# Potential Irreversible Catastrophic Shifts of the Assimilative Capacity of the Environment*

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February 8th, 2012

*We thank Valérie Nowaczyk for technical assistance.

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**Abstract**

Pollution accumulation may result in more or less severe losses of natural self-cleaning capacities. We study a polluting resource management problem submitted to a potential shift from a high to a low pollution self-regeneration regime be crossed some critical pollution stock threshold. We first describe the optimal resource exploitation policy absent the threshold. When at the threshold, the society has two options: either stabilizing the pollution level to avoid the loss of natural self-cleaning capacity or deliberately cross the threshold and switch to the low regeneration regime. We show under fairly general assumptions that there exists a unique critical pollution stock level such that thresholds located below this level will induce a switch from the high to the low regeneration regime while thresholds located above it will imply maintaining the high regime forever. We characterize the optimal policies in these two scenarios and show that triggering the low regeneration regime requires an upward jump of the resource consumption rate at the optimal switching time.

**JEL classification :** Q15, Q17

**Keywords :** natural regeneration capacity; regime shift; polluting resource; climate change
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# 1 Introduction

That the natural self-cleaning capacities of the environment could deteriorate in the long run is now a well identified problem having been received impressive research efforts both theoretically and empirically oriented during the last forty years, say from the pioneer theoretical works of Forster (1973, 1975), Keeler, Spencer and Zeckhauser (1973) and Plourde (1972) at the beginning of the ninety seventies.

Typically the deterioration process may be either progressive or abrupt that is leading to some "catastrophe", resulting *in fine* for the both progressive and catastrophic deterioration cases either in a partial or a complete loss of the assimilative capacity of the environment. The deterioration process may be modeled either as an exogenous process, that is purely time dependent, or linked to the flow of pollution generated by the economic activity and/or the stock of accumulated pollution, thus made be endogenous. As far as the process is endogenous, new options are opening : The society could use more or less polluting resources or resort to more or less complete abatement programs reducing the flow of pollution rejected in the environment. Last the structural relationships of the models may be either certain or include random components.

The effects of a damaged environment upon the wellbeing of the society may be too either progressive or catastrophic as pioneered by Cropper (1976), catastrophic wellbeing effects being not necessarily linked to catastrophic changes in the assimilative capacity. In most models of wellbeing catastrophes, including the seminal model of Cropper, the natural self-regeneration process is not affected by the catastrophe, and neither is triggered the wellbeing catastrophe by a sudden change in the environment quality. Symmetrically in many models of catastrophic changes of the self-regeneration regime, the regime switch does not result in catastrophic falls of welfare, at least in the short run. The reason is that a drastic change in the natural process of pollution accumulation does not translate into a discontinuity of the pollution stock path but rather in a discontinuity in its time derivative.

Most of the studies of the catastrophic loss of the self-regeneration power of the environment adopt a random model framework [1]. The time at which the catastrophe occurs is a random variable in time depending models. The instantaneous probability of disruption is a random function of the pollution stock in stock depending models, contrary to the main corpus of the literature devoted to the progressive deterioration in which the structural relationships are generally assumed to be certain [2]. A notable exception is the recent study of Prieur, Tidball and Withagen (2011) in which the assimilative power of the environment is totally annihilated once the pollution stock overshoots some definite and perfectly identified critical level. In the present paper we retain a similar framework, that is a partial equilibrium model of a closed economy without uncertainty. However we assume that the resource generating the pollution flow is inexhaustible and that the loss of self-regenerative capacities is only partial as in Polasky, de Zeew and Wagener (2011). Thus there can occur an irreversible shift of the self-regeneration regime once the pollution is gone beyond its critical level, but not necessarily an irreversible accumulation of the pollution stock since after the shift some assimilative capacity of the environment is preserved, although along optimal paths in many models, the pollution stock is permanently increasing for sufficiently low initial pollution stocks[3].

In this kind of model there exist two types of policies. The first one consists in avoiding to trigger the catastrophe by restricting the use of the polluting resource in order to keep the pollution stock within its safety limit. The second one consists in deliberately entering the new world of slowly self-regenerating environment and in choosing the date at which this new world must be visited. We show that under fairly standard assumptions determining which policy is the optimal one may be given a very simple formulation.

The paper is organized as follows. Section 2 introduces the model. In

[1]See Polasky, de Zeeuw and Wagener (2011) for a generous account of this literature. See also Athanassoglu and Xepapadeas (2012) for the case where precise probability laws are difficult to ascertain.

[2]See Cesar and de Zeeuw (1995), Farzin and Tahvonen (1996), Tahvonen (1995, 1997), Tahvonen and Salo (1996), Tahvonen and Withagen (1996), Toman and Withagen (1996) and Brock and Starret (2003) for progressive loss models with certain structural relationships.

[3]Most models are assuming stationary or increasing needs in energy services, thus generating a never decreasing pollution stock , clean renewable options and/or abatement options being not open in this type of models.

Section 3 we characterize the policies permitting to preserve the highly efficient self-regenerative power of the environment. In Section 4 we determine the conditions under which either policy is the optimal one. We examine the main characteristics of the policies triggering the regime switch in Section 5. We conclude in Section 6.

# 2 The model and preliminary results

## 2.1 Assumptions and notations

We consider an economy in which the consumption of ready to use services of some resource generates a net surplus for the final users.

The resource stock is infinite, we denote by $x(t)$ its instantaneous consumption rate. The average cost having to be engaged for supplying the services to the final users is constant and thus may be assumed to be nil without loss of generality.

Using the resource for providing ready to use services implies pollutant rejections in the environment. Let $\zeta$ be the unitary pollution content of the resource so that the current new pollution flow amounts to $\zeta x(t)$ at time $t$. This new pollution flow supplies the pollution stock present in the environment denoted by $Z(t)$. The environment has some self-regeneration capacity which can be destroyed and forever when the pollution stock overshoots some critical level $Z^i$. To keep matters as simple as possible we assume that before and after this bad event if any, the assimilative capacity of the environment is proportional to the current pollution stock falling from $\alpha_h$ down to $\alpha_\ell$ , $0 < \alpha_\ell < \alpha_h$ , once $Z(t) > Z^i$ so that the dynamics of $Z(t)$ is driven by :

$$\dot{Z}(t) = \begin{cases} \zeta x(t) - \alpha_h Z(t) \text{ , if } \quad Z(\tau) \leq Z^i \quad , \quad \tau \in [0, t] \\ \\ \zeta x(t) - \alpha_\ell Z(t) \qquad\qquad\qquad\qquad , \quad \text{other cases} \end{cases} \tag{2.1}$$

We evidently assume that the initial pollution stock denoted by $Z^0$ is lower than the critical stock $Z^i$ : $Z^0 < Z^i$.

The pollution stock $Z$ is generating instantaneous damages $h(Z(t))$ at time $t$ where $h$ is a $\mathcal{C}^2$ function strictly increasing and strictly convex, with $h(0) = 0$ , $h'(0) = 0$ , and $\lim_{Z\uparrow\infty} h(Z) = +\infty$ and $\lim_{Z\uparrow\infty} h'(Z) = +\infty$.

Let $u(x(t))$ be the instantaneous gross surplus resulting from the consumption of the services provided by $x(t)$. We assume that $u$ is a $\mathcal{C}^2$ function, strictly increasing and strictly concave with $\lim_{x\downarrow 0} u'(x) = +\infty$ and $\lim_{x\uparrow\infty} u'(x) = 0$ [4]. $u'$ is sometimes denoted by $p$ since the marginal surplus is the demand price of the services supplied to the users : $p(x) = u'(x)$ is the inverse of the direct demand function itself denoted by $x^d(p)$.

The social rate of discount $\rho$ is assumed to be constant through time and strictly positive.

The social welfare $W$ is the standard criterion, the sum of the discounted instantaneous well-beings:

$$W = \int_0^\infty \{u(x(t)) - h(Z(t))\} \, e^{-\rho t} dt.$$

## 2.2 The social planner problem

The social planner problem is to determine the resource consumption path $\{x(t),\ t \geq 0\}$ maximizing $W$ under the constraint (2.1), the initial condition $Z(0) = Z^0$, and the constraint $x(t) \geq 0$. Given the assumption $u'(0) = +\infty$, this last constraint will never be active and may be discarded.

We denote by $-\lambda$ the costate variable of $Z$ and by $\mathcal{H}$ the current value Hamiltonian of the problem :

$$\mathcal{H} = u(x(t)) - h(Z(t)) - \lambda(t)[\zeta x(t) - \alpha(t)Z(t)]$$

where $\alpha(t) \in \{\alpha_h, \alpha_\ell\}$ is given by the history of $Z$ according to (2.1).

The first order conditions are :

$$u'(x(t)) = \zeta\lambda(t) \tag{2.2}$$

[4]Note that we do not exclude negative gross surplus for low consumption rates.

and at any time $t$ at which $\lambda$ is differentiable :

$$\dot{\lambda} = (p + \alpha(t))\lambda(t) - h'(Z(t)) \tag{2.3}$$

Other conditions have to be taken into account when $Z(t) = Z^i$, we discuss later in Sections 3 and 5.

Last the transversality condition at infinity is :

$$\lim_{t \uparrow \infty} e^{-\rho t}\lambda(t)Z(t) = 0. \tag{2.4}$$

## 2.3 Benchmarks

It is useful for the clarity of the arguments developed in the following sections, to look at what could be the optimal consumption paths absent the regime switch problem in a system where the proportional self-regeneration rate is some fixed rate $\alpha \ : \alpha > 0$.

In such a system the dynamics of the shadow marginal cost of the resource services, or its optimal price, denoted by $p(t) \ : p(t) = \zeta\lambda(t)$, is obtained by differentiating this last equality and substituting (2.3) for $\dot{\lambda}(t)$ to obtain :

$$\dot{p}(t) = (\rho + \alpha)p(t) - \zeta h'(Z(t)). \tag{2.5}$$

The isocline $\dot{p} = 0$ in the $(p, Z)$ plane is defined as :

$$(p, Z) \ : \dot{p} = 0 \quad \Rightarrow \quad p = \frac{\zeta}{\rho + \alpha} \cdot h'(Z), \tag{2.6}$$

so that :

$$\left.\frac{dp}{dZ}\right|_{\dot{p}=0} = \frac{\zeta}{\rho + \alpha} \cdot h''(Z) > 0. \tag{2.7}$$

We denote by $p_k^p(Z), k \in \{h, \ell\}$, the function defining this locus $\dot{p} = 0$ under the regime $k$.

As for $Z$ its dynamics is given by :

$$\dot{Z}(t) = \zeta x^d(p(t)) - \alpha Z(t), \tag{2.8}$$

and the isocline $\dot{Z} = 0$ is defined by :

$$(p, Z) : \dot{Z} = 0 \quad \Rightarrow \quad x^d(p) = \frac{\alpha}{\zeta} Z \quad \Rightarrow \quad p = u'(\frac{\alpha}{\zeta} Z), \tag{2.9}$$

hence :

$$\left.\frac{dp}{dZ}\right|_{\dot{Z}=0} = \frac{\alpha}{\zeta} u'(\frac{\alpha}{\zeta}) < 0. \tag{2.10}$$

We denote by $p_k^Z(Z),\ k \in \{h, \ell\}$, the function defining the locus $\dot{Z} = 0$ under the regime $k$.

The asymptotic optimal stationary pollution stock $Z^*$ and price $p^*$ under the regime $\alpha$ are given as :

$$u'(\frac{\alpha}{\zeta}\ Z^*) = \frac{\zeta}{\rho + \alpha} \cdot h'(Z^*) \quad \text{and} \quad p^* = u'(\frac{\alpha}{\zeta}\ Z^*). \tag{2.11}$$

The pair $(p^*, Z^*)$ is a saddle point since $u^{''} < 0$ and $h^{''} > 0$.

*Sensitivity analysis of the long run steady state*

Differentiating (2.6) and (2.9) *w.r.t.* the self-regeneration rate $\alpha$ results in :

$$\left.\frac{\partial p}{\partial \alpha}\right|_{\dot{p}=0} \quad = \quad -\frac{\zeta}{(\rho + \alpha)^2} \cdot h'(Z) < 0, \tag{2.12}$$

and:

$$\left.\frac{\partial p}{\partial \alpha}\right|_{\dot{Z}=0} \quad = \quad \frac{1}{\zeta} u^{''}(\frac{\alpha}{(\zeta)Z} < 0. \tag{2.13}$$

Thus the higher is the assimilative capacity of the environment the lower are located the both isoclines $\dot{p} = 0$ and $\dot{Z} = 0$ in the $(p, Z)$ space, as illustrated in Figure 1 for $\alpha = \alpha_h$ and $\alpha = \alpha_\ell$ where for any $Z$ both $p_h^p(Z) < p_\ell^p(Z)$ and $p_h^Z(Z) < p_\ell^Z(Z)$. Figure 1 suggests that $p^*$ is decreasing with $\alpha$ while $Z^*$ may either increase or decrease.

Figure 1 here

Differentiating (2.11) *w.r.t.* $\alpha$, we obtain :

- first :

$$\frac{dp^*}{d\alpha} = \frac{\frac{1}{\rho+\alpha}u''[Z^* \cdot h'' + \frac{\alpha}{\zeta}u']}{-\frac{\alpha}{\zeta}u'' + \frac{\zeta}{\rho+\alpha}h''} < 0 \tag{2.14}$$

so that as suggested by Figure 1, $p^*$ is decreasing with $\alpha$ ;

- second :

$$\frac{dZ^*}{d\alpha} = \frac{\frac{1}{\zeta}Z^* \cdot u'' + \frac{\zeta}{(\rho+\alpha)^2}h'}{-\frac{\alpha}{\zeta}u'' + \frac{\zeta}{\rho+\alpha}h''} \tag{2.15}$$

the sign of which may be either positive or negative.

However the sign of $\frac{dZ^*}{d\alpha}$ may be given a very simple expression depending upon the elasticity of the demand for the resource services, the social rate of discount and the self-regenerating rate.

