On the Equivalence of Coalitional and Individual Strategy-Proofness Properties

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Abstract

In this paper, we introduce a sufficient condition on the domain of admissible preferences of a social choice mechanism under which the properties of individual and coalitional strategyproofness are equivalent. Then, we illustrate the usefulness of this general result in the case where a fixed budget has to be allocated among several pure public goods.

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1 Introduction

In social choice theory, we expect any collective decision within a group of individuals to reflect the preferences of its members over the feasible social alternatives. Since any individual is allowed to express freely his/her preference, it is however necessary to specify which alternative will be selected for each conceivable profile of preferences. This mapping is called hereafter a social choice mechanism. From a normative viewpoint, the concept is well defined since, as long as preferences are the only individual characteristics which matter, the mechanism simply describes which alternative *should* be selected in any possible set of circumstances. From a positive viewpoint, the mapping is truly a composed mapping, as an institution is an object which can be far more complicated than a direct revelation mechanism. An institution is described by a set of rules leading, from the perspective of the analysis, to a normal form game. The key observation is that when we account for equilibrium behavior in the setting describing that institution, we end up with a set of social alternatives which only depends upon the profile of preferences. This means that under the presumption that this set is nonempty and does not contain several alternatives, we can look at the composed map (amalgating the institution and the equilibrium behavioral responses) as a social choice mechanism.

In the context of a social choice mechanism, the strategic choice of an agent consists in reporting his/her preferences over the alternatives. In this revelation game, like in any game, the ultimate effect of his/her choice will depend upon his report together with the reports of the other individuals. There is no reason to assume that individuals will report the truth : if an agent can secure a better alternative by announcing preferences different from the truth, he/she may do so. These misreports can lead to a collective decision which has very little to do with the one based on true preferences and may turn out to be quite unsatisfactory. It becomes therefore important to identify which social choice mechanisms are immuned to such manipulations.

In this paper, we will focus on two notions describing the resistance of a social choice mechanism to manipulations. The first one is *strategyproofness* which is a very strong form of robustness against "misbehavior". A social choice mechanism is strategyproof or individual strategyproof (when we want to call the attention on the fact that only the behavior of individuals is taken into consideration) if telling the truth is a dominant strategy for every individual. This means that an individual does not need to solve the strategic uncertainty (attached, in principle to any game) to know what is his/her best strategy : no matter what the others do, a lie never pays out. This strong form of incentive compatibility is attractive but very demanding. In fact, an extremely dissapointing but fundamental result due to Gibbard (1973) and Satterthwaite (1975) states that if any preference can be reported, then only dictatorial mechanisms are strategyproof. In this paper, we are going to investigate the implication of strategyproofness for a class of environments where not every conceivable preference can be reported by an individual. We assume that an individual can report a preference from a prescribed subset of the all set of preferences, called the *set of admissible preferences*. Under that assumption of a restricted domain of preferences, the nihilist conclusion of the Gibbard-Satterthwaite impossibility theorem may disappear in the sense that there exist non dictatorial strategyproof social choice mechanisms. In this paper, we consider this general setting : for some domain of admissible preferences, strategyproofness leads to a very narrow class of mechanisms while for some others, the class may contain very satisfactory mechanisms. We dont touch the difficult and open question of characterizing the class of admissible domains of preferences leading to non dictatorial strategyproof social choice mechanisms.

The second notion of resistance to manipulation that we consider aims to incorporate the idea that besides individuals, groups (coalitions) of individuals may also play an active role, not captured by the notion of strategyproofness. Precisely, we want to consider a notion where the threats of coalitions are described and taken into consideration when designing the social choice mechanism. This calls for a precise definition of what a coalition can do if it forms[1] that its members cannot do on their own. This question is very controversial and before explaining the precise version that we will use here, it seems important to discuss briefly some issues related to it. We may ask first if coalitions can proceed in monetary transfers and make binding agreements in which case the apparatus of cooperative game theory with side payments could be useful to define the power of coalitions in contrast to individuals. In this paper, we will consider social environments where social alternatives are public in nature and in particular such that no monetary transfers are involved. This precludes the use of this approach to define coalitional behavior in our paper. Then, the unique role which is left to coalitions is to coordinate the reports of its members to attain a specific objective. Without side payments, the objective cannot be the sum of the payoffs of the members of the coalition and the all Pareto frontier has to ba taken into consideration. Precisely, the definition of coalitional strategyproofness that we consider in this paper is the following. A social choice mechanism is *coalitionally manipulable* if for some profile of preferences, there exist a

[1] We dont allude here to the direct cost(s) of forming a coalition. In principle; the trade off between these costs and the (expected) benefits resulting from the formation of the coalition should be explicitely formulated.

coalition such that when their members *jointly* misreport adequately their preferences, the mechanism selects an alternative that they all prefer to the one that would result all of them had reported their true preferences.A social choice mechanism is *coalitional strategyproof* if it is never coalitionally manipulable. This definition is the conventional[2] definition of coalitional strategyproofness used in the literature. From the perspective of the revelation game, it requires that the profile of truthful reports consitutes a *strong Nash equilibrium* (Aumann (1959)). This is extremely demanding requirement as it is difficult to guarantee existence of such equilibria. Note, in particular, that it implies that the profile is a Pareto efficient Nash equilibrium; this means that a Nash (in fact, dominant strategy) equilibrium like the one in a prisonner dilemna setting is excluded. In contrast, in settings like "pure coordination games" where there are many Nash equilibria which are ordered alike by the individuals, strong Nash equilibrium acts as a selection device.

