Option Pricing for Stocks with Dividends: An Analytic Approach by PDEs*

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Abstract

We study the Black-Scholes equations for pricing options on stocks by splitting it into two simpler PDEs that can be solved by analytically simpler and numerically faster methods than the original Black-Scholes PDE. We first use a deflator process β to arrive at a numeraire S^* (interest-neutral stock price) computed from the first equation and then obtain a simple Black-Scholes equation for the interest-neutral call option price P^* with no explicit dependence on the (instantaneous short) interest rate r. We also formulate two theorems on the solvability of these PDEs.

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1 Introduction

There have been a number of attemps to obtain and further investigate the solutions to various kinds of the Black-Scholes equations for pricing options on stocks (F. Black and M. Scholes [4]). These attempts are typically based on probabilistic, analytic, and numerical techniques, some of them including even explicit formulas, cf. Y. Achdou and O. Pironneau [1, Chapt. 2]. In our present work we study such a model of option pricing for stocks with dividends by rather abstract analytic methods from partial differential equations (PDEs, for short) and functional analysis. Our strategy is to *"split"* the Black-Scholes partial differential equation into *two simpler* PDEs that can be solved by analytically simpler and numerically faster methods than the original Black-Scholes PDE. In our approach we wish to use some arguments that are meaningful in Financial Mathematics. When using multiple integrals in probabilistic techniques, the introduction of ***numeraire*** typically reduces the dimension of the integration domain by one; see, e.g., T. Björk [2, Chapt. 24, pp. 348–367] and S. Benninga, T. Björk, and Z. Wiener [3]. We have decided to follow a similar idea with a numeraire, S^*, chosen in such a way that, relative to this numeraire, the option price P^* of a stock with dividend *does not explicitly depend* on the (instantaneous short) interest rate r. The dependence of P^* on the interest rate will be hidden in the dependence of the numeraire S^*. Also most of the difficulties in determining S^* will be concentrated in the Black-Scholes PDE for S^*. This makes the numeraire S^* into a kind of reliable, *interest-neutral* monetary unit. The Black-Scholes PDE for $P^* = P^*(S^*, t)$ as a function of S^* and time t will be considerably simpler, thus allowing fast numerical simulations among other advantages. Mathematically, this means that we need to eliminate the drift term from the Black-Scholes PDE for the option price $P = P(S,t)$ as a function of stock price S and time t; see eq. (1) in Section 2. Namely, the interest rate appears in the Black-Scholes PDE for the option price P *solely* in the drift term.

More specifically, we denote by

- $S = S(t)$ – the stock price at time t
- $S^* = S^*(S,t)$ – the *interest-neutral* stock price at time t, given the stock price S (Here, S^* should play the role of a ***numeraire***.)
- $P = P(S,t)$ – the (call) option price at time t, given the stock price S
- $P^* = P^*(S^*,t)$ – the *interest-neutral* (call) option price at time t, given the interest-neutral stock price S^*

The motivation for creating an ***interest-neutral*** stock price S^* is to work with a kind of ***terminalized*** stock price that is at any time $t \in [0,T]$ comparable with the *strike price* $S(T)$ at the *exercise date* T, regardless what the interest rate during the time interval $[0,T]$ is.

The *terminal condition* (at time $t = T$) imposed on $S^*(S,t)$, namely, $S^*(S,T) = S$ for all $S > 0$, is natural. Indeed, we have $S^*(S,t) - S \to 0$ as $t \nearrow T$, due to the vanishing effect of the deterministic factors, such as interest, dividend etc.

We replace the unknown function $S^* = S^*(S,t)$ of S and t by $\varphi = \varphi(S,t) \stackrel{\text{def}}{=} \ln(S^*/S)$ for all $S > 0$ and all $t \in [0,T]$; hence, we have $S^* = S\,\mathrm{e}^{\varphi(S,t)}$. Thus, the terminal condition reads $\varphi(S,T) = 0$ for all $S > 0$. If we need to vary also the exercise date T, we add the subscript (index) T to the function $\varphi(S,t)$, that is, we write $\varphi(S,t) \equiv \varphi_T(S,t)$ for $t \leq T$. We have $\varphi(S,t) > 0$, meaning a discount: The interest-neutral stock price S^* is discounted to $S = S^*\,\mathrm{e}^{-\varphi(S,t)}$, as the stock price S is terminalized (surcharged) to $S^* = S\,\mathrm{e}^{\varphi(S,t)}$. ***A nonnegative (positive) interest rate*** typically guarantees $\varphi(S,t) \geq 0$ for $0 \leq t \leq T$ ($\varphi(S,t) > 0$ for $0 \leq t < T$, respectively), for each $S > 0$.