Substituting $u'$ for $\frac{\zeta}{\rho+\alpha}\ h'$ (c.f. (2.11)) in the numerator of (2.15) we obtain $\frac{1}{\zeta}Z^*u'' + \frac{1}{\rho+\alpha}u'$. Hence the numerator is positive iff $\eta^* > \frac{\rho+\alpha}{\alpha}$ where :

$$\eta^* = -\frac{u'(\frac{\alpha}{\zeta}Z^*)}{u''(\frac{\alpha}{\zeta}Z^*)} \cdot \frac{1}{\frac{\alpha}{\zeta}Z^*} ,$$

is the absolute value, $\eta$, of the price elasticity of the resource services demand $x^d(p)$ at $p = p^* = u'(\frac{\alpha}{\zeta}Z^*)$.
Given that the denominator of (2.15) is positive, we conclude that :

$$\frac{dZ^*}{d\alpha} \lesseqqgtr 0 \quad \text{according to} \quad \eta^* \lesseqqgtr 1 + \frac{\rho}{\alpha}.$$

Similar sensitivity analysis *w.r.t.* the other parameters of the model, $\rho$ and $\zeta$, lead to the same qualitative conclusions. While the sign of $\frac{dp^*}{d\phi}$ , $\phi \in \rho, \zeta$ is unambiguous, the sign of $\frac{dZ^*}{d\phi}$ is strongly dependent on the price elasticity of the demand. These signs are compiled in the below Table 1 and the analytical expressions of the derivatives are given in Appendix A.1 .

|  | $\eta^* < 1$ | $\eta^* = 1$ | $1 < \eta^* < 1 + \frac{\rho}{\alpha}$ | $\eta^* = 1 + \frac{\rho}{\alpha}$ | $1 + \frac{\rho}{\alpha} < \eta^*$ |
|---|---|---|---|---|---|
| $dZ^*/d\alpha$ | $-$ | $-$ | $-$ | 0 | $+$ |
| $dZ^*/d\rho$ | $+$ | $+$ | $+$ | $+$ | $+$ |
| $dZ^*/d\zeta$ | $+$ | 0 | $-$ | $-$ | $-$ |
| $dp^*/d\alpha$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $dp^*/d\rho$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $dp^*/d\zeta$ | $+$ | $+$ | $+$ | $+$ | $+$ |

Table 1: Signs of the derivatives of $Z^*$ and $p^*$

Note that the demand elasticities at which the signs of $\frac{dZ^*}{d\alpha}$ and $\frac{dZ^*}{d\zeta}$ are reversed are not the same. Also remark that assuming a surplus function $u(x) = log\, x$, so easy to manipulate in some exercises, entails very specific implications concerning the sensitivity of $Z^*$ : The long run optimal pollution stock does not depend upon the pollution content of the resource and decreases with the proportional self-regeneration rate. The same kinds of remarks apply for quadratic surplus functions along which the demand elasticity is varying.

In Figure 1 the optimal paths are the trajectories $p_h^F(Z)$ and $p_\ell^F(Z)$ for respectively the high and low self-regeneration regimes. Figure 1 is drawn assuming that $\eta > 1 + \frac{\rho}{\alpha}$ since $Z_h^* > Z^* > Z_\ell^*$.

From now we assume that $Z^i$ is smaller than $Z_h^*$ whatever be $Z_\ell^*$, since for $Z^i > Z_h^*$ and $Z^0 < Z^i$, neglecting the switch possibility, the solution of the social planner problem is such that $Z^i$ is never overshot.

## 2.4 Determining the optimal policy

Assume that $Z^i$ has been attained at some date, then at this date the society is facing two options :

- the first one is to choose a policy maintaining the high natural self-regeneration rate $\alpha_h$, that is never overshoot $Z^i$;

- the second one is to engage the low self-regenerating process, but forever, by overshooting $Z^i$ [5].

The value of either policy when at $Z^i$ does not depend on what has been done before. Thus the whole structure of the optimal path is determined by the best choice when at $Z^i$. Once this best choice problem is solved it remains to determine the best way for attaining $Z^i$, best way which in turn depends upon the decision having to be made when at $Z^i$.

## 2.5 Notations

Although we are working alternatively with the high and the slow self-regeneration regime, the context is always sufficiently evident to be aware of which regime is under consideration thanks to the notations introduced in Sub-section 2.3 supra.

The only additional notations we cannot escape from are relative to the behavior of the different paths when attaining $Z^i$.

We denote by $t_i$ the time at which the critical pollution stock $Z^i$ is attained, and by $x^i$ the maximum consumption rate of the resource permitting to keep alive the high self-regeneration regime when at $Z^i$ : $x^i = \frac{\alpha_h}{\zeta} Z^i$.

When there is no discontinuity in a flow variable or a costate variable at $t_i$ we use the superscript $i$ for the value of the variable at this date $t_i$.

[5] Contrary to the case of a finite stock of resource it is never optimal to stay for some time at $Z^i$ before eventually switching to the low self-regenerating regime. In the present setting, while staying at $Z^i$, the state of the system does not change. In the case of a non-renewable resource there exists another state variable $X$, the current resource stock. Thus when staying at $Z^i$, one state variable $Z$ is left unchanged while $X$, the other state variable, decreases. In such a context, it may happen optimal to stay at $Z^i$ for a while because now the value of what has to be done depends upon both $Z$ and $X$, thus upon that has been done before attaining $Z^i$. See Prieur, Tidball and Withagen (2011) for further developments and simulations along these lines, in the more dramatic context where $\alpha_\ell = 0$.

For example $\lambda^i$ is the value of $\lambda(t_i)$ in this case. This is typically what occurs along the paths preserving the high self-regeneration capacities of the environment.

However along the paths triggering the regime shift generally the flow variables are discontinuous. Then we use $t_i^-$ and $t_i^+$ as the time argument for respectively the left and the right limits of the variable at $t_i$. For example $\lambda(t_i^-) = lim_{t\uparrow t_i}\lambda(t)$ and $\lambda(t_i^+) = lim_{t\downarrow t_i}\lambda(t)$. To simplify even more the notations we denote respectively by $\lambda_-^i$ and $\lambda_+^i$ the limits $\lambda(t_i^-)$ and $\lambda(t_i^+)$.

# 3 Best maintaining policies

In this section we determine the best policy preserving the initial high self-regenerating rate $\alpha_h$. We call these policies Best Maintaining Policies (B. M. P.).

The problem to be solved has the same objective function $W$, but since the highly efficient self-regenerating regime is permanently at work then the dynamics of $Z$ is given by:

$$\dot{Z}(t) \;=\; \zeta x(t) - \alpha_h Z(t)\ , \quad t \geq 0\ ,\ Z^i > Z^0 \geq 0\ , \tag{3.1}$$

to which must be added the following constraint in order that (3.1) holds for all $t \geq 0$:

$$Z^i - Z(t) \;\geq 0\ , \quad t \geq 0\ . \tag{3.2}$$

The current value Hamiltonian must be modified accordingly:

$$\mathcal{H} \;=\; u(x(t)) - h(Z(t)) - \lambda(t)\left[\zeta x(t) - \alpha_h Z(t)\right]\ .$$

Denote by $\nu$ the Lagrange multiplier associated to the constraint (3.2), then the current value Lagrangian denoted by $\mathcal{L}$ reads:

$$\mathcal{L} \;=\; \mathcal{H} + \nu(t)\left[Z^i - Z(t)\right]\ .$$

The first order condition relative to $x$, that is (2.2), does not change, neither the transversality condition (2.4).

The dynamics of $\lambda$ must now satisfy:

$$\dot{\lambda}(t) = (\rho + \alpha_h)\lambda(t) - h'(Z(t)) - \nu(t) \qquad (3.3)$$

$$\nu(t) \geq 0 \quad , \quad Z^i - Z(t) \geq 0 \ \text{ and } \nu(t)\left[Z^i - Z(t)\right] = 0 \ . \qquad (3.4)$$

Note that initially the constraint (3.2) does not bind since $Z^0 < Z^i$, hence $\nu(t) = 0$ and (3.3) reduces to (2.3). This is implying that the dynamics of $p$ and $Z$ are given by (2.5) and (2.6) with $\alpha = \alpha_h$, up to the time $t_i$ at which the critical stock $Z^i$ is approached. Thus the optimal path in the $(p, Z)$ plane is this trajectory $(M)$ in Figure **??** crossing the curve $\dot{Z} = 0$ at $Z = Z^i$ and tangent to the vertical through $Z^i$ at this point. From $t_i$ the constraint $Z^i - Z \geq 0$ begins to be active and forever: the system stays indefinitely in this state.

Figure 2 here

*Transversality condition at $t_i$*

Let $V^m(Z^i)$ be the continuation value of the B. M. P. when at $Z^i$:

$$V^m(Z^i) = \int_{t_i}^{\infty} \left[u(x^i) - h(Z^i)\right] e^{-\rho(t-t_i)} dt = \frac{1}{\rho}\left[u(x^i) - h(Z^i)\right] \ . \qquad (3.5)$$

For $Z^0 < Z^i$ and $Z^i \leq Z^*_h$, the B. M. P. path may be seen equivalently as the solution of:

$$\max_{\{x,t_i\}} \int_0^{t_i} \left[u(x(t)) - h(Z(t))\right] e^{-\rho t} dt + e^{-\rho t_i} V^m(Z^i)$$
$$s.t. \quad (3.1) \ \text{ and } \ (3.2)$$

Let $H(t)$ be the present value Hamiltonian: $H(t) = e^{-\rho t}\mathcal{H}(t)$. Then a well-known transversality condition at $t_i$ is:

$$\lim_{t \uparrow t_i} H(t) + \frac{\partial}{\partial t_i}\left[e^{-\rho t_i} V^m(Z^i)\right] = 0 \ .$$

Substituting for $H(t)$ and $V^m(Z^i)$ results in:

$$u(x(t_i)) - h(Z(t_i)) - \lambda(t_i)\left[\zeta x(t_i) - \alpha_h Z(t_i)\right] = u(x^i) - h(Z^i) \ ,$$

an equation clearly satisfied for $x(t_i) = x^i$ and $Z(t_i) = Z^i$ since then the last term into brackets of the L. H. S. of the equation is nil.

*Locating the maximum of* $V^m(Z^i)$

As it will become clear in Sub-section 4.2, it is important to determine the location of this critical stock, we denote by $\hat{Z}^i$, at which $V^m(Z^i)$ attains its maximum, within the range $(0, Z_h^*)$.

Let us show that the continuation value of the B. M. P. is first increasing and next decreasing within the interval $(0, Z_h^*)$. From $x^i = \alpha_h Z^i/\zeta$, we get:

$$\begin{aligned} \frac{dV^m}{dZ^i} &= \frac{1}{\rho}\left[\frac{\alpha_h}{\zeta}u'(x^i) - h'(Z^i)\right] \\ \frac{d^2V^m}{d(Z^i)^2} &= \frac{1}{\rho}\left[\left(\frac{\alpha_h}{\zeta}\right)^2 u''(x^i) - h''(Z^i)\right] < 0 \ . \end{aligned}$$

Let $\hat{Z}^i$ be that solution of:

$$u'\left(\frac{\alpha_h}{\zeta}Z^i\right) = \frac{\zeta}{\alpha_h}h'(Z^i) \ , \tag{3.6}$$

then:

$$\frac{dV^m}{dZ^i} \gtreqless 0 \quad \text{according to} \quad Z^i \lesseqgtr \hat{Z}^i \ .$$

The pollution stock $\hat{Z}^i$ at which $V^m$ attains its maximum is smaller than the long run pollution stock $Z_h^*$ towards which would converge the optimal stock of an efficient system free of the switching threat. The reason is illustrated in the below Figure 3. The L. H. S. of the equation (3.6) is nothing but than the function defining the isocline $\dot{Z} = 0$ for $\alpha = \alpha_h$, and the graph of the R. H. S. of (3.6), $\zeta h'(Z)/\alpha_h$, is located above the graph of $\zeta h'(Z)/(\rho + \alpha)$ defining the isocline $\dot{p} = 0$ for $\alpha = \alpha_h$.

Figure 3 here

For small critical pollution stocks $Z^i$ the blockaded consumption rate $x^i$ is also small meaning that $u'(x^i)$ is high while $h'(Z^i)$ is low, resulting in a positive marginal balance sheet $u'(x^i) - h'(Z^i)$. But $u'(x^i)$ is a decreasing function of $Z^i$ while $h'(Z^i)$ is an increasing function. Thus the marginal balance sheet is progressively deteriorating staying in the black as far as $Z^i$ stays under $\hat{Z}^i$ and going in the red for larger levels of the critical stock $Z^i$.

*Determining the stationary values of $\lambda$ and $\nu$ once blockaded at $Z^i$*

According to (3.3), the stationary value of $\lambda$ once $Z^i$ is attained, $\lambda^i$, must satisfy:

$$\dot{\lambda} \;=\; 0 \;=\; (\rho + \alpha_h)\lambda^i - h'(Z^i) - \nu^i \;.$$

According to the F.O.C. (2.2) relative to $x$, $\lambda^i = \frac{1}{\zeta}u'\left(\frac{\alpha_h}{\zeta}Z^i\right)$, hence[6]:

$$\nu^i_+ \;=\; \frac{\rho + \alpha_h}{\zeta}u'\left(\frac{\alpha_h}{\zeta}Z^i\right) - h'(Z^i) \;.$$

For $Z^i < Z^*_h$, $p^F_h(Z^i) > p^p_h(Z^i)$ (see Figure 2) meaning that $u'\left(\frac{\alpha_h}{\zeta}Z^i\right) > \frac{\zeta}{\rho+\alpha_h}h'(Z^i)$, hence $\nu^i_+ > 0$. For $Z^i = Z^*_h$, then $\nu^i = 0$.

Clearly:

$$\frac{d\lambda^i}{dZ^i} \;=\; \frac{\alpha_h}{\zeta^2}u''\left(\frac{\alpha_h}{\zeta}Z^i\right) < 0 \quad , \quad \frac{d\nu^i}{dZ^i} \;=\; \frac{\alpha_h(\rho + \alpha_h)}{\zeta^2}u''\left(\frac{\alpha_h}{\zeta}Z^i\right) - h''(Z^i) < 0 \;.$$

The higher is the critical pollution stock $Z^i$, the lower is the long run shadow marginal cost of the pollution stock and the lower is the Lagrange multiplier associated to the no-switch constraint, as expected.