We think that this definition is the most demanding conceivable definition of robustness against deviations by coalitions. This means that if a social choice mechanism is coalitional strategyproof in that sense, it is coalitional strategyproof in any other reasonable sense. If instead, a social choice mechanism fails to be coalitional strategyproof in that sense, this negative conclusion should be examined with caution. In particular, we may wonder if all the coalitional threats should be treated equally. Suppose indeed, that a profitable joint deviation by a coalition is identified leading to the conclusion that the profile of reports fails to pass the equilibrium test. If it turns to be the case that in the reduced game (the subset of players being the members of that coalition) some players or some subcoalitions of players find profitable to further deviate from the deviation, then this may just deter the initial deviation on credibility grounds. This type of criticism gave rise to several alternative and less demanding concepts of Nash equilibrium robust to coalitional deviations, among which the concept of coalition-proof Nash equilibrium (Bernheim, Peleg and Whinston (1987)). This concept is attractive because it is based on a consistent inductive definition of coalitional deviation but some other proposals have also been formulated. With such concept(s), there are, in principle, more coalitional strategyproof social choice mechanisms.

The main purpose of this paper is to study under which conditions the properties of (individual) strategyproofness and coalitional strategyproofness coincides. Strictly speaking, if a social choice mechanism is coalitional strategyproof then it is strategyproof but the converse does not need to hold true. Our main contribution is to identify a sufficient condition on the domain of admissible preferences for this equivalence to hold true. We call *rich*

[2]It comes often under different names : coalition, coalitionally and group strategyproofness are among the most spread.

domains such domains. Note that when this equivalence holds true, then the above discussion about the appropriate definition of coalitional threats becomes irrelevant as any other concept is nested between these two ones. In the second part of the paper, we illustrate the power of this result through the examination of a specific allocation environment. It is important to point out that behind our equivalence result, there is no hidden result like "if a domain is rich, then a strategyproof social choice mechanism is dictatorial". Indeed, as we show in that part, there are rich domains admitting non dictatorial strategyproof social choice mechanisms.

Related Literature

This paper is at the intersection of two branches of the literature. On one hand, we study the role of the domain of admissible preferences on the properties of a social choice mechanism satisfying some other conditions. On the other hand, we are mostly interested by the definition and implications of coalitional incentive compatibility constraints in the design of a social choice mechanism.

We are not the first to pay attention to the role of the domain in the characterization of strategyproof social choice mechanisms. As the Gibbard-Satterthwaite's theorem makes use of the universal domain condition, it was natural to investigate the responsability of that assumption in the derivation of the result. It was also useful since (besides voting) most environments of interest entail restricted preferences. This literature is nicely surveyed in Barbera (2001) and Sprumont (1995). To the best of our knowledge, very few general principles have been established and the research has consisted mostly in the detailed study of the implications of strategyproofness in some classes of problems. It is worth mentionning few of these general results. Dasgupta, Hammond and Maskin (1979) have introduced a general domain richness condition that they use as a generalization of the universal domain condition in the formulation of many results in the theory of implementation. In a general class of allocation environments (covering the cases with private components), Fleurbaey and Maniquet (1997) have introduced the notion of strict monotonic closedness and demonstrate that for such domains, strategyproofness, non-bossiness[3] and equal treatment of equals imply no-envy. For the same class of domains, Moulin (1993) demonstrates that coalitional strategyproofness and equal treatment of equals imply also no-envy. He also notes that coalition strategyproofness could be replaced by Maskin monotonicity which, as noted by Fleurbaey and Maniquet is a stronger requirement than strategyproofness and non-bossiness

[3] WThis property was introduced by Satterthwaite and Sonnenschein (1981). We dont define it precisely : it amounts to require that if by changing his report, an individual does not change his allocation, then he does not change the alocation of somebody else.

together, under the Dasgupta, Hammond and Maskin richness condition. These domain richness conditions are logically unrelated to our richness domain condition. Given two profiles satisfying some properties, these conditions ask for the existence of a third profile satisfying also some properties. In contrast, our condition asks that for any single profile satisfying some properties, there exists a second profile satisfying some properties.

We have already discussed some of the conceptual issues arising from the definition of coalitional incentive compatibility. The strong form considered here has been incorporated by many authors in axiomatic social choice. For instance, Moulin, in many (e.g. Moulin (1994), (1999)) of his works on axiomatic cost sharing, uses it as a key axiom. Of course, the question of coalitional incentive compatibility raises many challenging problems and have been formulated differently by other authors. Within the general theory of implementation, Maskin (1979) shows that the set of social choice mechanisms which can be implemented in strong Nash equilibrium is much smaller that the set of those which are simply Nash implementable. With the weaker concept of coalition-proof Nash equilibrium, Bernheim and Whinston (1987) derives similar conclusions. Based on a differential approach, Laffont and Maskin (1980) show the difficulty to conciliate strategyproofness and coalitional incentives. This line of research follows some early work by Green and Laffont (1979) emphasizing the impossibility of constructing Clarke-Groves mechanisms which are robust to the formation of coalitions. They assume[4] that side payments are possible among the members of the coalition and that coalitions do not face informational issues. When coalitions are confronted themselves to the issue of elicitating information about preferences, we open the door to a class of new and difficult problems where the games of side contracting have to be properly defined. Cremer (1996) revisits the family of Clarke-Groves mechanisms from that perpective and obtain few positive results. Several contributions in the traditional Bayesian mechanism design approach have paid attention to coalitional considerations in different settings ranging from auctions to general organizations. Laffont and Martimort (1997) have characterized quite generally the class of social choice mechanisms which are immuned to coalitional side contracting in this Bayesian setting.

2 Definitions and Notations

In this section, we present the class of social choice environments which we consider and we introduce the main definitions and notations which are going to be used throughout this paper.

[4] They call strongly coalition incentive compatible this class of Clarke-Groves mechanisms.

