Arbitrage and asset market equilibrium in infinite dimensional economies with risk-averse expected utilities *

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Abstract

We consider a model with an infinite numbers of states of nature, von Neumann - Morgenstern utilities and where agents have different probability beliefs. We show that no-arbitrage conditions, defined for finite dimensional asset markets models, are not sufficient to ensure existence of equilibrium in presence of an infinite number of states of nature. However, if the individually rational utility set $\mathcal{U}$ is compact, we obtain an equilibrium. We give conditions which imply the compactness of $\mathcal{U}$. We give examples of non-existence of equilibrium when these conditions do not hold.

Keywords: asset market equilibrium, individually rational attainable allocations, individually rational utility set, no-arbitrage prices, no-arbitrage condition.

JEL Classification: C62, D50, D81,D84,G1.

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1 Introduction

In finite dimensional markets with short-selling, conditions on agents' utilities insuring the existence of equilibria (or equivalent to the existence of equilibria) are by now well understood. In particular they can be interpreted as no-arbitrage conditions. In an uncertainty setting, where agents have different beliefs and different risk aversions, as originally shown by Hart (1974), the no-arbitrage conditions may be interpreted as compatibility of agent's risk adjusted beliefs. There is a huge literature on sufficient and necessary conditions for the existence of equilibria. In finite dimension, one can refer, for instance to Page (1987), Werner (1987), Nielsen (1989), Page and Wooders (1995, 1996), Allouch (1999), Allouch et al (2000). In infinite dimension asset markets, the no-arbitrage condition used for finite dimension do not imply existence of equilibrium. The standard assumption is to assume that the individually rational utility set is compact (see e.g. Cheng (1991), Brown and Werner (1995), Dana and Le Van (1996), Dana et al (1997), Dana and Le Van (2000), Le Van and Truong Xuan (2001).)
In this paper, we consider paper a model with an infinite number of states of nature, a finite number of agents and Von Neumann - Morgenstern utilities with different expectations.

More precisely, we consider a model where the utility of agent i is

$$U^i(x^i) = \sum_{s=1}^{\infty} \pi^i_s u^i(x^i_s)$$

where π^i is her belief and x^i is her consumption. The commodity space is $l^p(\pi)$ with $p \in \{1, \ldots, +\infty\}$.
When the number of states is finite, say K states, following Werner (1987), one can introduce for any agent i the set of useful vectors W^i to obtain the set of no-arbitrage prices denoted by S^i, which are defined as the set of vectors p which satisfy $p \cdot w > 0$ for any $w \in W^i \setminus \{0\}$. We say that the no-arbitrage holds if $\cap_i S^i \neq \emptyset$. When the utility functions are strictly concave, strictly increasing, this condition ensures the compactness of the individually rational allocations set. Dana and Le Van (2010) introduce for every agent i the convex cone P^i generated by the vectors $\{\pi^i_s u^{i\prime}(x^i_s)\}_{s=1,\ldots,K}$ where $x^i \in \mathbb{R}^K$ and $u^{i\prime}(+\infty) < u^{i\prime}(x^i_s) < u^{i\prime}(-\infty), \forall s, \forall i$. The no-arbitrage cone S^i is proved to be the interior of the cone P^i.
In this paper, following Dana and Le Van (2010), we define no-arbitrage prices p for agent i as follows: for any state s,

$$p_s = \lambda_i \pi^i_s u^{i\prime}(x^i_s)$$

where $\lambda_i > 0, x^i \in l^\infty$ and

$$u^{i\prime}(+\infty < u^{i\prime}(x^i_s) < u^{i\prime}(-\infty)$$

We say that the no-arbitrage condition (NA) holds if :

$$\lambda_i \pi^i_s u^{i\prime}(x^i_s) = \lambda_j \pi^j_s u^{j\prime}(x^j_s), \forall i, \forall j.$$

When the number of states is finite, as we said before, condition (NA) ensures existence of equilibrium. When the number of states is infinite, this condition only ensures the boundedness of the individually rational utility set. We give examples where (NA) is satisfied and no equilibrium exists. The strategy is therefore to give assumptions which imply the compactness of the individually rational utility set and hence existence of equilibrium. Our conditions might be considered as the weakest since we give also examples of non existence of equilibrium when these conditions do not hold.

The paper is organized as follows. In Section 2, we set up the model and define the equilibrium. In section 3, we introduce no-arbitrage conditions and relate them to the problem of existence of equilibrium. We show, through examples, that no-arbitrage conditions we introduce do not ensure existence of equilibrium. However if we assume the compactness of the individually rational utility set then we get an equilibrium. In Section 4, we give conditions for the compactness of the individually rational utility set. We give examples of non-existence of equilibrium when these conditions fail. Finally, proofs are put in Section Appendix. We mention that our methods of proofs are inspired by the ones in Le Van and Truong Xuan (2001). However, their model rules out the risk-neutral agents. That is not the case in our model.

2 The model

There are m agents indexed by $i = 1, \ldots, m$. The belief of agent i in state s is $\pi^i_s \geq 0$, and $\sum_{s=1}^{\infty} \pi^i_s = 1$. Let us denote by π the mean probability $\frac{1}{m}\sum_i \pi^i$. We first assume:

A0: π^i is equivalent to π^j for any i, j i.e. there exists a number $h > 0$ such that $h \leq \frac{\pi^i_s}{\pi^j_s} \leq \frac{1}{h}$ for all i, j, s.[1]

Under **A0**, without loss of generality, one can assume that $\pi^i_s > 0$ for any i, any s. In this paper, we always suppose that the condition **A0** is satisfied and $\pi^i_s > 0$ for any i, any s.

The consumption set of agent i is $X^i = l^p(\pi)$ with $p \in \{1, 2, \ldots, +\infty\}$ and agent i has an endowment $e^i \in l^p(\pi)$. We assume that for each agent i there

[1] We observe that when all agents have the same belief as in Cheng (1991), then **A0** is satisfied.

exists a concave, strictly increasing, differentiable function $u^i : \mathbb{R} \to \mathbb{R}$, such that, for any i, the function

$$U^i(x^i) = \sum_{s=1}^{\infty} \pi_s^i u^i(x_s^i)$$

is real-valued for any $x^i \in X^i$.

Agent i has $l^p(\pi)$ as consumption set, e^i as initial endowment and U^i as utility function, with $i = 1, \ldots, m$.

Definition 1 *An equilibrium is a list* $\big((x^{i*})_{i=1,\ldots,m}, p^*)\big)$ *such that* $x^{i*} \in X^i$ *for every* i *and* $p^* \in l_+^q(\pi) \setminus \{0\}$ *and*
(a) For any i, $U^i(x) > U^i(x^{i*}) \Rightarrow \sum_{s=1}^{\infty} p_s^* x_s > \sum_{s=1}^{\infty} p_s^* e_s^i$
(b) $\sum_{i=1}^{m} x^{i*} = \sum_{i=1}^{m} e^i$.

Define

$$\begin{aligned} a^i &= \inf_x u^{i\prime}(x) = u^{i\prime}(+\infty) \\ b^i &= \sup_x u^{i\prime}(x) = u^{i\prime}(-\infty). \end{aligned}$$

Let I_1 be the set of indexes i such that $a^i < b^i$, and I_2 be the set of indexes such that $a^i = b^i$ (the set of risk neutral agents).

We add:

Definition 2 *1. The individually rational attainable allocations set* $\mathcal{A}$ *is defined by*

$$\mathcal{A} = \{(x^i) \in (l^p(\pi))^m \mid \sum_{i=1}^{m} x^i = \sum_{i=1}^{m} e^i \ \ and \ \ U^i(x^i) \geq U^i(e^i) \ for \ all \ \ i\}.$$

2. The individually rational utility set $\mathcal{U}$ *is defined by*

$$\mathcal{U} = \{(v^1, v^2, ..., v^m) \in \mathbb{R}^m \mid \exists x \in \mathcal{A} \ \ s.t \ \ U^i(e^i) \leq v^i \leq U^i(x^i) \ \ for \ all \ i\}.$$

3 No-arbitrage condition and existence of equilibrium

We will first introduce a notion of no-arbitrage price based on useful vectors introduced by Werner.

A vector $w \in l^p(\pi)$ is *useful* for agent i if for any $x \in X^i$, we have

$$U^i(x + \lambda w) \geq U^i(x), \ \forall \lambda \geq 0, \ \forall x \in X^i$$

or equivalently

$$\sum_{s=1}^{\infty} \pi_s^i u^i(x_s + \lambda w_s) \geq \sum_{s=1}^{\infty} \pi_s^i u^i(x_s),\ \forall \lambda \geq 0,\ \forall x \in X^i.$$

Let W^i denote the set of useful vectors for agent i. It is easy to check that W^i is a closed convex cone.