Naturally, we wish to use the same numeraire S^* also for calculating the ratio between the interest-neutral (call) option price P^* and the (call) option price P, i.e., $P^* = P\,\mathrm{e}^{\varphi(S,t)}$ for all $S > 0$ and all $t \in [0,T]$. But then $\beta = \beta(S,t) = \mathrm{e}^{-\varphi(S,t)}$ is called a ***"deflator process"*** which measures the inflation from the present time t through the exercise date T, with $\beta(S,T) = 1$ for all $S > 0$; cf. T. Björk [2, Lemma 24.1, p. 350]. As the deflator process β works well for both, the stock and option prices, viz. $S^* = S/\beta$ and $P^* = P/\beta$, respectively, it is more advantageous to use β as a ***numeraire*** (T. Björk [2, p. 350]).

Our present work is organized as follows. In the next section (Section 2) we construct the mappings

$$(S,t) \longmapsto (S^*(S,t),t) \longmapsto P^*(S^*,t)$$

as already indicated above, by "splitting" the Black-Scholes PDE for the option price $P = P(S,t)$ into two simpler PDEs. In Section 3 we reformulate these two PDEs for the logarithmic stock prices $x = \ln S$ and $x^* = \ln S^*$ related by $x^* = x + \varphi(\mathrm{e}^x,t)$. The option price is then obtained from $P = SP^*/S^*$. These two parabolic PDEs are solved in Section 4.

2 Splitting the Black-Scholes equation

We further consider the usual Black-Scholes equation for the pricing of a (call) option on a given stock with a dividend as described in the Introduction (Section 1). We denote by

- $r = r(S,t)$ – the *(instantaneous short) interest rate* at time t, given the stock price S
- $\delta = \delta(S,t)$ – the *dividend rate* (*dividend*, for short) paid at time t, given the stock price S
- $\sigma = \sigma(S,t)$ – the volatility at time t, given the stock price S
- $K \equiv \text{const} > 0$ – the strike price (exercise price) of a European (call) option at exercise date T

For brevity, we call the ***dividend rate*** simply ***dividend***.

The ***(instantaneous short) interest rate*** is typically taken to be a positive constant $r(S,t) \equiv r > 0$.

The ***Black-Scholes equation*** for the unknown option price $P = P(S,t)$ takes the following form; see T. Björk [2, Prop. 16.7, p. 234]:

$$(1)\qquad \begin{cases} \dfrac{\partial P}{\partial t} - (r-\delta)\left(P - S\,\dfrac{\partial P}{\partial S}\right) - \delta P + \dfrac{1}{2}\,\sigma^2 S^2\,\dfrac{\partial^2 P}{\partial S^2} = 0 \\ \qquad\qquad \text{for } S > 0 \text{ and } t \in (0,T); \\ P(S,T) = (S-K)^+ \quad \text{for } S > 0. \end{cases}$$

We "split" this ***terminal-value problem*** into two simpler problems by constructing the mappings

$$(S,t) \longmapsto (S^*(S,t),t) \longmapsto P^*(S^*,t)$$

as already indicated in the Introduction (Section 1). To this end we need to calculate a few partial derivatives below:

Differentiating $\varphi(S,t) = \ln(S^*/S)$ (partially) with respect to S, we arrive at

$$(2)\qquad \frac{\partial\varphi}{\partial S} = \frac{1}{S^*}\,\frac{\partial S^*}{\partial S} - \frac{1}{S} = \frac{1}{S}\left(\frac{S}{S^*}\,\frac{\partial S^*}{\partial S} - 1\right)$$

which yields

$$(3)\qquad \frac{\partial S^*}{\partial S} = \frac{S^*}{S}\left(1 + S\,\frac{\partial\varphi}{\partial S}\right)$$

and

$$\frac{\partial\varphi}{\partial(\ln S)} = \frac{\partial(\ln S^*)}{\partial(\ln S)} - 1\,.$$

In contrast, differentiating $S = S^*\,\mathrm{e}^{-\varphi(S,t)}$ (partially) with respect to S^*, we get

$$(4)\qquad \frac{\partial S}{\partial S^*} = \mathrm{e}^{-\varphi(S,t)}\left(1 - S^*\,\frac{\partial\varphi}{\partial S}\,\frac{\partial S}{\partial S^*}\right) = \frac{S}{S^*} - S\,\frac{\partial\varphi}{\partial S}\,\frac{\partial S}{\partial S^*}$$

which yields

$$(5)\qquad \left(1 + S\,\frac{\partial\varphi}{\partial S}\right)\frac{\partial S}{\partial S^*} = \frac{S}{S^*}\,, \qquad \text{i.e.,} \qquad \frac{\partial S}{\partial S^*} = \frac{S}{S^*}\left(1 + S\,\frac{\partial\varphi}{\partial S}\right)^{-1}.$$

We combine (3) and (5) to get

$$(6)\qquad \frac{\partial S^*}{\partial S}\cdot\frac{\partial S}{\partial S^*} = 1\,.$$

Next, differentiating the expression

$$S^* - S\,\frac{\partial S^*}{\partial S} = -\,S^* S\,\frac{\partial\varphi}{\partial S} = -\,S^2\,\mathrm{e}^{\varphi(S,t)}\,\frac{\partial\varphi}{\partial S}$$

