

Majority Voting in Multidimensional Policy Spaces :

Kramer-Shepsle versus Stackelberg

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Abstract

We study majority voting over a bidimensional policy space when the voters' type space is either uni- or bidimensional. We show that a Condorcet winner fails to generically exist even with a unidimensional type space. We then study two voting procedures widely used in the literature. The Stackelberg (ST) procedure assumes that votes are taken one dimension at a time according to an exogenously specified sequence. The Kramer-Shepsle (KS) procedure also assumes that votes are taken separately on each dimension, but not in a sequential way. A vector of policies is a Kramer-Shepsle equilibrium if each component coincides with the majority choice on this dimension given the other components of the vector.

We study the existence and uniqueness of the ST and KS equilibria, and we compare them, looking e.g. at the impact of the ordering of votes for ST and identifying circumstances under which ST and KS equilibria coincide. In the process, we state explicitly the assumptions on the utility function that are needed for these equilibria to be well behaved. We especially stress the importance of single crossing conditions, and we identify two variants of these assumptions: a marginal version that is imposed on all policy dimensions separately, and a joint version whose definition involves both policy dimensions.

Keywords: Unidimensional and bidimensional type space, single crossing, one-sided separability

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1 Introduction

It is well known that majority voting suffers from what Bernheim and Slavov (2009) call the “curse of multidimensionality”: when the policy space is sufficiently rich, there is no policy option that gathers a majority of votes when faced with all other possible options –i.e., there is no Condorcet winner (see e.g. Plott 1967, Davis, DeGroot and Hinich 1972, McKelvey, Ordeshook and Ungar 1980, Banks, Duggan and Le Breton 2006 and Banks and Austen-Smith 1999).

To the best of our knowledge, all rigorous formal versions of this result assume that the space of voters’ types is multidimensional, with a probability distribution of voters’ types whose support is multidimensional as well. Also, the respective roles of the properties of the types’ distribution function and of utility functions are not clearly disentangled. For instance, in the spatial model of politics (where preferences are Euclidean), the symmetry of preferences is imposed and the focus is exclusively on the distribution of voters’ types. The first objective of this paper is to fill this gap and to study the existence of a Condorcet winner with a unidimensional type space.

Faced with this “curse of multidimensionality”, the applied political economy literature has adopted several approaches, including the obvious one of restricting the policy space to be unidimensional. In this paper, we adopt a bidimensional policy space and we focus on two widely used approaches having in common that votes never take place simultaneously on all dimensions.

The first approach assumes that citizens vote sequentially on each dimension. An exogenous ordering of the dimensions is considered and, at each voting stage, the outcomes of the preceding votes are known to the voters. For instance, when there are two dimensions, a first majority vote is organized on one of the policy dimensions and is followed by a second majority vote on the other dimension. We call Stackelberg (ST) equilibria the policies that can be supported at equilibrium for a particular ordering of the dimensions. This sequential resolution has been used by many authors in political economy models (see e.g. Alesina, Baqir and Easterly 1999,

Alesina, Baqir and Hoxby 2004, Cremer, De Donder and Gahvari 2004, Cremer *et al.* 2007, De Donder, Le Breton and Peluso 2009, Etro 2006, Gregorini 2009, Haimanko, Le Breton and Weber 2005).

The second approach assumes instead that there is no sequential ordering of the votes, but that they are taken separately on each dimension. Under the presumption that all dimensions except one have been settled, citizens cast their vote on the residual dimension. A solution is consistent if the vector of policies obtained through that procedure is self-supporting in a Nash-like manner. This idea has been independently developed by Kramer (1972) and Shepsle (1979) and hereafter we will call Kramer-Shepsle's equilibria (KS) the policy vectors meeting this consistency condition. More precisely, a vector is a Kramer-Shepsle's equilibrium if, for any dimension, the corresponding component in the vector coincides with the majority choice on this dimension given the other components of the policy vector. This concept has also been studied by the applied political economy literature, e.g. by De Donder and Hindriks (1998), Diba and Feldman (1984), Nechyba (1997), Sadanand and Williamson (1991). On the other hand, to the best of our knowledge, the only theoretical contributions are two unpublished papers by Banks and Duggan (2004) and Duggan (2001).

In this paper, we provide an analysis of the KS and ST equilibria in a general framework with a bidimensional policy space. We study their existence, uniqueness and we compare them, looking e.g. at the impact of the ordering of votes for ST and identifying circumstances under which ST and KS equilibria coincide. In the process, we state explicitly the assumptions on the utility function that are needed for these equilibria to be well-behaved. We especially stress the importance of single-crossing conditions, and we identify two variants of these assumptions: a "marginal" version that is imposed on all policy dimensions separately, and a "joint" version whose definition involves both policy dimensions. We perform this analysis first with a unidimensional type space, and then with a bidimensional type space.

