Equilibrium in the growth model with an endogenous labor-leisure choice*

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Abstract

We prove the existence of competitive equilibrium in an optimal growth model with elastic labor supply under general conditions. The proof of existence of equilibrium relies on recent results on existence of Lagrange multipliers. It requires less stringent assumptions than the literature which are often violated in applied macroeconomic problems. In particular, we do not require either the Inada conditions, strict concavity or differentiability. We give examples to illustrate the violation of the conditions used in earlier existence results and where there is a corner solution, but a competitive equilibrium can be shown to exist following the approach in this paper.

Keywords: Optimal growth, Competitive equilibrium, Lagrange multipliers, Elastic labor supply, Inada conditions.

JEL Classification: *C61, D51, E13, O41*

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1 Introduction

The optimal growth model is one of the main frameworks in macroeconomics. While variations of the model with inelastic labor supply are used widely in growth theory, the version with elastic labor supply is used in business cycle models, both for exogenous and endogenous fluctuations. Despite the central place of the model with elastic labor supply in dynamic general equilibrium models of macroeconomics, existence of competitive equilibrium in general settings has proved to be a challenge. Results of existence of equilibrium ([5], [7], [10], [13]) use strong conditions for existence. This paper establishes existence of equilibrium under very weak conditions: neither Inada conditions, nor strict concavity, nor constant returns to scale (or more generally, homogeneity), nor restrictions on cross-partials of the utility functions.

The approach taken in this paper is a direct method based on existence of Lagrange multipliers to the optimal problem and their representation as a summable sequence. The problem with inelastic labor supply was considered by [14]. This approach uses a separation argument where the multipliers are represented in the dual space $(\ell^{\infty})'$ of the space of bounded sequences ℓ^{∞}. While one would like the multipliers and prices to lie in ℓ^1, it is not the dual space ([8]). Previous work following [16], the representation theorems followed separation arguments applied to arbitrary vector spaces (see [3], [2], [6]). The Le Van and Saglam ([14]) approach also uses a separation argument but imposes restrictions on the asymptotic behavior of the objective functional and constraint functions which are easily shown to be satisfied in standard models. This is related to [9]. There is a difficulty in going from the inelastic labor supply to the elastic labor supply model: While one can show that the optimal capital stock is strictly positive, without assuming Inada conditions, one cannot be sure that the optimal labor supply sequence is strictly positive. Thus, the paper by [13] which took the approach of decentralizing the optimal solution via prices as marginal utilities had to make additional strong conditions on the utility function to ensure that the labor supply sequence remains strictly positive. As we show, following [14], that the Lagrange multipliers to the social planners problem are a summable sequence, we can directly use these to decentralize the optimal solution and not have to make strong assumptions to ensure interiority of the optimal plan. Thus, the Inada conditions do not have to be assumed. As the separation theorem does not require strict concavity or differentiability, these strong assumptions on utility functions can be dropped. This is especially important as an important specification of preferences in applied macroeconomics models is with linear utility of leisure where strict concavity is violated. This specification also results in the planners problem in models with indivisible labor ([11],[17]). The relaxation of Inada conditions is especially important as they may also be violated in utility and production functions of the CES class which are also widely used in the applied literature. Furthermore, there is no need to make any assumption on cross-partial derivatives of the utility function as in [1], [5], [7], [10] and [15].[1] Thus, whether labor supply is backward bending or not, and whether consumption is inferior or not plays no role in existence of equilibrium.

The organization of the paper is as follows. Section 2 provides the sufficient conditions on the objective function and the constraint functions so that Lagrangean multipliers can be represented by an ℓ^1_+ sequence of multipliers in optimal growth model with leisure in the utility function. In section 3, we prove the existence of competitive equilibrium in a model with a representative agent by using these multipliers as sequences of prices and wages. Section 4 provides examples where many of the conditions used to assert existence of an equilibrium are violated. In particular, there is a corner solution but show that a competitive equilibrium exists following the approach of the current paper. Section 5 concludes.

2 Lagrange multipliers in the optimal growth model

Consider an economy where the representative consumer has preferences defined over processes of consumption and leisure described by the utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t).$$

In each period, the consumer faces two resource constraints given by

$$\begin{aligned} c_t + k_{t+1} &\leq F(k_t, L_t) + (1-\delta)k_t, \\ l_t + L_t &= 1, \ \forall t \end{aligned}$$

where F is the production function, $\delta \in (0,1)$ is the depreciation rate of capital stock and L_t is labor. These constraints restrict allocations of commodities and time for the leisure.

Formally, the problem of the representative consumer is stated as follows:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\begin{aligned} \text{s.t. } c_t + k_{t+1} &\leq F(k_t, 1-l_t) + (1-\delta)k_t, \ \forall t \geq 0 \\ c_t &\geq 0, \ k_t \geq 0, \ l_t \geq 0, \ 1 - l_t \geq 0, \ \ \forall t \geq 0 \\ k_0 &\geq 0 \text{ is given.} \end{aligned}$$

[1] These papers essentially show the isomorphism of the dynamic problem with endogenous leisure to one without endogenous leisure, and the assumptions are used to show monotonicity of the optimal capital path which combined with the static labor-leisure choice gives existence in the original problem.

We make a set of assumptions on preferences and the production technology. The assumptions on the period utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ are:

Assumption U1: *u is continuous, concave, increasing on $\mathbb{R}^2_+$ and strictly increasing on $\mathbb{R}^2_{++}$.*

Assumption U2: $u(0,0) = 0$.

The assumptions on the production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ are as follows:

Assumption F1: *F is continuous, concave, increasing on $\mathbb{R}^2_+$ and strictly increasing on $\mathbb{R}^2_{++}$.*

Assumption F2: $F(0,0) = 0, \lim_{k\to 0} F_k(k,1) > \delta, \lim_{k\to+\infty} F_k(k,1) < \delta$.