*The B. M. P. path as the "most slowly" approach to $Z^i$*

To get a complete picture of the B. M. P. path it remains to determine $\lambda_0$, the initial level of $\lambda$. Given $Z^0 < Z^i$, for any $\lambda_0 \geq 0$, the system:

$$\dot{\lambda}(t) \;=\; (\rho + \alpha_h)\lambda(t) - h'(Z(t)) \tag{3.7}$$

$$\dot{Z}(t) \;=\; \zeta x^d(\lambda(t)) - \alpha_h Z(t) \tag{3.8}$$

[6]Note that $\nu(t)$ is discontinuous at $t_i$, nil for $t < t_i$, $\nu^i_- = 0$, and jumping up to $\nu^i_+$ at $t_i$.

has a unique solution $\left\{\left(\tilde{\lambda}(t,\lambda_0),\tilde{Z}(t,\lambda_0)\right)\ ,\ t\geq 0\right\}$ such that $\tilde{\lambda}(0,\lambda_0)=\lambda_0$ and $\tilde{Z}(0,\lambda_0)=Z^0$.

Let $\Lambda_0$ be the set of $\lambda_0$ values for which $\tilde{Z}(t,\lambda_0)$ attains the critical level $Z^i$. For any $\lambda_0\ \in\ \Lambda_0$ let us define $\tilde{t}_i(\lambda_0)$ as:

$$\tilde{t}_i(\lambda_0)\ =\ \inf\ t\ :\ \tilde{Z}(t,\lambda_0)\ \geq\ Z^i\ .$$

Then the value of $\lambda_0$ characterizing the optimal path is this value for which:

$$x^d\left(\zeta\tilde{\lambda}(\tilde{t}_i(\lambda_0),\lambda_0)\right)\ =\ x^i$$

This is the value of $\lambda_0$ corresponding to the trajectory $(M)$ in Figure 2.

Since the trajectories $\{\lambda, Z\}$ solution of the autonomous differential system (3.7)-(3.8) do not cross, then to a lower $Z^i$ corresponds a higher $\lambda_0$ and the same applies to a higher $Z^0$ (see Figure 2):

$$\frac{\partial\lambda_0}{\partial Z^i}\ <\ 0\quad\text{and}\quad\frac{\partial\lambda_0}{\partial Z^0}\ >\ 0\ . \tag{3.9}$$

A more stringent $Z^i$ constraint resolves into a higher penalty imposed upon the resource consumption before the $Z^i$ border is reached. A higher initial pollution stock has the same consequences on the resource consumption path.

According to a well-known property of the autonomous systems[7] the delay needed to go from $Z^0$ to $Z^i$ is the larger the more distant is the trajectory satisfying the F. O. C's from the saddle branch trajectory $p_h^F(Z)$ amongst those trajectories located above the saddle branch.

A first implication of this property is that, when the policy is the B. M. policy, the required delay for attaining $Z^i$ is the longest possible one. Maintaining the high self-regenerative capacity of the environment translates into delaying as far as possible the time at which the preserving constraint will begin to be binding. It is typically a case of 'most slowly' approach.

A second implication worth having to be pointed out is that, increasing the penalty: $dZ^i<0\Longrightarrow d\lambda_0>0$, thus inducing a reduced consumption rate

[7] See Caputo (2005), p 371-373, for a detailed argument.

before attaining $Z^i$ and a correlative slow down of the pollution accumulation rate, does not delay the attainment of the critical pollution stock: $dt_i/dZ^i > 0$.

To conclude:

**Proposition 1** *Assuming a sufficiently low triggering pollution stock $Z^i$: $Z^i < Z_h^*$, hence a no switch constraint $Z^i - Z \geq 0$ binding in the long run, then for any initial pollution stock $Z^0$ lower than the critical stock, the paths of resource consumption and pollution stock generated by the Best Maintaining Policies are two phases paths:*

- *a first phase of decreasing resource consumption and increasing pollution stock, ending at the time $t_i$ when the pollution stock attains the critical level $Z^i$ and simultaneously the resource consumption rate is reduced to this rate $x^i$ for which the pollution emission flow $\zeta x^i$ is balanced by the self-regeneration flow $\alpha_h Z^i$. The lower is the critical level $Z^i$ the shorter is the delay $t_i$ required for attaining $Z^i$.*
- *a second phase of infinite duration where the resource consumption rate is blockaded at $x^i$, the pollution stock being maintained at its critical level $Z^i$.*

# 4 Evaluation of the regime switch opportunity

Assume that the pollution stock is at its critical level $Z^i$ and that the society wants to enter the slow self-regenerating regime, that is wants to adopt what we call the Best Switching Policy (B. S. P.). Two problems arise, first how to engage the switch and second, what to do next.

Engaging the switch is not a true problem. That can be made by boosting the current use of the resource at any rate arbitrarily larger than $x^i$ during any arbitrarily short time interval, in brief by an arbitrarily small impulse without any noticeable implication upon the further development of the pollution stock through time.

Once the switch has been triggered, the society is facing a slow self-regenerating system with an initial condition $Z = Z^i$. Thus the best is to walk along the $p_l^F(Z)$ trajectory pointing towards $(p_l^*, Z_l^*)$.

Let us denote by $\lambda_+^i$ the value of $\lambda$ at the time $t_i^+$ when the economy begins to evolve along this path, that is (c.f. (2.2)):

$$\lambda_+^i \;=\; \frac{1}{\rho} p_l^F(Z^i) \;=\; \frac{1}{\rho} u'(x_+^i) \;, \tag{4.1}$$

where $x_+^i$ is solving $u'(x) = p_l^F(Z^i)$.

Define $\bar{Z}^i$ as the pollution stock level at which the curve $p_l^F(Z)$ is crossing the curve $p_h^Z(Z)$. Since we assume that $Z^i < Z_h^*$, then $\bar{Z}^i < \min\{Z_h^*, Z_l^*\}$ as illustrated in the below Figure 4:

Figure 4 here

## 4.1 Pro and con switch arguments

For $Z^i$ larger than $\bar{Z}^i$, the B. M. P. is more rewarding than the B. S. P. Consider for example an hypothetical critical pollution stock $Z_1^i$: $\bar{Z}^i < Z_1^i < Z_l^*$. At any time after $t_i$ the B. S. P. is generating a lower consumption rate since $p_h^Z(Z_1^i) < p_l^F(Z)$ for all $Z$, $Z \in (Z_1^i, Z_l^*)$, together with a larger pollution stock, $Z > Z_1^i$.

However the B. S. P. may present some advantages for critical pollution stocks smaller than $\bar{Z}^i$ but only for sufficiently small levels. Consider for example the critical level $Z_2^i$ in Figure 4. Along the B. S. P. path $p_l^F(Z)$ the resource consumption rate begins to be significantly higher than along the B. M. P. path because $p_l^F(Z_2^i) < p_h^Z(Z_2^i)$ while the pollution stock is only slightly higher than $Z_2^i$. Thus initially, that is within some time interval after the switch, the current net surplus is larger under the B. S. P. In the case of Figure **??**, $p_l^*$, the asymptotic price under the B. S. P. is lower than $p_h^Z(Z_2^i)$ meaning that the consumption rate advantage of the B. S. P. is maintained forever after the switch. Thus the cost-benefit balance of the switch is critically dependent upon the path of higher pollution damages induced by the switch. For a sufficiently low damage function the balance sheet is in the black but goes in the red for high damage functions.

However there exists more intricate cases in which $p_l^*$ is higher than $p_h^Z(Z^i)$ together with $p_h^Z(Z^i) > p_l^F(Z^i)$. In these cases the initial advantage of a higher consumption rate following the switch is progressively vanishing and eventually results in a lower consumption rate after some delay. In between the growing pollution stock is generating larger and larger damages as usual after the switch. This case is illustrated by the critical stock $Z_3^i$ in Figure 4. Let us define $\tilde{Z}_l^i$ as this pollution stock for which $p_h^Z(Z^i) = p_l^*$. Then $p_l^* > p_h^Z(Z_3^i) > p_l^F(Z_3^i)$ for any $Z_3^i$: $\tilde{Z}_l^i < Z_3^i < \bar{Z}^i$, so that, after the switch, there must exist a first phase during which the consumption rate is higher than the consumption rate at which the consumption is blockaded under the B. M. P., followed by a second infinite phase during which the consumption rate is lower contrary to the case $Z^i = Z_2^i$. But because $Z_3^i > Z_2^i$, then the asymptotic loss generated by higher and higher pollution stocks is lower in the case $Z^i = Z_3^i$ than in the case $Z^i = Z_2^i$: $h(Z_h^*) - h(Z_3^i) < h(Z_h^*) - h(Z_2^i)$.

Whatever tricky seem the above pro and con arguments, let us show that the relative advantage of the B. S. P. versus the B. M. P. may be given a very simple expression as a function of $Z^i$.

## 4.2 Determination of the optimal choice when at $Z^i$

Let us denote by $V^s(Z^i)$ the value of the B. S. P. when at $Z^i$, $Z^i \leq \bar{Z}^i$ $(< Z_h^*)$:

$$V^s(Z^i) \quad = \quad \int_{t_i}^{\infty} \left[u(x(t)) - h(Z(t))\right] e^{-\rho(t-t_i)} dt \ ,$$

where $\{(x(t), Z(t)) \ , \ t \geq t_i\}$ is the B. S. P. path after $t_i$.

According to the Hamilton-Jacobi-Bellman (H. J. B.) equation, $V^s(Z^i)$

may be given the equivalent following expression:

$$
\begin{aligned}
V^s(Z^i) &= \frac{1}{\rho}\left\{u(x^i_+) - h(Z^i) - \lambda^i_+\left[\zeta x^i_+ - \alpha_l Z^i\right]\right\} \\
&= \frac{1}{\rho}\left\{u(x^i_+) - h(Z^i) - \frac{1}{\zeta}u'(x^i_+)\left[\zeta x^i_+ - \alpha_l Z^i\right]\right\}
\end{aligned}
\tag{4.2}
$$

where $x^i_+$ is given by (4.1) as a function of $Z^i$. This is the expression with which we shall work. We show in Appendix A.2 that $V^s(Z^i)$ is decreasing as expected and concave:

$$
\frac{dV^s}{dZ^i} < 0 \quad \text{and} \quad \frac{d^2V^s}{d\left(Z^i\right)^2} < 0 \,. \tag{4.3}
$$

To determine the optimal policy we first show that[8]:

$$
V^m(0) < V^s(0) \quad \text{and } V^m(\bar{Z}^i) > V^s(\bar{Z}^i)\,,
$$

and second that $B(Z^i)$, the comparative advantage of the B. S. P., $B(Z^i) \equiv V^s(Z^i) - V^m(Z^i)$, is a decreasing function of $Z^i$ over the range $(0, \bar{Z}^i)$. Hence we may conclude that there exists a unique level of the critical pollution stock, we denote by $Z^i_R$: $0 < Z^i_R < \bar{Z}^i$, at which the comparative advantage of the switch is reversed, from the black to the red.

*Values of $V^m$ and $V^s$ at the boundaries of* $(0, \bar{Z}^i)$

For $Z^i = \bar{Z}^i$, then $x^i_+ = x^i$ and (4.2) may be rewritten as:

$$
\begin{aligned}
V^s(\bar{Z}^i) &= \frac{1}{\rho}\left\{u(x^i) - h(\bar{Z}^i) - \frac{1}{\zeta}u'(x^i)\left[\zeta x^i - \alpha_l \bar{Z}^i\right]\right\} \\
&= \frac{1}{\rho}\left[u(x^i) - h(\bar{Z}^i)\right] - \frac{1}{\rho\zeta}u'(x^i)\left[\zeta x^i - \alpha_l \bar{Z}^i\right] \\
&= V^m(\bar{Z}^i) - \frac{1}{\rho\zeta}u'(x^i)\dot{Z}(t^+_i)\,,
\end{aligned}
$$

where $\dot{Z}(t^+_i)$ is the rate of change of $Z(t)$ immediately after the switch under the B. S. P. Hence $\dot{Z}(t^+_i) > 0$ and:

$$
V^m(\bar{Z}^i) > V^s(\bar{Z}^i)\,. \tag{4.4}
$$

[8]Strictly speaking, in the below L. H. S. inequality $V^m(0)$ must be understood as $\lim_{Z^i \downarrow 0} V^m(Z^i)$.

Next, since $h(0) = 0$ and $x^i = \alpha_h Z^i/\zeta$, then:

$$V^m(0) \;=\; \lim_{Z^i \downarrow 0} \left\{ u\left(\frac{\alpha_h}{\zeta} Z^i\right) - h(Z^i) \right\} \;=\; \lim_{Z^i \downarrow 0} u\left(\frac{\alpha_h}{\zeta} Z^i\right) \;=\; u(0) \;,$$

and denoting by $x^i_+(0)$ the limit of $x^i_+$ as a function of $Z^i$ as $Z^i \to 0$:

$$\begin{aligned} V^s(0) \;&=\; \frac{1}{\rho} \lim_{Z^i \downarrow 0} \left\{ u\left(x^i_+(Z^i)\right) - h(Z^i) - \frac{1}{\zeta} u'\left(x^i_+(Z^i)\right) \left[\zeta x^i_+(Z^i) - \alpha_l Z^i\right] \right\} \\ &=\; \frac{1}{\rho} \left\{ u\left(x^i_+(0)\right) - u'\left(x^i_+(0)\right) x^i_+(0) \right\} \end{aligned}$$

Then by the concavity of $u$:

$$u'\left(x^i_+(0)\right)\left[x^i_+(0) - 0\right] < u\left(x^i_+(0)\right) - u(0) \;,$$

equivalently:

$$V^m(0) \;=\; \frac{1}{\rho} u(0) \;<\; \frac{1}{\rho} \left\{ u\left(x^i_+(0)\right) - u'\left(x^i_+(0)\right) x^i_+(0) \right\} \;=\; V^s(0) \;. \tag{4.5}$$

*Locating the maximum of $V^m(Z^i)$ with respect to the upper boundary $\bar{Z}^i$*

Would $V^m(Z^i)$ be a monotonously increasing function of $Z^i$ we could conclude from (4.4), (4.5) and $V^s(Z^i)$ decreasing that there exists a unique critical pollution stock at which the sign of $B(Z^i)$ is reversed.