Our results run as follows. Starting with a unidimensional type space, we show that the "curse of multidimensionality" (of the policy space) applies in this case as well: even when we

assume that the utility function satisfies both marginal and joint single-crossing, there is generically no Condorcet winner and, perhaps more surprisingly, in most cases and for any policy proposal, it is possible to find a direction that is favored by quasi all voters. We then study the KS and ST equilibria in this setting. We show that marginal single crossing guarantees the existence and unicity of the KS solution, and that assuming also strategic complementarity between policy dimensions results in the KS equilibrium to coincide with the unique ST equilibrium (independently of the ordering of votes). We then study a specific environment that has received a lot of attention in different literatures (e.g. on nation formation) and which does not satisfy the marginal single-crossing property. We show that the ST equilibrium studied in the literature corresponds to the KS equilibrium, but that the ST equilibrium with the opposite sequence of votes (which, to the best of our knowledge, has not been studied previously) is more complex, with the identity of the second-stage decisive voter being affected by the first-stage voting decision. We provide a thorough analysis of how first-stage voting is impacted in that case (i.e., how voters bias their first-stage voting choices when anticipating the impact on the second-stage decisive voter's identity).

We then move to a bidimensional type space. We do not study the existence of a simultaneous Condorcet winner in this setting since, in contrast to the one-dimensional types' space case, this analysis would be very similar to what has already been done in theoretical political science (Banks and Austen-Smith 1999, Ordeshook 1986). Instead, we focus on the analysis of the sets of Kramer-Shepsle and Stackelberg equilibria. We show that marginal single-crossing ensures the existence of KS equilibria, but not their uniqueness. We then move to the well known spatial framework, which is often used in the political science literature, and we show that KS and ST equilibria are unique and coincide when individuals differ in the (bidimensional) location of their most-preferred policy but share the same shape of their indifference curves. We provide an example with a discrete number of types differing both in the location and in the shape of their indifference curves and where i) there are multiple KS equilibria, ii) not all KS equilibria correspond to ST equilibria (whatever the ordering of the votes) and iii)

some KS equilibria do not correspond to any voter's most-preferred policy.

The paper is organized as follows: Section 2 presents the one-dimensional type general framework. Its first subsection analyzes simultaneous voting, the second subsection studies and compares Kramer-Shepsle and Stackelberg equilibria, while the third subsection is devoted to the analysis of a specific environment studied e.g. in the nation formation literature. Section 3 focuses on the case with two-dimensional types. General results about ST and KS equilibria are reported in section 3.1., while section 3.2. analyzes the standard set up of spatial voting with quadratic preferences. Section 4 concludes. Most proofs are relegated to Appendices.

2 One-Dimensional Types

Throughout the paper, we consider a population of voters who have to select a public policy in a two-dimensional policy space. A policy choice is therefore a vector $(x, y) \in X$, where the set of feasible policy choices X is assumed to be a rectangular subset of \mathfrak{R}^2 .¹ In this section, we assume that each voter is described by a one-dimensional type $\theta \in \mathfrak{R}$. The statistical distribution of types is given by a continuous cumulative distribution function F whose support is an interval $[\underline{\theta}, \bar{\theta}]$ of \mathfrak{R} , with f the corresponding density. The utility of a citizen of type θ for policy (x, y) is denoted by $V(\theta, x, y)$ that is assumed to be twice continuously differentiable and concave in x and y . The following examples illustrate the broad spectrum of applications covered by this framework.

Example 1 (Absolute Intensity of the Preference for Public Goods)

Let $X = \mathfrak{R}_+^2$ and $V(\theta, x, y) = \theta U(x, y) - (x + y)$ where U is a twice continuously differentiable, increasing and concave function of x and y . In this setting, x and y denote the quantities of two different pure public goods produced under constant returns to scale and financed through per capita taxation. The parameter θ reflects the intensity of the preference for the bundle (x, y) of public goods (aggregated through U) with respect to the private numeraire.

¹This implies that the choice over one dimension does not have any implication on the feasible choices over the other dimension. A more general case is the subject of Banks and Duggan (2004).

Example 2 (Proportional Income Taxation)

Let $X = \mathfrak{R}_+^2$ and $V(\theta, x, y) = U(x, y) - \frac{\theta}{\bar{\theta}}(x+y)$ where U is a twice continuously differentiable, increasing and concave function of x and y and $\bar{\theta} = \int_{\underline{\theta}}^{\bar{\theta}} \theta f(\theta) d\theta$. In this setting, x and y denote, once again, the quantities of two different pure public goods produced under constant returns to scale. The parameter θ denotes the income of a citizen. Under that interpretation, V is simply the indirect utility of a citizen with income θ when the contribution to the financing of the public good is proportional to income.

Example 3 (Absolute and Relative Intensities of the Preferences for Public Goods)

Let $X = \mathfrak{R}_+^2$ and $V(\theta, x, y) = Ax^{\phi(\theta)}y^{\psi(\theta)} - (x+y)$ where A is a positive parameter and ϕ and ψ are continuously differentiable and increasing functions with values in $]0, 1[$. The economic interpretation is the same as in Example 1 with preferences for public goods assumed to be Cobb-Douglas. However, the parameter θ now plays two roles: it describes both the absolute intensity of the preference for both public goods and it also determines the marginal rate of substitution between the two public goods.