(partially) with respect to S, we obtain

$$-\,\frac{\partial}{\partial S}\left(S^* S\,\frac{\partial\varphi}{\partial S}\right) = \frac{\partial}{\partial S}\left(S^* - S\,\frac{\partial S^*}{\partial S}\right) = -\,S\,\frac{\partial^2 S^*}{\partial S^2}$$

which can be calculated further as follows,

$$
\begin{aligned}
S\,\frac{\partial^2 S^*}{\partial S^2} &= \frac{\partial}{\partial S}\left(S^* S\,\frac{\partial\varphi}{\partial S}\right) = \frac{\partial}{\partial S}\left(S^2\,\mathrm{e}^{\varphi(S,t)}\,\frac{\partial\varphi}{\partial S}\right) \\
&= \mathrm{e}^{\varphi(S,t)}\left[2S\,\frac{\partial\varphi}{\partial S} + S^2\left(\frac{\partial\varphi}{\partial S}\right)^2 + S^2\,\frac{\partial^2\varphi}{\partial S^2}\right] \\
&= \frac{S^*}{S}\left[\left(1 + S\,\frac{\partial\varphi}{\partial S}\right)^2 - 1 + S^2\,\frac{\partial^2\varphi}{\partial S^2}\right].
\end{aligned} \tag{7}
$$

As an easy consequence, we derive

$$
\frac{1}{S^*}\left(S^* + S^2\,\frac{\partial^2 S^*}{\partial S^2}\right) = \left(1 + S\,\frac{\partial\varphi}{\partial S}\right)^2 + S^2\,\frac{\partial^2\varphi}{\partial S^2}. \tag{8}
$$

In contrast, recall that eq. (3) is equivalent with

$$
\frac{1}{S^*}\left(S^* - S\,\frac{\partial S^*}{\partial S}\right) = -\,S\,\frac{\partial\varphi}{\partial S}. \tag{9}
$$

Next, we recall that

$$
P^* = P(S,t)\,\mathrm{e}^{\varphi(S,t)} = \frac{S^*}{S}\,P. \tag{10}
$$

Similarly as above, we differentiate the logarithm $\ln P(S,t) = \ln P^*(S^*,t) - \varphi(S,t)$ (partially) with respect to S, thus arriving at

$$
\frac{1}{P}\,\frac{\partial P}{\partial S} = \frac{1}{P^*}\,\frac{\partial P^*}{\partial S^*}\,\frac{\partial S^*}{\partial S} - \frac{\partial\varphi}{\partial S} \tag{11}
$$

which yields

$$
S\,\frac{\partial P}{\partial S} = \frac{P}{P^*}\,\frac{\partial P^*}{\partial S^*}\,S\,\frac{\partial S^*}{\partial S} - PS\,\frac{\partial\varphi}{\partial S} = \frac{S}{S^*}\,\frac{\partial P^*}{\partial S^*}\,S\,\frac{\partial S^*}{\partial S} - PS\,\frac{\partial\varphi}{\partial S}
$$

and therefore, with a help from (3),

$$
\begin{aligned}
P - S\,\frac{\partial P}{\partial S} &= P\left(1 + S\,\frac{\partial\varphi}{\partial S}\right) - \frac{\partial P^*}{\partial S^*}\,S\left(1 + S\,\frac{\partial\varphi}{\partial S}\right) \\
= \left(P - S\,\frac{\partial P^*}{\partial S^*}\right)\left(1 + S\,\frac{\partial\varphi}{\partial S}\right) &= \left(P^* - S^*\,\frac{\partial P^*}{\partial S^*}\right)\mathrm{e}^{-\varphi(S,t)}\left(1 + S\,\frac{\partial\varphi}{\partial S}\right).
\end{aligned} \tag{12}
$$

Next, differentiating the expression $P - S\,\frac{\partial P}{\partial S}$ (partially) with respect to S, we obtain

$$
\frac{\partial}{\partial S}\left(P - S\,\frac{\partial P}{\partial S}\right) = -\,S\,\frac{\partial^2 P}{\partial S^2}
$$

and similarly

$$
\frac{\partial}{\partial S^*}\left(P^* - S^*\,\frac{\partial P^*}{\partial S^*}\right) = -\,S^*\,\frac{\partial^2 P^*}{(\partial S^*)^2}.
$$

It will be of importance below to notice that the expression $P^* - S^* \frac{\partial P^*}{\partial S^*}$ is a function of S^* and t only; moreover, only S^* depends on S, whereas t does not. Consequently, making use of eq. (12), we derive