However we have shown in Section 3 that in the $(p, Z)$ space, the locus of (price , pollution stock) pairs at which $V^m(Z^i)$ attains its maximum is defined by the condition (3.6):

$$p \;=\; u'\left(\frac{\alpha_h}{\zeta} Z^i\right) \;=\; \frac{\zeta}{\alpha_h} h'(Z^i) \;.$$

Thus in order that the maximum of $V^m$ be attained at the upper border $\bar{Z}^i$ of the interval $[0, \bar{Z}^i]$ it must be the case that (see Figure 5):

$$p^F_l(\bar{Z}^i) \;=\; \frac{\zeta}{\alpha_h} h'\left(\bar{Z}^i\right) \;. \tag{4.6}$$

Figure 5 here

Let $\bar{\alpha}_h$ be this value of $\alpha_h$ for which (4.6) is satisfied:

$$\bar{\alpha}_h \quad = \quad \zeta \frac{h'\left(\bar{Z}^i\right)}{p_l^F\left(\bar{Z}^i\right)}$$

Then:

$$\hat{Z}^i \gtreqqless \bar{Z}^i \quad \text{according to} \quad \alpha_h \lesseqqgtr \bar{\alpha}_h \ .$$

Thus, given all the other fundamentals of the model, for a sufficiently low $\alpha_h$, the maximum of $V^m(Z^i)$ is attained at some $Z^i$ outside the $(0, \bar{Z}^i)$ range. In such a case we may not conclude from the mere inequalities (4.4) and (4.5), the qualitative properties (4.3) of $V^s$ and the concavity of $V^m$, that there exists a unique critical pollution stock within the range $(0, \bar{Z}^i)$ at which the sign of $B(Z^i)$ is reversed. The below Figure 6 illustrates why, under the above characteristics of $V^m$ and $V^s$, the set of $Z^i$ values for which $B(Z^i)$ is positive may be made of disconnected intervals, $(0, Z_1^i)$ and $(Z_2^i, Z_3^i)$, $0 < Z_1^i < Z_2^i < Z_3^i < \bar{Z}^i$.

Figure 6 here

What has to be pointed out is that the qualification $\alpha_h < \bar{\alpha}_h$ is a qualification 'given all other fundamentals of the model'. Thus a 'low' $\alpha_h$ must be understood 'for a given $\alpha_l$', meaning that such cases could possibly appear only when the discrepancy between the most and the least efficient self-regenerating regimes is not too large. These cases are cases in which choosing either regime is not an easy cost-benefit assessment exercise, hence the necessity of proving the following property of the $B(Z^i)$ function.

*The comparative advantage $B(Z^i)$ is a decreasing function of $Z^i$*

To prove that $B(Z^i)$ is decreasing over $(0, \bar{Z}^i)$ we look at $B$ as the sum of two components, obtained from the H. J. B. expressions of both $V^m(Z^i)$ and $V^s(Z^i)$.

Consider for example the hypothetical critical level $Z_2^i$ in Figure 4. Choosing the B. S. P. rather than the B. M. P. may be seen as a move from

$x_2^i \equiv x^d\left(p_h^Z(Z_2^i)\right)$ up to $x_{+2}^i \equiv x^d\left(p_l^F(Z_2^i)\right)$ corresponding to two successive changes. The first change is a change of the self-regeneration regime at $x_2^i$ given. The second change, from $x_2^i$ to $x_{+2}^i$, is the change of the initial resource consumption rate of the path satisfying the F. O. C's under the low self-regeneration regime once the regime shift has been engaged.

Let us denote by $\Delta^-(Z^i)$ the loss (in absolute value) due to the regime switch at $x^i$ fixed and by $\Delta^+(Z^i)$ the gain resulting from a better adaptation, to the new self-regeneration regime, of the initial consumption rate $x_+^i$ rather than $x^i$.

From the H. J. B. expression of $V^m(Z^i)$:

$$V^m(Z^i) = \frac{1}{\rho}\left\{u(x^i) - h(Z^i) - \lambda^i\left[\zeta x^i - \alpha_h Z^i\right]\right\} \tag{4.7}$$

and the expression (4.2) of $V^s(Z^i)$, the loss induced by the regime switch absent any accommodation of the initial consumption rate $x^i$, is given by:

$$\begin{aligned}\Delta^-(Z^i) &= \frac{1}{\rho}\left\{u(x^i) - h(Z^i) - \lambda^i\left[\zeta x^i - \alpha_h Z^i\right]\right\} \\ &\quad -\frac{1}{\rho}\left\{u(x^i) - h(Z^i) - \lambda^i\left[\zeta x^i - \alpha_l Z^i\right]\right\}\end{aligned}$$

where $\lambda^i = u'(x^i)/\zeta$ is this value of $\lambda$ prevailing both:

- at $t_i$ along the constant path of the B. M. P. once $Z^i$ has been attained,

  and also:

- at $t_i^+$ along a path which would satisfy all the necessary conditions under the low self-regeneration regime starting with an initial consumption rate $x^i$.

Hence:

$$\Delta^-(Z^i) = \frac{1}{\zeta}u'(x^i)\left[\alpha_h - \alpha_l\right]Z^i \ .$$

Let us turn to $\Delta^+(Z^i)$. Using the derivative of the H. J. B. equation w.r.t. $x$, the gain resulting from the adaptation of the initial consumption rate to the low self-regeneration regime amounts to:

$$\Delta^+(Z^i) \quad = \quad -\frac{1}{\rho\zeta}\int_{x^i}^{x_+^i} u''(x)\left[\zeta x - \alpha_l Z^i\right] dx \ .$$

Integrating by parts, we obtain:

$$\begin{aligned}\Delta^+(Z^i) \quad = \quad & -\frac{1}{\rho\zeta}\left\{u'\left(x_+^i\right)\left[\zeta x_+^i - \alpha_l Z^i\right] - u'\left(x_+^i\right)\left[\zeta x^i - \alpha_l Z^i\right]\right. \\ & \left. -\zeta\int_{x^i}^{x_+^i} u'(x)dx\right\} \ .\end{aligned}$$

After some standard manipulations, the net benefit may be given the following expression:

$$\begin{aligned}B(Z^i) \quad &= \quad \Delta^+(Z^i) - \Delta^-(Z^i) \\ &= \quad \frac{1}{\rho\zeta}\left\{\zeta\int_{x^i}^{x_+^i} u'(x)dx - u'(x_+^i)\left[\zeta x_+^i - \alpha_l Z^i\right]\right\} \ ,\end{aligned}$$

hence:

$$\frac{dB}{dZ^i} \quad = \quad \frac{1}{\rho\zeta}\left\{-u''\left(x_+^i\right)\left[\zeta x_+^i - \alpha_l Z^i\right]\frac{dx_+^i}{dZ^i} + \alpha_l u'\left(x_+^i\right) - \alpha_h u'(x^i)\right\} < 0 \ .$$

We conclude as follows:

**Proposition 2** *There exists a unique reversing level of the critical pollution stock $Z_R^i$: $0 < Z_R^i < \bar{Z}^i$, under which the Best Switching Policy is the optimal policy and over which the Best Maintaining Policy is the optimal one. For $Z^i = Z_R^i$ the two policies are equivalent, meaning that once at $Z^i$, they generate the same continuation values.*

# 5 Characterizing the paths induced by the Best Switching Policies

In this section we characterize the paths of the consumption rate and the pollution stock induced by the B. S. P's. We show that over the range

$(0, \bar{Z}^i)$ choosing a B. S. P. always implies an upward jump of the resource consumption rate at the switching date $t_i$: $x^i_- < x^i_+$.

For characterizing the best paths from $Z^0$ to $Z^i$ it is useful to look at the social planner problem as the following problem:

$$
\begin{aligned}
\max_{x,t_i} \quad & \int_0^{t_i} \left[u(x(t)) - h\left(Z(t)\right)\right] e^{-\rho t} dt + e^{-\rho t_i} \max\left\{V^m(Z^i), V^s(Z^i)\right\} && (5.1)\\
s.t; \quad & (3.1) \text{ and } Z(t_i) = Z^i \ , && (5.2)
\end{aligned}
$$

to which must be added another constraint.

*Expliciting the maximum time delay constraint for attaining $Z^i$*

Whatever the value of the second term of the above objective function (5.1), as shown in Section 2 (see (2.5) with $\alpha = \alpha_h$ and (2.6)), the dynamics of $p$ and $Z$ before the time $t_i$ at which $Z^i$ is attained is given by the following system of autonomous differential equations:

$$
\begin{aligned}
\dot{p}(t) &= (\rho + \alpha_h)p(t) - \zeta h'\left(Z(t)\right) && (5.3)\\
\dot{Z}(t) &= \zeta x^d\left(p(t)\right) - \alpha_h Z(t) \ . && (5.4)
\end{aligned}
$$

In the $(p, Z)$ plane only these parts of the trajectories corresponding to (5.3)-(5.4) located under the locus $p^Z_h(Z)$ and permitting to attain $Z^i$ have to be considered, that is the part of the B. M. P. trajectory itself and the parts of all trajectories located below, as illustrated in Figure 7.

Figure 7 here

Given the properties of autonomous systems, the lower is the trajectory the shorter is the delay for attaining $Z^i$ starting from $Z^0 < Z^i$. Thus there exists a maximum delay for attaining $Z^i$ while respecting the F.O.C's having to be satisfied before $t_i$, the delay of the B. M. P. Let us denote by $t_i^m$ this delay[9]. The additional constraint having to be taken into account to fully specify the social planner problem reads:

$$
t_i^m - t_i \ \geq \ 0 \ , \qquad (5.5)
$$

[9]Clearly $t_i^m$ is depending upon $Z^i$.

equivalently:

$$p^i - p^i_- \;\geq\; 0 \quad \text{or} \quad x^i_- - x^i \;\geq\; 0\ . \tag{5.6}$$

We show that the constraint (5.5) is never effective for $Z^i \leq Z^i_R$ and is binding for $Z^i > Z^i_R$.

## 5.1 Optimal B. S. P. paths. Case: $Z^0 < Z^i < Z^i_R$

Since $Z^0 < Z^i < Z^i_R$ the B. S. P. is the optimal policy and the social planner problem is simply:

$$\begin{array}{ll} \max\limits_{x,t_i} & \displaystyle\int_0^{t_i} \left[u(x(t)) - h(Z(t))\right] e^{-\rho t} dt + e^{-\rho t_i} V^s(Z^i) \\ s.t. & (3.1)\ ,\ (5.2)\ ,\ (5.5)\ \text{ or } (5.6)\ . \end{array}$$

Assuming that (5.5) is not binding (we prove that it is the case later in this sub-section), then the transversality condition is:

$$\lim_{t \uparrow t_i} H(t) + \frac{\partial}{\partial t_i}\left[e^{-\rho t_i} V^s(Z^i)\right] \;=\; 0\ , \tag{5.7}$$

where $H(t)$ is the present value Hamiltonian. After substitution for $H$ and $V^s$ (c.f. (4.2)):

$$\begin{aligned} &\left\{u(x^i_-) - h(Z^i) - \lambda^i_-\left[\zeta x^i_- - \alpha_h Z^i\right]\right\} e^{-\rho t_i} \;=\; -\rho e^{-\rho t_i} V^s(Z^i) \\ &= \; e^{-\rho t_i}\left\{u(x^i_+) - h(Z^i) - \lambda^i_+\left[\zeta b x^i_+ - \alpha_l Z^i\right]\right\}\ . \end{aligned} \tag{5.8}$$

Substituting $u'(x^i_-)/\zeta$ and $u'(x^i_+)/\zeta$ for respectively $\lambda^i_-$ and $\lambda^i_+$ and simplifying, we get:

$$\begin{aligned} u(x^i_-) - u'(x^i_-)x^i_- \;&+\; u'(x^i_-)\frac{\alpha_h}{\zeta}Z^i \\ &= \; u'(x^i_+) - u'(x^i_+)x^i_+ + u'(x^i_+)\frac{\alpha_l}{\zeta}Z^i\ . \end{aligned} \tag{5.9}$$

Since $\alpha_h \neq \alpha_l$ the above equation (5.9) cannot be satisfied for a common value $x^s$ of the resource consumption rate at the switching date, we mean that, whatever be $x^s$, we cannot have: $x^i_- = x^s = x^i_+$.

Remark that in the equation (5.9) $x_+^i$ is known: $x_+^i = x^d\left(p_l^F(Z^i)\right)$. Thus the equation (5.9) can be seen as an equation in $x_-^i$ having to be determined given $x_+^i$. To determine $x_-^i$ let us proceed as follows.

Let $\Gamma_k(x, Z^i)$ be the function:

$$\Gamma_k(x, Z^i) \quad \equiv \quad u(x) - u'(x)x + u'(x)\frac{\alpha_k}{\zeta}Z^i \ , \ k \ \in \ \{h, l\}$$

hence:

$$\Gamma_h(x, Z^i) \quad > \quad \Gamma_l(x, Z^i) \tag{5.10}$$

$$\frac{\partial \Gamma_k}{\partial x} \quad = \quad -\frac{1}{\zeta}u''(x)\left[\zeta x - \alpha_k Z^i\right] \begin{cases} < 0 \quad , \quad 0 < x < \dfrac{\alpha_k}{\zeta}Z^i \\ \\ = 0 \quad , \quad x = \dfrac{\alpha_k}{\zeta}Z^i \\ \\ > 0 \quad , \quad \dfrac{\alpha_k}{\zeta}Z^i < x \end{cases} \tag{5.11}$$

$$\frac{\partial \Gamma_k}{\partial Z^i} \quad = \quad u'(x)\frac{\alpha_k}{\zeta} \ > \ 0 \tag{5.12}$$

For any $Z^i$, $\Gamma_k$ attains its minimum at $x = \alpha_k Z^i/\zeta$[10].