Example 4 (Spatial Politics with Differentiated weights)

Let $X = \mathfrak{R}^2$ and $V(\theta, x, y) = -\phi(\theta)(x - \theta)^2 - \psi(\theta)(y - \theta)^2$ where ϕ and ψ are two positive continuously differentiable functions. In this general framework, the parameter θ plays two roles. On one hand, it describes the favorite policy bundle of a citizen regardless of the specific features of ϕ and ψ . However, on the other hand, it also determines through these functions the respective weights placed by a citizen of the two dimensions. In the particular case where $\phi(\theta) = \psi(\theta) = 1$ for all θ in $[\underline{\theta}, \bar{\theta}]$, we obtain the spatial model of politics with the extra assumption that the support of the distribution is one dimensional (precisely the diagonal).

Example 5 (Local Jurisdictions, Nation Formation and “One and a Half Dimensional” Preferences)

Let $X = \mathfrak{R}_+ \times [0, 1]$ and $V(\theta, x, y) = v(x)\Psi(y - \theta) - x$ where v is increasing and concave, Ψ is increasing to the left of 0 and decreasing to the right of 0 and $\theta \in [0, 1]$. In this setting,

x denotes the quantity of a pure public good while y now denotes a horizontal characteristic of this public good. This policy environment has been analyzed by many authors, including Alesina, Baqir and Easterly (1999), Alesina, Baqir and Hoxby (2004), Perroni and Scharf (2001) in the analysis of local jurisdictions, and Etro (2006) and Gregorini (2009) in the exploration of models of nation formation. It is also reminiscent of the voting environment of Groseclose (2007) where the horizontal dimension denotes ideology while the other dimension represents valence (defined as an advantage that a candidate has due to a non-policy factor, such as incumbency or charisma). All voters have the same preference on the valence dimension (hence the term “one-and-a-half dimensional” used by Groseclose, 2007).

We first study the simultaneous voting game over the two dimensions before turning to sequential voting and the Kramer-Shepsle solution.

2.1 Simultaneous Voting

We now show that in the context of simultaneous voting over a bidimensional policy space with unidimensional voters’ types, the fact that utility functions satisfy reasonable “single-crossing” conditions does not guarantee the existence of a Condorcet winner. On the contrary, for any policy proposal, it is always possible to propose an alternative policy that is favored by quasi all voters. Throughout the paper, we assume the following monotonicity property:

Assumption 1 (Marginal Single-Crossing) *We assume that*

$$\frac{\partial^2 V(\theta, x, y)}{\partial \theta \partial x} \geq 0 \text{ and } \frac{\partial^2 V(\theta, x, y)}{\partial \theta \partial y} \geq 0$$

for all $(x, y) \in X$ and $\theta \in \mathfrak{R}$.

Assumption 1 simply states that the marginal utility of both dimensions increases monotonically with the type of the agent. This monotonicity assumption implies that the classical single-crossing condition (which states that “leftist voters tend to favor left policies more than voters who are rightist in political preferences” (Myerson, 1996, p.23)) is satisfied on each dimension separately, hence the term of marginal single-crossing assumption.

It is easy to see that Assumption 1 is satisfied in Examples 1 and 2,² and also in Example 3 if x and y are large enough. As for Example 4, we obtain $\frac{\partial^2 V(\theta, x, y)}{\partial \theta \partial x} = 2\phi(\theta) - 2\phi'(\theta)(x - \theta)$. The first term is always positive while the second term can take negative values. It is enough to bound the second term. Since $|x - \theta|$ is always less than 1, Assumption 1 holds as soon as $\phi'(\theta)$ is not too large. If we denote by m the minimum of $\phi(\theta)$ over $[\underline{\theta}, \bar{\theta}]$, then it will hold whenever $|\phi'(\theta)| < m$. The same analysis applies to $\frac{\partial^2 V(\theta, x, y)}{\partial \theta \partial y}$.

Assumption 1 does not hold for Example 5. We obtain $\frac{\partial^2 V(\theta, x, y)}{\partial \theta \partial x} = -v'(x)\Psi'(y - \theta)$ and $\frac{\partial^2 V(\theta, x, y)}{\partial \theta \partial y} = -\Psi''(y - \theta)$. The second-order derivative is always positive but the sign of the first-order derivative depends upon the position of y with respect to θ : its sign is positive iff $y > \theta$. This example is thus not covered by the results of this section and, given its importance in the literature, is analyzed separately in section 2.3.

We denote by $x(y, \theta)$ (respectively, $y(x, \theta)$) individual θ 's most-preferred value of x (resp., of y) for any given y (resp., given x). The following lemma (proved in Appendix 1) shows that concavity of the utility function together with Assumption 1 guarantee that i) the most-preferred value of x (respectively, of y) is increasing in θ , for any given y (resp., given x) and ii) the individual with the median type θ_{med} is decisive in both choices if they are taken separately.