The assumptions U1, U2, F1 are standard. Note we do not assume strict concavity, differentiability or Inada conditions for the utility and production functions. Assumption F2 is a weak assumption to ensure that there is a maximum sustainable capital stock, and thus the sequence of capital is bounded.

We have relaxed some important assumptions in the literature. [3] assumes that the production set is a convex cone (Theorem 3). [4] assumes the strictly positiveness of derivatives of utility functions on $\mathbb{R}^L_+$ (strictly monotonicity assumption). In our model, the utility functions may not be differentiable in $\mathbb{R}^2_+$.[2] [15] assumed the cross-partial derivative u^i_{cl} has constant sign, $u^i_c(x,x)$ and $u^i_l(x,x)$ are non-increasing in x, production function F is homogenous of degree $\alpha \le 1$ and $F_{kL} \ge 0$ (Assumptions U4, F4, U5, F5).[3]

We say that a sequence $\{c_t, k_t, l_t\}_{t=0,1,\dots,\infty}$ is feasible from k_0 if it satisfies the constraints

$$\begin{aligned} c_t + k_{t+1} &\le F(k_t, 1-l_t) + (1-\delta)k_t, \ \ \forall t \ge 0, \\ c_t &\ge 0, \ k_t \ge 0, \ l_t \ge, \ 1 - l_t \ge 0, \ \ \forall t \ge 0, \\ k_0 &> 0 \text{ is given.} \end{aligned}$$

It is easy to check that, for any initial condition $k_0 > 0$, a sequence $\mathbf{k} = \{k_t\}_{t=0}^{\infty}$ is feasible iff $0 \le k_{t+1} \le F(k_t, 1) + (1-\delta)k_t$ for all t. The class of feasible capital paths is denoted by $\Pi(k_0)$. A pair of consumption-leisure sequences $\{\mathbf{c}, \mathbf{l}\} = \{c_t, l_t\}_{t=0}^{\infty}$ is feasible from $k_0 > 0$ if there exists a sequence $\mathbf{k} \in \Pi(k_0)$ that satisfies $0 \le c_t + k_{t+1} \le F(k_t, 1-l_t) + (1-\delta)k_t$ and $0 \le l_t \le 1$ for all t.

Define $f(k_t, L_t) = F(k_t, L_t) + (1-\delta)k_t$. Assumption **F2** implies that

$$\begin{aligned} f_k(+\infty, 1) &= F_k(+\infty, 1) + (1-\delta) < 1 \\ f_k(0, 1) &= F_k(0, 1) + (1-\delta) > 1. \end{aligned}$$

From above, it follows that there exists $\overline{k} > 0$ such that: (i) $f(\overline{k}, 1) = \overline{k}$, (ii) $k > \overline{k}$ implies $f(k,1) < k$, (iii) $k < \overline{k}$ implies $f(k,1) > k$. Therefore for any $\mathbf{k} \in \Pi(k_0)$, we have $0 \le k_t \le \max(k_0, \overline{k})$. Thus, $\mathbf{k} \in \ell^{\infty}_+$ which in turn

[2] Let $F(k,L) = k^{\alpha}L^{1-\alpha}, \ \alpha \in (0,1)$. This function is not differentiable even in the extended real numbers at $(0,L)$ or $(k,0)$ for $L \ge 0, K \ge 0$. The assumptions in [4] that $u_c >> 0$, $u_l >> 0$, and D^2u is negative definite on R^2_+ are obviously violated.

[3] See section 5 for a further discussion of assumptions in the literature.

implies $\mathbf{c} \in \ell_+^\infty$, if $\{\mathbf{c}, \mathbf{k}\}$ is feasible from k_0. Denote $\mathbf{x} = \{\mathbf{c}, \mathbf{k}, \mathbf{l}\}$ and $\mathcal{F}(\mathbf{x}) = -\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$, $\Phi_t^1(\mathbf{x}) = c_t + k_{t+1} - f(k_t, 1 - l_t)$, $\Phi_t^2(\mathbf{x}) = -c_t$, $\Phi_t^3(\mathbf{x}) = -k_t$, $\Phi_t^4(\mathbf{x}) = -l_t, \Phi_t^5(\mathbf{x}) = l_t - 1, \forall t, \Phi_t = (\Phi_t^1, \Phi_t^2, \Phi_{t+1}^3, \Phi_t^4, \Phi_t^5)$, $\forall t$. The planning problem can be written as:

$$\min \mathcal{F}(\mathbf{x}) \quad \text{s.t. } \Phi(\mathbf{x}) \leq \mathbf{0}, \mathbf{x} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty \qquad (P)$$

$$\begin{aligned} \text{where } \mathcal{F} \quad &: \quad \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty \to \mathbb{R} \cup \{+\infty\} \\ \Phi \quad &= \quad (\Phi_t)_{t=0,\ldots,\infty} : \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty \to \mathbb{R} \cup \{+\infty\} \end{aligned}$$

$$\begin{aligned} \text{Let } C \quad &= \quad dom(\mathcal{F}) = \{\mathbf{x} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty | \mathcal{F}(\mathbf{x}) < +\infty\} \\ \Gamma \quad &= \quad dom(\Phi) = \{\mathbf{x} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty | \Phi_t(\mathbf{x}) < +\infty, \ \forall t\}. \end{aligned}$$

Proposition 1 *Let* $\mathbf{x}, \mathbf{y} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty, T \in \mathbb{N}$. *Define*

$$x_t^T(\mathbf{x}, \mathbf{y}) = \begin{cases} x_t & if \quad t \leq T \\ y_t & if \quad t > T \end{cases}.$$

Suppose that the two following assumptions are satisfied:

T1*: If* $\mathbf{x} \in C$, $\mathbf{y} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$ *satisfy* $\forall T \geq T_0$, $\mathbf{x}^T(\mathbf{x}, \mathbf{y}) \in C$, *then* $\mathcal{F}(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) \to \mathcal{F}(\mathbf{x})$ *when* $T \to \infty$.