$$
\begin{aligned}
S\,\frac{\partial^2 P}{\partial S^2} &= -\,\frac{\partial}{\partial S}\left(P^* - S^*\,\frac{\partial P^*}{\partial S^*}\right)\cdot \mathrm{e}^{-\varphi(S,t)}\left(1 + S\,\frac{\partial \varphi}{\partial S}\right) \\
&\quad - \left(P^* - S^*\,\frac{\partial P^*}{\partial S^*}\right)\cdot \frac{\partial}{\partial S}\left[\mathrm{e}^{-\varphi(S,t)}\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)\right] \\
&= S^*\,\frac{\partial^2 P^*}{(\partial S^*)^2}\,\frac{\partial S^*}{\partial S}\cdot \mathrm{e}^{-\varphi(S,t)}\left(1 + S\,\frac{\partial \varphi}{\partial S}\right) \\
&\quad - \left(P^* - S^*\,\frac{\partial P^*}{\partial S^*}\right)\cdot \frac{\partial}{\partial S}\left[\mathrm{e}^{-\varphi(S,t)}\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)\right] \\
&= S^*\,\frac{\partial^2 P^*}{(\partial S^*)^2}\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)^2 \\
&\quad - \left(P^* - S^*\,\frac{\partial P^*}{\partial S^*}\right)\mathrm{e}^{-\varphi(S,t)}\left[-\,\frac{\partial \varphi}{\partial S}\left(1 + S\,\frac{\partial \varphi}{\partial S}\right) + \frac{\partial \varphi}{\partial S} + S\,\frac{\partial^2 \varphi}{\partial S^2}\right]
\end{aligned}
$$

which yields, with a help from eq. (12) again,

$$
\begin{aligned}
S\,\frac{\partial^2 P}{\partial S^2} &= S^*\,\frac{\partial^2 P^*}{(\partial S^*)^2}\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)^2 \\
&\quad - \left(P - S\,\frac{\partial P}{\partial S}\right)\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)^{-1} S\left[\frac{\partial^2 \varphi}{\partial S^2} - \left(\frac{\partial \varphi}{\partial S}\right)^2\right]\,.
\end{aligned}
\tag{13}
$$

The (partial) time derivative of P is computed as follows, in analogy with eq. (11),

$$
\begin{aligned}
\frac{1}{P}\,\frac{\partial P}{\partial t} &= \frac{1}{P^*}\,\frac{\partial P^*}{\partial t} + \frac{1}{P^*}\,\frac{\partial P^*}{\partial S^*}\,\frac{\partial S^*}{\partial t} - \frac{\partial \varphi}{\partial t} \\
&= \frac{1}{P^*}\,\frac{\partial P^*}{\partial t} + \frac{1}{P^*}\,\frac{\partial P^*}{\partial S^*}\,S\,\mathrm{e}^{\varphi(S,t)}\,\frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial t} \\
&= \frac{1}{P^*}\left[\frac{\partial P^*}{\partial t} - \left(P^* - S^*\,\frac{\partial P^*}{\partial S^*}\right)\frac{\partial \varphi}{\partial t}\right]
\end{aligned}
$$

which yields, using eq. (12),

$$
\begin{aligned}
\frac{\partial P}{\partial t} &= \frac{S}{S^*}\left[\frac{\partial P^*}{\partial t} - \left(P - S\,\frac{\partial P}{\partial S}\right)\frac{S^*}{S}\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)^{-1}\frac{\partial \varphi}{\partial t}\right] \\
&= \frac{S}{S^*}\,\frac{\partial P^*}{\partial t} - \left(P - S\,\frac{\partial P}{\partial S}\right)\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)^{-1}\frac{\partial \varphi}{\partial t}\,.
\end{aligned}
\tag{14}
$$

The *Black-Scholes equation* (1) for the price of a (call) option on a stock with a dividend $\delta = \delta(S,t)$ (which is assumed to be paid continuously in time) thus reads

$$\begin{aligned}
0 &= \frac{\partial P}{\partial t} - (r-\delta)\left(P - S\,\frac{\partial P}{\partial S}\right) - \delta P + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 P}{\partial S^2} \\
&= \frac{S}{S^*}\,\frac{\partial P^*}{\partial t} - \left(P - S\,\frac{\partial P}{\partial S}\right)\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)^{-1}\frac{\partial \varphi}{\partial t} \\
&\quad - (r-\delta)\left(P - S\,\frac{\partial P}{\partial S}\right) - \delta P + \frac{1}{2}\,\sigma^2 S S^*\,\frac{\partial^2 P^*}{(\partial S^*)^2}\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)^2 \\
&\quad - \frac{1}{2}\,\sigma^2 S^2\left(P - S\,\frac{\partial P}{\partial S}\right)\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)^{-1}\left[\frac{\partial^2 \varphi}{\partial S^2} - \left(\frac{\partial \varphi}{\partial S}\right)^2\right] \\
&= \frac{S}{S^*}\left\{\frac{\partial P^*}{\partial t} - \delta P^* + \frac{1}{2}\left[\sigma\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)\right]^2 (S^*)^2\frac{\partial^2 P^*}{(\partial S^*)^2}\right\} \\
&\quad - \left(P - S\,\frac{\partial P}{\partial S}\right)\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)^{-1} \\
&\quad \times \left\{\frac{\partial \varphi}{\partial t} + (r-\delta)\left(1 + S\,\frac{\partial \varphi}{\partial S}\right) + \frac{1}{2}\,\sigma^2 S^2\left[\frac{\partial^2 \varphi}{\partial S^2} - \left(\frac{\partial \varphi}{\partial S}\right)^2\right]\right\},
\end{aligned} \tag{15}$$