Lemma 1 *Under Assumption 1,*

i) the most-preferred value of x (respectively, of y) is increasing in θ , for any given y (resp., given x):

$$\frac{\partial x(y, \theta)}{\partial \theta} \geq 0 \text{ and } \frac{\partial y(x, \theta)}{\partial \theta} \geq 0,$$

ii) there exists a majority voting equilibrium when voting over x for any given y (resp., over y for any given x)—i.e., a value of x such that there is no $x' \neq x$ that is favored by more than one half of the voters to x (and similarly for y). This equilibrium (called the one-dimensional Condorcet winner), which we denote by $x_m(y)$ (resp., $y_m(x)$), corresponds to the value of x

²Up to a change of sign in Example 2. This change of sign is innocuous here, as all our results in this section hold *mutatis mutandis* when marginal rates of substitution are monotonically decreasing in θ .

(resp., of y) that is most-preferred by the individual with the median type, θ_{med} :

$$\begin{aligned}x_m(y) &= x(y, \theta_{med}) \quad \forall y \in \mathfrak{R}, \\y_m(x) &= y(x, \theta_{med}) \quad \forall x \in \mathfrak{R}.\end{aligned}$$

The following proposition (proved in Appendix 2) shows that, if a Condorcet winner exists when voting simultaneously over the two dimensions, then it must be the most-preferred policy bundle of the individual with the median type, θ_{med} :

Proposition 1 *In the bidimensional majority voting setting with a unidimensional type space, under Assumption 1, the majority equilibrium (x^*, y^*) under simultaneous voting over both dimensions, if it exists, must be the unique ideal point of the median type voter θ_{med} :*

$$x^* = x(y^*, \theta_{med}) \text{ and } y^* = y(x^*, \theta_{med}).$$

We now investigate under which conditions a vector (x^s, y^s) (called the status quo hereafter) is preferred by a majority of voters to any local deviation. We establish the conditions under which an individual votes in favor of a motion moving away from the status quo in a (arbitrary) direction $d = (d_x, d_y) \in \mathfrak{R}^2$. The change in the utility of a voter of type θ induced by d is³

$$\varphi(\theta) \equiv \frac{\partial V(\theta, (x^s, y^s))}{\partial x} d_x + \frac{\partial V(\theta, (x^s, y^s))}{\partial y} d_y.$$

The population of voters who favor a move from the status quo in the direction d is composed of all the types for which $\varphi(\theta) > 0$. In particular we focus on the local Condorcet winner, defined as a policy pair (x^{**}, y^{**}) such that for any vector $(d_x, d_y) \in \mathfrak{R}^2$ and any $\varepsilon > 0$, the mass of citizens who strictly prefer $(x^{**} + \varepsilon d_x, y^{**} + \varepsilon d_y)$ to (x^{**}, y^{**}) is less than or at most equal to $\frac{1}{2}$. To test whether the policy pair (x^*, y^*) defined in Proposition 1 is a local Condorcet winner, we introduce the function

$$\Phi(d) = \int_{\{\theta \in [\underline{\theta}, \bar{\theta}]: \varphi(\theta) > 0\}} dF,$$

³The function φ also depends on d and on the status quo, but we simplify the notation by writing $\varphi(\theta)$.

which measures the proportion of voters favoring a deviation in direction d from the status quo (x^*, y^*) .

Observe that $\varphi'(\theta) \equiv \frac{\partial^2 V(\theta, x^*, y^*)}{\partial \theta \partial x} d_x + \frac{\partial^2 V(\theta, x^*, y^*)}{\partial \theta \partial y} d_y$, and that $\varphi(\theta_{med}) = 0$ since (x^*, y^*) is the most-preferred bundle of individual θ_{med} . By Assumption 1, we obtain that $\varphi'(\theta) > 0$ if $d_x > 0$ and $d_y > 0$, which means that all individuals favoring this direction d are such that $\theta > \theta_{med}$. By definition, this interval of types represents one half of the polity, so that $\Phi(d) = 1/2$. Similarly, $\varphi'(\theta) < 0$ if $d_x < 0$ and $d_y < 0$, so that all individuals with $\theta < \theta_{med}$ (and only them) favor the direction d , and $\Phi(d) = 1/2$. In words, if the deviation considered either increases or decreases both dimensions, then the individuals favoring this deviation are to be found only on one side of the median and are thus not numerous enough to defeat the status quo.

We now turn to deviations with both a positive and a negative component. Individuals with $\theta > \theta_{med}$ benefit from the positive component of the deviation but suffer from the negative component, and vice versa for the individuals with $\theta < \theta_{med}$. The set of voters who favor such a deviation may then be disjoint and could comprise both people above and below θ_{med} . We now characterize this set and study whether they represent more than one half of the electorate.