A vector p is a *weak no-arbitrage price* for agent i if it is in $l^q(\pi)$ and if it there exists $r \in l^q(\pi)$ s.t.:

$$\begin{aligned} \forall s,\ p_s &= \pi_s^i r_s \\ \text{and } \sum_{s=1}^{\infty} p_s w_s &> 0, \quad \text{for any } w \in W^i \setminus \{0\} \end{aligned}$$

In models with a finite number of states of nature, this is the Werner's definition of no-arbitrage prices.

Let σ^i be the set of weak no-arbitrage prices for agent i. A weak no-arbitrage (**WNA**) condition will be

$$\cap_i \sigma^i \neq \emptyset.$$

In the case of a finite number of states, the existence of a Werner's no-arbitrage price is sufficient to ensure the existence of equilibrium. In infinite dimension models, this property is not true.

We add another notion of arbitrage price. Let S^i be the set of vectors p in $l^q(\pi)$ which satisfy $\sum_{s\geq 1} \pi_s^i p(s) w(s) > 0$ for any $w \in W^i \setminus \{\emptyset\}$. In finite dimension, S^i is called the set of no-arbitrage prices of agent i. If W^i contains no line then S^i is open. In our case, since W^i contains $l_+^p(\pi)$, the set S^i has an empty interior if $p > 1$. However, one can show (see Dana and Le Van (2010)) that w is useful for agent i if and only if

$$\forall x \in X^i,\ \sum_{s=1}^{\infty} \pi_s^i u^{i\prime}(x_s) w_s \geq 0.$$

Observe that if u^i is strictly concave then for any $x \in X^i$ and $w \in W^i$, the function $\lambda \mapsto U^i(x + \lambda w)$ is strictly increasing. In this case, if $w \in W^i \setminus \{0\}$, then

$$\forall x \in X^i,\ \sum_{s=1}^{\infty} \pi_s^i u^{i\prime}(x_s) w_s > 0.$$

Following Dana and Le Van (2010), for any agent i, we consider the vectors $p \in l^q(\pi)$ defined by

$$\forall s,\ p_s = \lambda_i \pi_s^i u^{i\prime}(x_s^i).$$

Observe that $p \in \sigma^i$.

We introduce the assumption:

There exists $p \in l^\infty(\pi)$, $\overline{x} \in l^\infty(\pi)$, $\lambda_i > 0$ such that

$$p_s = \lambda^i \pi^i_s u^{i\prime}(x^i_s) \text{ for all } s.$$

This condition is equivalent to the following no-arbitrage condition:

$$(\mathbf{NA})\ \lambda^i \pi^i_s u^{i\prime}(x^i_s) = \lambda^j \pi^j_s u^{j\prime}(x^j_s) = \lambda^k a_k \pi^k_s,\ \forall s, \forall i \in I_1, \forall j \in I_1, \forall k \in I_2$$

with $x^i \in l^p(\pi), \forall i \in I_1$ and

$$\forall i \in I_1, a^i < \inf_s u^{i\prime}(x^i_s) < \sup_s u^{i\prime}(x^i_s) < b^i.$$

If we assume that for all $i \in I_1$, we have $a^i < u^{i\prime}(x) < b^i$, $\forall\ x$, then in finite dimension models, weak no-arbitrage (**WNA**) and no-arbitrage (**NA**) are equivalent and also equivalent to the existence of equilibrium. In our model with an infinitely countable set of natures, sufficient conditions for (**NA**) are given in the following proposition.

Proposition 1 *(i) If (**NA**) holds, then $\mathcal{U}$ is bounded*

(ii) Assume either $u^{i\prime}(-\infty) = +\infty$ for all $i \in I_1$ or $u^{i\prime}(+\infty) = 0$ for all $i \in I_1$.

*(ii.1) If $I_2 = \emptyset$, then no arbitrage condition (**NA**) holds.*

*(ii.1) When $I_2 \neq \emptyset$, no arbitrage condition (**NA**) holds if, and only if, $\pi^i = \pi^j, \forall i \in I_2, \forall j \in I_2$.*

Proof: See Appendix. ■

No-arbitrage condition (**NA**) does not warrant existence of equilibrium in presence of an infinite number of states of nature. We give an example of an economy with two agents, and with an infinitely countable number of states of nature, where the no-arbitrage condition (**NA**) is satisfied, and there exists no equilibrium.

Example 1 *Consider an economy with two agents ($i = 1, 2$), with endowments equal to 0. The probabilities are equivalent: $\pi^1_s = \left(\frac{1}{2}\right)^s$, $\pi^2_s = \frac{1}{S_\alpha}\frac{1-\alpha^s}{2^s}$, where $1 < \alpha < 1$, and $S_\alpha = \sum_s \frac{1-\alpha^s}{2^s}$.*

The rewarded utilities satisfy

$$\begin{aligned}
u^{1\prime}(x) &= b^1,\ \forall x \le 0 \\
u^{1\prime}(+\infty) &= 0 \\
u^1(0) &= 0 \\
u^{2\prime}(x) &= a^2,\ \forall x \ge 0 \\
u^{2\prime}(-\infty) &= +\infty \\
u^2(0) &= 0
\end{aligned}$$

There exists $z > 0$ *with* $u^{1\prime}(z) < b^1$. *Let* $\overline{x}_s^1 = z, \forall s$. *Since* $u^{2\prime}(-\infty) = +\infty$, *there exists* $\overline{x}_s^2 < 0$ *which satisfies*

$$u^{2\prime}(0) = (1-\alpha^s)u^{2\prime}(\overline{x}_s^2)$$

One can check that

$$\lambda \pi_s^1 u^{1\prime}(\overline{x}_s^1) = \pi_s^2 u^{2\prime}(\overline{x}_s^2), \forall s$$

with $\lambda = \frac{u^{2\prime}(0)}{u^{1\prime}(z)} \times \frac{1}{S_\alpha}$. *Since*

$$0 = u^{1\prime}(+\infty) < u^{1\prime}(z) = u^{1\prime}(\overline{x}_s^1) < b^1$$

$$a^2 = u^{2\prime}(0) < \frac{u^{2\prime}(0)}{1-\alpha^s} = u^{2\prime}(\overline{x}_s^2) < u^{2\prime}(-\infty) = +\infty$$

*no-arbitrage condition (**NA**) is satisfied.*

We now show that no equilibrium exists. Assume there exists an equilibrium $(p,(x^1,x^2))$ *with* $x_s^1 = x_s = -x_s^2$. *We have*

$$\begin{aligned}
\forall s,\ \lambda_1 \pi_s^1 u^{1\prime}(x_s) &= \lambda_2 \pi_s^2 u^{2\prime}(-x_s) \\
\text{or } \lambda \pi_s^1 u^{1\prime}(x_s) &= \pi_s^2 u^{2\prime}(-x_s),\ \text{with } \lambda = \frac{\lambda_1}{\lambda_2}
\end{aligned}$$

For all s:

$$\lambda' \frac{1}{2^s} u^{1\prime}(x_s) = \frac{1-\alpha^s}{2^s} u^{2\prime}(-x_s)$$

or equivalently

$$\lambda' = (1-\alpha^s)\frac{u^{2\prime}(-x_s)}{u^{1\prime}(x_s)}, \forall s$$

with $\lambda' = \lambda S_\alpha$. *Since* $\sum_s p_s x_s = 0$ *and* $p_s > 0$ *for any* s, *one must have* s_0 *with* $x_{s_0} \le 0$. *In this case*

$$\lambda' = (1-\alpha^{s_0})\frac{a^2}{b^1}$$

and for any $s \ne s_0$, $x_s > 0$. *We then obtain*

$$\frac{u^{2\prime}(-x_{s_0+1})}{u^{1\prime}(x_{s_0+1})} > \frac{u^{2\prime}(0)}{u^{1\prime}(0)} = \frac{a^2}{b^1}$$

since $x_{s_0+1} > 0$. *Now, because* $1-\alpha^{s_0+1} > 1-\alpha^{s_0}$ *we obtain, on the one one hand:*

$$\begin{aligned}\lambda' &= (1-\alpha^{s_0+1})\frac{u^{2\prime}(-x_{s_0+1})}{u^{1\prime}(x_{s_0+1})}\\ &> (1-\alpha^{s_0})\frac{u^{2\prime}(-x_{s_0+1})}{u^{1\prime}(x_{s_0+1})}\\ &> (1-\alpha^{s_0})\frac{a^2}{b^1}\end{aligned}$$

and on the other hand

$$\lambda' = (1-\alpha^{s_0})\frac{a^2}{b^1}$$

which is a contradiction. Then there exists no equilibrium.