where we have employed eqs. (12), (13), and (14). The Black-Scholes equation (15), now taking the form

$$\begin{aligned}
0 &= \frac{\partial P}{\partial t} - (r-\delta)\left(P - S\,\frac{\partial P}{\partial S}\right) - \delta P + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 P}{\partial S^2} \\
&= \frac{S}{S^*}\left\{\frac{\partial P^*}{\partial t} - \delta P^* + \frac{1}{2}\left[\sigma\left(1 + S\,\frac{\partial \varphi}{\partial S}\right)\right]^2 (S^*)^2\frac{\partial^2 P^*}{(\partial S^*)^2}\right\} \\
&\quad - \frac{S}{S^*}\left(P^* - S^*\,\frac{\partial P^*}{\partial S^*}\right) \\
&\quad \times \left\{\frac{\partial \varphi}{\partial t} + (r-\delta)\left(1 + S\,\frac{\partial \varphi}{\partial S}\right) + \frac{1}{2}\,\sigma^2 S^2\left[\frac{\partial^2 \varphi}{\partial S^2} - \left(\frac{\partial \varphi}{\partial S}\right)^2\right]\right\}
\end{aligned} \tag{16}$$

for $S > 0$ and $t \in (0,T)$, is supplemented by the terminal condition

$$P(S,T) = (S-K)^+ \quad \text{for } S > 0. \tag{16_T}$$

This terminal-value problem possesses the following natural "splitting" into two independent problems as follows:

The new ***interest-neutral Black-Scholes equation*** for the price of a (call) option on a stock with a dividend $\delta = \delta(S,t)$ (paid continuously in time) reads

$$\frac{\partial P^*}{\partial t} - \delta P^* + \frac{1}{2}\left[\sigma\left(1 - S\,\frac{\partial \varphi}{\partial S}\right)\right]^2 (S^*)^2\frac{\partial^2 P^*}{(\partial S^*)^2} = 0 \tag{17}$$

with the terminal condition

$$P^*(S^*,T) = (S^*-K)^+ \quad \text{for } S^* > 0, \tag{17_T}$$

thanks to $\varphi(S,T) = 0$ for all $S > 0$. Here, $S = S(S^*, t)$ is an *implicit* solution to the equation $S\,\mathrm{e}^{\varphi(S,t)} = S^*$ for each $t \in [0,T]$, and $\varphi = \varphi(S,t)$ satisfies the *nonlinear* partial differential equation

$$\frac{\partial\varphi}{\partial t} + (r-\delta)\left(1 + S\,\frac{\partial\varphi}{\partial S}\right) + \frac{1}{2}\,\sigma^2 S^2 \left[\frac{\partial^2\varphi}{\partial S^2} - \left(\frac{\partial\varphi}{\partial S}\right)^2\right] = 0 \tag{18}$$

for all $S > 0$ and all $t \in [0,T)$, with the terminal condition

$$\varphi(S,T) = 0 \quad \text{ for } S > 0, \tag{18_T}$$

thanks to $\varphi(S,T) = 0$ for all $S > 0$. This is a ***backward diffusion equation.*** Notice that $S(S^*, T) = S^*$ for all $S^* > 0$.

In conclusion, we have thus constructed the desired mappings

$$(S,t) \;\longmapsto\; (S^*(S,t),t) \;\longmapsto\; P^*(S^*,t)$$

by taking $S^*(S,t) = S\,\mathrm{e}^{\varphi(S,t)}$ with $\varphi = \varphi(S,t)$ from problem (18), (18_T) and $P^* = P^*(S^*,t)$ from problem (17), (17_T). The (call) option price $P = P^*\,\mathrm{e}^{-\varphi(S,t)}$ for all $S > 0$ and all $t \in [0,T]$ follows from eq. (10).

In order to solve the *nonlinear* terminal value problem (18), (18_T), we transform it into the following equivalent *linear* terminal value problem for the deflator process $\beta = \mathrm{e}^{-\varphi(S,t)} > 0$. The *linear* PDE for the unknown deflator process $\beta = \beta(S,t)$ takes the following simple form which is analogous to the *standard linear* ***Black-Scholes equation***; see T. Björk [2, Theorem 7.7, p. 97]:

$$\left\{\begin{aligned} &\frac{\partial\beta}{\partial t} - (r-\delta)\left(\beta - S\,\frac{\partial\beta}{\partial S}\right) + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2\beta}{\partial S^2} = 0 \\ &\qquad\qquad\qquad\qquad\qquad \text{for } S > 0 \text{ and } t \in (0,T); \\ &\qquad\qquad\qquad\qquad\qquad \beta(S,T) = 1 \quad \text{ for } S > 0. \end{aligned}\right. \tag{19}$$

