition is preserved through reduction of the filtration. Definition 3.1 requires a stronger property, i.e. the identity of the two decompositions. For the unicity of the decomposition of z_t, we obtain immediately the following lemma :

Lemma 3.1 *The three following properties are equivalent :*
i) $(\mathcal{F}_t)_t$ does not weakly instantaneously cause z_t given $(\mathcal{G}_t)_t$ in the Granger sense.
ii) H_t^* is $(\mathcal{F}_t)_t$-predictable.
iii) M_t^* is a local $(\mathcal{F}_t)_t$-martingale.

Definition 3.2 *$(\mathcal{F}_t)_t$ does not strongly instantaneously cause z_t given $(\mathcal{G}_t)_t$ in the Granger sense if any $(\mathcal{Z}_t)_t$-adapted $(\mathcal{G}_t)_t$-special semi-martingale is a $(\mathcal{F}_t)_t$-special semi- martingale with an identical decomposition with respect to the two filtrations.*

It follows immediately from the previous definition that the strong concept implies the weak one. Global and instantaneous concepts of non-causality are connected by the following relations:

Theorem 3.1
(i) Strong global non-causality implies strong instantaneous non-causality in the Granger sense.
(ii) If $\mathcal{G}_t = \mathcal{Z}_t$, the strong instantaneous non-causality implies the strong global non-causality.

■

This theorem is an immediate application of the corollary (2.6). However there is in general no equivalence between the weak concepts, except for particular but very important processes.

4 Examples

We briefly apply our previous concepts to three classes of processes.

Example I : Counting processes.
Let $N_t = (N_t^1, N_t^2)$ be a bivariate counting process characterized by $N_t^1 = \sum_{n>0} \mathbb{I}(\tau_n^1 \leq t)$ and $N_t^2 = \sum_{n>0} \mathbb{I}(\tau_n^2 \leq t)$, where τ_n^1 and τ_n^2 are two increasing sequences of positive random variables, which represent the jump times of N_t^1 and N_t^2, respectively. We define $\mathcal{Z}_t = \mathcal{G}_t$ as the filtration generated by N_t^1 and $\mathcal{F}_t$ the filtration generated by N_t^2. In this case, the four previous definitions of non-causality are equivalent.

A precise statement of this theorem and its proof is given in the appendix.

Example II : Continuous invertible moving average (CIMA) processes.
Following Comte and Renault (1996), let x_t be a continuous time gaussian process admitting a CIMA representation :

$$x_t = \int_0^t A(t,s)dW_s + m_t$$

where W_s is a multidimensional Brownian motion, $A(t,s)$ is a deterministic triangular matrix function of class C^1 and $m(t)$ is a deterministic function. The matrix A is chosen canonically, i.e. such that $A(t,t)$ is lower triangular.

This process is a semimartingale with decomposition :

$$x_t = x_0 + H_t + M_t$$

where :

$$M_t = \int_0^t A(s,s)dW_s \quad \text{and} \quad H_t = m_t - m_0 + \int_0^t ds \int_0^s \frac{\partial A}{\partial s}(s,u)dW_u.$$

Let us now decompose x_t into a vector (y_t, z_t), so as A is partitioned accordingly to this decomposition and A_{zy} is the block of A corresponding to this partition. Then it follows immediately that, if $A_{zy} = 0$, the filtration $(\mathcal{F}_t)_t$ generated by x_t does not cause strongly globally the z_t process.

Moreover, using properties of gaussian processes, Comte and Renault show that if $A_{zy} = 0$, y_t does not weakly instantaneously cause z_t. They deduce the equivalence of the four definitions of non-causality in the case of CIMA processes.

Example III : Markov processes.

