*Title Page

Majority Voting and the Single-Crossing Property when Voters Belong to Separate Groups

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Abstract

We provide conditions under which a Condorcet winner exists when voters are exogenously distributed in groups, with preferences satisfying the single-crossing property separately inside each group. We also show that the majority voting social preference is acyclic.

Keywords: Spence-Mirrlees condition, unidimensional policy space, median voter, anchors

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1 Motivation

The single-crossing property (SCP) of preferences is a well known sufficient condition for the existence of a majority voting equilibrium on unidimensional choice domains: see for instance Gans and Smart (1996). SCP requires that, if a voter's type prefers a larger option to a smaller one, then so do all voters with a larger type. Unlike single-peakedness, SCP guarantees that the majority voting social preference is transitive, and corresponds to the median voter's.

The popularity of this concept is due to the large set of environments satisfying Gans and Smart (1996)'s premises, where voters care about two dimensions linked by some budget constraint. Take for instance the canonical example inspired by Roberts (1977). A continuum of agents have the same quasi-linear utility function $c-V(l)$ where c is consumption, l labor supply, with the disutility from supplying labor, $V(.)$, increasing and convex. Agents differ in productivity: the maximum productivity is denoted by $\bar{\theta}$ and an agent with type $\theta \in [0, \bar{\theta}]$ choosing to work l enjoys the consumption level $c = (1-t)(\bar{\theta}-\theta)l+T$, where $t \in [0, 1]$ is a proportional tax on income while T is a lump-sum transfer received from the government. Agents vote over t and then choose how much labor l to supply. With quasi-linear preferences, individual labor supply only depends on the type θ and the tax rate t, and is denoted by $l^*(t, \theta)$. Plugging it in the utility function, we obtain the indirect utility of an individual of type θ, $V(t, T) = (1-t)(\bar{\theta}-\theta)l^*(t, \theta) + T - V(l^*(t, \theta))$. The marginal rate of substitution between t and T is given by

$$-\frac{\partial V(t,T)/\partial t}{\partial V(t,T)/\partial T} = l^*(t,\theta)(\bar{\theta}-\theta) \qquad (1)$$

and is monotone (decreasing) in θ if $l^*(t, \theta)$ is also monotone (decreasing) in θ, a condition called "hierarchical adherence" by Roberts (1977). Using the government budget constraint to establish that T is a function of t, we obtain from Gans and Smart (1996) the equivalence between the monotonicity of the marginal rate of substitution and the SCP over $t \in [0, 1]$.

The objective of this note is to study the setting where voters are exogenously distributed in groups ($i = 1, ..., N$), with the SCP satisfied inside each group but not necessarily across groups (when voters from all groups are pooled together). For instance, in the previous example, agents may receive a different lump-sum transfer according to the group they belong to, so that $T = \gamma_i G(t)$ where $\gamma_i > 0$ for all i and where $G(t)$ is the total tax proceeds raised by the government. In other words, an agent's type is now bidimensional and consists of her productivity index θ and the group i she belongs to. Preferences clearly satisfy the SCP property inside any group i, but when all groups are pooled together it is unclear how to aggregate the bidimensional type (θ, i) to verify whether the SCP holds or not.[1]

We now propose a more generic model together with assumptions that are sufficient to guarantee the existence of a majority voting equilibrium.

2 The generic model

A continuum of voters are distributed among N groups, with μ_i denoting the proportion of voters who belong to group $i = 1, ..., N$. Voters are identified by the group i they belong to and by their type, denoted by θ. The distribution of types in group i is given by the distribution function F_i, with density f_i over $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$. The policy space is unidimensional and is the same across groups. That is, agents in all groups have to choose by majority voting one option (for the whole society) x in the set $X = [\underline{x}, \overline{x}]$. The utility function $u_i(x, \theta)$ represents the preferences of voters of type $\theta \in \Theta_i$ belonging to group i for any option $x \in X$.

We assume that these preferences satisfy the Single-Crossing Property inside each group i:

Assumption 1

$$\begin{aligned} \forall i = \; & 1, ..., N, \forall x', x \in X \textit{ with } x' > x, \forall \theta', \theta \in \Theta_i \textit{ with } \theta' > \theta, \\ & u_i(x', \theta) \geq u_i(x, \theta) \Rightarrow u_i(x', \theta') \geq u_i(x, \theta'). \end{aligned}$$

[1] See Remark 2 for a more formal statement.