Consider without loss of generality the case where d is such that $d_x > 0$ and $d_y < 0$. Recall that voters who favor a direction d are such that $\varphi(\theta) > 0$. Denoting by

$$MRS(\theta) = \frac{\frac{\partial V(\theta, x^*, y^*)}{\partial x}}{\frac{\partial V(\theta, x^*, y^*)}{\partial y}}$$

the (absolute value of)⁴ the marginal rate of substitution between x and y at (x^*, y^*) for individual θ , we obtain that voters who favor the direction d are such that $\theta > \theta_{med}$ together with $MRS(\theta) > -d_x/d_y$ (i.e., those for whom the utility gain from a larger value of x is bigger than the utility loss from the lower value of y), or such that $\theta < \theta_{med}$ together with $MRS(\theta) < -d_x/d_y$ (i.e., those for whom the utility gain from a smaller value of y is bigger

⁴Note that, under Assumption 1, the marginal rate of substitution at (x^*, y^*) is negative for all individuals since $\partial V(\theta, x^*, y^*)/\partial x > 0$ and $\partial V(\theta, x^*, y^*)/\partial y > 0$ for all $\theta > \theta_{med}$ while $\partial V(\theta, x^*, y^*)/\partial x < 0$ and $\partial V(\theta, x^*, y^*)/\partial y < 0$ for all $\theta < \theta_{med}$. Slightly abusing notation, we denote by $MRS(\theta_{med})$ the limit, as θ tends towards θ_{med} , of $MRS(\theta)$.

than the utility loss from the larger value of x). The identification of the coalition of citizens $\Theta(d_x, d_y)$ supporting the deviation is illustrated on Figure 1 below, where we represent the MRS measured at (x^*, y^*) as a function of θ . It is important to note that this coalition need not be connected.⁵

Insert Figure 1 about here

The construction itself shows that the circumstances for having (x^*, y^*) undefeated are very exceptional. Indeed, given the choice of $-d_x/d_y$, if the set $\{\theta \in [\underline{\theta}, \bar{\theta}] : MRS(\theta) = -d_x/d_y\}$ has measure 0 for F , then it must be the case that the coalition $\Theta(d_x, d_y)$ and its complement $[\underline{\theta}, \bar{\theta}] \setminus \Theta(d_x, d_y)$ have both a measure equal to $\frac{1}{2}$ with respect to F for the policy (x^*, y^*) to be a local Condorcet winner. This may happen for some specific value of $-d_x/d_y$ but then a small perturbation of d_x/d_y is likely to destroy this property.⁶

We then impose further structure on the problem in the hope of finding circumstances under which a local Condorcet winner exists. An interesting benchmark, often used in the political economy literature, is the case where the utility function exhibits the single-crossing or Spence-Mirrlees's condition (Gans and Smart 1996, Rothstein 1990)—i.e., where the marginal rate of substitution is monotone⁷ in θ :

Assumption 2 (Joint Single-crossing) *We assume that*

$$\frac{\partial V(\theta, x, y)/\partial x}{\partial V(\theta, x, y)/\partial y} \text{ is increasing in } \theta$$

for all $(x, y) \in X$ and $\theta \in \mathfrak{R}$.

⁵If we consider for instance the specific frameworks of Examples 3 and 4 for the bundle $(x^*, y^*) = (\theta_{med}, y_{med})$, we obtain that $MRS(\theta) = \frac{\phi(\theta)}{\psi(\theta)}$. This function can display any sort of chaotic behavior for convenient selection of the functions ϕ and ψ .

⁶This reasoning does not hold when $MRS(\theta)$ is constant since, for any given directional deviation, the society is always divided equally. We provide an example of such a case after Proposition 2 below.

⁷The subsequent analysis would carry through to the case where the MRS is monotone decreasing in type. Note that there is no logical connection between the two single-crossing conditions that we study (Assumptions 1 and 2) in the sense that neither implies nor precludes the other.

We then obtain the following result.

Proposition 2 *In the bidimensional majority voting setting with a unidimensional type space, under Assumptions 1 (marginal single-crossing) and 2 (joint single crossing):*

a) *There is no majority equilibrium: the policy bundle (x^*, y^*) is defeated at the majority by most of the deviations d such that $d_x d_y < 0$.*

b) *Moreover, there exists a deviation $\tilde{d} = (\tilde{d}_x, \tilde{d}_y)$ with $\tilde{d}_x \tilde{d}_y < 0$ that is preferred by all voters (except θ_{med}) to (x^*, y^*) .*

In order to prove Proposition 2, we use Figure 2, where we make use of Assumption 2. The first panel depicts the case where $MRS(\theta_{med}) < -d_y/d_x$. In that case, all individuals below θ_{med} prefer the deviation. This is also the case for individuals with $\theta > \theta_{med}$ who are such that $MRS(\theta) > -d_y/d_x$. A strict majority favors d if this second group is not empty, which is the case provided that $MRS(\bar{\theta}) > -d_y/d_x$ -i.e., that d_y is not too large or d_x not too small (in absolute values). Figure 2(b) illustrates the case where $MRS(\theta_{med}) > -d_y/d_x$. In that case, all people with $\theta > \theta_{med}$ favor the deviation, together with individuals with $\theta < \theta_{med}$ for which $MRS(\theta) < -d_y/d_x$. As soon as this second group is not empty (which is the case if $MRS(\underline{\theta}) < -d_y/d_x$ -i.e., that d_y is not too small or d_x not too large, in absolute values), a strict majority of voters favor the deviation. Moreover, as explained above, if strictly less than 50% favor the deviation d , and if the set of individuals with $MRS(\theta) = -d_x/d_y$ is not too large, then strictly more than 50% of the voters favor the deviation $-d$. This proves Proposition 2 (a).