Relations between non-causality and Markov assumptions were considered in discrete time in Florens et alii (1993). In this third example, we extend some of the results given in their paper to continuous-time Markov processes, but homogeneity and stationarity assumptions will be introduced in order to characterize non-causality by properties of the canonical semigroup and of the infinitesimal generator of the process (see e.g. Hansen and Scheinkman (1995)). Let $x_t = (y_t, z_t) \in \mathbb{R}^n (y_t \in \mathbb{R}^p, z_t \in \mathbb{R}^q, p+q=n)$ be an homogeneous stationary vector valued Markov process whose marginal distribution is denoted Q. We denote by L^2_x the Hilbert space of Q square integrable random variables on $\mathbb{R}^n$ and by L^2_z the subspace of L^2_x of functions which depend on the last q coordinates only (or equivalently of $\mathbb{R}^p \times \mathcal{B}_q$ measurable functions where $\mathcal{B}_q$ is the Borelian σ-field of $\mathbb{R}^q$).

Let us briefly summarize definitions given in Hansen and Scheinkman (1995) :

i) $\forall u \geq 0$, T_u is a linear operator from L^2_x to L^2_x defined by $T_u(\varphi) = \psi$ where $\varphi \in L^2_x$ and $\psi(\xi) = E(\varphi(x_u)|x_0 = \xi)$. The collection T_u satisfies the properties $T_0(\varphi) = \varphi$ and $T_u(T_v) = T_{u+v}$ and is the canonical semigroup associated to the process.

We assume that T_u satisfies a continuity condition :

$$\lim_{u \downarrow 0} T_u(\varphi) = \varphi \quad \forall \varphi \in L^2_x$$

ii) The infinitesimal generator is defined by the limit :

$$A(\varphi) = \lim_{u \downarrow 0} \frac{1}{u}(T_u(\varphi) - \varphi) = \frac{d}{du} T_u(\varphi)|_{u=0}$$

and D_x is the (dense) subset of L^2_x in which A is defined. We denote $D_z = D_x \cap L^2_z$. The generator characterizes the semi group. If A is bounded, this characterization is obtained through the exponential formula:

$$T_u = e^{uA} = \sum_{k \geq 0} \frac{A^k u^k}{k!}$$

The generator deduced from Markov processes is in general unbounded and the exponential formula must be extended using Yosida approximation A_λ of the generator :

$$T_u = \lim_{\lambda \to \infty} e^{uA_\lambda}$$

where $A_\lambda = \lambda A(\lambda I - A)^{-1}$ (I is the identity operator). (See Pazy (1983) section 1.3).

In this context, the filtration $(\mathcal{F}_t)_t$ is generated by the x_t-process and $(\mathcal{Z}_t)_t$ by the z_t-process only. We restrict our attention to the case $\mathcal{Z}_t = \mathcal{G}_t$. Moreover let $(\mathcal{Y}_t)_t$ be the filtration generated by the y_t-process. A (minor) technical assumption is required. Let $\sigma(x_t)$ (resp. $\sigma(z_t)$) be the σ-field generated by x_t (resp z_t). We assume that $\mathcal{Z}_t$ and $\sigma(x_t)$ are measurably separated given $\sigma(z_t)$. This means (see Florens et alii (1993)) that any $\mathcal{Z}_t$-measurable function a.s. equal to a $\sigma(x_t)$-measurable function is a.s. equal to a $\sigma(z_t)$-measurable function. This assumption concerns the null sets of the product of $\mathcal{Z}_t$ and $\sigma(x_t)$. In particular, if the joint distribution of $(z_s)_{s\in[0,t]}$ and y_t is equivalent to a distribution such that they are independent, the measurability condition is satisfied. Then we get the following characterizations of non-causality :

- Property 1 : $\mathcal{F}_t$ does not weakly globally cause $\varphi(z_t)(\varphi \in L^2_z)$ if and only if $T_u(\varphi) \in L^2_z \quad \forall u \geq 0$.

- Property 2 : $(\mathcal{F}_t)_t$ does not strongly globally cause z_t if and only if $T_u(L^2_z) \subset L^2_z \quad \forall u \geq 0$.