T2*: If* $\mathbf{x} \in \Gamma$, $\mathbf{y} \in \Gamma$ *and* $\mathbf{x}^T(\mathbf{x}, \mathbf{y}) \in \Gamma$, $\forall T \geq T_0$, *then*

a) $\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) \to \Phi_t(\mathbf{x})$ *as* $T \to \infty$

b) $\exists M$ *s.t.* $\forall T \geq T_0$, $\|\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y}))\| \leq M$

c) $\forall N \geq T_0$, $\lim_{t \to \infty} [\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) - \Phi_t(\mathbf{y})] = 0$.

Let $\mathbf{x}^*$ *be a solution to* (P) *and* $\mathbf{x}^0 \in C$ *satisfies the Slater condition:*

$$\sup_t \Phi_t(\mathbf{x}^0) < 0.$$

Suppose $\mathbf{x}^T(\mathbf{x}^*, \mathbf{x}^0) \in C \cap \Gamma$. *Then, there exists* $\mathbf{\Lambda} \in \ell_+^1 \backslash \{0\}$ *such that*

$$\mathcal{F}(\mathbf{x}) + \mathbf{\Lambda}\Phi(\mathbf{x}) \geq \mathcal{F}(\mathbf{x}^*) + \Lambda\Phi(\mathbf{x}^*), \ \forall \mathbf{x} \in (C \cap \Gamma)$$

and $\Lambda\Phi(\mathbf{x}^*) = 0$.

Proof: It is easy to see that $\ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$ is isomorphic with ℓ_+^∞, since, for example, there exists an isomorphism

$$\Pi : \ell_+^\infty \to \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty,$$

$$\Pi(\mathbf{x}) = ((x_0, x_3, x_6, \ldots)(x_1, x_4, x_7, \ldots), (x_2, x_5, x_8, \ldots))$$

and

$$\Pi^{-1}(\mathbf{u}, \mathbf{v}, \mathbf{s}) = (u_0, v_0, s_0, u_1, v_1, s_1, u_2, v_2, s_2, \ldots).$$

Thus, there exists an isomorphism $\Pi^{'}: (\ell^{\infty}_{+} \times \ell^{\infty}_{+} \times \ell^{\infty}_{+})^{'} \to (\ell^{\infty}_{+})^{'}$. It follows from Theorem 1 in [14] that there exists $\overline{\mathbf{\Lambda}} \in (\ell^{\infty}_{+} \times \ell^{\infty}_{+} \times \ell^{\infty}_{+})^{'}$. Let $\Lambda = \Pi^{'}(\overline{\mathbf{\Lambda}}) \in (\ell^{\infty}_{+})^{'}$. Then, the results are derived by the analogous arguments where a standard separation theorem used[4] as in the Theorem 2 in [14]. ■

Note that ***T1*** holds when F is continuous in the product topology. ***T2c*** is satisfied if there is asymptotically insensitivity, i.e. if x is changed only on a finitely many values the constraint value for large t does not change that much ([9]). ***T2c*** is the asymptotically non-anticipatory assumption and requires Φ to be nearly weak-* continuous ([9]). ***T2b*** holds when when $\Gamma = \mathrm{dom}(\Phi) = \ell^{\infty}$ and Φ is continuous (see [9], [14]).

Proposition 2 *If* $\mathbf{x}^* = (\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*)$ *is a solution to the following problem*[5]*:*

$$\min -\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \tag{Q}$$

$$s.t.\ c_t + k_{t+1} - f(k_t, 1 - l_t) \le 0,$$
$$-c_t \le 0,\ -k_t \le 0,\ 0 \le l_t \le 1,$$

then there exists $\lambda = (\lambda^1, \lambda^2, \lambda^3, \lambda^4, \lambda^5) \in \ell^1_+ \times \ell^1_+ \times \ell^1_+ \times \ell^1_+ \times \ell^1_+, \lambda \neq \mathbf{0}$ *such*

[4] As the Remark 6.1.1 in [6], assumption $f_k(0,1) > 1$ is equivalent to the Adequacy Assumption in [3] and this assumption is crucial to have equilibrium prices in ℓ^1_+ since it implies that the production set has an interior point. Subsequently, it allows using a separation theorem in the infinite dimensional space to obtain Lagrange multipliers.

[5] A solution exists following a standard argument which is sketched for completeness. Observe that the feasible set is in a fixed ball of ℓ^{∞} which is weak*-(ℓ^{∞}, ℓ^1) compact. We show that the function $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$ is continuous in this topology on the feasible set. Since the weak* topology is metrizable on any ball, we can take a feasible sequence $(c_t(n), l_t(n))_n$ converging to some (c_t, l_t) in the feasible set. Since any feasible consumptions sequence is uniformly bounded by a number depending only on k_0, for any $\epsilon > 0$ there exists T_0 such that for any $T \ge T_0$, for any n, we have

$$\sum_{t \ge T} \beta^t u(c_t(n), l_t(n)) \le \epsilon,\ \sum_{t \ge T} \beta^t u(c_t, l_t) \le \epsilon$$

Hence,

$$\left| \sum_{t=0}^{+\infty} \beta^t [u(c_t(n), l_t(n)) - u(c_t, l_t)] \right| \le \sum_{t=0}^{T-1} \beta^t |u(c_t(n), l_t(n)) - u(c_t, l_t)| + 2\epsilon.$$

Since weak* convergence implies pointwise convergence, the result is established.