We are interested in a situation where a society (group) described by a finite set $N = \{1, 2, ..., n\}$ of individuals must decide which alternative to select out of a set X of feasible alternatives. Each individual $i \in N$ is described by his/her preference R_i over X, which is assumed to be a complete preorder. We will denote respectively by P_i and I_i the strict preference and the indifference relations induced by R_i. Sometimes, we will represent a preference R_i by a utility function U_i, but the reader should keep in mind that, in our framework, two utility functions representing the same preference will always be considered as equivalent. A profile of preferences is a vector $\pi \equiv (R_1, R_2, ..., R_n)$ describing the preferences of each individual in the society. If π is a profile of preferences and $S \subseteq N$ is a subset of individuals, then π_S denotes the subprofile $(R_i)_{i \in S}$; when $S = N \backslash \{i\}$ for some $i \in N$, we denote π_{-i} for π_S. If π and π' are two profiles of preferences and $S \subseteq N$, then $\pi'' \equiv \left(\pi_S, \pi'_{N \backslash S}\right)$ denotes the profile such that $\pi''(i) = \pi(i)$ if $i \in S$ and $\pi''(i) = \pi'(i)$ if $i \notin S$.

Definition 1 *Let* Π *be a subset of profiles. A social choice mechanism with domain* Π *is a mapping* C *from* Π *into* X.

If Π consists of all possibles profiles, the domain is said to be *universal*. Otherwise, it is said to be *restricted*. The notion of domain is central in our paper as the results are driven by assumptions which will be formulated on the domain. We will limit our investigation to Cartesian domains i.e. domains such that $\Pi = \prod_{i=1}^{n} D_i$ where for all $i \in N$, D_i is a subset of complete preorders over X.

The social choice mechanism reflects the aspirations and properties that this society wants to take into account to proceed in selecting a social alternative. The input of such mechanism is a profile of preferences. This means that once we know the diversity of opinions in the society, conflicts but also areas of agreement, we have, in principle, everything needed to pick up a compromise. To operate, the mechanism needs this input, but in most cases, this input is not known or verifiable with certainty by all members of the society. Confronted with this difficulty, we could then consider a broader class of social choice mechanisms where the domain would be now a Cartesian set $M = \prod_{i=1}^{n} M_i$ where for all $i \in N$, M_i is an abstract set of messages or reports that can be sent by individual i. As every individual is ultimately interested by the social alternative that will be selected, such a mechanism together with the profile π of preferences generates a normal form game among the individuals which are assumed to be rational players : for each individual i, the choice of the message m_i to be sent, constitutes a strategic choice.

Using the apparatus of game theory, we can predict the equilibrium behavior of the individuals and therefore the social outcome. In this paper, we focus on social choice mechanisms admitting equilibria in dominant strategies i.e. such that, for all profile π in the domain Π, each individual i has a dominant strategy. It is well known[5] that for such equilibrium concept, there is no loss of generality in restricting attention to the class of social choice mechanisms introduced in definition 1 and to impose that the report of the truth is a dominant strategy for every individual in every possible circumstance.

Definition 2 *A social choice mechanism C with domain Π is manipulable by individual i at profile π if there exists $R_i' \in D_i$ such that $C(\pi_{-i}, R_i') P_i C(\pi)$. A social choice mechanism C with domain Π is strategyproof if there is no individual i and no profile $\pi \in \Pi$ such that C is manipulable by i at π.*

This property reflects the necessity to provide incentives to individuals to make sure that they report the right information. Strategyproofness is a strong form of incentive compatibility as it requires the existence of dominant strategies. From the perspective of constructing the social choice mechanism, this property acts as a constraint in the design of the rule.

Some few more definitions and notations are needed. From now on, we limit our attention[6] to the case where $D_i \equiv D$ for all $i \in N$. This assumption is not innocuous as it implies that there are no intrinsic ex ante differences among individuals. This rules out social environments with private dimensions. Let $D^* \equiv \cup_{x \in X} D_x$ where :

$$D_x \equiv \{R \in D : xPy \text{ for all } y \in X \backslash \{x\}\}$$

D_x is the set of preferences for which the alternative x is uniquely best. Finally, let $X^* \equiv \{x \in X \text{ such that } D_x \neq \varnothing\}$ be the set of alternatives which may appear on top for the domain of preferences D which is considered and let C^* be the restriction of C to the subdomain $(D^*)^n$.

The following result which will be used in some proofs. Let :

$$C(\Pi) \equiv \{x \in X : x = C(\pi) \text{ for some } \pi \in \Pi\}$$

be the range of the mechanism C.

Lemma 1 *Let C be a strategyproof social choice mechanism with domain D. For all $\pi \in D^n$ and all $x \in C(\Pi)$, if $R_i \in D_x$ for all $i \in N$, then $C(\pi) = x$.*

Definition 3 *A social choice mechanism C with domain Π is regular if $C(\Pi) \subseteq X^*$.*

[5] This is the so called revelation principle (Dasgupta, Hammond and Maskin (1979). The reader may consult Jackson (2001) for a nice overview of implementation theory.

[6] In what follows, we will often abusely use the expression domain for both Π and D. Under this uniformity assumption, we dont see any risk of confusion.

To the best of our knowledge, this property is new. It requires that the range of the mechanism is contained in the subset of alternatives which appear on the top of an admissible preference. It is certainly controversial in any environment where an alternative which could be considered as a good social compromise is disregarded simply because at best, it appears on second position in any indididual preference. In this paper, we will consider environments where the property of regularity does not raise any problem. The following simple lemma will be useful.

Lemma 2 *Let C be a strategyproof and regular social choice mechanism with domain D. Then, $C(\Pi) = C^*(\Pi)$.*

Proof : Since $C^*(\Pi) \subseteq C(\Pi)$, we are left to prove that $C(\Pi) \subseteq C^*(\Pi)$. Let $x \in C(\Pi)$. Since C is regular, $x \in X^*$. Let $\pi \in D^n$ be such that $R_i \in D_x$ for all $i \in N$. By lemma 1, $x = C(\pi) = C(\pi)$ and hence $x \in C^*(\Pi)$ □

The property described in the following definition has been introduced by Barbera et Peleg (1990).