Now $x_-^i$ appears as the solution of:

$$\Gamma_h(x, Z^i) \quad = \quad \Gamma_l(x_+^i, Z^i) \tag{5.13}$$

The solution of this equation is illustrated in the below Figure 8:

Figure 8 here

[10]Note that:

$$\frac{\partial^2 \Gamma_k}{\partial x^2} \quad = \quad -\frac{1}{\zeta}u''(x)\left[\zeta x - \alpha_k Z^i\right] - u''(x) \ ,$$

hence:

$$\left.\frac{\partial^2 \Gamma_k}{\partial x^2}\right|_{x=\frac{\alpha_k}{\zeta}Z^i} \quad = \quad -u''\left(\frac{\alpha_k}{\zeta}Z^i\right) \ > \ 0 \ .$$

Figure 8 has to be read starting from the known consumption rate $x^i_+$: $x^i_+ > \alpha_l Z^i/\zeta$[11]. Thus following the direction of the arrow, first $\Gamma_l(x^i_+, Z^i)$ is fixed, $\left(x^i_+, \Gamma_l(x^i_+, Z^i)\right)$ being on the increasing part of the $\Gamma_l$ curve. Next switching from the $\Gamma_l$ curve to the $\Gamma_h$ curve at the common value $\Gamma_l(x^i_+, Z^i)$, $x^i_-$ is identified as this value of $x$ for which (5.13) is satisfied[12].

Thus along an optimal B. S. P. path, when at $t_i$ and the pollution stock approaches its critical level $Z^i$, then the consumption rate of the resource jumps from $x^i_-$ up to $x^i_+$, equivalently the full marginal cost of the resource services must fall from $p^i_-$ down to $p^i_+ = p^F_l(Z^i)$.

This scheme works as far as $x^i_-$ is at least equal to $x^i$, i.e. as far as (5.13) has a solution, equivalently as far as (5.5) or (5.6) are not binding. To prove it is the case for $Z^i < Z^i_R$, we show that:

a. If $x^i_+ > x^i$, then $x^i$ is a decreasing function of $Z^i$;

b. Assuming $x^i_- = x^i$ implies that $Z^i = Z^i_R$.

Since $x^i$ is an increasing function of $Z^i$ ($x^i = \alpha_h Z^i/\zeta$), then $x^i - x^i_-$ is a decreasing function of $Z^i$ down to 0 at $Z^i = Z^i_R$.

*a. $x^i_i$ is a decreasing function of $Z^i$ if $x^i_- > x^i$*

Let us differentiate the transversality condition $H(t_i) = \rho V^s(Z^i)$, holding when $x^i_- > x^i$, i.e. when (5.6) is not binding:

$$\frac{\partial \Gamma_h}{\partial x^i_-} dx^i_- + \frac{\partial \Gamma_h}{\partial Z^i} dZ^i - h'(Z^i) dZ^i \quad = \quad \rho \frac{dV^s(Z^i)}{dZ^i} dZ^i \ .$$

Remark that $dV^s/dZ^i = -\lambda^i_+ = -u'(x^i_+)/\zeta$. Thus using the expressions

[11] Since $Z^i < \bar{Z}^i$, then $p^F_l(Z^i) < u'(x^i)$ implies together with $u'(x^i) = p^Z_h(Z^i) < p^Z_l(Z^i) = u'\left(\alpha_l Z^i/\zeta\right)$ that $x^i_+ = x^d\left(p^F_l(Z^i)\right) > \alpha_l Z^i/\zeta$.

[12] In Figure 8, the equation (5.10) has two solutions $x^i_-$ and $x^{i'}_-$: $x^{i'}_- < x^i < x^i_-$. Clearly $x^{i'}_-$ is violating the constraint (5.6), while $x^i_-$ is satisfying it.

(5.11) and (5.12) for the partial derivatives of $\Gamma_h$, we get:

$$\begin{aligned} -\frac{1}{\zeta}u''(x^i_-)\left[\zeta x^i_- - \alpha_h Z^i\right]dx^i_- + u'(x^i_-)\frac{\alpha_h}{\zeta}dZ^i - h'(Z^i)dZ^i \\ = -\frac{\rho}{\zeta}u'(x^i_+)dZ^i\ . \end{aligned}$$

Since $x^i_- > x^i$, then $\zeta x^i_- - \alpha_h Z^i = \dot{Z}(t_i^-) > 0$, and:

$$\frac{dx^i_-}{dZ^i} = \frac{1}{u''(x^i_-)}\frac{\rho u'(x^i_+) + \alpha_h u'(x^i_-) - \zeta h'(Z^i)}{\dot{Z}(t_i^-)}\ .$$

The numerator of the second R.H.S. term may be rewritten as:

$$\left[(\rho+\alpha_l)u'(x^i_+) - \zeta h'(Z^i)\right] + \left[\alpha_h u'(x^i_-) - \alpha_l u'(x^i_+)\right]\ .$$

Since $u'(x^i_+) = p^i_+$ then the first term into brackets of the above sum is nothing but than $\dot{p}(t_i^+)$: $\dot{p}(t_i^+) > 0$ for $Z^i < \bar{Z}^i \leq Z^*_h$. Since $\alpha_l < \alpha_h$, and $x^i_+ > x^i_-$ implies $u'(x^i_+) < u'(x^i_-)$, then the second term is also positive. Hence:

$$\frac{dx^i_-}{dZ^i} = \frac{1}{u''(x^i_-)}\frac{\dot{p}(t_i^+) + \alpha_h u'(x^i_-) - \alpha_l u'(x^i_+)}{\dot{Z}(t_i^-)} < 0\ . \qquad (5.14)$$

*b. Assuming $x^i_- = x^i$ implies that $Z^i = Z^i_R$*

Let us assume that $x^i_- = x^i$ so that $\zeta x^i_- - \alpha_h Z^i = 0$. Then:

$$\begin{aligned} H(t_i^-) &= \left\{u(x^i_-) - h(Z^i) - \lambda^i_-\left[\zeta x^i_- - \alpha_h Z^i\right]\right\}e^{-\rho t_i} \\ &= \left\{u(x^i) - h(Z^i)\right\}e^{-\rho t_i} = -\frac{\partial}{\partial t_i}\left[e^{-\rho t_i}V^m(Z^i)\right]\ . \end{aligned}$$

But by (5.7) we have also $H(t_i) = -\frac{\partial}{\partial t_i}\left[e^{-\rho t_i}V^s(Z^i)\right]$, thus $V^m(Z^i) = V^s(Z^i)$ an equality which is satisfied iff $Z^i = Z^i_R$ according to the Proposition 2.

Although the size of the jump of the resource consumption rate at $t_i$ cannot be easily characterized as a function of $Z^i$, the fall in the full marginal cost of the resource services can be shown to increase with $Z^i$: $Z^i < Z^i_R$.

From $dp^i_-/dZ^i = u''(x^i_-)dx^i_-/dZ^i$ and (5.14) we get:

$$\frac{dp^i_-}{dZ^i} = \frac{\dot{p}(t_i^+) + \alpha_h u'(x^i_-) - \alpha_l u'(x^i_+)}{\dot{Z}(t^i_-)} > 0\ ,$$

as expected since $dx^i_-/dZ^i < 0$ and $x^d(p)$ is a standard demand function decreasing with the price of the good.

Next, from:

$$\frac{dp^i_+}{dZ^i} = \left.\frac{dp^F_l}{dZ}\right|_{Z=Z^i} = \frac{\dot{p}(t_i^+)}{\dot{Z}(t_i^+)}$$

we obtain:

$$\frac{d}{dZ^i}\left[p^i_- - p^i_+\right] = \frac{\dot{p}(t_i^+)}{\dot{Z}(t_i^-)} + \frac{\alpha_h u'(x^i_-) - \alpha_l u'(x^i_+)}{\dot{Z}(t_i^-)} - \frac{\dot{p}(t_i^+)}{\dot{Z}(t_i^+)} .$$

Since $x^i_- < x^i_+$ and $\alpha_h > \alpha_l$, then: $\dot{Z}(t_i^-) < \dot{Z}(t_i^+)$ which implies:

$$\dot{p}(t_i^+)\left(\frac{1}{\dot{Z}(t_i^-)} - \frac{1}{\dot{Z}(t_i^+)}\right) > 0 ,$$

and we conclude that:

$$\frac{d}{dZ^i}\left[p^i_+ - p^i_-\right] > 0 \tag{5.15}$$

The resource services price fall increases with $Z^i$ and attains its maximum at $Z^i = Z^i_R$ when $p^i_- = p^i$, i.e. when $x^i_- = x^i$.

The trajectories of the B. M. P. and B. S. P. paths for the case $Z^0 < Z^i < Z^0_R$, in which the B. S. P. is the optimal policy are illustrated in the below Figure 9.

Figure 9 here

## 5.2 Non optimal B. S. P. paths. Case $Z^0 < Z^i$ and $Z^i_R < Z^i < \bar{Z}^i$

In this case the B. S. P. is solving the same problem than in the preceding sub-section 5.1, excepted that here its solution is not the solution of the social planner problem since at $t_i$ the bad option is enforced. The problem is reminiscent of the Kydland and Prescott (1977) work on the temporal consistency of policies and/or strategies, the 'rule' being to choose the self-regeneration switch when at $Z^i$ while the 'discretion' would command to maintain the high self-regeneration regime[13].

We know from the analysis of the preceding sub-section that neglecting the constraint (5.5) (or (5.6)) would result in a transversality condition at $t^i$ which cannot be satisfied. Thus the constraint (5.5) must now be taken into account and be given its due care. Let $\mu$ be the multiplier associated to the constraint $t_i^m - t_i \geq 0$, so that the transversality condition has to be written as follows:

$$\lim_{t \uparrow t_i} H(t) \;=\; \rho V^s(Z^i) + \mu$$

Equivalently $x_-^i$ and $\mu$ must together solve:

$$\begin{aligned} &\Gamma_h(x, Z^i) \;=\; \Gamma_l(x_+^i, Z^i) + \mu \ , \ x_+^i \ \text{ given : } x_+^i > x^i \\ &\mu \geq 0 \ , \ x_-^i - x^i \geq 0 \ \text{ and } \mu \left[x_-^i - x^i\right] \;=\; 0 \ . \end{aligned}$$

The solution $x_-^i = x^i$ and $\mu = \Gamma_h(x^i, Z^i) - \Gamma_l(x_+^i, Z^i)$ is illustrated in the below Figure 10.

Figure 10 here

Since $p_-^i = p^i$ is decreasing with $Z^i$ while $p_+^i$ is increasing because $Z^i < \bar{Z}^i < Z_h^*$, the size of the full marginal cost falls when at $t_i$, $p_-^i - p_+^i = p^i - p_+^i$, is now decreasing down to 0 when $Z^i$ tends to $\bar{Z}^i$ after having attained its maximum at $Z^i = Z_R^i$.

The B. S. P. trajectory when $Z^i$ is higher than $Z_R^i$ is illustrated in the below Figure 11. Since $p_-^i = p^i$, then the initial part of the path from $t = 0$ to $t = t_i$, equivalently from $Z^0$ to $Z^i$, is the same than the initial part of the B. M. P. path. The only difference is now that when at $Z^i$ the bad decision is enforced[14].

[13] The same remark holds for the B. M. P. path when $Z^i < Z_R^i$, the rule being to maintain the high self-regeneration rate $\alpha_h$ rather than switch in this case.

[14] Note that when the B. M. P. path is chosen when $Z^i < Z_R^i$, the other case of bad decision when at $Z^i$, then the first path of the B. M. P. path is not the same than the first path of the B. S. P. path since in this case $p_-^i < p^i$.

Figure 11 here

Summing up:

**Proposition 3** *Along the Best Switching Policies paths, whatever be $Z^i$: $0 < Z^i < \bar{Z}^i$, the resource consumption path is discontinuous at $t = t_i$ when the pollution stock attains its critical level $Z^i$, jumping from $x^i_-$, $x^i_- \geq x^i$, up to $x^i_+$, $x^i_+ > x^i_-$, meaning that the shadow marginal cost of the pollution stock is falling down. The pollution stock is continuously increasing, but its instantaneous growth rate $\dot{Z}(t)$ is discontinuous, jumping upwards at the switch date. Furthermore:*

*a. For $Z^i$ lower than $Z^i_R$, the Best Switching Policy is the optimal policy and $x^i_-$ is decreasing with $Z^i$ while $x^i$ is increasing so that the gap between the B. S. P. and the B. M. P. consumption rates when attaining $Z^i$, $x^i_- - x^i$, is decreasing with $Z^i$ down to 0 when $Z^i$ tends to $Z^i_R$, simultaneously the size of the energy services price fall, $p^i_- - p^i_+$, increases with $Z^i$ and attains its maximum when $Z^i = Z^i_R$ and $p^i_- = p^i$;*

*b. For $Z^i$ higher than $Z^i_R$, then $x^i_-$ is blockaded at $x^i$ and now the upward jump $x^i_+ - x^i$ is decreasing with $Z^i$ down to 0 when $Z^i = \bar{Z}^i$; the same holds for the price fall $p^i - p^i_+$ now decreasing with $Z^i$ down to 0 at $Z^i = \bar{Z}^i$.*

# 6 Conclusion

The management of environmental catastrophes has received a lot of attention in the literature, such catastrophes being usually described as uncertain events impacting negatively welfare. We depart from this approach by assuming first that the catastrophe is known to arise with certainty if some critical threshold in terms of pollution concentration is overshot. Second, we describe the catastrophe as an irreversible loss of the self-regenerating capacity of the environment, thus constraining the ability of the economy to generate welfare after the catastrophe, an indirect welfare effect rather than the direct welfare effect postulated in the earlier literature.

We have first examined the nature of this indirect effect by comparing the optimal management policy of a polluting resource when the self-cleaning capacity of the environment is either low or high. We show that higher self-regeneration capacities imply in all cases a lower opportunity cost of pollution and a higher consumption rate of the polluting resource in the long run. But lower self-cleaning capacities of the environment does not necessarily imply a higher long run pollution stock, this issue depending crucially upon the resource demand function elasticity. A higher discount rate has the same qualitative positive effect over the long run resource consumption than a better assimilative capacity of the environment, but contrary to the ambiguous effect of the self-cleaning capacity, it always increases the long run pollution stock.

We have next considered the problem of a society having accumulated pollution up to the catastrophic threshold and contemplating what to do next: either maintain the high regeneration capacity of the environment of either choosing to switch irreversibly toward the low self-regeneration regime. We discuss the pro and con of these two options. Maintaining the high regeneration regime allows to avoid an increased accumulation of damages in the future but imposes a strong constraint upon consumption possibilities. On the other hand, a higher trend of damages in the future may itself be compensated by a higher consumption rate if the switch to the low regeneration regime is chosen, making the choice of the best option a non trivial exercise.