that: $\forall \mathbf{x} = (\mathbf{c}, \mathbf{k}, \mathbf{l}) \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$

$$\begin{aligned}
& \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*) - \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* + k_{t+1}^* - f(k_t^*, 1 - l_t^*)) \\
& + \sum_{t=0}^{\infty} \lambda_t^2 c_t^* + \sum_{t=0}^{\infty} \lambda_t^3 k_t^* + \sum_{t=0}^{\infty} \lambda_t^4 l_t^* + \sum_{t=0}^{\infty} \lambda_t^5 (1 - l_t^*) \\
\geq \; & \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) - \sum_{t=0}^{\infty} \lambda_t^1 (c_t + k_{t+1} - f(k_t, 1 - l_t)) \\
& + \sum_{t=0}^{\infty} \lambda_t^2 c_t + \sum_{t=0}^{\infty} \lambda_t^3 k_t + \sum_{t=0}^{\infty} \lambda_t^4 l_t + \sum_{t=0}^{\infty} \lambda_t^5 (1 - l_t) \qquad (1)
\end{aligned}$$

$$\lambda_t^1 (c_t^* + k_{t+1}^* - f(k_t^*, 1 - l_t^*)) = 0, \;\; \forall t \geq 0 \qquad (2)$$

$$\lambda_t^2 c_t^* = 0, \;\; \forall t \geq 0 \qquad (3)$$

$$\lambda_t^3 k_t^* = 0, \;\; \forall t \geq 0 \qquad (4)$$

$$\lambda_t^4 l_t^* = 0, \;\; \forall t \geq 0 \qquad (5)$$

$$\lambda_t^5 (1 - l_t^*) = 0, \;\; \forall t \geq 0 \qquad (6)$$

$$0 \in \beta^t \partial_1 u(c_t^*, l_t^*) - \{\lambda_t^1\} + \{\lambda_t^2\}, \;\; \forall t \geq 0 \qquad (7)$$

$$0 \in \beta^t \partial_2 u(c_t^*, l_t^*) - \lambda_t^1 \partial_2 f(k_t^*, L_t^*) + \{\lambda_t^4\} - \{\lambda_t^5\}, \;\; \forall t \geq 0 \qquad (8)$$

$$0 \in \lambda_t^1 \partial_1 f(k_t^*, L_t^*) + \{\lambda_t^3\} - \{\lambda_{t-1}^1\}, \;\; \forall t \geq 0 \qquad (9)$$

where $\partial_i u(c_t^*, l_t^*), \partial_i f(k_t^*, L_t^*)$ *respectively denote the projection on the* i^{th} *component of the subdifferential of the function* u *at* (c_t^*, l_t^*) *and the function* f *at* (k_t^*, L_t^*).

Proof: We first check that the Slater condition holds. Indeed, since $f_k'(0,1) > 1$, then for all $k_0 > 0$, there exists some $0 < \widehat{k} < k_0$ such that: $0 < \widehat{k} < f(\widehat{k}, 1)$ and $0 < \widehat{k} < f(k_0, 1)$. Thus, there exists two small positive numbers ε, ε_1 such that:

$$0 < \widehat{k} + \varepsilon < f(\widehat{k}, 1 - \varepsilon_1) \text{ and } 0 < \widehat{k} + \varepsilon < f(k_0, 1 - \varepsilon_1).$$

Denote $\mathbf{x}^0 = (\mathbf{c}^0, \mathbf{k}^0, \mathbf{l}^0)$ such that $\mathbf{c}^0 = (\varepsilon, \varepsilon, ...)$, $\mathbf{k}^0 = (k_0, \widehat{k}, \widehat{k}, ...)$, $\mathbf{l}^0 = (\varepsilon_1, \varepsilon_1, ...)$. We have

$$\begin{aligned}
\Phi_0^1(\mathbf{x}^0) &= c_0 + k_1 - f(k_0, 1 - l_0) \\
&= \varepsilon + \widehat{k} - f(k_0, 1 - \varepsilon_1) < 0 \\
\Phi_1^1(\mathbf{x}^0) &= c_1 + k_2 - f(k_1, 1 - l_1) \\
&= \varepsilon + \widehat{k} - f(\widehat{k}, 1 - \varepsilon_1) < 0 \\
\Phi_t^1(\mathbf{x}^0) &= \varepsilon + \widehat{k} - f(\widehat{k}, 1 - \varepsilon_1) < 0, \; \forall t \geq 2
\end{aligned}$$

$$\Phi_t^2(\mathbf{x}^0) = -\varepsilon < 0, \; \forall t \geq 0, \;\; \Phi_0^3(\mathbf{x}^0) = -k_0 < 0$$

$$\Phi_t^3(\mathbf{x}^0) = -\widehat{k} < 0, \quad \forall t \geq 1, \;\; \Phi_t^4(\mathbf{x}^0) = -\varepsilon_1 < 0, \; \forall t \geq 0$$

$$\Phi_t^5(\mathbf{x}^0) = \varepsilon_1 - 1 < 0, \; \forall t \geq 0.$$

Therefore, the Slater condition is satisfied. Now, it is obvious that, $\forall T$, $\mathbf{x}^T(\mathbf{x}^*, \mathbf{x}^0)$ belongs to $\ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$. As in [14], Assumption **T2** is satisfied. We now check Assumption **T1**. For any $\widetilde{\mathbf{x}} \in C$, $\widetilde{\widetilde{\mathbf{x}}} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$ such that for any T, $\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}}) \in C$ we have

$$\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) = -\sum_{t=0}^{T} \beta^t u(\widetilde{c}_t, \widetilde{l}_t) - \sum_{t=T+1}^{\infty} \beta^t u(\widetilde{\widetilde{c}}_t, \widetilde{\widetilde{l}}_t).$$

As $\widetilde{\widetilde{\mathbf{x}}} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$, $\sup_t |\widetilde{\widetilde{c}}_t| < +\infty$, there exists $m > 0, \forall t, |\widetilde{\widetilde{c}}_t| \leq m$. Since $\beta \in (0,1)$ we have

$$\sum_{t=T+1}^{\infty} \beta^t u(m,1) = u(m,1) \sum_{t=T+1}^{\infty} \beta^t \to 0 \text{ as } T \to \infty.$$

Hence, $\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) \to \mathcal{F}(\widetilde{\mathbf{x}})$ when $T \to \infty$. Taking account of Proposition 1, we get (1)-(6).