Our objective is to prove the existence of a Condorcet winner (an option preferred by a majority of voters across groups to any other option in X) when all voters in all groups vote simultaneously.

The set of most-preferred outcomes in X of an individual of type θ belonging to group i is denoted by

$$M_i(\theta) = \arg\max_{x \in X} u_i(x, \theta).$$

We assume that

Assumption 2 *$M_i(\theta)$ is a function that is continuous and strictly increasing in θ, $\forall i = 1, ..., N$ and $\forall \theta \in \Theta_i$,*

and that

Assumption 3 *The image of Θ_i under the function M_i is $[\underline{x}, \overline{x}]$ in all groups $i = 1, ..., N$.*

The crucial part of Assumption 2 is continuity: we come back in the conclusion to the assumption that $M_i(\theta)$ is a function and that it is strictly increasing.[2] De Donder (2012) shows how the technical Assumption 3 can be weakened.

Remark 1 *We have already shown that section 1's example satisfies Assumption 1 when labor supply is monotone decreasing in θ. Assuming that $G(t)$ is a strictly concave function is sufficient to guarantee Assumption 2. To satisfy Assumption 3 with $[\underline{x}, \overline{x}] = [0, \hat{t}]$, denote by $\hat{t}$ the value of $t \leq 1$ that maximizes $G(t)$, so that $M_i(\overline{\theta}) = \hat{t}$ in all groups i, and assume that the marginal rate of substitution (1) of an agent with $\theta = 0$ at $t = 0$ is larger than $\gamma_i G'(0)$, $\forall i$, so that $M_i(0) = 0$ in all groups.*

[2] Theorem 4 of Milgrom and Shannon (1994) guarantees that M_i is weakly increasing in θ.

Remark 2 *Take section 1's example and assume that labor supply is monotone decreasing in* θ *for all* t*, that* G *is strictly concave in* t *and order groups by increasing value of* γ_i*. We then have that* $M_i(\theta)$ *is increasing in both* θ *and* i*. We can then choose* θ' *in group* i *and* θ *in group* j *with* $\theta' > \theta$*,* $i < j$ *and* $t' = M_i(\theta') < t = M_j(\theta)$*. Although Assumption 1 is satisfied in all groups, we have* $u_j(t, \theta) > u_j(t', \theta)$ *and* $u_i(t, \theta') < u_i(t', \theta')$ *so that SCP is violated when we pool voters from all groups and identify them only by their productivity parameter* θ*.*

Since $M_i(\theta)$ is continuous and strictly increasing in θ (Assumption 2) with the same image $[\underline{x}, \overline{x}]$ in all groups (Assumption 3), we know from the intermediate value theorem that, for any $x \in [\underline{x}, \overline{x}]$, there exists a unique type $\theta_i^*(x)$ in *all* groups who prefers x to any other feasible option.

We now prove that

Proposition 1 *Under Assumptions 1, 2 and 3,*
(a) there exists at least one option $x \in X$*, which we denote* x^{CW}*, such that*

$$\sum \mu_i F_i(\theta_i^*(x^{CW})) = 1/2, \tag{2}$$

(b) x^{CW} *is a Condorcet winner in the set* X*,*
(c) Although the majority voting social preference over X *need not correspond to that of any individual's type, it is acyclic.*

Proof: (a) Observe that $\theta_i^*(x)$ is continuous and strictly increasing in x . Then there exists at least one value $x^{CW} \in [\underline{x}, \overline{x}]$ that is such that

$$\sum \mu_i F_i(\theta_i^*(x^{CW})) = 1/2.$$

(b) We now prove that x^{CW} gathers at least one half of the votes when faced with any other option $y \in X$. Since $x^{CW} \in [\underline{x}, \overline{x}]$, we know that there exists a type θ_i^{CW} in all groups i who is such that $\theta_i^{CW} = \theta_i^*(x^{CW})$— i.e., that

$u_i(x^{CW}, \theta_i^{CW}) \geq u_i(y, \theta_i^{CW})$, $\forall y \in X$. Assume that $y < x^{CW}$. This in turns means, using Assumption 1, that, inside each group i, we have that

$$\forall \theta \geq \theta_i^{CW}, u_i(x^{CW}, \theta) \geq u_i(y, \theta).$$

This guarantees that at least a fraction $1 - F_i(\theta_i^{CW})$, in each group i, will support x^{CW} when faced against y. By definition of x^{CW} (see equation (2)), this support aggregates to one half over all groups, so that x^{CW} can not be defeated at the majority by y.