Definition 4 *A social choice mechanism C with domain D^n satisfies the modified strong positive association*[7] *property if for all $\pi, \pi' \in D^n$, all $i \in N$* and all $x \in C(\Pi)$*, if $C(\pi) = x$* and *xP'_iy for all $y \in C(\Pi) \backslash \{x\}$* such that xR_iy, then $x = C\left(\pi_{-i}, R'_i\right)$.

The following lemma due to Barbera and Peleg will be useful.

Lemma 3 *A strategyproof social choice mechanism C with domain Π satisfies the modified strong positive association property.*

The notion of strategyproofness describes individual incentives to report the truth. The next notion deals with the behavior of coalitions.

Definition 5 *A social choice mechanism C with domain Π is manipulable by coalition S at profile π if there exists $\pi' \in \Pi$ such that $C\left(\pi'_S, \pi_{N \backslash S}\right) P_i C(\pi)$* for all *. A social choice mechanism C with domain Π is coalitional strategyproof if there is no coalition S and no profile $\pi \in \Pi$ such that C is manipulable by S at π.*

Coalitional strategy proofness is obviously more demanding than strategyproofness. It requires that there are no profitable deviations from reporting the truth not only by individuals but also by groups of individuals. This property is very demanding as it does not impose to the deviating coalition to be credible. When a coalition S deviates, it is faced with a reduced game where the players are the members of S and it is natural to restrict

[7] Muller and Satterthwaite (1977) have proved the equivalence of strategyproofness and strong positive association over the universal domain of complete orders. The modified positive association property is neccesary but not sufficient in general for strategyproofness. Tanaka (2002) exhibits an association property which is both necessary and sufficient on the domain of continuous preferences?

attention to patterns of plays which pass some equilibrium test in this reduced game. Besides profitability, coalitional strategyproofness does not include such type of restrictions. Doing so would lead to a less demanding notion of coalitional strategyproofness. So, in some sense, the notion considered here is the more demanding one within this class of notions and if a social choice mechanism verifies that version, it verifies any other property of coalitional strategyproofness that may be reasonably considered.

We conclude this section with an important notion that will be used in subsequent proofs and a technical lemma. Given a social choice mechanism C with domain D^n, a profile $\pi \in D^n$ and a coalition $S \subseteq N$, we denote by $C_S\left[\pi_{N\backslash S}\right]$ the social choice mechanism defined over the subsociety S with domain D^S by :

$$C_S\left[\pi_{N\backslash S}\right](\pi'_S) = C\left(\pi'_S, \pi_{N\backslash S}\right) \text{ for all } \pi'_S \in D^S$$

The range of the mechanism $C_S\left[\pi_{N\backslash S}\right]$ will be denoted $A_S\left[\pi_{N\backslash S}\right]$: it describes the set of alternatives (options) attainable by coalition S given the subprofile $\left[\pi_{N\backslash S}\right]$ of reports by individuals outside coalition S. These sets , called *option sets* by Barbera and Peleg (1990)[8] will play a critical role in the rest of the paper. For all $i \in N$ and $\pi \in D^n$, the option set of coalition $\{i\}$ will be denoted $A_i\left[\pi_{-i}\right]$.

Lemma 4 *Let X be a metric space and D be a subset of the set of continuous preferences over X . If C is a strategyproof social choice mechanism with domain D^n then for all $\pi \in D^n$ and all $S \subseteq N$, $A_S\left[\pi_{N\backslash S}\right] \cap X^*$ is a closed subset of X^*.*

Proof : Let $\pi \in D^n$, $S \subseteq N$ and $x \in X^* \backslash A_S\left[\pi_{N\backslash S}\right]$. We claim that there exists $\varepsilon > 0$ such that :

$$B(x,\varepsilon) \cap A_S\left[\pi_{N\backslash S}\right] = \varnothing$$

Suppose on the contrary that for all $\varepsilon > 0$, there exists $z_\varepsilon \in B(x,\varepsilon)$ such that $z_\varepsilon \in A_S\left[\pi_{N\backslash S}\right]$. Since $x \in X^*$, there exists $R^* \in D^n$ such that $\in D_x$. Let $\pi' \equiv (R^*, R^*, ..., R^*)$ $y \equiv C\left(\pi'_S, \pi_{N\backslash S}\right)$. Since $y \neq x$ and $R^* \in D_x$ we deduce : xP^*y . Since preferences are continuous, we deduce that there exists $\delta > 0$ such that for all z in the ball $B(x,\delta) : zP^*y$. Select such a $\varepsilon > 0$ and $R^\varepsilon \in D$ such that such $R^\varepsilon \in D_{z_\varepsilon}$. Let $\pi^\varepsilon \equiv (R^\varepsilon, R^\varepsilon, ..., R^\varepsilon)$ and

$$w = C(\pi^\varepsilon_S, \pi_{N\backslash S})$$

Since $z_\varepsilon \in A_S\left[\pi_{N\backslash S}\right]$, we deduce from lemma 1 that $w = z_\varepsilon$.

[8]This technique has been pionnered by Laffond (1980).