- Property 3 : $(\mathcal{F}_t)_t$ does not weakly instantaneously cause $\varphi(z_t)(\varphi \in D_z)$ if and only if $A(\varphi) \in L^2_z$.

- Property 4 : $(\mathcal{F}_t)_t$ does not strongly instantaneously cause $\mathcal{Z}_t$ if and only if $A(D_z) \subset L^2_z$.

In this Markov case, the general theorem of equivalence between strong instantaneous and strong global non-causality is an elementary corrolary of the (exact or approximated) exponential formula, as first remarked by Schweder (1970) in the finite state case.

Contrary to the previous examples, weak instantaneous non-causality does not imply strong non-causality. However, for particular processes, property 4 may be weakened. For example, let us assume that x_t is a multivariate diffusion model defined by the stochastic differential equation

$$dx_t = \mu dt + \Sigma^{\frac{1}{2}} dW_t$$

where μ and Σ are functions of x_t and W_t is the multivariate Wiener process. Regularity conditions implying the existence of a stationary markovian solution (See Hansen and Scheinkman (1995) example 4.4) are assumed to be satisfied and the infinitesimal generator is :

$$A(\varphi) = \mu\partial\varphi + \frac{1}{2} tr\Sigma\partial^2\varphi$$

$\partial\varphi$ and $\partial^2\varphi$ denoted the vector of partial derivatives and the matrix of second order derivatives respectively. The set D_x contains at least fonctions $\varphi \in L^2_x$ which $\partial\varphi$ and $\partial^2\varphi$ are continuous. Let us assume that the coordinate functions $C_{z_i} : x = (y_1, \cdots, y_p, z_1, \cdots, z_q) \to z_i$ $(i = 1, \cdots, q)$ are square integrable respectively to Q (the second order moments of the z_t process exist). Then $\mathcal{F}_t$ does not globally cause z_t if and only if $A(C_{z_i})$ and $A(C_{z_i}C_{z_j})$ are in L_z^2 ($\forall i, j \in \{1, \cdots, q\}$).

This last example extends the analysis of non causality "in mean" and "in variance" introduced by Comte and Renault (1996).

APPENDIX

Proof of theorem 2.3

We use the following notations :

- If $\mathcal{M}$ is a sub-σ-field of $\mathcal{A}$:

- $[\mathcal{M}]$ denotes the set of $\mathcal{M}$-measurable numerical functions,
- $[\mathcal{M}]_b$ denotes the set of $\mathcal{M}$-measurable and bounded numerical functions,
- $[\overline{\mathcal{M}}]$ denotes the set of numerical functions which are almost surely equal to $\mathcal{M}$-measurable functions.

A - (i) implies definition 3.2 because $\mathcal{Z}_t \subset \mathcal{Z}_\infty$, using elementary properties of conditional independence.

The implication "definition 3.2 $\Rightarrow$ (i)" may be obtained by a monotone class argument, or by the following elementary application of the martingale theorem :

$$\forall t \in \mathbb{R}, \forall f \in [\mathcal{F}_s]_b, \quad E[f|\mathcal{Z}_\infty \vee \mathcal{G}_s] = \lim_{t\to\infty} E(f|\mathcal{Z}_t \vee \mathcal{G}_s)$$

$$= E[f|\mathcal{G}_s] \qquad \text{a.s.}$$

The first equality is due to :

$$\bigvee_{t\geq 0} (\mathcal{Z}_t \bigvee \mathcal{G}_s) = \mathcal{Z}_\infty \bigvee \mathcal{G}_s$$

and the second one to the conditional independence :

$$\mathcal{Z}_t \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_s.$$

The equality :

$$\forall f \in [\mathcal{F}_s]_b, E[f|\mathcal{Z}_\infty \bigvee \mathcal{G}_s) = E(f|\mathcal{G}_s) \qquad \text{a.s.}$$

is a characteriza on of (i).