The case where $y > x^{CW}$ is proved likewise, using the contrapositive of Assumption 1.

(c) Take any 3 options $\{a, b, c\} \in X$ all differing from x^{CW}, with $a > b > c$. Agents $\theta_i^*(x^{CW})$ need not share the same preferences when comparing these options, so it is always possible to construct examples where any type $\theta_i^*(x^{CW})$ is in the minority when comparing two of them. Hence, the social preference need not correspond to the preferences of any type $\theta_i^*(x^{CW})$ (or of any other type). We now show that this does not imply that majority voting cycles exist. Given Assumptions 1 to 3, there exist, in all groups i, types θ_i^{ab}, θ_i^{ac} and θ_i^{bc} that are indifferent, respectively, between options a and b, a and c, and b and c. Also, Assumption 2 implies that $\theta_i^{ab} > \theta_i^{ac} > \theta_i^{bc}$. All individuals with $\theta < \theta_i^{bc}$ then prefer c to b and b to a (and thus c to a), all individuals with $\theta_i^{bc} < \theta < \theta_i^{ac}$ prefer b to c to a, all agents with $\theta_i^{ac} < \theta < \theta_i^{ab}$ prefer b to a to c, while all agents with $\theta > \theta^{ab}$ prefer a to b to c. It is easy to see that, with such preferences, there cannot be a majority voting cycle encompassing a, b and c, even if the distribution of types and the identity of the threshold types differ from one group to another. Note also that this precludes cycles between any four options, since such cycles would imply the existence of at least one cycle between three options. Applying repeatedly this observation to cycles of larger length, we obtain that there is no majority voting cycle of any length.

□

Given the continuity and strict monotonicity of $M_i(\theta)$ postulated in Assumption 2, for all $x \in X$, there exists one type of voters in *every* group that most-prefers this option x to any other feasible option. This type plays a role of "anchor" in the group, which allows to apply the separation argument at the heart of the usual, one-group, median voter theorem with SCP. Observe that these anchors need not correspond to the same type θ in all groups, and that they need not be the median θ voters inside each group separately. Also, there is no need for these anchors to be actually represented in all groups: we need not impose that $f_i(\theta_i^*(x^{CW})) > 0$ for anchor θ_i in any group i. It is easy to see that the Condorcet winner is unique if, for all $x \in X$, $f_i(\theta_i^*(x)) > 0$ for at least one group i.

These anchors need not be decisive for *all* pairwise majority comparisons — i.e., when comparing any two options x and y other than x^{CW}. It is indeed easy to construct examples where any anchor is in the minority for some vote pairing options different from x^{CW}. The separation argument then shows that, in certain groups, individuals with a type lower than the anchor's prefer x to y, while in other groups types larger than the anchor's prefer x to y. Since anchors need not be the median θ voters of their own group, there is no way at this level of generality to assess whether x is majority preferred to y. This line of reasoning suggests that majority cycles (not including x^{CW} of course) may exist, since no type need be decisive in all pairwise majority comparisons. We show that it is *not* the case: Assumptions 1 to 3 impose sufficient structure on individual preferences to prevent the existence of a majority voting cycle of any length. In other words, with multiple groups SCP guarantees that the majority voting social preference is acyclic, although it need not correspond to the preference of any specific individual, including the anchors.

3 Conclusion

We show in De Donder (2012) that our results extend to the case where $M_i(\theta)$ is a correspondence and increases weakly with θ for all i. Several anchors may most-prefer the same option in any group, and the Condorcet winner may not be unique even if, for all $x \in X$, $f_i(\theta_i^*(x)) > 0$ for at least one group i. We also show there that the continuity assumption of $M_i(\theta)$ is crucial to our results since it ensures that anchors exist for all values of x.

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