Without loss of generality, let $S \equiv \{1, ..., s\}$ where $s \equiv \#S$ and consider the finite sequence of profiles $\left(\widetilde{\pi}^j\right)_{0 \leq j \leq s}$ defined as follows :

$$\widetilde{R}_i^j \equiv \begin{cases} R_i \text{ for all } i \notin S \\ R^\varepsilon \text{ for all } i \in \{1, ..., j\} \\ R^* \text{ for all } i \in \{j+1, ..., s\} \end{cases}$$

Since C is strategyproof, we deduce :

$$C(\widetilde{\pi}_S^j, \pi_{N \setminus S}) R^* C(\widetilde{\pi}_S^{j+1}, \pi_{N \setminus S}) \text{ for all } j = 0, ..., s-1$$

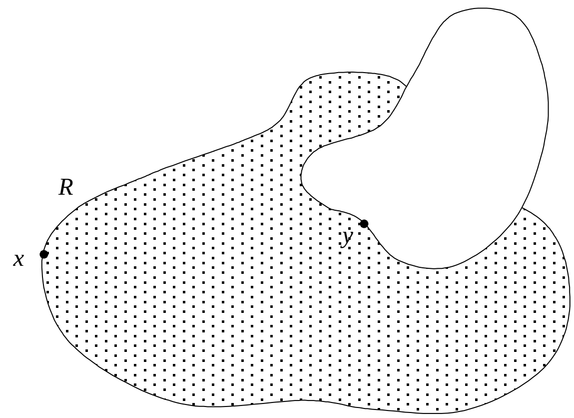
Since $C(\widetilde{\pi}_S^0, \pi_{N \setminus S}) = y$ and $C(\widetilde{\pi}_S^s, \pi_{N \setminus S}) = w = z_\varepsilon$, we deduce from above and transitivity of R^* that $y R^* z_\varepsilon$, a contradiction to $z_\varepsilon P^* y$ □

3 Rich Domains

In the preceding section, we have introduced two notions of strategyproofness. Individual strategyproofness takes into consideration profitable dishonest reports by individuals while coalitional strategyproofness extends the requirement to all coalitional dishonest joint reports. As already pointed out, the two notions are nested : coalitional strategyproofness is more demanding than strategyprofness. It is not difficult to produce environments for which it is strictly more demanding. The purpose of this section is to identify a class of social environments for which the two notions coincide. Precisely, we introduce a condition on the domain D of preferences which is sufficient for this equivalence to hold true. This class of domains, that we call *rich domains* hereafter, is defined as follows.

Definition 6 *A domain $\Pi = D^n$ is rich if for all $R \in D$ and $x, y \in X$ such that yPx and $y \in X^*$, there exists $R' \in D$ such that $R' \in D_y$ and for all $z \neq x$ such that xRz, we have $xP'z$.*

To be rich, a domain must contain enough preferences. Of course, the universal domain is rich but there are also many restricted domains which meet this richness requirement. Intuitively, when a domain is rich we are able to consider transformations of individual preferences where the positions of two given alternatives are improved in the process. Precisely the alternative y which was best among the two is now best among all and the other one x still strictly dominates the alternatives that it was strictly dominating before but now x also strictly dominates the alternatives belonging to its former indifference curve. This is illustrated on figures 1 and 2 in the case where alternatives are vectors in the two dimensional Euclidean space. On figure 1, we have drawn the upper contour sets of x and y for

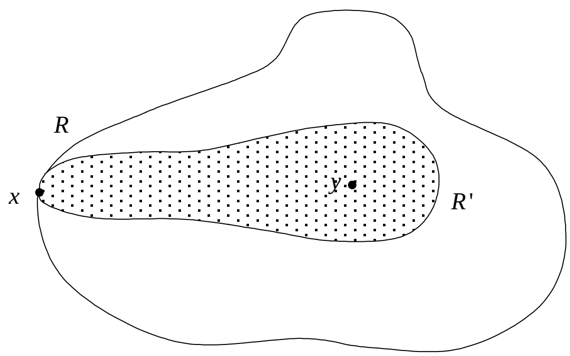


the preference R. On figure 2, we have reproduced the upper contour set of x for R and drawn, as a dotted curve, the upper countour set of x for R'. The upper contour set of y for R' consists exclusively of y.

It should be transparent from this illustration that for a domain to be rich, we must have enough degrees of freedom to deform preferences. If not, the richness condition is likely to be violated. Consider for instance the traditionnal Euclidean environment popular in formal political science i.e. the setting where X is some Euclidean space $\Re^m$ and D is the subset of Euclidean preferences : a preference R over X is Euclidean if there exists $p \in \Re^m$ such that xRy iff $\| x - p \| \leq \| y - p \|$. The upper contours sets are the spheres centered on p. The set of Euclidean preferences is not rich. To see why, consider the case where $m = 2$. On figure 3, we have drawn the upper contour set of x with y inside but different from p. As we can see immediately, necessarily, the circle centered on y and containing x, has points outside the first disk.

We will see two important examples of rich domains in the next section. Besides these environments, we can also prove[9] that the set of continuous preferences over a metric space considered by Barbera and Peleg (1990) is rich. However, it is important to call the attention on the fact that there are general properties of preferences which preclude the richness

[9] A proof is available upon request from the authors.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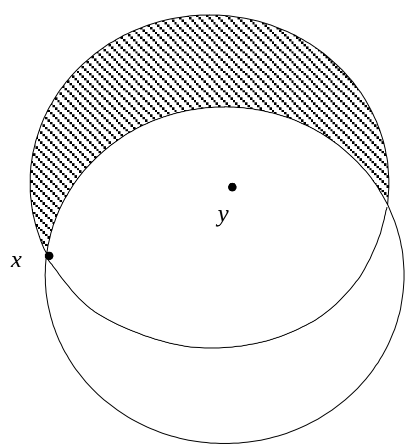
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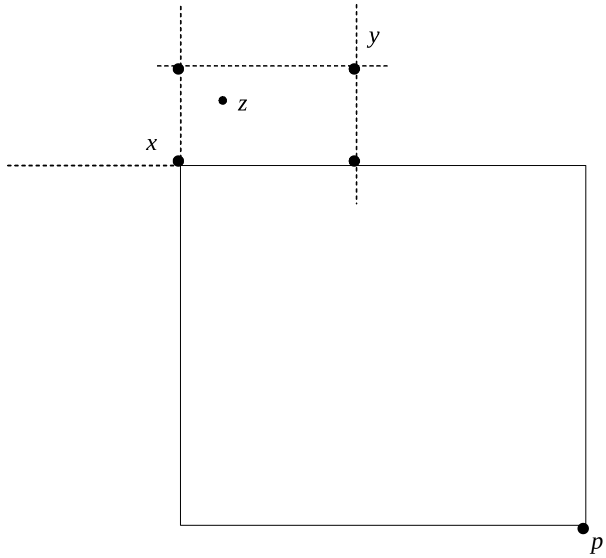