In infinite dimension, with a vector space L as commodity space, Brown and Werner [4], Dana, Le Van and Magnien [9] assume the compactness of $\mathcal{U}$ and get existence of equilibrium with prices in L'. For our model, we will prove that when the commodity space is l^∞, we get an equilibrium with prices in $l^1(\pi)$.

Theorem 1 *Assume* ***A0****. Our model has an equilibrium if we add the assumption that* $\mathcal{U}$ *is compact. If* X^i *is* $l^p(\pi)$ *with* $1 \le p < +\infty$ *then the equilibrium price* p^**is in* $l^q(\pi)$*. If* $p = +\infty$*, then* $p^* \in l^1(\pi)$*.*

Proof: Since $\mathcal{U}$ is compact and X^i is $l^p(\pi)$ there exists an equilibrium $((x^{i*}), p^*)$ (see Dana and al (1997)) with $x^{i*} \in l^p(\pi)$.

When $1 \le p < +\infty$, the price p^* belongs to $l^q(\pi)$. When $p = \infty$ we will show that the equilibrium price belongs to $l^1(\pi)$. The equilibrium price can be written as $p^* + \phi$ where $p^* \in l^1(\pi)$ and ϕ is a purely finitely additive function. For any i, the equilibrium allocation x^{i*} solves the problem:

$$\max \sum_{s=1}^{\infty} \pi_s^i u^i(x_s^i)$$

$$\text{s.t. } \sum_{s=1}^{\infty} p_s^* x_s^i + \phi(x^i) = \sum_{s=1}^{\infty} p_s^* e_s^i + \phi(e^i)$$

From Theorem V.3.1, page 91, in Arrow-Hurwicz-Uzawa in [2], for any i, there exists ζ_i s.t.

$$\sum_{s=1}^{\infty} \pi_s^i u^i(x_s^{*i}) - \zeta_i(\sum_{s=1}^{\infty} p_s^* x_s^{*i} + \phi(x^{i*})) \ge \sum_{s=1}^{\infty} \pi_s^i u^i(x_s) - \zeta_i(\sum_{s=1}^{\infty} p_s^* x_s + \phi(x^i)).$$

Suppose that $\phi \ne 0$. Since $\phi \ge 0$, then $\phi(\mathbf{1}) > 0$, with $\mathbf{1} = (1, 1, 1, \ldots)$. Define $x^i(N)$ as:

$$x_s^i(N) = x_s^{i*} \text{ with } s = 1, 2, \ldots, N.$$
$$x_s^i(N) = x_s^{i*} - 1 \text{ with } s \geq N+1.$$

Observe that $x^i(N) \in l^\infty(\pi)$. We have:

$$\sum_{s=1}^{\infty} \pi_s^i u^i(x_s^{i*}) - \zeta_i(\sum_{s=1}^{\infty} p_s^* x_s^{*i} + \phi(x^{i*})) \geq \sum_{s=1}^{\infty} \pi_s^i u^i(x_s^i(N)) - \zeta_i(\sum_{s=1}^{\infty} p_s^* x_s^i(N) + \phi(x^i(N))).$$

$\Rightarrow$

$$\sum_{s \geq N+1} \pi_s^i u^i(x_s^{i*}) - \zeta_i(\sum_{s \geq N+1} p_s^* x_s^{*i} + \phi(x^{i*})) \geq \sum_{s \geq N+1} \pi_s^i u^i(x_s^{i*} - 1) - \zeta_i(\sum_{s \geq N+1} p_s^*(x_s^{i*} - 1) + \phi(x^{i*} - \mathbf{1})).$$

$\Rightarrow$

$$\sum_{s \geq N+1} \pi_s^i u^i(x_s^{i*}) - \sum_{s \geq N+1} \pi_s^i u^i(x_s^{i*} - 1) - \zeta^i \sum_{s \geq N+1} p_s^* \geq \zeta^i(\phi(x^{i*}) - \phi(x^{i*} - \mathbf{1})) = \zeta^i \phi(\mathbf{1}).$$

Let $N \to \infty$, the LHS converges to 0. This implies $\phi(\mathbf{1}) \leq 0$: a contradiction. Hence $\phi = 0$. ■

Notice that if $\mathcal{A}$ is compact for the $l^1(\pi)$ topology, then $\mathcal{U}$ is compact. And we get an equilibrium. The proof of the compactness of $\mathcal{U}$ uses the fact that for all i, U^i is upper semi-continuous on the projection of $\mathcal{A}$ on the i^{th}-component.

4 Sufficient conditions to obtain the compactness of $\mathcal{U}$

Proposition 2 *Assume (**A0**). If $b^i = +\infty$ for all i, the allocation set $\mathcal{A}$ is $l^1(\pi)$-compact.*

Proof: It is given in Appendix. We prove that $\mathcal{A}$ which is bounded in $l^1(\pi)$ satisfies Dunford-Pettis criterion. Hence it is $\sigma(l^1(\pi), l^\infty(\pi))$-compact. We prove however that a bounded set in $l^1(\pi)$ is compact for the $l^1(\pi)$-topology if and only if it is $\sigma(l^1(\pi), l^\infty(\pi))$-compact. ■

Proposition 3 *Assume (**A0**). If $a^i = 0$ for all i, then $\mathcal{U}$ is compact.*

Proof: See Appendix. ■

Remark 1 *In Example 1, we have a model with two agents. Agent 1 has $a^1 = 0$, $b^1 < +\infty$. Agent 2 has $a^2 > 0$, $b^2 = +\infty$. The assumptions of Propositions 2 and 3 are not satisfied. We still have no-arbitrage condition and we have no equilibrium in this model.*

Proposition 4 *Assume (**A0**). Assume $I_2 \neq \emptyset$. If $a^i = 0$ and $b^i = +\infty$ for all $i \in I_1$ and (**NA**) holds, then $\mathcal{U}$ is compact.*

Proof: See Appendix. ■

We can be surprised that in presence of risk-neutral agents we have to impose $u^{i\prime}(+\infty) = 0$ and $u^{i\prime}(-\infty) = +\infty$ for any agent $i \in I_1$. We give below an example with two agents. The first is risk-neutral while the second is risk-averse. The utility of the latter agent only satisfies either the marginal utility at $+\infty$ equals 0 or the marginal utility at $-\infty$ is $+\infty$. In this example there exists no equilibrium.

Example 2 *Consider an economy with two agents ($i = 1, 2$), with endowments equal to 0. The probabilities are equivalent: $\pi^1_s = \left(\frac{1}{2}\right)^s$, $\pi^2_s = \frac{1}{S_\alpha}\frac{1-\alpha^s}{2^s}$, where $1 < \alpha < 1$, and $S_\alpha = \sum_s \frac{1-\alpha^s}{2^s}$.*

Case 1

The rewarded utilities satisfy

$$
\begin{aligned}
u^{1\prime}(x) &= 1, \ \forall x \in \mathbb{R} \\
u^{2\prime}(x) &= a^2, \ \forall x \geq 0 \\
u^{2\prime}(-\infty) &= +\infty \\
u^{2\prime}(x) &> a^2 \ \forall x < 0.
\end{aligned}
$$

Assume there exists an equilibrium $(p, (x^1, x^2))$. Then $x^1_s = -x^2_s = x_s$ for any s. There exists $\lambda > 0$ such that

$$
\begin{aligned}
\frac{\lambda}{2^s} u^{1\prime}(x_s) &= \frac{1-\alpha^s}{S_\alpha 2^s} u^{2\prime}(-x_s) \forall s \\
\Leftrightarrow u^{2\prime}(-x_s) &= \lambda \frac{S_\alpha}{1-\alpha^s} \forall s.
\end{aligned}
$$

Since $\sum_s p_s x_s = 0$ and $p_s > 0$ for all s, there exists $x_{s_0} \leq 0$, i.e $-x_{s_0} \geq 0$. Then

$$
a^2 = \lambda \frac{S_\alpha}{1-\alpha^{s_0}}
$$

and $\forall s \neq s_0$, $x_s > 0$. Hence $u^{2\prime}(-x_{s_0+1}) > a^2$. This implies

$$
\lambda \frac{S_\alpha}{1-\alpha^{s_0+1}} > \lambda \frac{S_\alpha}{1-\alpha^{s_0}} \Rightarrow \alpha^{s_0+1} > \alpha^{s_0}.
$$

A contradiction.