Notice that the difference $r - \delta$ has replaced the interest rate r in the standard linear Black-Scholes equation. Eq. (19) follows from (18), (18_T) with a help from the following calculations:

$$\frac{\partial\beta}{\partial t} = -\,\mathrm{e}^{-\varphi(S,t)}\,\frac{\partial\varphi}{\partial t}\,, \quad \frac{\partial\beta}{\partial S} = -\,\mathrm{e}^{-\varphi(S,t)}\,\frac{\partial\varphi}{\partial S}\,, \quad \text{and}$$
$$\frac{\partial^2\beta}{\partial S^2} = -\,\mathrm{e}^{-\varphi(S,t)}\left[\frac{\partial^2\varphi}{\partial S^2} - \left(\frac{\partial\varphi}{\partial S}\right)^2\right].$$

3 Reformulation of the Black-Scholes equations

This section is devoted to reformulating the two linear parabolic terminal value problems from the previous section: eqs. (17), (17_T) and eq. (19). We perform this reformulation in a standard way, cf. Y. Achdou and O. Pironneau [1], Chapt. 2, §2.2.3, p. 26.

First, we remove the singularities in the diffusion coefficients in eqs. (17) and (19). We begin with the latter, eq. (19).

Upon the logarithmic substitution $x = \ln S \in \mathbb{R}$ we get

$$\frac{\partial \beta}{\partial S} = \frac{1}{S}\frac{\partial \beta}{\partial x} \quad \text{and} \quad \frac{\partial^2 \beta}{\partial S^2} = \frac{1}{S^2}\left(\frac{\partial^2 \beta}{\partial x^2} - \frac{\partial \beta}{\partial x}\right)$$

and, thus, eq. (19) becomes

$$\frac{\partial \beta}{\partial t} - (r-\delta)\left(\beta - \frac{\partial \beta}{\partial x}\right) + \frac{1}{2}\sigma^2\left(\frac{\partial^2 \beta}{\partial x^2} - \frac{\partial \beta}{\partial x}\right) = 0 \tag{20}$$

for all $x \in \mathbb{R}$ and all $t \in [0,T)$, with the terminal condition

$$\beta(x,T) = 1 \quad \text{for all } x \in \mathbb{R}. \tag{20_T}$$

Of course, here we mean $r = r(S,t) = r(\mathrm{e}^x,t)$ and $\delta = \delta(S,t) = \delta(\mathrm{e}^x,t)$ as functions of $x \in \mathbb{R}$ and $t \in (0,T)$. Now we substitute $\tilde{\beta}(x,t) = \beta(x,t) - 1$ above, thus arriving at the terminal value problem

$$\frac{\partial \tilde{\beta}}{\partial t} - (r-\delta)\left(\tilde{\beta} - \frac{\partial \tilde{\beta}}{\partial x}\right) + \frac{1}{2}\sigma^2\left(\frac{\partial^2 \tilde{\beta}}{\partial x^2} - \frac{\partial \tilde{\beta}}{\partial x}\right) = r - \delta \quad \text{for all } x \in \mathbb{R} \text{ and all } t \in [0,T); \tag{21}$$

$$\tilde{\beta}(x,T) = 0 \quad \text{for all } x \in \mathbb{R}. \tag{21_T}$$

Analogously, upon the substitution $x^* = \ln S^* \in \mathbb{R}$ we get

$$\frac{\partial P^*}{\partial S^*} = \frac{1}{S^*}\frac{\partial P^*}{\partial x^*} \quad \text{and} \quad \frac{\partial^2 P^*}{(\partial S^*)^2} = \frac{1}{(S^*)^2}\left(\frac{\partial^2 P^*}{(\partial x^*)^2} - \frac{\partial P^*}{\partial x^*}\right)$$

and, thus, eq. (17) becomes

$$\frac{\partial P^*}{\partial t} - \delta P^* + \frac{1}{2}(\sigma^*)^2\left(\frac{\partial^2 P^*}{(\partial x^*)^2} - \frac{\partial P^*}{\partial x^*}\right) = 0 \tag{22}$$

for all $x^* \in \mathbb{R}$ and all $t \in [0,T)$, with the terminal condition (cf. (17_T))

$$P^*(x^*,T) = (\exp(x^*) - K)^+ \quad \text{for all } x^* \in \mathbb{R}, \tag{22_T}$$

where

$$\sigma^*(S^*,t) \stackrel{\text{def}}{=} \sigma(S,t)\left(1 - S\frac{\partial \varphi}{\partial S}\right) \tag{23}$$

is the new ***interest-neutral volatility***, i.e.,

$$\sigma^*(\exp(x^*),t) = \sigma(\mathrm{e}^x,t)\left(1 - \frac{\partial \varphi}{\partial x}\right).$$