Finally, we obtain (7)-(9) from the Kuhn-Tucker first-order conditions. ■

3 Competitive equilibrium

Definition 1 *A competitive equilibrium consists of an allocation* $\{\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*, \mathbf{L}^*\} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$*, a price sequence* $\mathbf{p}^* \in \ell_+^1$ *for the consumption good, a wage sequence* $\mathbf{w}^* \in \ell_+^1$ *for labor and a price* $r > 0$ *for the initial capital stock* k_0 *such that:*

i) $\{\mathbf{c}^*, \mathbf{l}^*\}$ *is a solution to the problem*

$$\begin{array}{ll} \max & \displaystyle\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ s.t. & \mathbf{p}^*\mathbf{c} \leq \mathbf{w}^*\mathbf{L} + \pi^* + rk_0 \end{array}$$

where π^* *is the maximum profit of the firm.*

ii) $\{\mathbf{k}^*, \mathbf{L}^*\}$ *is a solution to the firm's problem*

$$\begin{array}{rcl} \pi^* & = & \max \displaystyle\sum_{t=0}^{\infty} p_t^*[f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{\infty} w_t^* L_t - rk_0 \\ s.t. \;\; 0 & \leq & k_{t+1} \leq f(k_t, L_t), \;\; L_t \geq 0, \forall t. \end{array}$$

iii) Markets clear

$$\begin{array}{rcl} c_t^* + k_{t+1}^* & = & f(k_t^*, L_t^*) \;\; \forall t \\ l_t^* + L_t^* & = & 1 \;\; \forall t \\ \text{and } k_0^* & = & k_0 \end{array}$$

Theorem 1 *Let* $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*\}$ *solve Problem* (Q)*. Take*

$$p_t^* = \lambda_t^1 \text{ for any } t \text{ and } r > 0.$$

There exists $f_L(k_t^*, L_t^*) \in \partial_2 f(k_t^*, L_t^*)$ *such that* $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r\}$ *is a competitive equilibrium with* $w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*)$*.*

Proof: Consider $\lambda = \{\boldsymbol{\lambda^1}, \lambda^{\mathbf{2}},\ \lambda^{\mathbf{3}}, \lambda^{\mathbf{4}}, \lambda^{\mathbf{5}}\}$ of Proposition 2. Conditions (7), (8), (9) in Proposition 2 show that $\partial u(c_t^*, l_t^*)$ and $\partial f(k_t^*, L_t^*)$ are nonempty and there exist $u_c(c_t^*, l_t^*) \in \partial_1 u(c_t^*, l_t^*)$, $u_l(c_t^*, l_t^*) \in \partial_2 u(c_t^*, l_t^*)$, $f_k(k_t^*, L_t^*) \in \partial_1 f(k_t^*, L_t^*)$ and $f_L(k_t^*, L_t^*) \in \partial_2 f(k_t^*, L_t^*)$ such that $\forall t$

$$\beta^t u_c(c_t^*, l_t^*) - \lambda_t^1 + \lambda_t^2 = 0 \tag{10}$$

$$\beta^t u_l(c_t^*, l_t^*) - \lambda_t^1 f_L(k_t^*, L_t^*) + \lambda_t^4 - \lambda_t^5 = 0 \tag{11}$$

$$\lambda_t^1 f_k(k_t^*, L_t^*) + \lambda_t^3 - \lambda_{t-1}^1 = 0 \tag{12}$$

Define $w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*) < +\infty$.

First, we claim that $\mathbf{w}^* \in \ell_+^1$.

We have

$$+\infty > \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*) - \sum_{t=0}^{\infty} \beta^t u(0,0) \geq \sum_{t=0}^{\infty} \beta^t u_c(c_t^*, l_t^*) c_t^* + \sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*) l_t^*,$$

which implies

$$\sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*) l_t^* < +\infty, \tag{13}$$

and

$$+\infty > \sum_{t=0}^{\infty} \lambda_t^1 f(k_t^*, L_t^*) - \sum_{t=0}^{\infty} \lambda_t^1 f(0,0) \geq \sum_{t=0}^{\infty} \lambda_t^1 f_k(k_t^*, L_t^*) k_t^* + \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^*$$

which implies

$$\sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* < +\infty. \tag{14}$$

Given T, we multiply (11) by L_t^* and sum up from 0 to T. Observe that

$$\forall T, \sum_{t=0}^{T} \beta^t u_l(c_t^*, l_t^*) L_t^* = \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* + \sum_{t=0}^{T} \lambda_t^5 L_t^* - \sum_{t=0}^{T} \lambda_t^4 L_t^*. \tag{15}$$

$$0 \leq \sum_{t=0}^{\infty} \lambda_t^5 L_t^* \leq \sum_{t=0}^{\infty} \lambda_t^5 < +\infty. \tag{16}$$

$$0 \leq \sum_{t=0}^{\infty} \lambda_t^4 L_t^* \leq \sum_{t=0}^{\infty} \lambda_t^4 < +\infty. \tag{17}$$

Thus, since $L_t^* = 1 - l_t^*$, from (15), we get

$$\begin{aligned}\sum_{t=0}^{T} \beta^t u_l(c_t^*, l_t^*) &= \sum_{t=0}^{T} \beta^t u_l(c_t^*, l_t^*) l_t^* + \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* \\ &\quad + \sum_{t=0}^{T} \lambda_t^5 L_t^* - \sum_{t=0}^{T} \lambda_t^4 L_t^*.\end{aligned}$$