Case 2

The rewarded utilities satisfy

$$\begin{array}{rcl} u^{1\prime}(x) & = & b^1,\ \forall x \leq 0 \\ u^{1\prime}(+\infty) & = & 0 \\ u^{1\prime}(x) & < & b^1\ \forall x > 0 \\ u^{2\prime}(x) & = & 1\ \forall x \in \mathbb{R}. \end{array}$$

Assume there exists an equilibrium $(p,(x^1,x^2))$. *Then* $x^1_s = -x^2_s = x_s$ *for any* s. *There exists* $\lambda > 0$ *such that*

$$u^{1\prime}(x_s) \ = \ \lambda \frac{1-\alpha^s}{S_\alpha} \forall s.$$

Since $\sum_s p_s x_s = 0$ *and* $p_s > 0$ *for all* s, *there exists* $x_{s_0} \leq 0$, *i.e* $-x_{s_0} \geq 0$. *Then*

$$b^1 = \lambda \frac{1-\alpha^{s_0}}{S_\alpha}$$

and $\forall s \neq s_0$, $x_s > 0$. *Hence* $u^{1\prime}(x_{s_0+1}) < b^1$. *This implies*

$$\lambda \frac{1-\alpha^{s_0+1}}{S_\alpha} < \lambda \frac{1-\alpha^{s_0}}{S_\alpha} \Rightarrow \alpha^{s_0+1} > \alpha^{s_0}.$$

A contradiction.

5 Appendix

The following Lemma is required for the proofs of propositions 1, 2, 3, 4. It basically shows that if (**NA**) is satisfied then, on the one hand, the projection $\mathcal{A}^i$ of $\mathcal{A}$ on the $i^{th}-$ component is bounded for any $i \in I_1$ and, on the other hand, agents in I_2 have the same belief.

Lemma 1 *Assume (**NA**).*

(i) For all $i, j \in I_2$ *we have* $\pi^i = \pi^j$.

(ii) Denote by π^I *the same probability of belief of agents in* I_2. *There exists a constant* $C > 0$ *which depends only on* $\overline{p}, \overline{x}, \overline{e}$ *such that for any* $(x^1, \ldots, x^m) \in \mathcal{A}$ *we have*

$$\sum_{s=1}^{\infty} \pi^i_s |x^i_s| \leq C \text{ for all } i \in I_1$$

and

$$\sum_{s=1}^{\infty} \pi^I_s |\sum_{i \in I_2} x^i_s| \leq C.$$

(iii) $\mathcal{U}$ *is bounded.*

Proof:

Take any (**NA**) price p. There exists $(\overline{x}^i)$, $\{\lambda_i > 0\}_i$ such that for all i, s: $p_s = \lambda_i \pi^i_s u^{i\prime}(\overline{x}^i_s)$.

(i) For $i, j \in I_2$ we have $\lambda_i a^i \pi^i_s = \lambda_j a^j \pi^j_s$, $\forall s$. This implies

$$\sum_{s=1}^{\infty} \lambda_i a^i \pi^i_s = \sum_{s=1}^{\infty} \lambda_j a^j \pi^j_s.$$

This implies $\lambda_i a^i = \lambda_j a^j$, hence $\pi^i = \pi^j$.

(ii) We firstly prove that the exists $C > 0$ such that:

$$\sum_{i \in I_1} \sum_{s=1}^{\infty} \pi^i_s |x^i_s| \leq C.$$

Define $e' = \overline{e} - \sum_{i \in I_2} x^i = \sum_{i \in I_1} x^i \in l^1(\pi)$.

For $i \in I_1$, since $a^i < \inf_s u^{i\prime}(\overline{x}^i_s) \leq \sup_s u^{i\prime}(\overline{x}^i_s) < b^i$, we have $\overline{x}^i \in l^\infty(\pi)$. Observe that $p \in l^\infty(\pi)$.

Choose $\eta > 0$ such that

$$a^i < u^{i\prime}(\overline{x}^i_s)(1+\eta) < b^i \text{ for} \tag{1}$$

for all $i \in I_1$. Then we define the price q as follows: $\forall i, j \in I_1$,

$$q_s = p_s(1+\eta) = \lambda_i \pi^i_s u^{i\prime}(\overline{x}^i_s)(1+\eta) = \lambda_j \pi^j_s u^{j\prime}(\overline{x}^j_s)(1+\eta).$$

It follows from (1) that, for each $i \in I_1$, there exist $\overline{z}^i$ such that $\forall s$, $q_s = \lambda_i \pi^i_s u^{i\prime}(\overline{z}^i_s)$. Observe that $a^i < \inf_s u^{i\prime}(z^i_s) \leq \sup_s u^{i\prime}(z^i_s) < b^i$, so $\overline{z}^i \in l^\infty(\pi)$. Observe also that $\forall s,\ p_s < q_s$.

Denote

$$x^+ \ :=\begin{cases} x \ \text{ if } \ x > 0 \\ 0 \text{ if } x \leq 0 \end{cases}$$

$$x^- \ :=\begin{cases} -x^i \ \text{ if } \ x < 0 \\ 0 \text{ if } x \geq 0 \end{cases}$$

Notice that $x = x^+ - x^-$, $|x| = x^+ + x^-$ and $u(x) = u(x^+) + u(-x^-) - u(0)$.

Now we fix $N \in \mathbb{N}$. For $i \in I_1$, from the concavity of the utility function u^i we have

$$\lambda_i \sum_{s=1}^{N} \pi^i_s u^i(\overline{x}^i_s) - \lambda_i \sum_{s=1}^{N} \pi^i_s u^i(x^{i+}_s) \ \geq \ \lambda_i \sum_{s=1}^{N} \pi^i_s u^{i\prime}(\overline{x}^i_s)(\overline{x}^i_s - x^{i+}_s)$$

$$\lambda_i \sum_{s=1}^{N} \pi^i_s u^i(\overline{z}^i_s) - \lambda_i \sum_{s=1}^{N} \pi^i_s u^i(-x^{i-}_s) \ \geq \ \lambda_i \sum_{s=1}^{N} \pi^i_s u^{i\prime}(\overline{z}^i_s)(\overline{z}^i_s + x^{i-}_s).$$

Therefore,

$$\begin{aligned}\lambda_i \sum_{s=1}^{N} \pi_s^i u^{i\prime}(\overline{z}_s^i) x_s^{i-} \quad &\leq \quad \lambda_i \sum_{s=1}^{N} \pi_s^i [u^i(\overline{z}_s^i) + u^i(\overline{x}_s^i) - u^i(x_s^{i+}) - u^i(-x_s^{i-})] \\ & \quad -\lambda_i \sum_{s=1}^{N} \pi_s^i u^{i\prime}(\overline{z}_s^i)\overline{z}_s^i + \lambda_i \sum_{s=1}^{N} \pi_s^i u^{i\prime}(\overline{x}_s^i) x_s^{i+} - \lambda_i \sum_{s=1}^{N} \pi_s^i u^{i\prime}(\overline{x}_s^i)\overline{x}_s^i.\end{aligned}$$

Define $U_N^i(x) := \sum_{s=1}^N \pi_s^i u^i(x_s)$. Note that $\lim_{N\to\infty} U_N^i(x) = U^i(x)$. The above inequality implies

$$\begin{aligned}\sum_{s=1}^{N} q_s x_s^{i-} \quad &\leq \quad \lambda_i[U_N^i(\overline{z}^i) + U_N^i(\overline{x}^i) - U_N^i(x^i) - U_N^i(0)] \\ & - \quad \sum_{s=1}^{N} q_s \overline{z}_s^i + \sum_{s=1}^{N} p_s x_s^{i+} - \sum_{s=1}^{N} p_s \overline{x}_s^i \\ & \leq \quad \lambda_i[U_N^i(\overline{z}^i) + U_N^i(\overline{x}^i) - U_N^i(x^i) - U_N^i(0)] \\ & - \quad \sum_{s=1}^{N} p_s^i \overline{x}_s^i - \sum_{s=1}^{N} q_s \overline{z}_s^i + \sum_{s=1}^{N} p_s x_s^{i+} \\ = & \qquad C_N^i + \sum_{s=1}^{N} p_s x_s^{i+}\end{aligned}$$

where $C_N^i = \lambda_i[U_N^i(\overline{z}^i) + U_N^i(\overline{x}^i) - U_N^i(x^i) - U_N^i(0)] - \sum_{s=1}^N p_s^i \overline{x}_s^i - \sum_{s=1}^N q_s \overline{z}_s^i$.

Observe that since $\overline{x}^i$ et $\overline{z}^i$ belong to $l^\infty(\pi)$, the limit $\lim_N C_N^i$ exists.