condition. For instance, if D is a subset of the set of separable preferences over a Cartesian set of alternatives[10], then D cannot be rich. To see why, consider the specific case[11] where $X = \Re^2$ and D is the subset of separable preferences with single peaked marginal preferences as defined by Barbera, Gul and Stachetti (1993) and Border and Jordan (1981). The domain D contains preferences R such that p is best, yPx and xPz where the respective positions of p, x and y and z are represented on figure 4. The key features of this pattern are that y does not belong to the rectangle generated by x and p and that z belongs to the rectangle generated by x and y. This is illustrated on figure 4 below. From the definition of D, it follows that any preference $R' \in D$ such that y is on top for R' implies that any alternatives w in the rectangle generated by x and y is prefered to x according to R'. In particular, we have $zR'x$.

Our main result on rich domains is the following.

Theorem 1 *Let* C *be a social choice mechanism with domain* $\Pi = D^n$*. If* D *is rich, then* C *is strategyproof if and only* C *is coalitional strategyproof.*

Proof : Let C be a strategyproof social choice mechanism on a domain D^n assumed to

[10] Intuitively, a preference over a product space is separable if preferences over each factor of the product are defined without ambiguity. Such well defined projections are then called marginal preferences.

[11] A similar conclusion holds true for the domain of continuous and separable preferences considered by Le Breton and Weymark (1999).

be rich. We now prove that C is coalitional strategyproof. assume on the contrary that C is not coalitional strategyproof. Then, there exists $S \subseteq N$ and $\pi, \pi' \in D^n$ such that for all $i \in S$:

$$x \equiv C(\pi'_S, \pi_{N \backslash S}) P_i C(\pi) \equiv y$$

Since D is rich, there exists $\pi'' \in D^n$ such that for all $i \in S$:

$$R''_i \in D_x \text{ and for all } z \neq y : yR_i z \Rightarrow yP''_i z$$

and for all $i \notin S$:

$$R''_i = R_i$$

Given the construction of π'' and since $C(\pi) = y$, a repeated application of lemma 3 leads to :

$$C(\pi'') = y \tag{1}$$

On the other hand, note that since C is strategyproof, the restricted social choice mechanism $C_S\left[\pi_{N \backslash S}\right] = C_S\left[\pi''_{N \backslash S}\right]$ is also strategyproof. Since $x \in A_S\left[\pi''_{N \backslash S}\right]$ and $R''_i \in D_x$ for all $i \in S$, we deduce from lemma 1 that $C_S\left[\pi''_{N \backslash S}\right](\pi''_S) = C(\pi'') = x$ in contradiction to (1) □

While less important, there are also some other implications of the richness condition that we would like to report as they will be used as auxilliary results in the next section.

Definition 7 *A social choice mechanism C with domain D^n is dictatorial if there exists an individual* $i \in N$ such that for all $\pi \in D^n$ and all $x, y \in C(\Pi)$, if $xP_i y$, then $C(\pi) \neq y$.

A dictatorial social choice mechanism ignores the preferences of all but one individual : the most prefered alternative of this individual, called the dictator, is selected to be the social outcome.

Lemma 5 *Let C* be a *regular social choice mechanism with domain* $\Pi = D^n$. *If D is rich, then C is dictatorial if and only C^* is dictatorial.*

Proof : Assume that C^* is dictatorial and let us prove that C is dictatorial too. Let i be the dictator for C^* and assume on the contrary that i is not a dictator for C. Then there exist $\pi \in D^n$ and $x, y \in C(\Pi)$ such that $C(\pi) = x$ and $yP_i x$. Since D is rich, there exists $R'_i \in D$ such that :

$$R'_i \in D_y \text{ and for all } z \neq x : xR_i z \Rightarrow xP'_i z$$

Further, since C is regular, for all $j \in N \backslash \{i\}$, there exists R'_j such that :

$$R'_j \in D_x$$

A repeated application of lemma 3 leads to $C(\pi') = x$. But, on the other hand, since $C(\pi') = C^*(\pi')$ and i is a dictator for C^*, we deduce that $C(\pi') = y$ in contradiction to the earlier statement □

4 Applications

The main purpose of this section is to illustrate the usefulness of theorem 1 though a detailed examination of a specific[12] but important environment. When a domain of preferences Π is rich, the analysis of the implications of strategyproofness in the construction of social choice mechanisms is considerably simplified as we know that the mechanism is in fact coalitional strategyproof. Note in particular that if a mechanism C is coalitional strategyproof, then it is Pareto efficient over the range $C(\Pi)$ i.e. there does not exist $\pi \in \Pi$ and $x \in C(\Pi) : xP_iC(\pi)$ for all $i \in N$.

Since Pareto Efficiency put some constraints on the subset of social outcomes that may be considered, this information can be exploited to simplify the analysis of the mechanism C.

4.1 Allocation of a Budget Across Several Different Pure Public Goods

The allocation environment considered in this section has been examined first by Zhou (1991) and is defined as follows. An exogeneous monetary budget of size normalized to 1 is to be allocated across m different pure public goods. The set X of alternatives is therefore the unitary m-dimensional simplex :

$$\left\{ x \in \Re^m_+ : \sum_{k=1}^{m} x_k = 1 \right\}$$

We assume that each individual $i \in N$ has a preference over the m-dimensional positive orthant $\Re^m_+$ which is assumed to be strictly monotonic and strictly convex. The set D is the set of restrictions of such preferences to the set X. It is straightforward to show that

[12] Without aiming to provide general guidelines for the user.

a preference R is in D iff its upper contour sets are strictly convex. Theorem 3 stated below holds true for all n and all $m \geq 3$ but for the sake of simplicity, we will limit our investigation to the case where $m = 3$ and $n = 2$. The case where $m = 2$ is considered in the next subsection.