Using (13), (14), (16), (17) and letting $T \to \infty$, we obtain

$$\begin{aligned}0 &\leq \sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*) = \sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*) l_t^* + \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* \\ &\quad + \sum_{t=0}^{\infty} \lambda_t^5 L_t^* - \sum_{t=0}^{\infty} \lambda_t^4 L_t^* < +\infty.\end{aligned}$$

Consequently, from (11), $\sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) < +\infty$ i.e. $\mathbf{w}^* \in \ell_+^1$. So, we have $\{\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*, \mathbf{L}^*\} \in \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty \times \ell_+^\infty$, with $\mathbf{p}^* \in \ell_+^1$ and $\mathbf{w}^* \in \ell_+^1$.

We now show that $(\mathbf{k}^*, \mathbf{L}^*)$ is solution to the firm's problem.

Since $p_t^* = \lambda_t^1$, $w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*)$, we have

$$\pi^* = \sum_{t=0}^{\infty} \lambda_t^1 [f(k_t^*, L_t^*) - k_{t+1}^*] - \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) \ L_t^* - r k_0$$

Let :

$$\begin{aligned}\Delta_T &= \sum_{t=0}^{T} \lambda_t^1 [f(k_t^*, L_t^*) - k_{t+1}^*] - \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*) \ L_t^* - r k_0 \\ &\quad - \left(\sum_{t=0}^{T} \lambda_t^1 [f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*) \ L_t - r k_0 \right).\end{aligned}$$

From the concavity of f, we get

$$\begin{aligned}\Delta_T &\geq \sum_{t=1}^{T} \lambda_t^1 f_k(k_t^*, L_t^*)(k_t^* - k_t) - \sum_{t=0}^{T} \lambda_t^1 (k_{t+1}^* - k_{t+1}) \\ &= [\lambda_1^1 f_k(k_1^*, L_1^*) - \lambda_0^1](k_1^* - k_1) + \ldots \\ &\quad + [\lambda_T^1 f_k(k_T^*, L_T^*) - \lambda_{T-1}^1](k_T^* - k_T) - \lambda_T^1 (k_{T+1}^* - k_{T+1}).\end{aligned}$$

By (4) and (12), we have: $\forall t = 1, 2, \ldots, T$

$$[\lambda_t^1 f_k(k_t^*, L_t^*) - \lambda_{t-1}^1](k_t^* - k_t) = -\lambda_t^3 (k_t^* - k_t) = \lambda_t^3 k_t \geq 0.$$

Thus,

$$\Delta_T \geq -\lambda_T^1 (k_{T+1}^* - k_{T+1}) = -\lambda_T^1 k_{T+1}^* + \lambda_T^1 k_{T+1} \geq -\lambda_T^1 k_{T+1}^*.$$

Since $\lambda^1 \in \ell_+^1$, $\sup_T k_{T+1}^* < +\infty$, we have

$$\lim_{T \to +\infty} \Delta_T \geq \lim_{T \to +\infty} - \lambda_T^1 k_{T+1}^* = 0.$$

We have proved that the sequences $(\mathbf{k}^*, \mathbf{L}^*)$ maximize the profit of the firm. We now show that $\mathbf{c}^*$ solves the consumer's problem.

$$\text{Let } \{\mathbf{c}, \mathbf{L}\} \text{ satisfy } \sum_{t=0}^{\infty} \lambda_t^1 c_t \leq \sum_{t=0}^{\infty} w_t^* L_t + \pi^* + r k_0. \tag{18}$$

By the concavity of u, we have:

$$\Delta = \sum_{t=0}^{\infty} \beta^t u(c_t^*, l_t^*) - \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$\geq \sum_{t=0}^{\infty} \beta^t u_c(c_t^*, l_t^*)(c_t^* - c_t) + \sum_{t=0}^{\infty} \beta^t u_l(c_t^*, l_t^*)\ (l_t^* - l_t).$$

Combining (3), (6), (10), (11) yields

$$\Delta \geq \sum_{t=0}^{\infty} (\lambda_t^1 - \lambda_t^2)(c_t^* - c_t) + \sum_{t=0}^{\infty} (\lambda_t^1 f_L(k_t^*, 1 - l_t^*) + \lambda_t^5 - \lambda_t^4)(l_t^* - l_t)$$

$$= \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* - c_t) + \sum_{t=0}^{\infty} \lambda_t^2 c_t - \sum_{t=0}^{\infty} \lambda_t^2 c_t^* + \sum_{t=0}^{\infty} (w_t^* + \lambda_t^5)(l_t^* - l_t)$$
$$- \sum_{t=0}^{\infty} \lambda_t^4 l_t^* + \sum_{t=0}^{\infty} \lambda_t^4 l_t$$

$$\begin{aligned}
\geq\ & \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* - c_t) + \sum_{t=0}^{\infty} (w_t^* + \lambda_t^5)(l_t^* - l_t) = \\
& \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* - c_t) + \sum_{t=0}^{\infty} w_t^* (l_t^* - l_t) + \sum_{t=0}^{\infty} \lambda_t^5 (1 - l_t) \\
\geq\ & \sum_{t=0}^{\infty} \lambda_t^1 (c_t^* - c_t) + \sum_{t=0}^{\infty} w_t^* (L_t - L_t^*).
\end{aligned}$$

Since

$$\pi^* = \sum_{t=0}^{\infty} \lambda_t^1 c_t^* - \sum_{t=0}^{\infty} w_t^* L_t^* - r k_0,$$

it follows from (18) that

$$\begin{aligned}
\Delta\ \geq\ & \sum_{t=0}^{\infty} p_t^* c_t^* - \sum_{t=0}^{\infty} w_t^* L_t^* - r k_0 - \left(\sum_{t=0}^{\infty} p_t^* c_t - \sum_{t=0}^{\infty} w_t^* L_t - r k_0 \right) \\
\geq\ & \pi^* - \pi^* = 0
\end{aligned}$$

Consequently, $\Delta \geq 0$ that means c^* solves the consumer's problem.