Hence, $\forall i$

$$\sum_{s=1}^{N} (q_s - p_s) x_s^{i-} \leq C_N^i + \sum_{s=1}^{N} p_s x_s^i.$$

Thus, we have

$$\sum_{i\in I_1} \sum_{s=1}^{N} (q_s - p_s) x_s^{i-} \leq \sum_{i\in I_1} C_N^i + \sum_{i\in I_1} \sum_{s=1}^{N} p_s x_s^i = \sum_{i\in I_1} C_N^i + \sum_{s=1}^{N} p_s e'_s.$$

Since $e' \in l^1(\pi)$, $\sum_{i\in I_1} C_N^i + \sum_{s=1}^N p_s e'_s$ converges. Now let N tends to infinity. Notice that $U_N^i(x) \to U^i(x)$ for all x, and recall that $U^i(x^i) \geq U^i(e^i)$, with $\overline{x}^i, \overline{z}^i \in l^\infty(\pi)$. We then have

$$\limsup_{N\to\infty} C_N^i \leq \lambda_i[U^i(\overline{z}^i) + U^i(\overline{x}^i) - U^i(e^i) - U^i(0)] - \sum_{s=1}^{\infty} q_s \overline{z}_s^i - \sum_{s=1}^{\infty} p_s \overline{x}_s^i =: C^i.$$

Thus,

$$\sum_{i\in I_1} \sum_{s=1}^{\infty} (q_s - p_s) x_s^{i-} \leq \sum_{i\in I_1} C^i + \sum_{s=1}^{\infty} p_s e'_s =: C_1 + \sum_{s=1}^{\infty} p_s e'_s.$$

We also have

$$\sum_{i\in I_1}\sum_{s=1}^{\infty}(q_s-p_s)(x_s^{i+}-x_s^{i-})=\sum_{i\in I_1}\sum_{s=1}^{\infty}(q_s-p_s)x_s^i=\sum_{s=1}^{\infty}(q_s-p_s)e'_s$$

which implies

$$\begin{aligned}\sum_{i\in I_1}\sum_{s=1}^{\infty}(q_s-p_s)x_s^{i+} &= \sum_{s=1}^{\infty}(q_s-p_s)e'_s+\sum_{i\in I_1}\sum_{s=1}^{\infty}(q_s-p_s)x_s^{i-}\\ &\leq C_1+\sum_{s=1}^{\infty}p_se'_s+\sum_{s=1}^{\infty}(q_s-p_s)e'_s\\ &= C_1+\sum_{s=1}^{\infty}q_se'_s.\end{aligned}$$

Thus for $i\in I_1$

$$\begin{aligned}\sum_{s=1}^{\infty}(q_s-p_s)|x_s^i| &\leq 2C_1+\sum_{s=1}^{\infty}(p_s+q_s)e'_s\\ &= 2C_1+(2\eta+1)\sum_{s=1}^{\infty}p_se'_s\\ &= 2C_1+(2\eta+1)\sum_{s=1}^{\infty}p_s\overline{e}_s-(2\eta+1)\sum_{i\in I_2}\sum_{s=1}^{\infty}p_sx_s^i\\ &= 2C_1+(2\eta+1)\sum_{s=1}^{\infty}p_s\overline{e}_s-(2\eta+1)\sum_{i\in I_2}\lambda_ia^i\sum_{s=1}^{\infty}\pi_s^ix_s^i\\ &= 2C_1+(2\eta+1)\sum_{s=1}^{\infty}p_s\overline{e}_s-(2\eta+1)\sum_{i\in I_2}\lambda_iU^i(x^i)\\ &\leq 2C_1+(2\eta+1)\sum_{s=1}^{\infty}p_s\overline{e}_s-(2\eta+1)\sum_{i\in I_2}\lambda_iU^i(e^i)\\ &= C^2.\end{aligned}$$

then

$$\eta\sum_{s=1}^{\infty}p_s|x_s^i|\leq C^2.$$

Let $\mu_i:=\inf_s u^{i\prime}(\bar{x}_s^i)>0$, and $\mu:=\min_i\mu_i$. Then $\sum_{s=1}^{\infty}p_s|x_s^i|\geq\mu\sum_{s=1}^{\infty}\pi_s^i|x_s^i|$ which implies for all $i\in I_1$

$$\sum_{s=1}^{\infty}\pi_s^i|x_s^i|\leq D_1$$

with $D_1=C^2/(\eta\mu)$.

For I_2 we have:

$$\begin{aligned}\sum_{s=1}^{\infty}\pi_s^I|\sum_{i\in I_2}x_s^i| &\leq \sum_{s=1}^{\infty}\pi_s^I|\bar{e}_s| + \sum_{i\in I_1}\sum_{s=1}^{\infty}\pi_s^i|x_s^i| \\ &\leq D_2\end{aligned}$$

with $D_2 = \sum_{s=1}^{\infty}\pi_s^I|\bar{e}_s| + |I_1|D_1$.

We take $C = \max\{D_1, D_2\}$.

(iii) The utility set $\mathcal{U}$ is bounded by Jensen inequality. ∎

Proof of Proposition 1

(i) The proof of the boundedness of $\mathcal{U}$ comes from Lemma 1.

(ii.1) Consider the case where $I_2 = \emptyset$.

(a) Assume $u^{i\prime}(-\infty) = +\infty$, for any i. Let a satisfy $a^1 < u^{1\prime}(a)$. For $i > 1$, let

$$\zeta_s^i = \frac{\pi_s^1}{\pi_s^i}u^{1\prime}(a), \forall s$$

Then $\frac{1}{h}u^{1\prime}(a) \geq \zeta_s^i \geq hu^{1\prime}(a)$. One can find λ^i s.t. $\frac{\zeta_s^i}{\lambda^i} \geq \alpha^i > a^i$. Define

$$u^{i\prime}(x_s^i) = \frac{\zeta_s^i}{\lambda^i}, \forall s$$

Then

$$a^i < \alpha^i \leq u^{i\prime}(x_s^i) \leq \frac{1}{\lambda^i h}u^{1\prime}(a), \forall s$$

Since $b^i = +\infty$, we have $x^i \in l^\infty$. Obviously

$$\lambda^i\pi_s^i u^{i\prime}(x_s^i) = \pi_s^1 u^{1\prime}(a), \forall s.$$

(b) Assume $u^{i\prime}(+\infty) = 0$ for all i. Let a satisfy $0 < u^{1\prime}(a) < b^1$. Define ζ_s^i as before. We have $\zeta_s^i \leq \frac{1}{h}u^{1\prime}(a)$. Choose λ^i s.t. $\frac{\zeta_s^i}{\lambda^i} \leq \beta^i < b^i$. Then define x_s^i as before in **A**.1. Using the same arguments, we have

$$\lambda^i\pi_s^i u^{i\prime}(x_s^i) = \pi_s^1 u^{1\prime}(a), \forall s.$$

(ii.2) Now we consider the case where $I_2 \neq \emptyset$. If (**NA**) holds, from Lemma 1,$\pi^k = \pi^l, \forall k \in I_2, \forall l \in I_2$.

Conversely, assume that $\pi^k = \pi^l = \pi', \forall k \in I_2, \forall l \in I_2$. Assume $1 \in I_2$. For $i \in I_2, i \neq 1$, choose λ^i such that $\lambda^i a_i = a_1$.
Consider the case $u^{i\prime}(-\infty) = +\infty, \forall i \in I_1$.
For $i \in I_1$, choose as before ζ_s^i:

$$\frac{1}{h}a_1 \geq \zeta_s^i = \frac{a_1\pi_s'}{\pi_s^i} \geq a_1 h$$

There exists λ^i s.t.

$$\frac{\zeta_s^i}{\lambda^i} \geq \alpha^i > a^i$$

and

$$a^i < u^{i\prime}(x_s^i) = \frac{\zeta_s^i}{\lambda^i} \leq \frac{1}{h} a_1.$$

Since $b^i = +\infty$, we have $x^i \in l^\infty$.

The same argument as before if $u^{i\prime}(+\infty) = 0, \forall i \in I_1$. ■

Lemma 2 *A closed, bounded set B in $l^1(\pi)$ is compact if and only if B satisfies the following property: For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in B$ we have*

$$\sum_{s \geq N} \pi_s |x_s| < \epsilon.$$

Proof: Suppose that B is compact and there exists a subsequence $\{x(n)\}_n$ of B, $\epsilon > 0$ such that

$$\sum_{s=n}^{\infty} \pi_s |x_s(n)| > \epsilon, \forall n.$$

Without loss of generality, we can assume that $x(n)$ converges to x in $l^1(\pi)$ or $\|x(n) - x\|_{l^1(\pi)} \to 0$.