Finally, we substitute

$$\tilde{P}^*(x^*,t) = P^*(x^*,t)\cdot\exp(-\alpha x^*),$$

where $\alpha > 1$ is a suitable constant, thus arriving at the terminal value problem for $\tilde{P}^*$,

$$\frac{\partial \tilde{P}^*}{\partial t} - \delta \tilde{P}^* + \frac{1}{2}(\sigma^*)^2 \left(\frac{\partial^2 \tilde{P}^*}{(\partial x^*)^2} + (2\alpha - 1)\frac{\partial \tilde{P}^*}{\partial x^*} + (\alpha^2 - 1)\tilde{P}^* \right) = 0 \quad \text{for all } x^* \in \mathbb{R} \text{ and all } t \in [0,T]; \tag{24}$$

$$\tilde{P}^*(x^*, T) = [1 - K \cdot \exp(-x^*)]^+ \cdot \exp(-(\alpha - 1)x^*) \quad \text{for all } x^* \in \mathbb{R}. \tag{24_T}$$

Clearly, the terminal values satisfy $\tilde{P}^*(\,\cdot\,, T) \in L^p(\mathbb{R})$, $1 \le p \le \infty$, by virtue of

$$\tilde{P}^*(x^*, T) = 0 \quad \text{for all } x^* \le \ln K, \qquad \text{and}$$
$$0 \le \tilde{P}^*(x^*, T) \le \exp(-(\alpha - 1)x^*) \quad \text{for all } x^* \ge \ln K,$$

where $\alpha - 1 > 0$.

Notice that $x = x(x^*, t)$ is an *implicit* solution to the equation $x + \varphi(x,t) = x^*$ for each $t \in [0,T]$, where $\varphi = -\ln \beta$, and $\beta = \beta(x,t)$ satisfies the *linear* partial differential equation (20) with the terminal condition (20_T).

4 Main results: solutions in $L^\infty(\mathbb{R})$ and $L^2(\mathbb{R})$

This section is devoted to solving the two linear parabolic terminal value problems from the previous section: the former, (21), (21_T), in the Hölder space $C^{2+\alpha,1}(\mathbb{R} \times [0,T])$, $0 < \alpha < 1$, the latter, (24), (24_T), in the abstract Hölder space $C^\theta([0,T] \to L^2(\mathbb{R}))$, $0 < \theta < 1$. The interpolation-type Hölder spaces $C^{\alpha,\theta}(\mathbb{R} \times [0,T])$, for $0 \le \alpha < \infty$ and $\theta = 0, 1$ or $\alpha/2$ are defined in A. Lunardi [9], §5.1, pp. 175–177.

Our first solvability result, for problem (21), (21_T), reads as follows.

Theorem 4.1 *Let all $r, \delta, \sigma : (0,\infty) \times [0,T] \to \mathbb{R}$ be continuous functions, such that both functions*

$$(r - \delta) \circ (\exp, \mathrm{id}) : \mathbb{R} \times [0,T] \to \mathbb{R} : (x,t) \mapsto r(\mathrm{e}^x, t) - \delta(\mathrm{e}^x, t) \quad \textit{and}$$
$$\sigma \circ (\exp, \mathrm{id}) : \mathbb{R} \times [0,T] \to \mathbb{R} : (x,t) \mapsto \sigma(\mathrm{e}^x, t)$$

are bounded and belong to $C^{\alpha,0}(\mathbb{R} \times [0,T])$, for some $\alpha \in (0,1)$, and

$$\sigma_0 \stackrel{\text{def}}{=} \inf_{\substack{S>0 \\ 0 \le t \le T}} \sigma(S,t) > 0\,.$$

The the terminal value problem (21), (21_T) possesses a unique classical solution $\tilde{\beta} \in C^{\alpha+2,1}(\mathbb{R} \times [0,T])$. *In particular, eq. (21) holds in the classical sense with $\tilde{\beta}$ and all partial derivatives $\partial\tilde{\beta}/\partial t$, $\partial\tilde{\beta}/\partial x$, and $\partial^2\tilde{\beta}/\partial x^2$ being continuous in all of $\mathbb{R} \times [0,T]$.*

In A. Lunardi [9], §4.1, Definition 4.1.1, pp. 123–124, such a solution is called a ***strict solution***; her ***strong*** or ***classical solutions*** are somewhat weaker.

Proof. This theorem follows immediately from A. LUNARDI [9], §5.1.1, Theorem 5.1.9, p. 189. ■

Before turning towards our second theorem, we need to study the implicit function equation

$$S\,(1+\tilde\beta(\ln S,t))^{-1} \equiv S\,\beta(\ln S,t)^{-1} = S^* \in (0,\infty)$$

for the unknown function $S = S(S^*,t) \in (0,\infty)$ of $S^* \in (0,\infty)$, at every fixed time $t \in [0,T]$. Equivalently, setting $x = \ln S$ and $x^* = \ln S^*$, at every fixed time $t \in [0,T]$ we need to determine the unknown function $x = x(x^*,t) \in \mathbb{R}$ of $x^* \in \mathbb{R}$ from the equation

$$\mathrm{e}^x\,(1+\tilde\beta(x,t))^{-1} \equiv \mathrm{e}^x\,\beta(x,t)^{-1} = \mathrm{e}^{x^*}$$