Despite these potential ambiguities, we show that the choice problem may be given a simple answer. There exists a unique critical level of the catastrophic threshold, such that for thresholds below this tipping point, it is better to overshoot the threshold and enter the low regime, while the contrary happens above the tipping point, maintaining the high regime being the society best choice.

We have next characterized the optimal policy in these two scenarios. Starting from any initial pollution stock below the catastrophic threshold, they are two phases paths, a first phase of convergence towards the threshold with a declining use of the polluting resource and an increasing trend of damages resulting from pollution accumulation, and a second phase of either constant consumption and damages level if the preservation option is chosen, or either a declining consumption trend and an increasing damages trend if the switching option is chosen. We show that switching requires an upward jump of the resource consumption rate at the switching time. If preservation is the optimal choice when attaining the threshold, the path converging to it may be described as the 'slowest approach' towards the threshold, the time delay for attaining it being the largest possible delay among the paths satisfying the optimality conditions and attaining the threshold. If the threshold is lower than the tipping point, the optimal switching option will induce a faster convergence towards the threshold than the best conservative transition path (a non optimal path in this case), being characterized by a higher consumption rate and a faster trend of damages increase. Last we show that in value terms, the fall down in resource implicit price at the switching time increases with the level of the threshold when switching is the optimal option.

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# Appendix

## A.1 Appendix 1: Derivatives of $Z^*$ and $p^*$ with respect to $\rho$ and $\zeta$.

Totally differentiating (2.11) and substituting $u'(\alpha Z/\zeta)$ for $\zeta h'(Z^*)/(\rho+\alpha)$ results in:

$$\begin{aligned}
\frac{dZ^*}{d\rho} &= \frac{u'}{-\frac{\alpha(\rho+\alpha)}{\zeta}u'' - \zeta h''} \;>\; 0 \\
\frac{dZ^*}{d\zeta} &= \frac{(\rho+\alpha)\left[\frac{\alpha}{Z^*}u'' + u'\right]}{\alpha(\rho+\alpha)u'' - \zeta^2 h''} \;,
\end{aligned}$$

hence:

$$\frac{dZ^*}{d\zeta} \gtreqless 0 \quad \text{according to:} \quad \eta^* \lesseqgtr 1 \;.$$

where $u'$ and $u''$ are evaluated at $x^* = \alpha Z^*/\zeta$ and $\eta^*$ is the demand elasticity evaluated at $p^* = u'(\alpha Z^*/\zeta)$.

Totally differentiating $p^* = u'(\alpha Z^*/\zeta)$ and substituting the above expressions for $dZ^*/d\rho$ and $dZ^*/d\zeta$, we get:

$$\begin{aligned}
\frac{dp^*}{d\rho} &= \frac{\alpha u'\, u''}{\zeta^2 h^2 - \alpha(\rho+\alpha)u''} \;<\; 0 \\
\frac{dp^*}{d\zeta} &= \frac{\alpha u''\left[\frac{1}{\zeta}h'' + u'\right]}{\zeta\left\{\alpha(\rho+\alpha)u'' - \zeta^2 h''\right\}} \;>\; 0 \;.
\end{aligned}$$

## A.2 Appendix 2: Properties of the $V^s(Z^i)$ function

Let us start from:

$$V^s(Z^i) \;=\; \frac{1}{\rho}\left\{u'(x^i_+) - h(Z^i) - \frac{1}{\zeta}u'(x^i_+)\left[\zeta x^i_+ - \alpha_l Z^i\right]\right\} \equiv \mathcal{V}^s(x^i_+, Z^i) \;,$$

where $x_+^i$ solves $u'(x) = p_l^F(Z^i)$.

Thus:

$$\begin{aligned}
\frac{\partial \mathcal{V}^s}{\partial Z^i} &= \frac{1}{\rho\zeta}\left[-\zeta h'(Z^i) + \alpha_l u'(x_+^i)\right] \ , \\
\frac{\partial \mathcal{V}^s}{\partial x_+^i} &= -\frac{1}{\rho\zeta}u''(x_+^i)\left[\zeta x_+^i - \alpha_l Z^i\right] \ = \ -\frac{1}{\rho\zeta}u''(x_+^i)\dot{Z}(t_i^+) \ > \ 0 \ , \\
\frac{dx_+^i}{dZ^i} &= \frac{1}{u''(x_+^i)}\left.\frac{dp_l^F}{dZ}\right|_{Z=Z^i} \ = \ \frac{1}{u''(x_+^i)}\left.\frac{\dot{p}}{\dot{Z}}\right|_{t=t_i^+} \ < \ 0 \ ,
\end{aligned}$$

hence:

$$\begin{aligned}
\frac{dV^s}{dZ^i} &= \frac{\partial \mathcal{V}^s}{\partial x_+^i}\frac{dx_+^i}{dZ^i} + \frac{\partial \mathcal{V}^s}{\partial Z^i} \\
&= \left[-\frac{1}{\rho\zeta}u''(x_+^i)\dot{Z}(t_i^+)\right]\left[\frac{1}{u''(x_+^i)}\frac{\dot{p}(t_i^+)}{\dot{Z}(t_i^+)}\right] \\
&\qquad + \frac{1}{\rho\zeta}\left[-\zeta h'(Z^i) + \alpha_l u'(x_+^i)\right] \\
&= \frac{1}{\rho\zeta}\left[-\dot{p}(t_i^+) - \zeta h'(Z^i) + \alpha_l u'(x_+^i)\right] \ . \qquad \text{(A.2.1)}
\end{aligned}$$

Remember that (c.f. (2.5)):

$$\dot{p}(t_i^+) \ = \ (\rho + \alpha_l)p(t_i^+) - \zeta h'(Z^i) \ = \ (\rho + \alpha_l)u'(x_+^i) - \zeta h'(Z^i) \ .$$

Hence substituting for $\dot{p}(t_i^+)$ in (A.2.1), we get after simplifications:

$$\frac{dV^s}{dZ^i} \ = \ -\frac{1}{\zeta}u'(x_+^i) \ < \ 0 \ ,$$

and:

$$\frac{d^2V^s}{d\left(Z^i\right)^2} \ = \ -\frac{1}{\zeta}u'(x_+^i)\frac{dx_+^i}{dZ^i} \ = \ -\frac{1}{\zeta}\left.\frac{\dot{p}}{\dot{Z}}\right|_{t=t_i^+} \ < \ 0 \ .$$