Finally, the market clears at every period, since $\forall t, c_t^* + k_{t+1}^* = f(k_t^*, L_t^*)$ and $1 - l_t^* = L_t^*$. ∎

4 Examples

We give two parametric example illustrating the violation of standard assumptions made in the literature and that there is a corner solution - in the first there is zero labor and in the second, zero capital - so that the existing results cannot be applied to establish existence of equilibrium. For this example, using our results we calculate the competitive equilibrium.

4.1 Example 1: Competitive equilibrium with $L_t^* = 0, l_t^* = 1$

Consider an economy with a good that can either be consumed or invested as capital, one firm and one consumer. The consumer has preferences defined over processes of consumption and leisure described by the utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t) = \sum_{t=0}^{\infty} \beta^t (c_t + m l_t),$$

The firm produces capital good by using capital k_t and labor $L_t = 1 - l_t$. The production function $f(k_t, L_t) = (k_t^\alpha + L_t)^{1/\theta}, 0 < \alpha < \theta, 0 < \beta < 1 < \theta, f$ is concave and increasing. Assume that $m = \frac{1}{\theta}\left(\frac{\theta}{\beta\alpha}\right)^{\frac{\alpha(\theta-1)}{\theta-\alpha}}$. The planning problem is

$$\begin{aligned}
& \max \sum_{t=0}^{\infty} \beta^t (c_t + m l_t) \\
\text{s.t. } c_t + k_{t+1} &\le (k_t^\alpha + L_t)^{1/\theta}, \ \forall t \ge 0 \\
L_t + l_t &= 1, \ \forall t \ge 0 \\
c_t &\ge 0, k_t \ge 0, l_t \ge 0, 1 - l_t \ge 0, \ \forall t \ge 0 \\
k_0 &\ge 0 \text{ is given.}
\end{aligned}$$

Inada conditions are not satisfied for both the utility and production functions. The utility function is also not strictly concave.[6]

Let $\lambda_t = (\lambda_t^i)_{i=1}^5, \lambda_t \neq 0$ denote the Lagrange multipliers. The Lagrangean is

$$\begin{aligned}
\mathcal{H} &= \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) - \sum_{t=0}^{\infty} \lambda_t^1 (c_t + k_{t+1} - f(k_t, 1 - l_t)) \\
&\quad + \sum_{t=0}^{\infty} \lambda_t^2 c_t + \sum_{t=0}^{\infty} \lambda_t^3 k_t + \sum_{t=0}^{\infty} \lambda_t^4 l_t + \sum_{t=0}^{\infty} \lambda_t^5 (1 - l_t)
\end{aligned}$$

[6] From the example it will be clear that we make utility linear in consumption only for ease of calculation.

It follows from Kuhn-Tucker necessary conditions that, $\forall t \geq 0$

$$\begin{aligned}
0 &= \beta^t - \lambda_t^1 + \lambda_t^2 \\
0 &= \beta^t m - \frac{1}{\theta}\lambda_t^1 (k_t^\alpha + 1 - l_t)^{\frac{1-\theta}{\theta}} + \lambda_t^4 - \lambda_t^5 \\
0 &= \frac{\alpha}{\theta}\lambda_t^1 k_t^{\alpha-1}(k_t^\alpha + 1 - l_t)^{\frac{1-\theta}{\theta}} + \lambda_t^3 - \lambda_{t-1}^1 \\
0 &= \lambda_t^1 (c_t + k_{t+1} - (k_t^\alpha + L_t)^{1/\theta}) \\
\lambda_t^2 c_t &= 0, \lambda_t^3 k_t = 0, \lambda_t^4 l_t = 0, \lambda_t^5 (1 - l_t) = 0.
\end{aligned}$$

It is easy to check that, the above system of equation has a solution:

$$\begin{aligned}
\lambda_t^{*1} &= \beta^t, \lambda_t^{*2} = \lambda_t^{*3} = \lambda_t^{*4} = \lambda_t^{*5} = 0, \\
k_t^* &= (\frac{\beta\alpha}{\theta})^{\frac{\theta}{\theta-\alpha}} := k_s \in (0,1), \\
c_t^* &= (k_s)^{\alpha/\theta} - k_s > 0, \\
l_t^* &= 1, \\
L_t^* &= 0.
\end{aligned}$$

As we know from section 3, if we define the sequence price $p_t^* = \lambda_t^{*1} = \beta^t$ for the consumption good and $w_t^* \in \lambda_t^{*1}\partial_2 f(k_t^*, L_t^*) = \lambda_t^{*1} f_L(k_s, 0) = \beta^t \frac{1}{\theta} k_s^{\frac{\alpha(1-\theta)}{\theta}}$ then $p_t^* \in \ell_+^1$, $w_t^* \in \ell_+^1$ and $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r\}$ is a competitive equilibrium.