The third panel of Figure 2 shows that, if the deviation \tilde{d} is such that $MRS(\theta_{med}) = -d_y/d_x$, all voters (except of course θ_{med}) favor this deviation, proving part b) of Proposition 2.

Insert Figure 2 about here

While the reader may not be surprised by part a) of Proposition 2, part b) is more surprising, since in that case there is a unanimity against the median voter's most-preferred policy, even under marginal and joint single crossing conditions. The fact that single crossing is not

conducive to the existence of an equilibrium can be shown when studying one family of preferences (not satisfying Assumption 2) where we can guarantee the existence of a Condorcet winner, whatever the distribution function F . Assume that the function V is symmetrical with respect to the variables x and y :

$$V(\theta, x, y) = V(\theta, y, x) \text{ for all } \theta, x \text{ and } y \text{ in } \mathfrak{R}.$$

In that case, $x^* = y^*$ and $MRS(\theta) = 1$ for all $\theta \neq \theta_{med}$. This means that individuals with $\theta > \theta_{med}$ favor all deviations d such that either $d_x > 0$ and $d_y > 0$, or that $d_x d_y < 0$ together with $|d_y| < |d_x|$, and oppose all other deviations. Likewise, individuals with $\theta < \theta_{med}$ favor all deviations d such that either $d_x < 0$ and $d_y < 0$, or that $d_x d_y < 0$ together with $|d_y| > |d_x|$, and oppose all other deviations. We then obtain that $\Phi(d) = 1/2$ for all d , so that (x^*, y^*) is a Condorcet winner whatever the distribution F of types. The intuition for this result is that assuming a perfectly symmetrical utility function brings us back essentially to the realm of one-dimensional policies, where the classical median voter theorem applies.

The take home message of this section is then that, except in very peculiar circumstances such as a perfectly symmetrical utility function, there is little hope of finding a Condorcet winner when voting simultaneously over the two dimensions, even when the type space is unidimensional and single crossing conditions are satisfied.

We now move to the other equilibrium concepts studied in this paper, those proposed by Kramer and Shepsle, and by Stackelberg.

2.2 Kramer-Shepsle and Stackelberg equilibria

Let us examine first the Kramer-Shepsle equilibria.

Definition 1 *A Kramer-Shepsle (or KS) equilibrium is a policy vector (x^{KS}, y^{KS}) such that*

$$x^{KS} = x_m(y^{KS}) \text{ and } y^{KS} = y_m(x^{KS}).$$

With X convex⁸, and since the functions $x_m(\cdot)$ and $y_m(\cdot)$ are continuous, we deduce from Brouwer's fixed point theorem that there always exists a KS equilibrium. Under Assumption 1, if (x^{KS}, y^{KS}) is a KS equilibrium then we have

$$\frac{\partial V(\theta_{med}, x^{KS}, y^{KS})}{\partial x} = 0 \text{ and } \frac{\partial V(\theta_{med}, x^{KS}, y^{KS})}{\partial y} = 0, \quad (1)$$

i.e., (x^{KS}, y^{KS}) is an optimal policy for a θ_{med} type voter. Since this point is unique, there exists a unique KS equilibrium in such a setting. We have thus proved the following.

Proposition 3 *In the bidimensional majority voting setting with a unidimensional type space, under Assumption 1, there exists a unique KS equilibrium (x^{KS}, y^{KS}) that coincides with the optimal policy of the θ_{med} type voter.*

Let us now move to the set of Stackelberg (or ST) equilibria that arise when there is a sequence of two votes. We assume (without loss of generality at this stage) that individuals are first called to vote over x and then, after having observed the voting outcome of this first round, that they vote over y . We solve for these ST equilibria and compare them with both the SK equilibrium and with the ST equilibria under the opposite sequence (where voters choose first y and then x).

Solving backward, we know from Lemma 1 that, for any outcome x in the first stage, the majority voting equilibrium in the second stage is the most-preferred value of y of the median type θ_{med} , so that

$$\begin{aligned} y_m(x) &= \arg \max_y V(\theta_{med}, x, y) \\ &= y(\theta_{med}, x). \end{aligned}$$

This implies that, in the first stage, the reduced utility of a citizen of type θ for x is equal to

$$U(\theta, x) = V(\theta, x, y_m(x)).$$

⁸Since X has been assumed to be rectangular, this implies that the projections of X on the two axis are intervals.

Definition 2 A Stackelberg (or ST) equilibrium when voters choose first x and then y is a policy vector (x^{ST}, y^{ST}) such that

$$\int_{\{\theta \in [\underline{\theta}, \bar{\theta}]: U(\theta, x) - U(\theta, x^{ST}) > 0\}} f(\theta) d\theta \leq \frac{1}{2} \text{ for all } x \in \mathbb{R}$$

$$\text{and } y^{ST} = y_m(x^{ST}).$$

Of course, the first part of the definition of ST is not easy to test in general. Under the presumption that the function $U(\theta, x)$ is concave in x for all θ ,⁹ the first-order condition describing the optimal first-stage choice of a citizen of type θ is given by

$$\begin{aligned} x^F(\theta) &= \arg \max_x V(\theta_{med}, x, y_m(x)) \\ &= \arg \max_x U(\theta_{med}, x) \end{aligned}$$

and satisfies

$$\frac{\partial U(\theta, x)}{\partial x} = \frac{\partial V(\theta, x, y_m(x))}{\partial x} + \frac{\partial V(\theta, x, y_m(x))}{\partial y} \frac{dy_m(x)}{dx} = 0. \quad (2)$$