By choosing N large enough such that $\|x(n) - x\|_{l^1(\pi)} < \frac{\epsilon}{2}$ and $\sum_{s \geq n} \pi_s |x_s| < \frac{\epsilon}{2}$ for all $n \geq N$. And for all $n \geq N$ we have

$$\begin{aligned}
\sum_{s=n}^{\infty} \pi_s |x_s| &\geq \sum_{s=n}^{\infty} \pi_s |x_s(n)| - \sum_{s=n}^{\infty} \pi_s |x_s(n) - x_s| \\
&> \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}.
\end{aligned}$$

A contradiction.

Now we suppose that for any $\epsilon > 0$, there exists N such that $\forall x \in B$, $\sum_{s=N}^{\infty} \pi_s |x_s| < \epsilon$.

We have to prove that for any sequence $x(n) \in B$, there exists a convergent subsequence of $x(n)$ in $l^1(\pi)$.

Since B is bounded in $l^1(\pi)$, there exists $a > 0$ such that, $\forall\, x \in B$, $\sum_{s \geq 1} \pi_s |x_s| \leq a$.

Since $\{x(n)\}_n$ belong to a compact set for the product topology, there exists a subsequence $\{x(n_k)\}_k$ converges to $\overline{x}$ for the product topology. In particular for all s, $x_s(n_k)$ converges to $\overline{x}_s$ when $k \to \infty$.

Fix $\epsilon > 0$. We will prove that for k, l big enough, $\|x(n_k) - x(n_l)\|_{l^1(\pi)} < \epsilon$. Choose $N > 0$ such that for all $x \in B$, $\sum_{s \geq N} \pi_s |x_s| < \frac{\epsilon}{4}$. Choose M such that for all $n_k > M$ we have $\sum_{s=1}^{M} \pi_s |x_s(n_k) - \overline{x}_s| < \frac{\epsilon}{4}$. For all $n_k \geq N_0, n_l \geq N_0$

where $N_0 = \max\{N, M\}$ we have $\sum_{s=1}^{N_0} \pi_s |x_s(n_k) - x_s(n_l)| \leq \sum_{s=1}^{N_0} \pi_s |x_s(n_k) - \overline{x}_s| + \sum_{s=1}^{N_0} \pi_s |x_s(n_k) - \overline{x}_s| < \frac{\epsilon}{2}$.

Then

$$\begin{aligned}\sum_{s\geq 1} \pi_s |x_s(n_k) - x_s(n_l)| &\leq \sum_{s=1}^{N_0-1} \pi_s |x_s(n_k) - x_s(n_l)| + \sum_{s\geq N_0} \pi_s |x_s(n_k)| + \sum_{s\geq N_0} \pi_s |x_s(n_l)| \\ &< \epsilon\end{aligned}$$

Hence $\{x(n_k)\}_k$ is a Cauchy sequence, then it converges in $l^1(\pi)$ topology. So B is compact in $l^1(\pi)$ topology. ■

Corollary 1 *1. A closed, bounded set B in $l^1(\pi)$ is compact for $l^1(\pi)$-topology if and only if it is compact for the weak topology $\sigma(l^1(\pi)), l^\infty(\pi))$.*
2. A closed, bounded set B in $l^p(\pi)$, $p > 1$ is compact for $l^1(\pi)$.

Proof: 1. Since Lemma 2 is equivalent to the Dunford-Pettis criterion, the result follows.
2. For $p > 1$, a closed bounded set is $\sigma(l^p, l^q)$-compact. But it is also $\sigma(l^1, l^\infty)$-compact, since $l^p(\pi) \subset l^1(\pi)$ and $l^\infty \subset l^q(\pi)$. Apply statement 1. ■

Proof of Proposition 2

The idea of the proof is that, if the attainable allocation sequence does not belong to a weakly compact set, then for some state s, there will be an agent i such that x_s^i tends to $+\infty$ and an agent j such that x_s^j tends to $-\infty$. Then by reducing x_s^i and increasing x_s^j, the value of $U^i(x^i)$ does not diminish very much. Because $b^j = +\infty$, the value of $U^j(x^j)$ will become very large, even tends to infinity, and that leads us to a contradiction with the bounded property of $\mathcal{U}$.

Assume the contrary: $\mathcal{A}$ is not compact. Then from Lemma 1, there exists a sequence $\{(x^1(n), x^2(n), ..., x^m(n))\}_n \subset \mathcal{A}$, an agent i and a constant $\epsilon > 0$ such that

$$\forall\, n, \sum_{s=n}^{\infty} \pi_s^i |x_s^i(n)| > \epsilon.$$

Denote for all k, $v^k := \limsup_{n\to\infty} U^k(x^k(n))$.

By Lemma 1, $\mathcal{A}$ is bounded in $l^1(\pi)$. We can suppose, without loss generality, that $\sum_{s=n}^{\infty} \pi_s^i |x_s^i(n)| \to c^i > 0$ when $n \to \infty$. This implies $\lim_n \sum_{s=n}^{\infty} \pi_s^i x_s^{i+}(n) - \lim_n \sum_{s=n}^{\infty} \pi_s^i x_s^{i-}(n) = c^i$. The limits of these two sums exist because $x^i \in l^1(\pi)$. We know that $\sum_{j\neq i} x_s^i(n) = \overline{e}_s - x_s^i(n)$. So, for every s, $\exists j$ such that $x_s^j(n) \leq -\frac{x_s^i(n) - |\overline{e}_s|}{m-1}$. Since there is a finite number of agents $j \neq i$, we can assume that, for simplicity, there exist i and j which satisfy two properties:

1. $\exists\ E^i_n \subset \mathbb{N} \cap \{s \geq n\}$, $x^i_s > 0$ for all $s \in E^i_n$ and

$$\lim_n \sum_{s \in E^i_n} \pi^i_s x^i_s(n) = c^i > 0.$$

2. For all $s \in E^i_n$

$$x^j_s(n) \leq -\frac{x^i_s(n) - |\overline{e}_s|}{m-1}.$$

With each $M > 0$, define the set $S^i_n \subset E^i_n$ as follows

$$S^i_n = \{s : x^i_s(n) > |\overline{e}_s| + M(m-1)\}. \tag{2}$$

We have an observation: $\lim_n \sum_{E^i_n \setminus S^i_n} \pi^i_s x^i_s(n) = 0$. Indeed

$$\begin{aligned} \lim_{n\to\infty} \sum_{s \in E^i_n \setminus S^i_n} \pi^i_s x^i_s(n) &\leq \sum_{s \in E^i_n \setminus S^i_n} \pi^i_s \left(|\overline{e}_s| + M(m-1)\right) \\ &\leq \sum_{s=n}^{\infty} \pi^i_s |\overline{e}_s| + M(m-1) \sum_{s=n}^{\infty} \pi^i_s \end{aligned}$$

which tends to zero, since $\overline{e} \in l^1(\pi)$.

Hence we have $S^i_n \neq \emptyset$ for all n big enough, and

$$\lim_{n\to\infty} \sum_{s \in S^i_n} \pi^i_s x^i_s(n) = c^i.$$

We have

$$x^j_s(n) \leq \frac{|\overline{e}_s| - x^i_s(n)}{m-1} < -M. \tag{3}$$

Since π^i and π^j are equivalent, we can assume that $\lim_n \sum_{s \in S^i_n} \pi^j_s x^i_s(n) = c^j > 0$. Notice that these limits do not depend on M.

Define $\alpha := \min(v^k, v^i - \frac{u^{i\prime}(0)c^i}{m-1}) - 1, (k = 1, \ldots, m)$. Define $\mathcal{A}_\alpha$ the set of $(x^k) \in l^1(\pi)$ satisfies $U^k(x^k) \geq \alpha\ \forall k$ and $\sum x^k = \overline{e}$. From Lemma 1 we know that there exists $C > 0$ such that $U^j(x^j) < C$ for all $(x^1, \ldots, x^m) \in \mathcal{A}_\alpha$. Notice that our sequence $(x^k(n)) \in \mathcal{A}_\alpha$ for n large enough.

Since $b^j = +\infty$ we can choose M very big such that

$$v^j + \frac{u^{j\prime}(-M)c^j}{m-1} > C.$$

Now consider the sequence $(y^1(n), y^2(n), \ldots, y^m(n))$ defined as follows

$$y^i_s(n) := x^i_s(n) - \frac{x^i_s - |\overline{e}_s|}{m-1} + M \text{ with } s \in S^i_n,$$

$$y^j_s(n) := x^j_s(n) + \frac{x^i_s - |\overline{e}_s|}{m-1} - M \text{ with } s \in S^i_n.$$

Let $y^k_s = x^k_s$ with every $k \neq i, j$ or $s \notin S^i_n$.