4.2 Example 2: Competitive equilibrium with $k_t^* = 0$

Now consider the production function $f(k_t, L_t) = (k_t + L_t^\alpha)^{1/\theta}$ where $0 < \alpha < \theta, 0 < \beta < 1 < \theta$ and the utility function

$$u(c_t, l_t) = c_t + \frac{1}{\theta}\left(\frac{\beta}{\theta}\right)^{\frac{\alpha-\theta}{\alpha(\theta-1)}} l_t$$

We obtain the Kuhn-Tucker conditions, $\forall t \geq 0$

$$\begin{aligned}
0 &= \beta^t - \lambda_t^1 + \lambda_t^2 \\
0 &= \frac{1}{\theta}(\frac{\beta}{\theta})^{\frac{\alpha-\theta}{\alpha(\theta-1)}}\beta^t - \frac{\alpha}{\theta}\lambda_t^1 L_t{}^{\alpha-1}(k_t + L_t^\alpha)^{\frac{1-\theta}{\theta}} + \lambda_t^4 - \lambda_t^5 \\
0 &= \frac{1}{\theta}\lambda_t^1 (k_t + L_t^\alpha)^{\frac{1-\theta}{\theta}} + \lambda_t^3 - \lambda_{t-1}^1 \\
0 &= \lambda_t^1 (c_t + k_{t+1} - (k_t + L_t^\alpha)^{1/\theta}) \\
\lambda_t^2 c_t &= 0, \lambda_t^3 k_t = 0, \lambda_t^4 l_t = 0, \lambda_t^5 (1 - l_t) = 0.
\end{aligned}$$

The system of equation has a corner solution $k_t^* = 0$ and

$$\begin{aligned}
\lambda_t^{*1} &= \beta^t, \lambda_t^{*2} = \lambda_t^{*3} = \lambda_t^{*4} = \lambda_t^{*5} = 0 \\
L_t^* &= (\frac{\beta}{\theta})^{\frac{\theta}{\alpha(\theta-1)}} := L_s \in (0,1) \\
c_t^* &= (L_s)^{\alpha/\theta} > 0 \\
l_t^* &= 1 - L_s
\end{aligned}$$

4.3 Violation of some assumptions in the literature

The firm produces the capital good, which is used as an input, by using capital k_t and labor L_t. Define the production set

$$Y_t = \{(z_t, -L_t) \in \mathbb{R} \times \mathbb{R}_- : z_t + k_t \leq f(k_t, L_t), k_t \geq 0, L_t \geq 0\}$$

where $f(k_t, L_t) = (k_t^\alpha + L_t)^{1/\theta}$, $0 < \alpha < \theta, 1 < \theta$.

We show that $\{0\} \neq Y \cap (-Y)$.

Let $y_t = z_t + k_t$ denote the output. In this economy, $z_t = y_t - k_t$ is the net output. Then

$$Y_t = \{(y_t - k_t, -L_t) \in \mathbb{R} \times \mathbb{R}_- : y_t \leq f(k_t, L_t), k_t \geq 0, L_t \geq 0\}$$

For any $\bar{k}_t \in (0,1)$, $f(\bar{k}_t, 0) = \bar{k}_t^{\frac{\alpha}{\theta}}$ and $0 < \bar{k}_t < \bar{k}_t^{\frac{\alpha}{\theta}} = f(\bar{k}_t, 0)$. There exists $\epsilon > 0$ such that $\epsilon + \bar{k}_t < f(\bar{k}_t, 0)$. Let denote $y_t^\epsilon = \epsilon + \bar{k}_t$. We have $\mathbf{x} = (\varepsilon, 0) = (y_t^\epsilon - \bar{k}_t, 0) \in Y$ since $y_t^\epsilon \leq f(\bar{k}_t, 0)$. Clearly $-\epsilon + \bar{k}_t < f(\bar{k}_t, 0)$ so $-x = (-\varepsilon, 0) = ((-\varepsilon + \bar{k}_t) - \bar{k}_t, 0) \in Y$. This implies $0 \neq x \in Y \cap (-Y)$.

Moreover, for any $\lambda > 0$, if $\lambda.\mathbf{x} \in Y$ then $\lambda\epsilon + \bar{k}_t < f(\bar{k}_t, 0)$. Let $\lambda \to +\infty$, a contradiction. Thus Y is not a cone (Assumption in [3], Theorem 3).

5 Discussion and Conclusion

This paper studies existence of equilibrium in the optimal growth model with elastic labor supply. This model is the workhorse of dynamic general equilibrium theory for both endogenous and real business cycles. The results on existence of equilibrium have assumed strong conditions which are violated in some specifications of applied models.

This paper uses a separation argument to obtain Lagrange multipliers which lie in ℓ^1. As the separation argument relies on convexity, strict convexity can be relaxed; this also means that assumptions on cross partials of utility functions are not needed (as in [1], [5], [7], [10] and [15]); and homogeneity of production is not needed. These above papers assume normality of leisure (rule out backward bending labor supply curves) to show that the capital path is monotonic but this is inessential to show existence of a competitive equilibrium. The representation theorem involves assumptions on asymptotic properties of the constraint set (which are weaker than Mackey continuity (see [3] and [9]). The assumptions ensure that the either the optimal sequence $\{c_t, l_t\}_{t=0}^\infty$ is either always strictly interior or always equal to zero. Thus, one does not have to impose strong conditions, either Inada or $\lim_{\epsilon \to 0} \dfrac{u(\epsilon, \epsilon)}{\epsilon} \to +\infty$ as in [13] to ensure that the sequence of labor is strictly interior. This later condition is not satisfied, for example, in homogeneous period one utility functions. The existence result also does not employ any differentiability assumptions. Thus, it covers both Leontief utility and production functions $Y = min(K/v, L/u)$ and $Y/L = (1/v)K/L$.

This implies that the intensive production function, $y = f(k)$ where $y = Y/L$ and $k = K/L$ is effectively a straight line with slope $1/v$ up to the capital-labor ratio $k^* = K^*/L^*$ and is horizontal thereafter. Another well known model where differentiability is violated is the Intensive Activity Analysis Production Function but existence follows from our results.

A careful reader will observe that we can introduce tax and other distortions for the existence of a competitive equilibrium as long as concavity is maintained in line with the results in the literature. For monotonicity results, stronger results need to be imposed.

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