B - Definition 3.2 or (i) implies trivially (ii).

The reciprocal follows from a monotone class argument (see Dellacherie and Meyer, 1980a, p.19) ; we want to prove the property :

$$\forall \psi \in [\mathcal{Z}_\infty]_b, E(\psi|\mathcal{F}_s) = E(\psi|\mathcal{G}_s) \qquad \text{a.s.}$$

and we know that this property is true for the bounded functions depending on a finite number of coordinates only.

The assumptions (22.1), (a) and (b), of Dellacherie and Meyer (1980a, p.22) are trivially satisfied, which implies the result.

C - (iii) implies (i) by taking σ fixed and equal to s.

Reciprocally let us start from :

$$\mathcal{Z}_\infty \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_s.$$

which is equivalent to :

$$(*) \qquad (\mathcal{Z}_\infty \bigvee \mathcal{G}_s) \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_s.$$

Let us consider $\varphi \in [\mathcal{Z}_\infty]_b$. We have to prove that :

$$E(\varphi|\mathcal{F}_\sigma) \in [\overline{\mathcal{G}_\sigma}]$$

or equivalently that :

$$\forall s, \mathbb{I}\{\sigma \leq s\} E(\varphi|\mathcal{F}_\sigma) \in [\bar{\mathcal{G}}_\sigma].$$

As

$$\mathbb{I}\{\sigma \leq s\} \in \mathcal{G}_\sigma \subset \mathcal{F}_\sigma,$$

$$\begin{aligned} \mathbb{I}\{\sigma \leq s\} E(\varphi|\mathcal{F}_\sigma) &= E(\mathbb{I}\{\sigma \leq s\}\varphi|\mathcal{F}_\sigma) \\ &= E(\mathbb{I}(\{\sigma \leq s\}\varphi|\mathcal{F}_s). \end{aligned}$$

This last equality follows from :

$$\begin{aligned} &\forall g \in [\mathcal{F}_\sigma]_b, \qquad \forall \varphi \in [\mathcal{Z}_\infty]_b, \\ &E(g\varphi \mathbb{I}\{\sigma \leq s\}) = E(g\ E(\varphi \mathbb{I}\{\sigma \leq s\}|\mathcal{F}_\sigma)). \\ &= E(g \mathbb{I}\{\sigma \leq s\} E(\varphi|\mathcal{F}_s)) = E(g\ E(\mathbb{I}\{\sigma \leq s\}\varphi|\mathcal{F}_s)). \end{aligned}$$

The first and second equalities define the conditional expectation w.r.t. $\mathcal{F}_\sigma$ or $\mathcal{F}_s$, using :

$$g \mathbb{I}\{\sigma \leq s\} \in [\mathcal{F}_s]_b.$$

We also use the fact that : $\mathbb{I}\{\sigma \leq s\} \in [\mathcal{F}_s]$ (because σ is a stopping time adapted to $\mathcal{G}_s \subset \mathcal{F}_s$) and the property : $\mathbb{I}\{\sigma \leq s\} \in [\mathcal{F}_\sigma]$.

Finally, as $\mathbb{I}\{\sigma \leq s\}\varphi \in [\mathcal{Z}_\infty \bigvee \mathcal{G}_s]_b$,

$$E(\mathbb{I}\{\sigma \leq s\}\varphi|\mathcal{F}_s) \in [\overline{\mathcal{G}_s}], \quad \text{using } (*).$$

D - (iii) implies (iv) because :

$$\forall \tau, \mathcal{Z}_\tau \subset \mathcal{Z}_\infty,$$

and (iv) implies definition 3.2, by substituting fixed times to stopping times. ∎