Notice that $\sum_i y^i(n) = \overline{e}$, and $y^i_s(n) \le x^i_s(n)$, $y^j_s(n) \ge x^j_s(n)$ for all s. We will prove that $\{U^l(y^l(n))\}_{l=1,m}$ is bounded below by α, but $U^j(y^j(n))$ is not bounded above by C. And this is a contradiction.

Indeed,

$$\begin{aligned}
U^i(y^i(n)) - U^i(x^i(n)) &= \sum_{s\in S^i_n} \pi^i_s (u^i(y^i_s(n)) - u^i(x^i_s(n))) \\
&\ge \sum_{s\in S^i_n} \pi^i_s u^{i\prime}(x^i_s(n) - \frac{x^i_s(n) - |\overline{e}_s|}{m-1} + M)(-\frac{x^i_s(n) - |\overline{e}_s|}{m-1} + M) \\
&\ge \sum_{s\in S^i_n} \pi^i_s u^{i\prime}(M)(-\frac{x^i_s(n)}{m-1}) + u^{i\prime}(M)(\frac{|\overline{e}_s|}{m-1} + M) \sum_{s\in S^i_n} \pi^i_s \\
&\ge -\frac{u^{i\prime}(M)}{m-1} \sum_{s\in S^i_n} \pi^i_s x^i_s(n) + u^{i\prime}(M)(\frac{|\overline{e}_s|}{m-1} + M) \sum_{s\in S^i_n} \pi^i_s.
\end{aligned}$$

When $n \to \infty$, the second term of the right hand side term in the inequality above tends to zero while first term tends to $-\frac{u^{i\prime}(M)c^i}{m-1}$. Thus,

$$\limsup_{n\to\infty} U^i(y^i(n)) \ge v^i - \frac{u^{i\prime}(M)c^i}{m-1} \ge v^i - \frac{u^{i\prime}(0)c^i}{m-1} > \alpha.$$

For n large enough, $U^k(y^k(n))$ is bounded below by α,$\forall k \neq j$. Then we can estimate the limit of $U^j(y^j(n))$ when $n \to \infty$,

$$\begin{aligned}
U^j(y^j(n)) - U^j(x^j(n)) &= \sum_{s\in S^i_n} \pi^j_s (u^j(y^j_s(n)) - u^j(x^j_s(n))) \\
&\ge \sum_{s\in S^i_n} \pi^j_s u^{j\prime}(x^j_s(n) + \frac{x^i_s(n) - |\overline{e}_s|}{m-1} - M)(\frac{x^i_s(n) - |\overline{e}_s|}{m-1} - M)
\end{aligned}$$

$$\begin{aligned}
U^j(y^j(n)) - U^j(x^j(n)) &\ge \sum_{s\in S^i_n} \pi^j_s u^{j\prime}(-M)(\frac{x^i_s(n) - |\overline{e}_s|}{m-1} - M) \\
&\ge \frac{u^{j\prime}(-M)}{m-1} \sum_{s\in S^i_n} \pi^j_s x^i_s(n) - M u^{j\prime}(-M) \sum_{s\in S^i_n} \pi^j_s - \frac{u^{j\prime}(-M)}{m-1} \sum_{s\in S^i_n} |\overline{e}_s| \pi^j_s.
\end{aligned}$$

Take the limit

$$\limsup_{n\to\infty} U^j(y^j(n)) \ge v^j + \frac{u^{j\prime}(-M)c^j}{m-1} > C.$$

A contradiction. Hence $\mathcal{A}$ is $l^1(\pi)$-compact. ■

For the proof of Proposition 3, we require the following lemma

Lemma 3 *Suppose that $\mathcal{A}$ is bounded and $(v^1, v^2, \ldots, v^m)$ is in the closure of $\mathcal{U}$. Suppose that there exists a sequence $\{x(n)\}_n \subset \mathcal{A}$ such that there exists i such that $\lim_n U^i(x^i(n)) > v^i$, and for all $j \neq i$, $\lim_n U^j(x^j(n)) \geq v^j$. Then $(v^1, v^2, \ldots, v^m) \in \mathcal{U}$.*

Proof: Fix $t \in \mathbb{N}$ arbitrarily. Let $C > 0$ be the upper bound of $\mathcal{A}$ in $l^1(\pi)$, we know that $|x^j_t(n)| < \frac{C}{\pi^j_t}$ for all j and all n. Fix some j. We define the sequence $\{y^k(n)\}_{k=1,\ldots,m}$ as follows

$$\begin{aligned}
y^k(n) &= x^k(n) \text{ if } k \neq i, j \\
y^i_s(n) &= x^i_s(n) \text{ if } s \neq t \\
y^i_t(n) &= x^i_t(n) - \epsilon \\
y^j_t(n) &= x^j_t(n) + \epsilon
\end{aligned}$$

For $k \neq i, j$, $\lim_n U^k(y^k(n)) = v^k$. And we have

$$\begin{aligned}
U^i(y^i(n)) - U^i(x^i(n)) &= \pi^i_t(u^i(y^i_t(n)) - u^i(x^i_t(n))) \\
&\geq \pi^i_t(-\epsilon)u^{i\prime}(x^i_t(n) - \epsilon) \geq -\epsilon\pi^i_t u^{i\prime}(-\frac{C}{\pi^i_t} - \epsilon)
\end{aligned}$$

and

$$\begin{aligned}
U^j(y^j(n)) - U^j(x^j(n)) &= \pi^j_t(u^j(y^j_t(n)) - u^j(x^j_t(n))) \\
&\geq \pi^j_t \epsilon u^{j\prime}(x^j_t(n) + \epsilon) \geq \epsilon\pi^j_t u^{j\prime}(\frac{C}{\pi^j_t} + \epsilon).
\end{aligned}$$

Since $\liminf_n U^i(x^i(n)) > v^i$, by choosing ϵ small enough, the sequence $\{y(n)\}_n$ will satisfy $\liminf_n U^i(y^i(n)) > v^i$ and $\liminf_n U^j(y^j(n)) > v^j$.

By induction we can find a sequence $\{z^k(n)\}_n \subset \mathcal{A}$ which satisfies $\lim_n U^k(z^k(n)) > v^k$ for all $k = 1, 2, \ldots, m$. Hence $(v^1, v^2, \ldots, v^m) \in \mathcal{U}$. ■

Proof of Proposition 3

Since the (**NA**)condition holds, from Proposition 1, we know that $\mathcal{U}$ is bounded. We will prove that $\mathcal{U}$ is closed. Suppose that $(v^1, \ldots, v^m)$ belong to the closure of $\mathcal{U}$ and the sequence $\{x(n)\}_n \subset \mathcal{A}$ such that $\lim_n U^i(x^i(n)) = v^i$.

If the sequence $\{x(n)\}_n$ belongs to a compact set of $l^1(\pi)$, without loss of generality, we can suppose that $\lim_n x^i(n) = x^i$ in this topology. Since U^i is continuous, we have $U^i(x^i) \geq v^i$ for all i. Thus $(v^1, \ldots, v^m) \in \mathcal{U}$.

If the sequence $\{x(n)\}_n$ does not belong to a compact set, we can suppose that there exists $c > 0$ such that for an agent i

$$\lim_{n\to\infty} \sum_{s=n}^{\infty} \pi^i_s |x^i_s(n)| = c.$$

As in the proof of Proposition 2, we can choose a pair (i, j) which satisfies the two properties:

1. $\exists\ E_n^i \subset \mathbb{N} \cap \{s \geq n\}$, $x_s^i > 0$ for all $s \in E_n^i$ and

$$\lim_n \sum_{s \in E_n^i} \pi_s^i x_s^i(n) = c^i > 0.$$

2. For all $s \in E_n^i$

$$x_s^j(n) \leq -\frac{x_s^i(n) - |\overline{e}_s|}{m-1}.$$

With each $M > 0$, define the set $S_n^i \subset E_n^i$ as follows

$$S_n^i = \{s : x_s^i(n) > |\overline{e}_s| + M(m-1)\}. \tag{4}$$

We have an observation: $\lim_n \sum_{E_n^i \setminus S_n^i} \pi_s^i x_s^i(n) = 0$. Indeed

$$\begin{aligned}\lim_{n\to\infty} \sum_{s \in E_n^i \setminus S_n^i} \pi_s^i x_s^i(n) &\leq \sum_{s \in E_n^i \setminus S_n^i} \pi_s^i \left(|\overline{e}_s| + M(m-1)\right) \\ &\leq \sum_{s=n}^{\infty} \pi_s^i |\overline{e}_s| + M(m-1) \sum_{s=n}^{\infty} \pi_s^i\end{aligned}$$

which tends to zero, since $\overline{e} \in l^1(\pi)$.