The first term of (2) describes the direct effect of varying x on the individual's utility, while the second term describes the indirect effect through variations in the second-stage voting outcome. To be able to sign this derivative, we will make an extensive use of the following assumption:

Assumption 3 (Strategic complementarity) *We assume that the two policy dimensions are strategic complements:*

$$\frac{\partial^2 V(\theta, x, y)}{\partial x \partial y} \geq 0. \quad (3)$$

From this assumption, we deduce the following lemma.

Lemma 2 *With strategic complements, we have both (a) $dy_m(x)/dx \geq 0$ and (b) $dx_m(y)/dy \geq 0$.*

⁹The first part of Appendix 6 studies the concavity of $U(\theta, x)$.

Proof. The proof of (a) comes from the concavity of $V(\theta, x, y)$ in y together with

$$\frac{dy_m(x)}{dx} = -\frac{\frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial x \partial y}}{\frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial y^2}}. \quad (4)$$

The proof of (b) is obtained similarly. ■

Applying the implicit function theorem to (2), we get

$$\frac{dx^F(\theta)}{d\theta} = -\frac{\frac{\partial^2 V(\theta, x^F(\theta), y_m(x^F(\theta)))}{\partial x \partial \theta} + \frac{\partial^2 V(\theta, x^F(\theta), y_m(x^F(\theta)))}{\partial y \partial \theta} \frac{dy_m(x^F(\theta))}{dx}}{\frac{\partial^2 U(\theta, x^F(\theta))}{\partial x^2}}.$$

Since, from the second-order condition, the denominator of this ratio is negative, we deduce from Assumption 1 and Lemma 2(a) that $\frac{dx^F(\theta)}{d\theta} \geq 0$. Recall that Assumption 1 states that the marginal utility obtained from both dimensions x and y increases with θ . Coupled with strategic complementarity of x and y , we then obtain that the most-preferred first-period value of x increases with θ (since a larger type reaps a larger direct benefit from an increase in x and also benefits more from the increase in the equilibrium value of y that a larger x generates).

Since we have assumed that $U(\theta, x)$ is concave in x for all θ , we can apply the median voter theorem and we obtain from the monotonicity $\frac{dx^F(\theta)}{d\theta} \geq 0$ that the majority equilibrium is $x^F(\theta_{med})$. But from (2), we know that

$$\frac{\partial V(\theta_{med}, x^F(\theta_{med}), y_m(x^F(\theta_{med})))}{\partial x} + \frac{\partial V(\theta_{med}, x^F(\theta_{med}), y_m(x^F(\theta_{med})))}{\partial y} \frac{dy_m(x^F(\theta_{med}))}{dx} = 0.$$

Since we have also showed that

$$\frac{\partial V(\theta_{med}, x^F(\theta_{med}), y_m(x^F(\theta_{med})))}{\partial y} = 0, \quad (5)$$

we deduce that

$$\frac{\partial V(\theta_{med}, x^F(\theta_{med}), y_m(x^F(\theta_{med})))}{\partial x} = 0. \quad (6)$$

In words, the median voter anticipates in the first stage that he will remain decisive in the second stage as well. In his first-stage choice of x , he then ignores (by an envelope theorem argument) the indirect effect of x on his utility, and chooses the optimal value of x given the value of y that will result in the second stage. The resulting policy bundle $(x^F(\theta_{med}), y_m(x^F(\theta_{med})))$

constitutes the unique Stackelberg equilibrium. Moreover, comparing equations (1), (5) and (6) shows that this policy pair is the same as the Kramer-Shepsle equilibrium. We can state the following:

Proposition 4 *In the bidimensional majority voting setting with a unidimensional type space, under Assumptions 1 (marginal single crossing) and 3 (strategic complementarity) and assuming that the function $U(\theta, x)$ is concave in x for all θ , the unique Stackelberg equilibrium where people vote first over x and then over y coincides with the ideal two-dimensional policy of the median citizen θ_{med} , and with the Kramer-Shepsle equilibrium. The same equilibrium also emerges with the opposite sequence of vote (provided that the reduced first-period utility function is concave in y).*

So, if both dimensions are strategic complements, the order of the vote does not matter and the ST equilibrium is moreover identical to the KS equilibrium. If both dimensions are not strategic complements, then the most-preferred first-stage value of x need not be monotone in θ . In that case, it is necessary to consider the decreasing rearrangement \tilde{x} of x . Then the median outcome x_{med} is the solution to the equation

$$F(\theta : \tilde{x}(\theta) \leq x_{med}) = \frac{1}{2},$$

and in general $x_{med} \neq x(\theta_{med})$. Also, in that case, the order of the votes matters, since the ST equilibrium will typically differ according to whether people vote first over x or over y .

To go beyond these generalities, we need to put more structure on the utility function. In the next section, we concentrate on a family of utility functions that has been studied at length, for instance in the nation formation literature.