Proof of theorem 2.4

i) Let ξ_t be a $(\mathcal{Z}_t)_t$-adapted and $(\mathcal{G}_t)_t$-martingale process. One has $E(\xi_t|\mathcal{G}_s) = \xi_s$ a.s. by the martingale property and $E(\xi_t|\mathcal{G}_s) = E(\xi_t|\mathcal{F}_s)$ a.s. by the non-causality assumption. Then $E(\xi_t|\mathcal{F}_s) = \xi_s$ a.s. and ξ_t is a $(\mathcal{F}_t)_t$-martingale.

ii) Let ξ be an integrable $\mathcal{Z}_\infty$-measurable random variable and let us define $\xi_t = E(\xi|\mathcal{Z}_t)$. This process is a $(\mathcal{Z}_t)_t$-martingale and by assumption is a $(\mathcal{F}_t)_t$-martingale. It follows that :

$\forall s \leq t \quad , \quad E(\xi|\mathcal{Z}_s) = E(\xi_t|\mathcal{Z}_s) = \xi_s = E(\xi_t|\mathcal{F}_s)$ a.s.

As ξ_t may be any integrable $\mathcal{Z}_t$-measurable random variable, the previous equalities imply $\mathcal{Z}_t \perp\!\!\!\perp \mathcal{F}_s|\mathcal{Z}_s$ for any $s \leq t$, which is equivalent to the non-causality property. ■

Proof of theorem 2.5

i) First let us note that the non-causality hypothesis implies $\forall t \in [0, t_o], \forall s \leq t, \mathcal{Z}_\infty \perp\!\!\!\perp \mathcal{F}_t \vee \mathcal{Z}_{t_0}|\mathcal{Z}_{t_0} \vee \mathcal{F}_s$. This follows from $\mathcal{Z}_\infty \perp\!\!\!\perp \mathcal{F}_{t_0}|\mathcal{Z}_{t_0}$ and from elementary properties of conditional independence (see Florens, Mouchart, Rolin, 1990, chap. II).

Let η_t be a $\mathcal{Z}_{t_0} \vee \mathcal{F}_t$-martingale. Using this definition it follows that $E(\eta_t|\mathcal{Z}_{t_0} \vee \mathcal{F}_s) = \eta_s$ a.s.. The previous conditional independence implies $E(\eta_t|\mathcal{Z}_{t_0} \vee \mathcal{F}_s) = E(\eta_t|\mathcal{Z}_\infty \vee \mathcal{F}_s)$ a.s. and then $E(\eta_t|\mathcal{Z}_\infty \vee \mathcal{F}_s) = \eta_s$ a.s.

ii) Let η be an integrable $\mathcal{F}_{t_o}$-measurable random variable and $\eta_t = E(\eta|\mathcal{Z}_{t_0} \vee \mathcal{F}_t), t \in [0, t_0]$. This process is a $(\mathcal{Z}_{t_0} \vee \mathcal{F}_t)_t$-martingale and by assumption becomes a $(\mathcal{Z}_\infty \vee \mathcal{F}_t)_t$-martingale. Then :

$$E(\eta_t|\mathcal{Z}_{t_0} \vee \mathcal{F}_s) = E(\eta_t|\mathcal{Z}_\infty \vee \mathcal{F}_s) = \eta_s \quad \text{a.s.}$$

In particular, if $t = t_0$, we get :

$$E(\eta|\mathcal{Z}_{t_0} \vee \mathcal{F}_s) = E(\eta|\mathcal{Z}_\infty \vee \mathcal{F}_s) \quad \text{a.s.} \quad \forall \eta.$$

This equality is equivalent to :

$$\mathcal{Z}_\infty \perp\!\!\!\perp \mathcal{F}_{t_0}|\mathcal{Z}_{t_0} \vee \mathcal{F}_s \quad \forall\, 0 \leq s \leq t_0.$$

The previous conditional independence is satisfied for any s and t_0 and then implies

$$\mathcal{Z}_\infty \perp\!\!\!\perp \mathcal{F}_t|\mathcal{Z}_t \vee \mathcal{F}_0 \quad \forall t.$$

If moreover, $\mathcal{Z}_\infty \perp\!\!\!\perp \mathcal{F}_0|\mathcal{Z}_t$, we obtain

$$\mathcal{Z}_\infty \perp\!\!\!\perp \mathcal{F}_t|\mathcal{Z}_t \quad \forall t,$$

using the basic properties of conditional independence (see for example, Florens and Mouchart [1982], appendix).