Lemma 6 *Let $m = 3$ and $n = 2$. Then the set of preferences D of preferences with strictly convex upper contour sets is rich.*

Proof : Let R be a preference in D such that yPx for some $x, y \in X$. Let A be the upper contour set of x with respect to R i.e.

$$A = \{z \in X : zRx\}$$

A is a closed and strictly convex subset of X with $y \in$ Interior A. Let A' be a closed and strictly convex subset of A such that $y \in$ Interior A' and Boundary $A \cap$ Boundary $A' = \{x\}$. The construction of such subset is illustrated on figure 5.

Let J be the jauge of $(A' - \{y\})$ with respect to y i.e. the function defined by :

$$J(w) = \underset{w-y \in \lambda(A'-\{y\})}{Inf} \lambda$$

It is well known[13] that J is a continuous and convex (here strictly convex) function such that :

$$J(w) = 1 \text{ iff } w \in \text{Boundary } A'$$

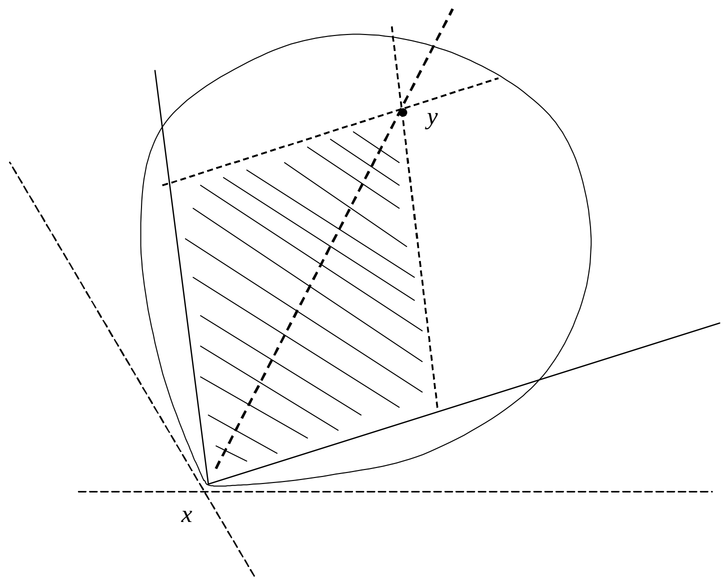
Let R' be the preference generated by $-J$. By construction, $R' \in D_y$ and for all $z \neq x$, xRz implies $xP'z$ □

Lemma 7 *Let $m = 3$ and $n = 2$. Let C be a strategyproof social choice mechanism over D such that $C(D^n) = X$. Then, for all $x \in X$ and all R_1, $R_1' \in D_x$, $A_2(R_1) = A_2(R_1')$.*

Proof : Assume on the contrary that there exists $z \in A_2(R_1)$ such that $z \notin A_2(R_1')$. We construct a preference R_2 as follows. On one hand, since from lemma 4, $A_2(R_1')$ is closed, there exists a ball $B(z, \varepsilon)$ where $\varepsilon > 0$ such that $B(z, \varepsilon) \cap A_2(R_1') = \varnothing$. On the other hand, from lemma 1, we deduce that $x \in A_2(R_1')$. Let :

$$w \equiv \text{Boundary } B(z, \varepsilon) \cap [x, z]$$

[13] See e.g. Rockafellar (1970).



Since R_1 is strictly convex : wP_1z. Since R_1 is continuous, we deduce therefore that there exists a ball $B(w, \delta)$ where $\delta > 0$ such that for all $u \in B(w, \delta) : uP_1z$. Let $\{u', u''\} \equiv$ Boundary $B(z, \varepsilon) \cap$ Boundary $B(w, \delta)$. Consider the two half- lines with origin x and going respectively through u' and u'' and the convex set S as on figure 6.

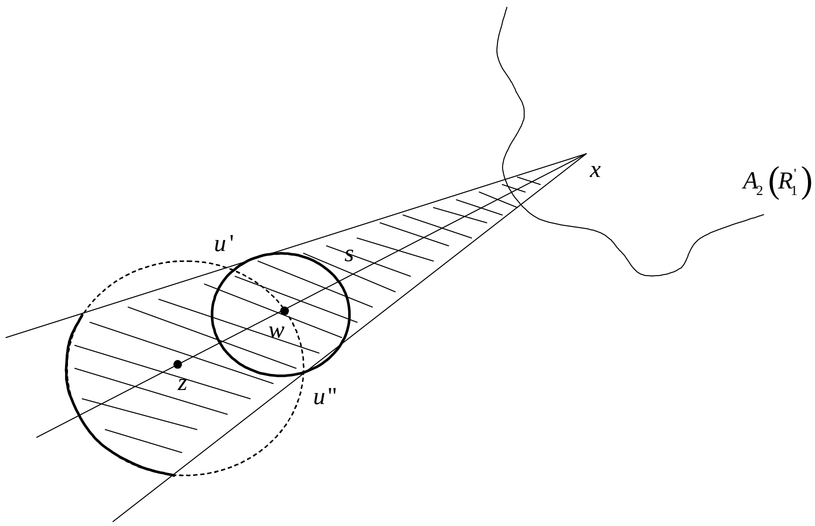
Proceeding as in the proof of lemma 6, let H be defined over X as the jauge of S with respect to z and R_2 be the preference generated by H. We deduce that R_2 is strictly convex. Further, z is the unique best element and the boundary of S is the indifference curve going through x.