Hence we have $S_n^i \neq \emptyset$ for all n big enough, and

$$\lim_{n\to\infty} \sum_{s \in S_n^i} \pi_s^i x_s^i(n) = c^i.$$

We have

$$x_s^j(n) \leq \frac{|\overline{e}_s| - x_s^i(n)}{m-1} < -M. \tag{5}$$

Since π^i and π^j are equivalent, we can assume that $\lim_n \sum_{s \in S_n^i} \pi_s^j x_s^i(n) = c^j > 0$. Notice that these limits do not depend on M.

Define $\alpha := \min(v^k, v^i - \frac{u^{i\prime}(0)c^i}{m-1}) - 1, (k = 1, \ldots, m)$. Define $\mathcal{A}_\alpha$ the set of $(x^k) \in l^1(\pi)$ satisfies $U^k(x^k) \geq \alpha\ \forall k$ and $\sum x^k = \overline{e}$. From Lemma 1 we know that there exists $C > 0$ such that $U^j(x^j) < C$ for all $(x^1, \ldots, x^m) \in \mathcal{A}_\alpha$. Notice that our sequence $(x^k(n)) \in \mathcal{A}_\alpha$ for n large enough. Fix $\epsilon > 0$. Since $u^{i\prime}(+\infty) = 0$ we can choose $M > 0$ such that $u^{i\prime}(M) < (m-1)\epsilon/c$. By similar arguments as in the proof of Proposition 2, we can construct the sequence $(y^k(n))$ such that:

$$\liminf_{n\to\infty} U^i(y^i(n)) \geq v^i - \frac{u^{i\prime}(M)c^i}{m-1}$$

$$\liminf_{n\to\infty} U^j(y^j(n)) \geq v^j + \frac{u^{j\prime}(-M)c^j}{m-1}$$

$$\liminf_{n\to\infty} U^k(y^k(n)) = v^k \text{ for all } k \neq i, j$$

with $c^i, c^j > 0$ and $c^i < c$ and c^i and c^j do not depend on M.

So, for n large enough, $U^i(y^i(n)) > v^i - \epsilon$, and for all $k \neq i, j$, $U^k(y^k(n)) > v^k - \epsilon$ whereas $\lim_n U^j(y^j(n)) = v^j + \frac{u^{j\prime}(-M)c^j}{m-1} > v^j + \frac{u^{j\prime}(0)c^j}{m-1} > v^j$. Let $\epsilon \to 0$ and by applying the Lemma 3, we have $(v^1, v^2, \dots, v^m) \in \mathcal{U}$. ■

Proof of Proposition 4

We proceed by two steps.

Step 1 We assume $I_2 = \{i_0\}$. Suppose that the sequence $\{x_n\}$ satisfies $\lim_n U^i(x^i_n) = v^i$. We prove that $(v^1, \dots, v^m) \in \mathcal{U}$. If $\{x_n\}$ belongs to a compact set of $l^1(\pi)$ topology, then $\liminf_n U^i(x^i) \geq v^i$, hence $v \in \mathcal{U}$.

Suppose that the sequence $\{x_n\}$ does not belong to a compact set. By using the same argument in the proof of Proposition 2, there exist i and j which satisfy two properties:

1. $\exists\, E^i_n \subset \mathbb{N} \cap \{s \geq n\}$, $x^i_s > 0$ for all $s \in E^i_n$ and
$$\lim_n \sum_{s \in E^i_n} \pi^i_s x^i_s(n) = c^i > 0.$$

2. For all $s \in E^i_n$
$$x^j_s(n) \leq -\frac{x^i_s(n) - |\bar{e}_s|}{m-1}.$$

If $j \neq i_0$, since $b^j = +\infty$, by using the same arguments in the proof of Theorem 1 we have a contradiction.

We consider now the case $j = i_0$. Since $i \neq i_0$, $a^i = 0$, by using the same arguments in the proof of Proposition 2, we have $(v^1, \dots, v^m) \in \mathcal{U}$.

Hence $\mathcal{U}$ is compact.

Step 2

Claim Let $f^1, \dots, f^n$, be n vectors in $l^p(\pi)$, and $p \in l^\infty(\pi)$. Take any $x \in l^p(\pi)$ such that $p \cdot x = p \cdot \sum_i f^i$. Then there exists $x^1, \dots, x^n$ in $l^p(\pi)$ such that $\sum_i x^i = x$ and $p \cdot x^i = p \cdot f^i$ for all i.

Proof of the claim. This is true for $n = 1$. Suppose that the claim is true for $n-1$. Take any x^1 such that $p \cdot x^1 = p \cdot f^1$. We have $p \cdot (x - x^1) = p \cdot \sum_{i=2}^n f^i$. Using the hypothesis of induction, there exists $x^2, \dots, x^n$ such that $\sum_{i=2}^n x^i = x - x^1$ and $p \cdot x^i = p \cdot f^i$ for all i. ■

In the proof of Proposition 2, we have $\pi^i = \pi^j = \pi^I$ and $\lambda_i a^i = \lambda_j a^j = \zeta$, for all $i, j \in I_2$. For $x^I \in l^1(\pi)$ satisfying $x^I = \sum_{i \in I_2} x^i$, with $x^i \in l^1(\pi)$, define
$$U^I(x^I) = \sum_{i \in I_2} \lambda_i U^i(x^i).$$

We consider now the economy $\mathcal{E}^I$ with $|I_1|+1$ agents, $|I_1|$ agents who are risk averse, with endowment e^i, utility function U^i, and the last agent (denoted by I) with endowment $e^I = \sum_{i\in I_2} e^i$, utility function U^I. It is easy to verify that agent I is risk neutral, with

$$U^I(x_s^I) = \zeta \sum_{s=1}^{\infty} \pi_s^I x_s^I,$$

and the new economy satisfies (**NA**) condition. By Theorem 1, this economy has an equilibrium, denote by (p^*, x^*). For all $i \in I_1$, x^{*i} is the solution to

$$\begin{aligned} &\max \sum_{s=1}^{\infty} \pi_s^i u^i(x_s^i) \\ \text{s.t. } &\sum_{s=1}^{\infty} p_s^* x_s^i = \sum_{s=1}^{\infty} p_s^* e_s^i. \end{aligned}$$

and x^{*I} is the solution to

$$\begin{aligned} &\max U^I(x^I) \\ \text{s.t. } &\sum_{s=1}^{\infty} p_s^* x_s^I = \sum_{s=1}^{\infty} p_s^* e_s^I. \end{aligned}$$

If $U^I(x^i) > U^I(x^{*I})$, then $p^* \cdot x^i > p^* \cdot e^I$.

By the same arguments in the proof of the Theorem 1, there exists $\lambda_i^* > 0$, $\lambda_I > 0$ such that for $i \in I_1$, $p_s^* = \lambda_i \pi_s^i u^{i\prime}(x_s^{*i})$. For I, we have $p_s^* = \zeta \pi_s^I = \lambda_i a^i \pi_s^I$, $\forall i \in I_2$, $\forall s$. The function U^I is strictly increasing, so

$$p^* \cdot x^{*I} = p^* \cdot e^{*I} = p^* \cdot \sum_{i\in I_2} e^i.$$

By the claim, for all $i \in I_2$ there exist $x^{*i} \in l^p(\pi)$ such that $\sum_{i\in I_2} x^{*i} = x^{*I}$, and $p^* \cdot x^{*i} = p^* \cdot e^i$, $\forall i$.

Fix $i \in I_2$. Take x^i such that $U^i(x^i) > U^i(x^{*i})$. We prove that $p^* \cdot x^i > p \cdot e^i$. Indeed, we have

$$U^i(x^i) = \sum_{s=1}^{\infty} a^i \pi_s^i x_s^i = \frac{1}{\lambda_i} \sum_{s=1}^{\infty} \lambda_i a^i \pi_s^i = \frac{1}{\lambda_i} \sum_{s=1}^{\infty} p_s^* x_s^i.$$

Hence $U^i(x^i) > U^i(x^{*i})$ implies $p^* \cdot x^i > p^* \cdot x^{*i} = p^* \cdot e^i$.

We have proved that $(p^*, (x^{*i})_i)$ is an equilibrium of the model. ■

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