Proof of corollary 2.6

We analyze the extension to local martingales of theorem 2.5 only. The structure of the proof of theorem 2.6 is identical and the extension could be done analogously.

Let us first consider the second part of theorem 2.4. If the assumption is true for any local martingale, then it is true for martingales (because martingales are local martingales) and the conclusion still applies. However we have to take into account the regularity conditions which are prevalent for local martingales. The introduction of such conditions does not change the results for two reasons

1) The martingale considered in the proof is the family of conditional expectation $E(\xi|\mathcal{Z}_t)$ of an integrable random variable. Under the "conditions habituelles", such a family may be chosen as verifying the cadlag condition (see Protter [1990], chapter 1, section 2) and is uniformly integrable by the martingale convergence theorem.

2) The conditional independence $\mathcal{Z}_t \perp\!\!\!\perp \mathcal{F}_s | \mathcal{F}_s$ is equivalent to the condition $E(\xi_t|\mathcal{F}_s) = E(\xi_t|\mathcal{Z}_s)$ a.s. for any cadlag process ξ_t under the "conditions habituelles".

Let us now examine the first part of the theorem. If ξ_t is a $(\mathcal{Z}_t)_t$-adapted $(\mathcal{G}_t)_t$-local martingale with respect to the sequence $(\sigma_n)_{n\geq 0}$ of stopping times, we can reproduce the proof of theorem 2.4 i) using the theorem 2.3 iv), from which the non-causality condition implies $\mathcal{Z}_{t\wedge\sigma_n} \perp\!\!\!\perp \mathcal{F}_{s\wedge\sigma_n} | \mathcal{G}_{s\wedge\sigma_n}$.

Proof of example I

Let $N_t = (N_t^1, N_t^2) = (\sum_{n\geq 1} \mathbb{I}(\tau_n^1 \leq t), (\sum_{n\geq 1}) \mathbb{I}(\tau_n^2 \leq t))$ be a bivariate counting process where $(\tau_n^i)_{n\geq 1}$ is the increasing sequence of jump times of N_t^i. The distribution of this process is described by the family of joint survivor functions :

$$S_{n_1,n_2}(t_1^1, \cdots t_{n_1}^1, t_1^2, \cdots t_{n_2}^2)$$

$$= prob(\tau_1^1 > t_1^1, \cdots \tau_{n_1}^1 > t_{n_1}^1, \tau_1^2 t_1^2, \cdots, \tau_{n_2}^2 > t_{n_2}^2),$$

It is assumed that these joint survivor functions are continuously differentiable and that these processes admit stochastic intensities. Let us denote $\mathcal{F}_t$ the filtration generated by the whole process N_t and $\mathcal{G}_t = \mathcal{Z}_t$ the one generated by N_t^1 only. Now let h_t and h_t^* be the stochastic intensities of N_t^1 with respect to $\mathcal{F}_t$ and $\mathcal{Z}_t$, respectively.

The equivalence between the different concepts of non-causality follows from the property : $h_t = h_t^*$ a.s. implies that N_t^2 does not strongly globally cause N_t^1 (i.e. $\mathcal{G}_\infty \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_s$).