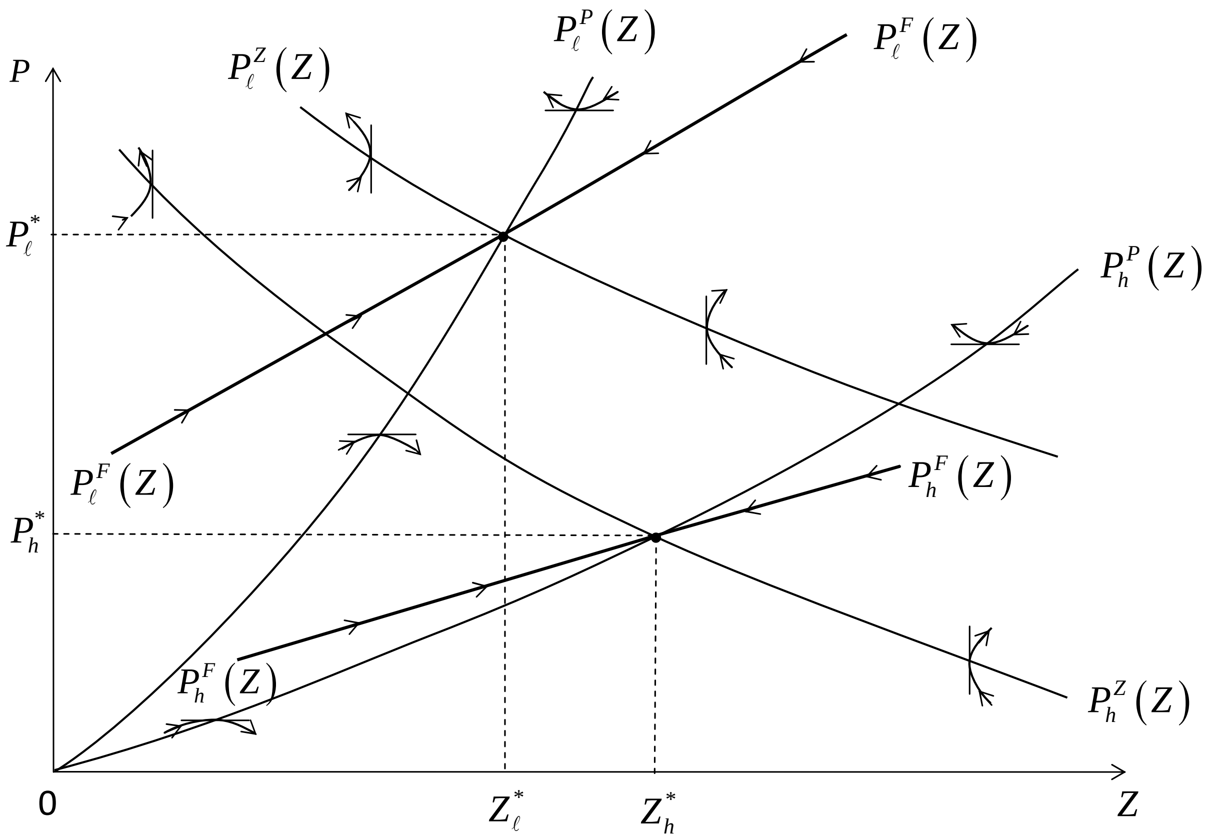


Figure 1 : Optimal Paths under Speedy and Slow Selfregenerating Regimes

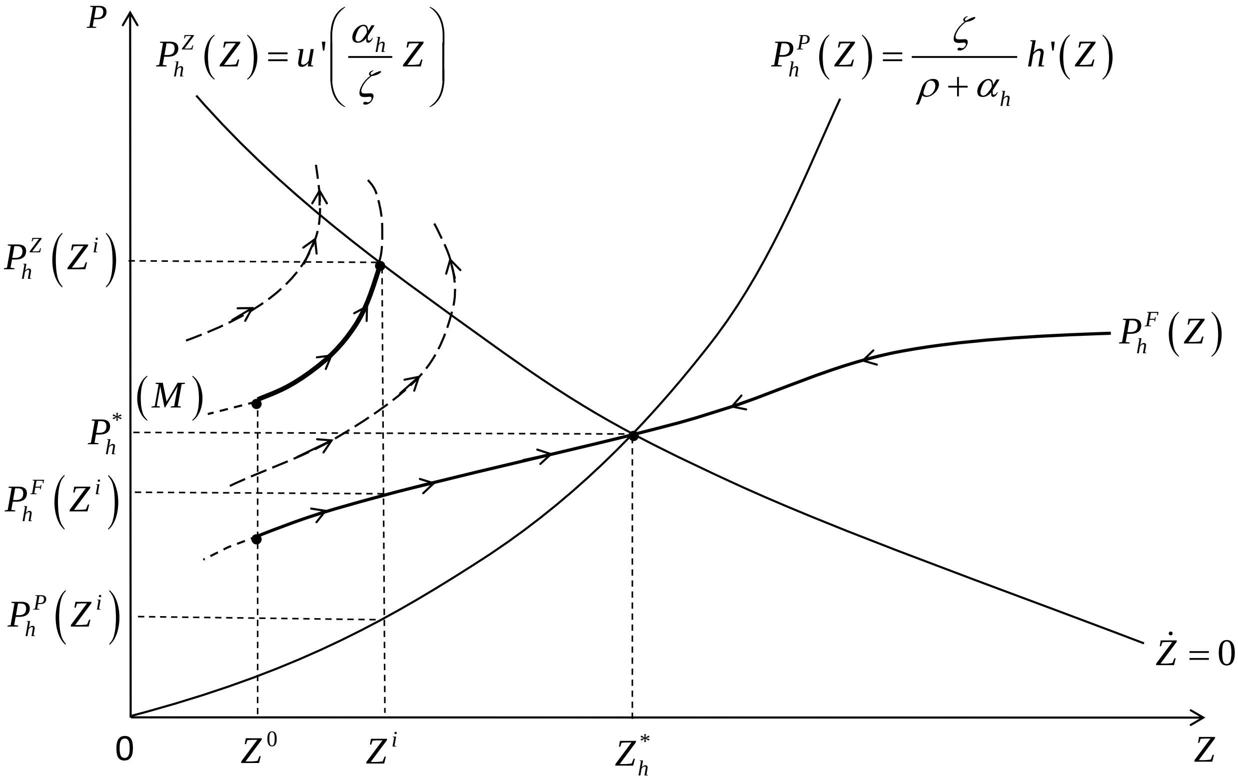


Figure 2 : Best Maintaining Path: Path $(M)$

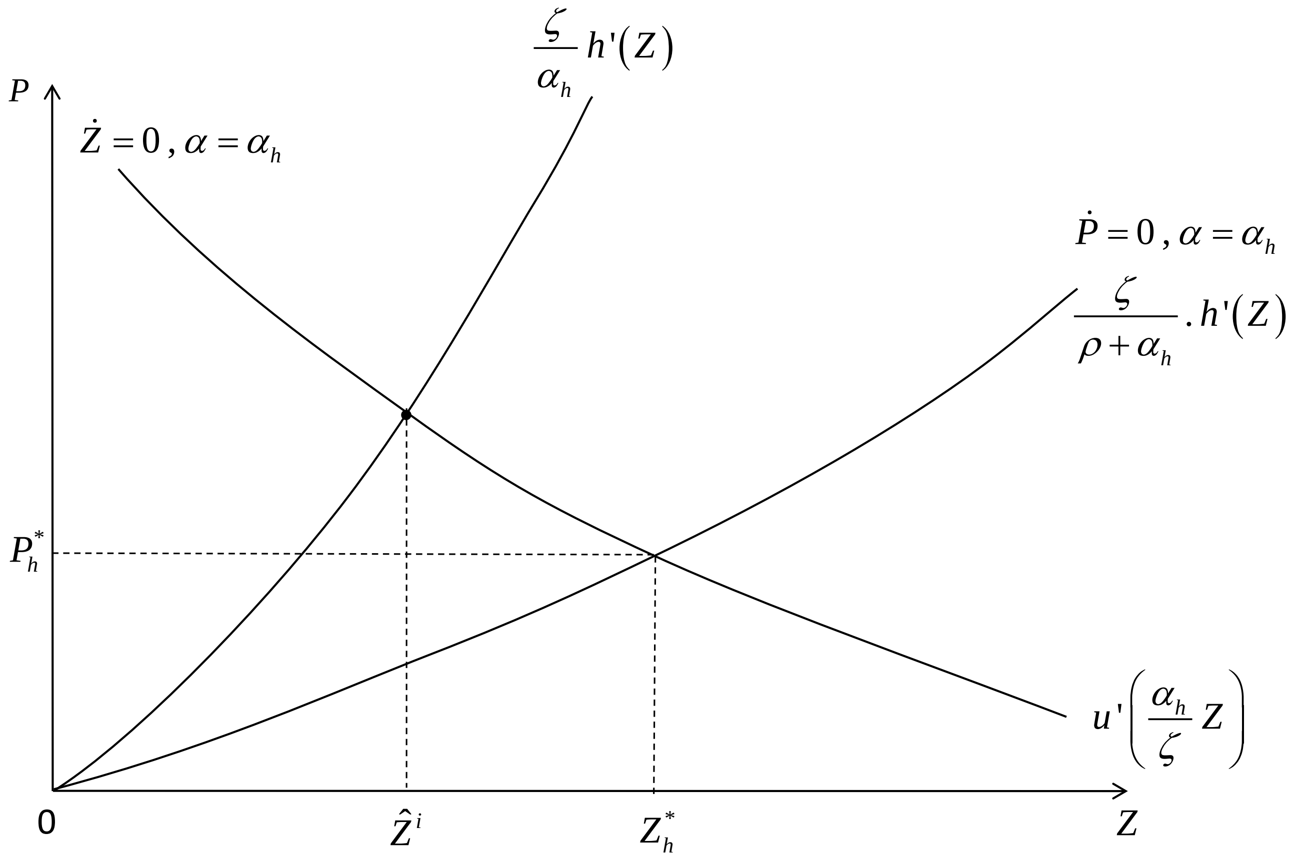


Figure 3 : Determination of $\hat{Z}^i$

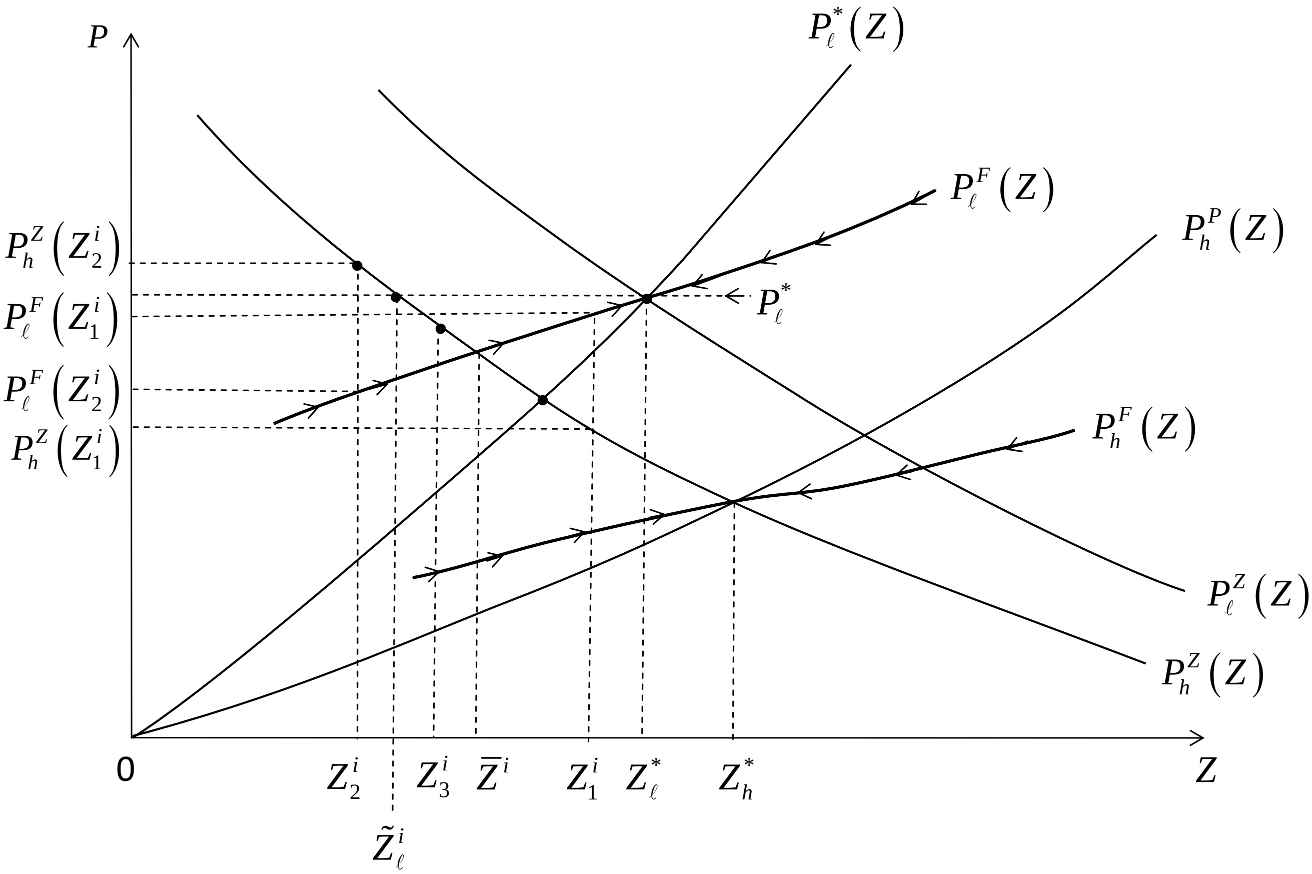


Figure 4 : Determination of the Range $\left(0, \bar{Z}^i\right)$ of Non Trivial Choices between Best Maintaining and Best Switching Policies

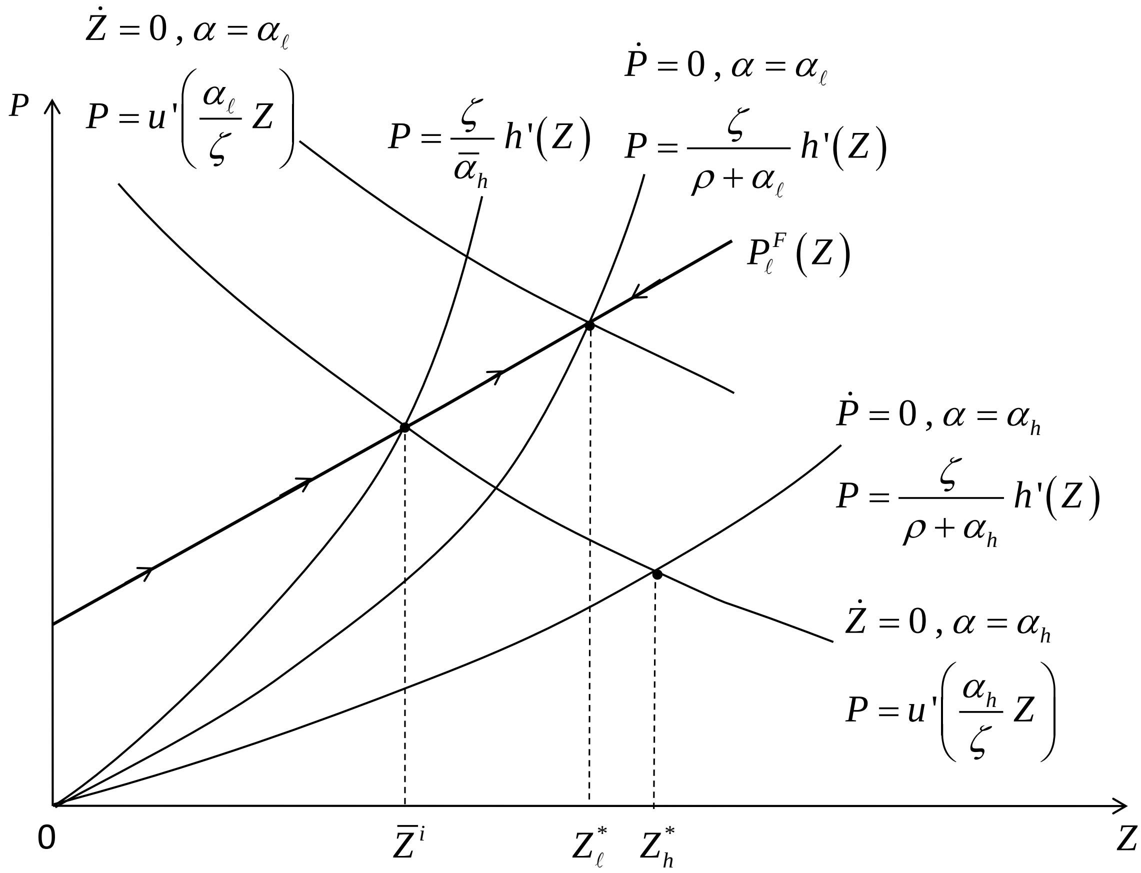


Figure 5 : Determination of $\bar{\alpha}_h$

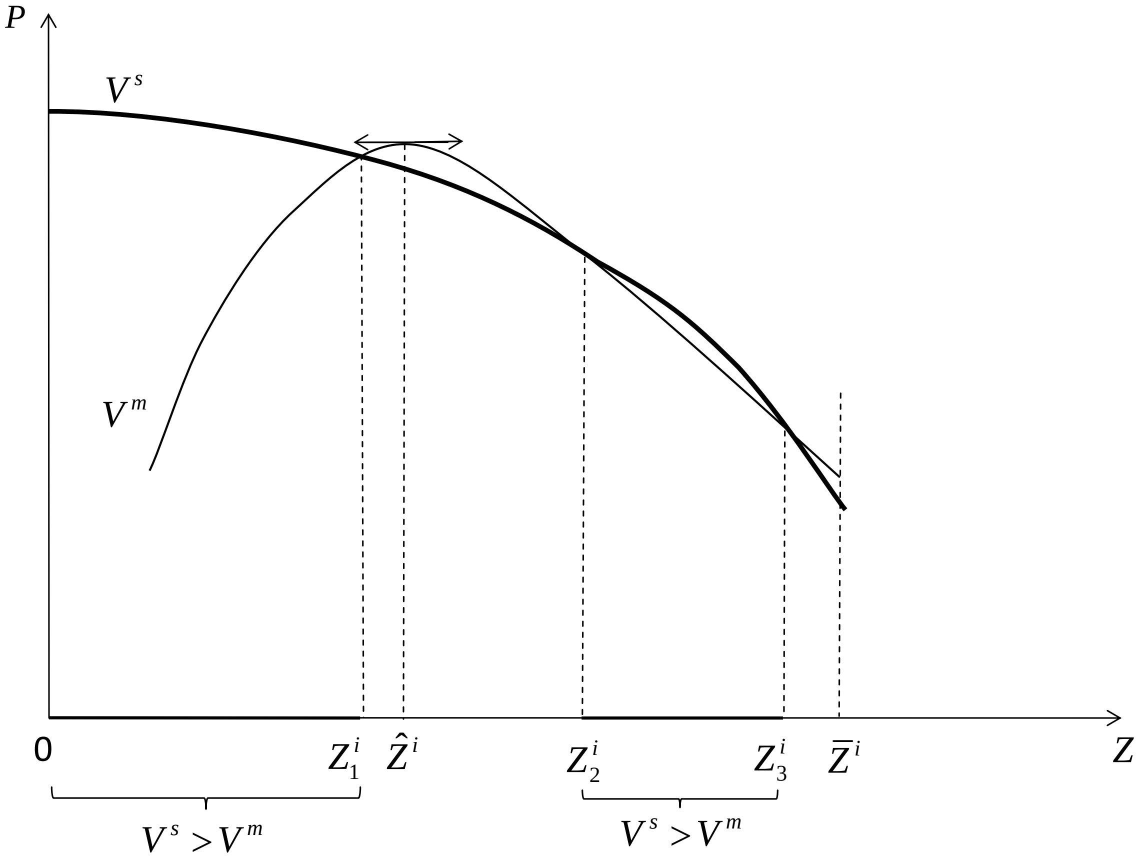


Figure 6 : Example of Disconnected Sets of $Z^i : V^s\left(Z^i\right) > V^m\left(Z^i\right)$ when $\hat{Z}^i < \overline{Z}^i$

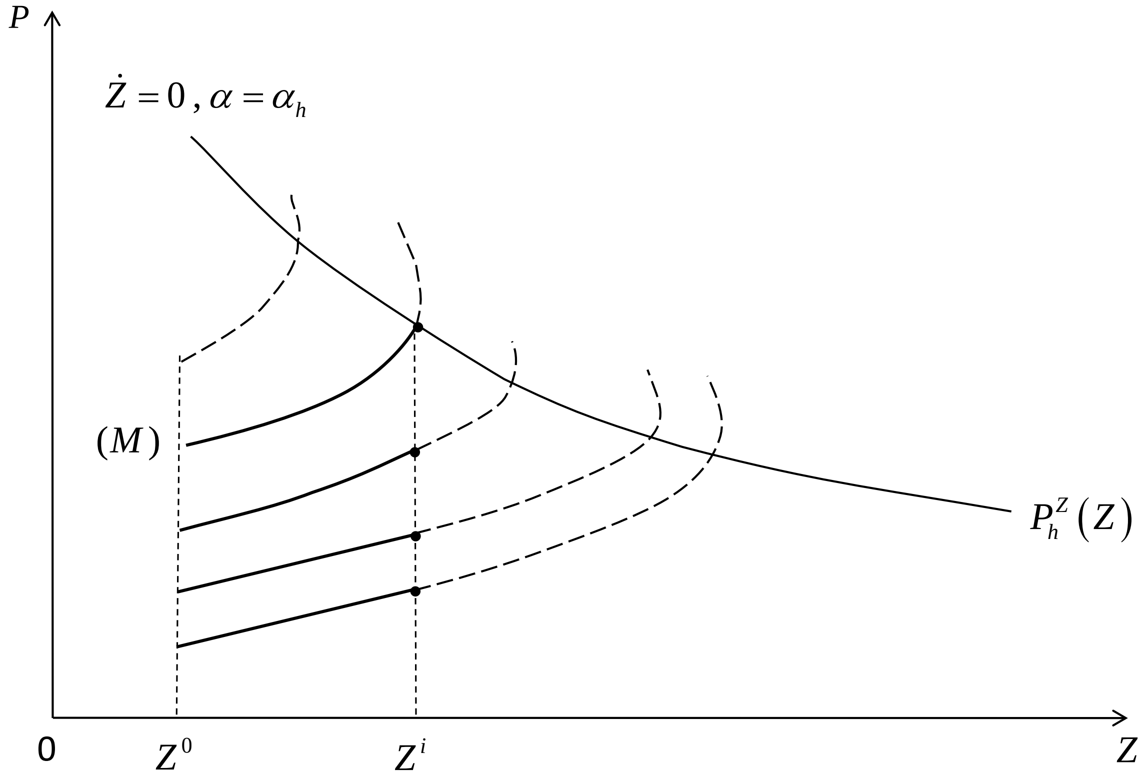


Figure 7 : Paths from $Z^0$ to $Z^i$ Satisfying the F.O.C.'s.
The higher is the path the more delayed is $t_i$.