Since, by assumption, $z \in A_2(R_1)$, there exists $R_2' \in D$ such that $z = C(R_1, R_2')$. Since C is strategyproof, we deduce therefore that :

$$C(R_1, R_2) = z \tag{2}$$

Now, let B be the set[14] of best alternatives of R_2 over $A_2(R_1')$. By construction of $B(z, \varepsilon)$, $B \cap B(z, \varepsilon) = \varnothing$. Also, by construction of S and since $x \in A_2(R_1') : B \subset S$. Further, since C is strategyproof, we deduce therefore that there exist $b \in B$ such that :

[14]Since the set $A_2(R_1')$ is compact, the set B is nonempty. Note however, that,since $A_2(R_1')$ is not necessarily convex, the set B may contain more than one alternative.



$$C(R_1', R_2) = b \qquad (3)$$

From the construction of S and the position of b in S, we deduce from the strict convexity of R_1 that bP_1z. Comparing (2) and (3), this implies then that C is manipulable by individual 1 at the profile $\pi = (R_1, R_2)$ in contradiction to our assumption that C is strategyproof □

We are now in position to prove the main result of this section. To proceed, we will use a result proved by Bordes, Laffond and le Breton (1990) for the domain of Euclidean preferences over X. Let $\widehat{D}$ be the subset of Euclidean preferences over $\Re^2$ such that their ideal point belongs to X. Without any risk of confusion, we identify $\widehat{D}$ with X.

Theorem 2 *Let $m = 3$ and $n = 2$. Let C be a coalitional strategyproof social choice mechanism over $\widehat{D}$ such that $C(\widehat{D}^n) = X$. Then, C is dictatorial.*

Theorem 3 *Let $m = 3$ and $n = 2$. Let C be a strategyproof social choice mechanism over D such that $C(D^n) = X$. Then, C is dictatorial.*

Proof : From lemma 7, D is rich and therefore, from theorem 1, C is coalitional strategyproof. Let $\widehat{C}$ be the restriction of C to $\widehat{D}^n$. Then, $\widehat{C}$ is also coalitional strategyproof. We deduce from theorem 2 that $\widehat{C}$ is dictatorial. Without loss of generality, let individual 1 be the dictator for $\widehat{C}$. We now prove that 1 is also a dictator for C. This is equivalent to show that for all $\pi \in D^2$, the option set $A_2(R_1)$ is equal to the unique best element of R_1.

Let :

$$\widehat{A}_2(R_1) \equiv \left\{x \in X : x = C(R_1, R_2) \text{ for some } R_2 \in \widehat{D}\right\}$$

From lemma 3, we deduce that if $z = C(R_1, R_2)$, then $z = C(R_1, R_2')$ where $R_2' \in \widehat{D}_z$. From lemma 6, $\widehat{A}_2(R_1) = A_2(x_1)$ where x_1 denotes both the best alternative for R_1 and the Euclidean preference with ideal point R_1. By combining both claims, we obtain that :

$$A_2(R_1) = A_2(x_1) = \widehat{A}_2(x_1)$$

But, since 1 is a dictator for C, $\widehat{A}_2(x_1) = \{x_1\}$ and the conclusion follows $\square$

Theorem 3 can be extended to domains larger than D. Since here $X^* = X$, any social choice mechanism is trivially regular. Therefore from lemma 4, we deduce that C is dictatorial iff C^* is dictatorial. Therefore, it is enough to prove that C is dictatorial. A careful examination of the proof of lemma 6, shows that we dont exploit the full force of the strict convexity of R_1. What is truly needed is the strict monotonicity along any half-line with the best alternative x_1 as origin. This implies that any domain $\widetilde{D}$ such that $\widetilde{D}^*$is contained in this subset of convex preferences leads to the same conclusion.

The structure of the proof of theorem 3 is quite instructive. Once we know that the social choice mechanism C is coalitional strategyproof, we can exploit the simple fact that any restriction of C to a subdomain is also coalitional strategyproof. On these subdomains, the geometry of the Pareto set is sometimes easy to derive. For instance, in the case where the subdomain consists of the subset of Euclidean preferences, the Pareto set is the convex hull of the ideal points of the two individuals. The proof of theorem 2 based on the technique of option sets uses this property. Once we know what happens on a subdomain, it remains of course to extend the result to the all domain. The key step[15], which corresponds here to lemma 7, is a "top only" property asserting that strategyproofness implies that only the top alternatives of the two individuals matter in calculating the social outcome.

Theorem 3 is a slightly weaker version of an impossibility result established for this environment by Zhou (1991). His setting is identical to the one considered here but instead of us, Zhou does not assume that the range of the mechanism C coincides with X and demonstrates his result under the weaker assumption that the range of C is two dimensional.

[15] This "tops only" property is a familiar cornerstone in this area.

It is straightforward to see that the conclusion of theorem 3 holds true for the social environment where $X = \Re^2_+$ and D is the set of preferences with compact and strictly convex upper contour sets. However, this conclusion does not hold true for the domain $\widehat{D}$. In fact, as demonstrated by Le Breton and Sen (1999) when a domain D consists of separable preferences over a product set, strategyproofness implies decomposability. The class of decomposable strategyproof social choice mechanisms contains non dictatorial mechnisms. However, these mechanisms are not Pareto efficient. If we insist on Pareto efficiency, then the class of strategyproof social choice mechanisms collapses on dictatorial mechanisms.

4.2 Single Peakedness

Theorem 3 was derived under the assumption that there are at least three different public goods. When there are only two public goods, the set X is an interval. A preference in D over that interval is single peaked. We know that for this social environment there are many non dictatorial strategyproof social choice mechanisms, on top of which the so called median mechanism. The general family of strategyproof social choice mechanisms has been characterized by Moulin (1980). The domain D is rich and a shorter proof of the characterization result exploiting theorem 1 could be provided. But more importantly, this setting is interesting as it ilustrates the fact that there are domains where individual and coalitional strategyproofness are equivalent without being equivalent to dictatoriality.

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