The proof may be decomposed in three steps :

i) Firstly, remark that it is sufficient to verify that :

$$\mathcal{G}_{\tau^1_{n+1}} \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_s, \forall n, \forall s : \tau^1_n \le s \le \tau^1_{n+1}. \tag{a}$$

Indeed, by an induction argument, (a) implies :

$$\mathcal{G}_\infty \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_s \quad , \quad \forall s. \tag{b}$$

Let us assume the property (a) is true. We want to show that :

$$\mathcal{G}_{\tau^1_{n+p}} \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_s \quad , \quad \forall p \ge 1, \forall s. \tag{c}$$

By (a), (c) is true for $p = 1$ and we have :

$$\mathcal{G}_{\tau^1_{n+p+1}} \perp\!\!\!\perp \mathcal{F}_{\tau^1_{n+p}} | \mathcal{G}_{\tau^1_{n+p}} \tag{d}$$

because $\tau^1_{n+p} \le \tau^1_{n+p+1}$ and, in (a), the time s may be replaced by a stopping time, using the same argument as in the proof of theorem 2.1 (iii). Moreover (d) implies :

$$\mathcal{G}_{\tau^1_{n+p+1}} \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_{\tau^1_{n+p}} \bigvee \mathcal{G}_s \tag{e}$$

because $\mathcal{G}_s \subset \mathcal{G}_{\tau^1_{n+p}}$ and $\mathcal{F}_s \subset \mathcal{F}_{\tau^1_{n+p}}$. Then, using the fundamental properties of the conditional independence (see, e.g., Florens and Mouchart (1982), theorem A1), (c) and (e) are jointly equivalent to

$$\mathcal{G}_{\tau^1_{n+p+1}} \perp\!\!\!\perp (\mathcal{F}_s \bigvee \mathcal{G}_{\tau^1_{n+p}}) | \mathcal{G}_s \tag{f}$$

which implies (c) for $p + 1$. Using a monotone class argument, (c) then implies :

$$\bigvee_{p=1}^{\infty} \mathcal{G}_{\tau^1_{n+p+1}} \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_s \quad , \quad \forall s \tag{g}$$

and (b) is demonstrated.

Intuitively, in this part of the proof, we have shown that the short-run prediction property (a) is equivalent to the long-run prediction property (b).

ii) From the equality between the two Doob-Meyer decompositions of N_t^1 given the two filtrations $(\mathcal{G}_t)_t$ and $(\mathcal{F}_t)_t$, we deduce that the decomposition with respect to a third filtration $(\mathcal{G}^s_t)_t$, verifying $\mathcal{G}_t \subset \mathcal{G}^s_t \subset \mathcal{F}_t$, $\forall t$, is also equal to the original decomposition. This result is an obvious consequence of the definition of the compensator and of the property of unicity (see also Bremaud (1981), section II4).

In the sequel of the proof, $\mathcal{G}_t^s$ will be :

$$\forall s \leq t, \quad \mathcal{G}_t^s = \mathcal{G}_t \bigvee \sigma\{N_u^2, 0 \leq u \leq s\}. \tag{h}$$

iii) We now consider the three stochastic intensities h_t, h_t^s and h_t^* with respect to the three filtrations $(\mathcal{F}_t), (\mathcal{G}_t^s), (\mathcal{G}_t)$ and, by hypothesis, $h_t = h_t^*$ which implies $h_t^* = h_t^s (s \leq t)$ ((recall that the equalities between stochastic processes are up to an evanescent process). We have :

$$h_t^s = -\frac{\partial}{\partial t} \log S_{n,p}^c(t, s | (\tau_i^1 = t_i^1)_{i=1,\cdots,n}, (\tau_j^2 = t_j^2)_{j=1,\cdots,p}) \tag{i}$$

where $\qquad \tau_n^1 \leq s < t < \tau_{n+1}^1,$

and $\qquad \tau_n^2 \leq s < t < \tau_{p+1}^1,$

$$h_t^* = -\frac{\partial}{\partial t} \log S_n^c(t | (\tau_i^1 = t_i^1)_{i=1,\cdots,n}) \tag{j}$$

where $\qquad \tau_n^1 \leq t \leq \tau_{n+1}^1.$

Then :