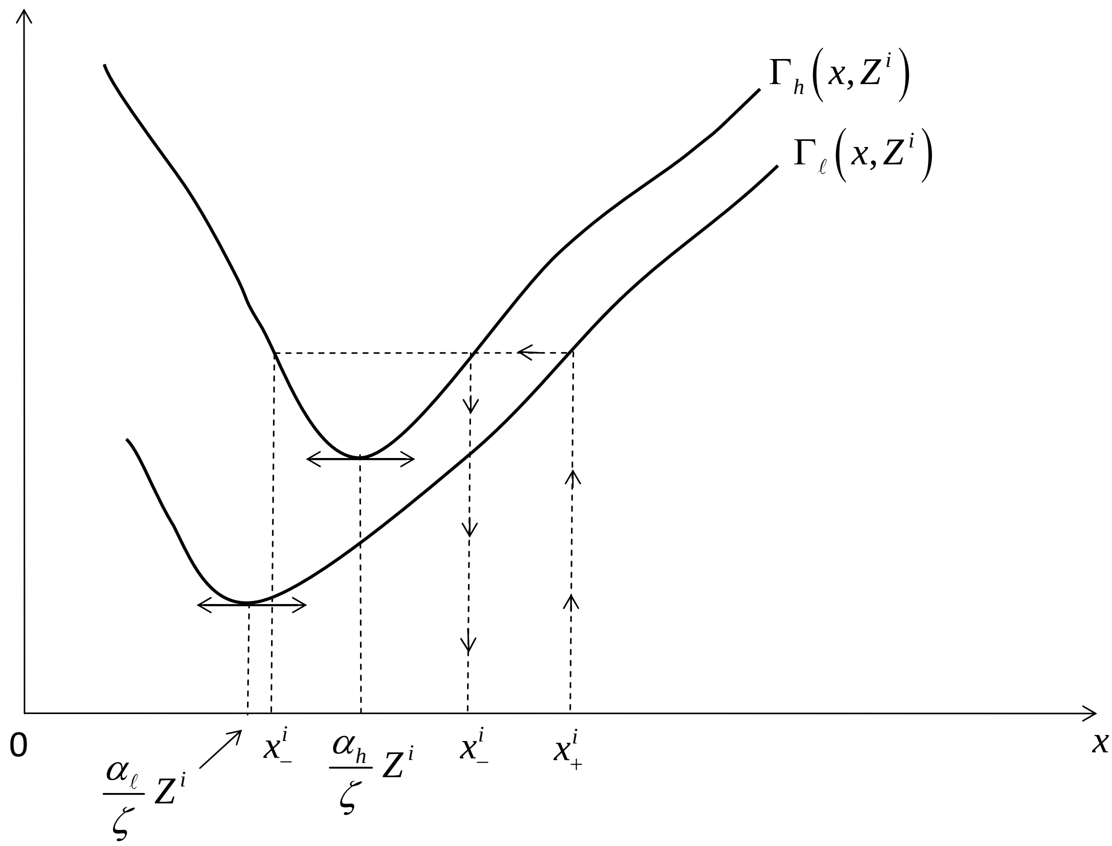


Figure 8 : Determination of $x_-^i$ Given $x_+^i$ . Case $Z^i < Z_R^i$

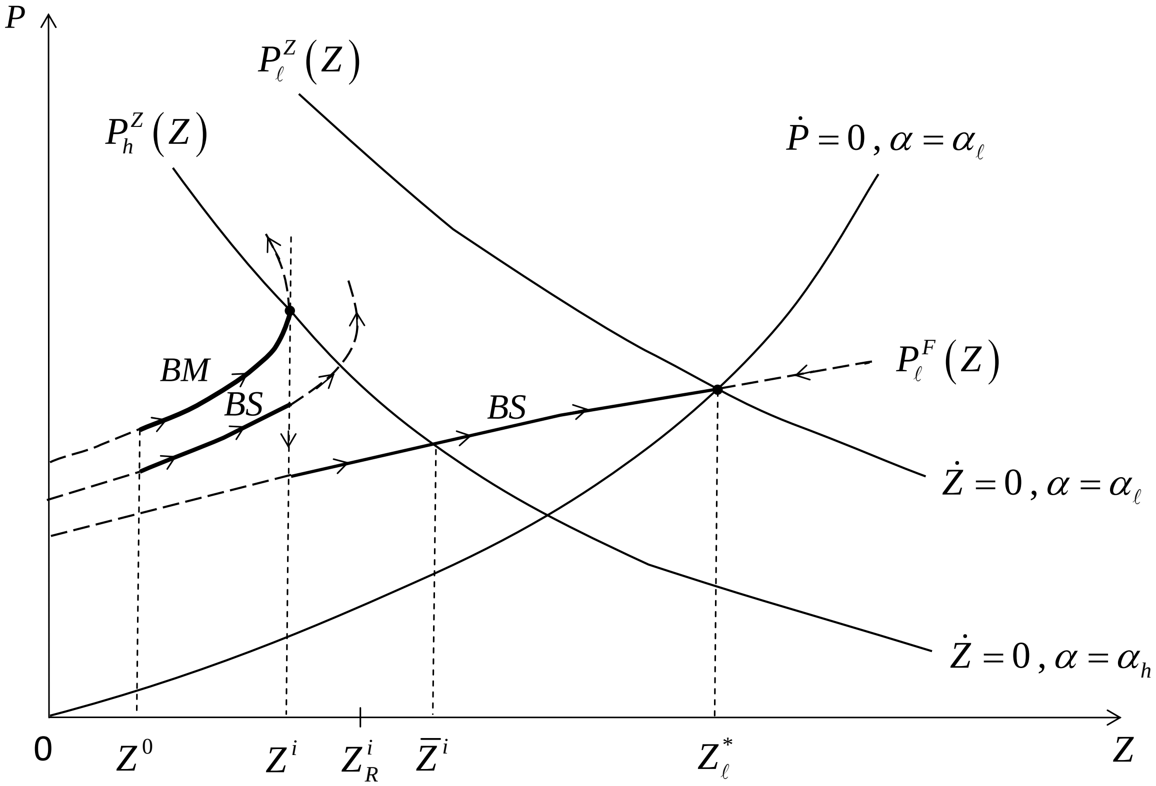


Figure 9 : Best Maintaining and Switching Paths. Case : $0 < Z^i < Z_R^i$
B. M: Best Maintaining Path ; B.S: Best Switching Path.

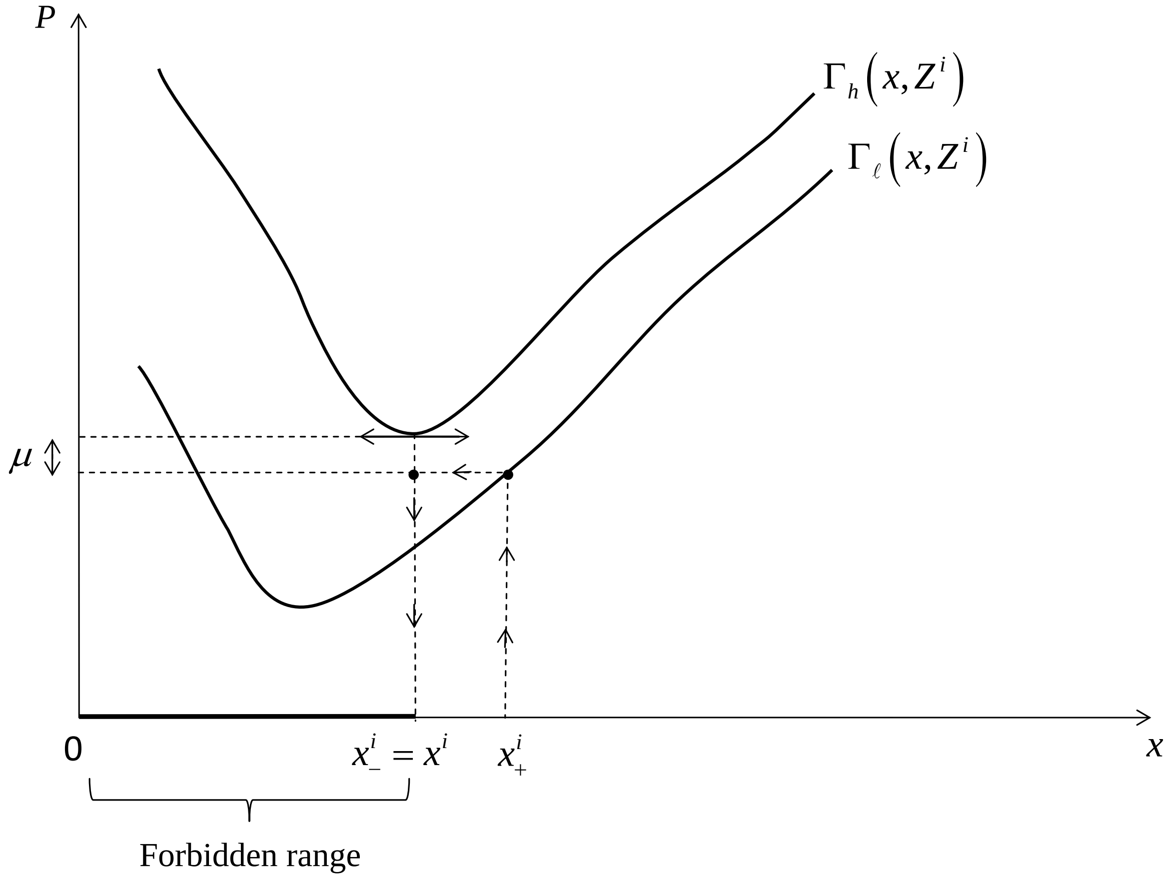


Figure 10 : Determination of $x_-^i$ Given $x_+^i$ . Case $Z_R^i < Z^i < \overline{Z}^i$

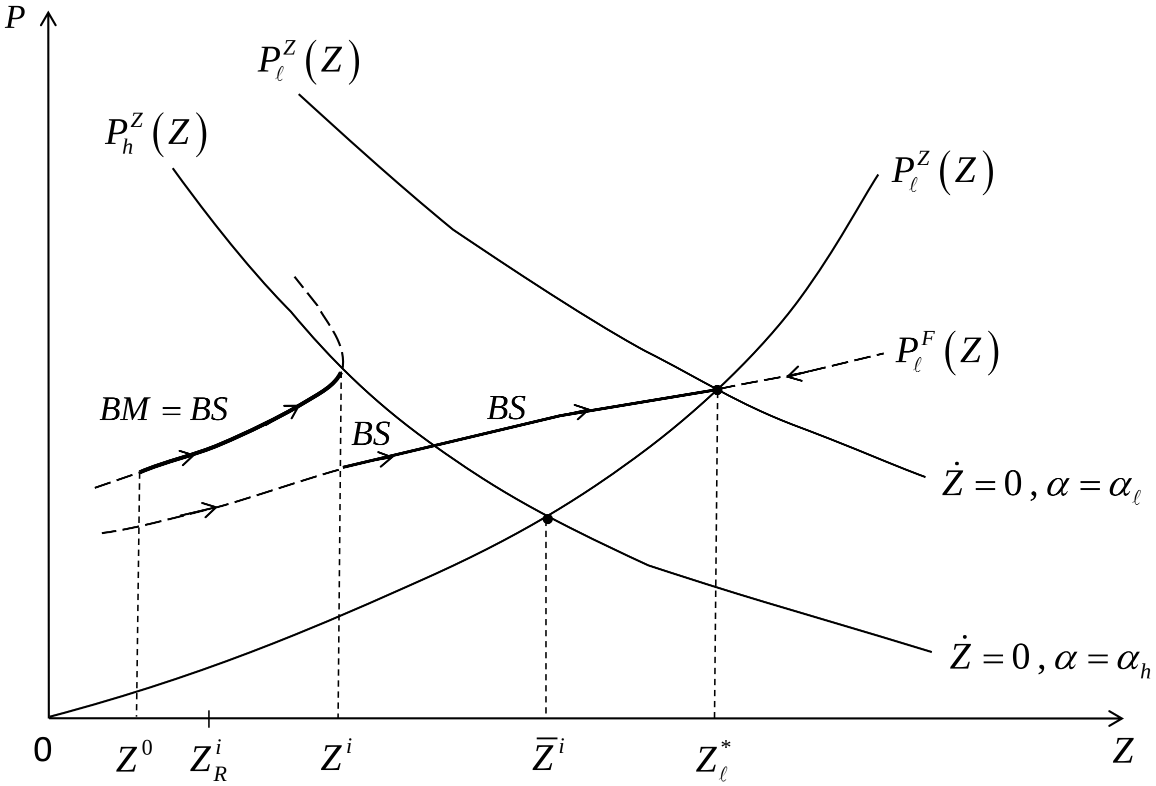


Figure 11 : Best Maintaining and Switching Paths. Case : $Z_R^i < Z^i < \bar{Z}^i$
B. M: Best Maintaining Path ; B.S: Best Switching Path.

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