or, equivalently,

$$x - \ln(1+\tilde\beta(x,t)) = x^*\,.$$

The left-hand side has the partial derivative

$$\frac{\partial}{\partial x}\left(x - \ln(1+\tilde\beta(x,t))\right) = 1 - (1+\tilde\beta(x,t))^{-1}\,\frac{\partial\tilde\beta}{\partial x}$$

which, for $\beta(x,t) \equiv 1+\tilde\beta(x,t) > 0$, is positive if and only if

$$\frac{\partial\tilde\beta}{\partial x} < 1+\tilde\beta(x,t)\,. \tag{25}$$

Thanks to $\tilde\beta(x,T) = 0$ for all $x \in \mathbb{R}$ in (21_T), condition (25) holds for every $t \in [0,T]$ close to T, by $\tilde\beta \in C^{\alpha+2,1}(\mathbb{R}\times[0,T])$. If this is the case, we may apply the implicit function theorem to conclude that the mapping $(x^*,t) \mapsto x(x^*,t) : \mathbb{R}\times[T-\eta,T] \to \mathbb{R}$ is of class C^1, for some $\eta \in (0,T]$ small enough.

In the formulation of our second theorem we will assume that $\eta = T$ above, that is to say, the mapping $(x^*,t) \mapsto x(x^*,t) : \mathbb{R}\times[0,T] \to \mathbb{R}$ ***exists and is of class*** C^1**.** We now consider $\delta \equiv \delta(S^*,t) = \delta(\mathrm{e}^{x^*},t)$ and $\sigma^* \equiv \sigma^*(S^*,t) = \sigma^*(\mathrm{e}^{x^*},t)$ to be functions of $(S^*,t) \in (0,\infty)\times[0,T]$. Our second solvability result, for problem (24), (24_T), is as follows.

Theorem 4.2 *Let $\delta,\sigma^* : (0,\infty)\times[0,T] \to \mathbb{R}$ be bounded continuous functions, such that both functions*

$$\delta\circ(\exp,\mathrm{id}) : \mathbb{R}\times[0,T] \to \mathbb{R} : (x^*,t) \mapsto \delta(\mathrm{e}^{x^*},t) \quad \textit{and}$$

$$\sigma^*\circ(\exp,\mathrm{id}) : \mathbb{R}\times[0,T] \to \mathbb{R} : (x^*,t) \mapsto \sigma(\mathrm{e}^{x^*},t)$$

belong to $C^{0,\theta}(\mathbb{R}\times[0,T])$, for some $\theta \in (0,1)$, and

$$\sigma_0 \stackrel{\mathrm{def}}{=} \inf_{\substack{S^*>0\\ 0\le t\le T}} \sigma(S^*,t) > 0\,.$$

Then the terminal value problem (24), (24_T) possesses a unique classical solution

$$\tilde P^* \in C^\theta\left([0,T] \to L^2(\mathbb{R})\right) \cap C^{1+\theta}\left((0,T] \to L^2(\mathbb{R})\right) \cap C^\theta\left((0,T] \to W^{2,2}(\mathbb{R})\right)\,.$$

In particular, eq. (24) holds in the L^2_{loc}-sense in $\mathbb{R}\times(0,T)$ with $\tilde P^* \in C^\theta\left([0,T] \to L^2(\mathbb{R})\right)$ and all weak partial derivatives $\partial\tilde P^*/\partial t$, $\partial\tilde P^*/\partial x^*$, and $\partial^2\tilde P^*/(\partial x^*)^2$ belonging to $C^\theta\left((0,T] \to L^2(\mathbb{R})\right)$.

Also in A. Lunardi [9], §4.1, Definition 4.1.1, pp. 123–124, such a solution is indeed called a ***classical solution***.

Proof. This theorem follows immediately from A. Lunardi [9], §6.1, Theorem 6.1.4, p. 216. ■

5 Discussion of main results

Our initial objective was to split the Black-Scholes PDE (1) into *two simpler* PDEs, eqs. (17) (with the terminal condition (17_T)) and (19). The advantage of the latter, eq. (19), relative to the original problem (1), is that in (19) one is left with the difference $r-\delta$ only, which has replaced the interest rate r in the standard linear Black-Scholes equation (1). Moreover, in the former, eq. (17), the dependence on the interest rate r is only implicit through the interest-neutral volatility $\sigma^* = \sigma^*(S^*, t)$ defined in (23). An unpleasant consequence of this transformation, after the introduction of the logarithmic stock prices $x = \ln S$ and $x^* = \ln S^*$ in Section 3, is that the drift terms

$$\left(r-\delta-\tfrac{1}{2}\,\sigma^2\right)\frac{\partial\tilde{\beta}}{\partial x} \quad \text{and} \quad -\tfrac{1}{2}\,(\sigma^*)^2\,\frac{\partial P^*}{\partial x^*}$$

appear in eqs. (21) and (22), respectively.

The drift term can hardly be eliminated from eq. (21), but from eq. (22) and, analogously, from eq. (24) it can be eliminated by a suitable choice of function $\hat{\beta} = P/\tilde{P}^*$ satisfying an equation analogous to (21) in place of $\tilde{\beta}$. We leave the computational details to an interested reader.

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