$$\int_s^t h_u^s du = \int_s^t h_u^* du \tag{k}$$

which implies :

$$\begin{aligned} &\frac{S_{n,p}^c(t, s | (\tau_i^1 = t_i^1)_{i=1,\cdots,n}, (\tau_j^2 = t_j^2)_{j=1,\cdots,p})}{S_{n,p}^c(s, s | (\tau_i^1 = t_i^1)_{i=1,\cdots,n}, (\tau_j^2 = t_j^2)_{j=1,\cdots,p})} \\ &= \frac{S_n^c(t | (\tau_i^1 = t_i^1)_{i=1,\cdots,n})}{S_n^c(s | (\tau_i^1 = t_i^1)_{i=1,\cdots,n})} \end{aligned} \tag{ℓ}$$

or equivalently :

$$P(\tau_{n+1}^1 \leq t | (\tau_i^1 = t_i^1)_{i=1,\cdots,n}, \tau_{n+1}^1 > s, (\tau_j^2 = t_j^2)_{j=1,\cdots,p}, \tau_{p+1}^2 > s)$$

$$= P(\tau_{n+1}^1 \leq t | (\tau_i^1 = t_i^1)_{i=1,\cdots,n}, \tau_{n+1}^1 > s). \tag{m}$$

Then :

$$\sigma(\tau_{n+1}^1) \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_s, \quad \forall s : \tau_n^1 \leq s < \tau_{n+1}^1 \tag{n}$$

and :

$$\mathcal{G}_{\tau_{n+1}^1} \perp\!\!\!\perp \mathcal{F}_s | \mathcal{G}_s, \quad \forall s : \tau_n^1 \leq s < \tau_{n+1}^1 \tag{o}$$

because $\mathcal{G}_{\tau^1_{n+1}} = \mathcal{G}_s \bigvee \sigma(\tau^1_{n+1})$ (see Karr (1986), section 2.1).

Then using the first part of the proof, the demonstration is complete ■

Proof of example III

- Proof of property 1 : Non causality assumption is defined by

$$E(\varphi(z_t)|\mathcal{F}_s) = E(\varphi(z_t)|\mathcal{Z}_s) \quad \text{a.s.}$$

and Markovian hypothesis implies :

$$E(\varphi(z_t)|\mathcal{F}_s) = E(\varphi(z_t)|X_s) \quad \text{a.s.}$$

Then, using measurable separability, $E(\varphi(z_t)|\mathcal{F}_s)$ is a.s. equal to a function $\psi(z_s)$. From homogeneity assumption we get :

$$E(\varphi(z_u)|x_0 = \xi) = \psi(\zeta) \quad \text{a.s. } \xi \in I\!R^n, \zeta \in I\!R^q$$

and then $T_u(\varphi) \in L^2_z$ where $\varphi \in L^2_z$.

Reciprocally, the property $T_u(\varphi) \in L^2_z$ implies that $E(\varphi(z_t)|X_s)$ is a.s. equal to a function $\psi(z_s)$. Moreover, the Markovian property implies that :

$$E(\varphi(z_t)|\mathcal{F}_s) \quad = \quad E(\varphi(z_t)|X_s) \qquad \text{a.s.}$$

Then $E(\varphi(z_t)|\mathcal{F}_s)$ is a.s. equal to $\psi(z_s)$ and so is $\mathcal{Z}_s$-measurable, which defines the non-causality.

- Property 2 is proved by repeating the previous argument for any $\varphi \in L^2_z$.
- Property 3 is essentially based on the relation between the semimartingale decomposition and the infinitesimal generator. This relation is considered in Revuz and Yor (1991, chap. VII, prop. 1.6) which implies that if $\varphi \in D_z$, the predictable component of $\varphi(z_t)$ is equal to :

$$H_t = \int_0^t \psi(z_s)ds$$

where $\psi = A(\varphi)$.

- Property 4 follows from the same argument than property 3.

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