2.3 One-sided Separability

In this subsection, we focus on the environment described in Example 5, which has received a great deal of attention in different fields. This setting is characterized by both a horizontal and

a vertical dimension. As already pointed out, Assumption 1 (marginal single crossing) is not satisfied so that this setting calls for a separate tailored treatment.

Let us assume that $\theta \in [0, 1]$ and

$$V(\theta, x, y) = v(x)\Psi(y - \theta) - x \text{ where } x \in \mathfrak{R}_+ \text{ and } y \in [0, 1]. \quad (7)$$

We assume that v is increasing and concave, and that Ψ is a function with values in \mathfrak{R}_{++} , symmetrical with respect to 0 and increasing to the left of 0.¹⁰ We also assume that the function Ψ is differentiable everywhere, so that $\Psi'(0) = 0$.¹¹ This general form describes the situation of a public policy program with a vertical dimension x (the quantity or quality level of a public good) and a horizontal dimension y (a characteristic of the public good, such as its color, location,...). The type θ of a voter represents her most-preferred public good variant y among all feasible options : any departure from this ideal choice decreases her utility for any value of x . Also, for any fixed type of public good y , each voter derives a gross benefit from this public good consumption which increases with x . We assume that the unit cost of production of the public good is one, that there is a mass one of consumers, and that public provision is financed with a lump sum tax. We thus have to subtract x from the gross utility to obtain the net utility of the public good. Note that the function $V(\cdot)$ is concave in x but not necessarily in y , as we make no concavity assumption on the function $\Psi(\cdot)$.

We start by looking at the ST procedure where citizens vote first over x and then over y . This is the sequence the jurisdiction and nation formation literatures have focused on. Note first that the majority choice over y does not depend upon x , while the converse is not true, as an individual's willingness to pay for the public good depends on its location. We dub this property *one-sided separability*. Whatever the value of x , the majority choice over y , which we denote by y_{med} , is given by

$$y_m(x) = y(x, \theta_{med}) = y_{med} = \theta_{med}.$$

¹⁰Therefore, it is decreasing to the right of 0. Alesina, Baqir and Easterly (1999), Etro (2006) and Gregorini (2009) consider the specific case where $\Psi(y, \theta) = \lambda - |\theta - y|$ where λ is a parameter larger than 1.

¹¹This differentiability assumption is not necessary for our arguments but allows to significantly simplify some proofs.

Given y_{med} , the reduced utility function takes the form

$$U(\theta, x) = v(x)\Psi(y_{med} - \theta) - x.$$

Given our assumptions on v and Ψ , U is a concave function of x with a peak at $x(\theta)$ where $x(\theta)$ is the unique solution x to the equation

$$v'(x)\Psi(y_{med} - \theta) = 1, \tag{8}$$

which is the familiar rule equating the marginal utility from the public good to its marginal taxation cost for individual θ . It is clear that this peak decreases continuously as θ moves away from θ_{med} , both to the left and to the right of θ_{med} . As the function $U(\cdot)$ is concave in x , we can apply the median voter theorem and assert that there exists a majority equilibrium value of x , which corresponds to the median most-preferred value of x when $y = y_{med}$. As should be obvious from (8), this decisive individual is **not** the individual with the median location θ_{med} , since this individual is the one with the largest willingness to pay for the public good, but rather the individual with the median distance to the median (i.e., the median value of $|y_{med} - \theta|$, since the function $\Psi(\cdot)$ is symmetrical around zero). We explain in Appendix 3 how to solve for the median optimal value of x , which we denote by x_{med} .

From the above arguments, we deduce that (x_{med}, y_{med}) is the unique ST equilibrium when voting first over x and then over y . It is also clear that this policy pair is the unique KS equilibrium as well, since $y_m(x) = y_{med}$ whatever the value of x . We thus have the following Proposition.

Proposition 5 *Given the utility function (7), the policy (x_{med}, y_{med}) is the (unique) Kramer-Shepsle equilibrium and it coincides with the Stackelberg equilibrium when people vote first over x and then over y .*

We now study the Stackelberg equilibrium when we reverse the vote sequence. Given an arbitrary value of y from the first vote, consider the second stage of the game— i.e., the vote over x . Since the utility function (7) is concave in x , we can apply the median voter theorem to learn

that the majority chosen x is the median most-preferred value of x given y . The most-preferred value of x of individual θ given y is

$$x(y, \theta) = (v')^{-1} \left(\frac{1}{\Psi(y - \theta)} \right),$$

which is symmetrical in θ around y , and decreasing as θ moves away from y . Assume without loss of generality that $y \leq y_{med}$. Two cases can materialize. In the first one, the decisive voters are the individuals located at a distance δ from y (to the left or to the right) and such that

$$F(y + \delta) - F(y - \delta) = \frac{1}{2}, \quad (9)$$

i.e., such that exactly 50% of the polity is located at a distance at most equal to δ from y (and thus prefer a larger value of x than $x(y, y \pm \delta)$). Note that equation (9) has a solution provided that $\delta \leq y$ – i.e., that y is such that $F(2y) \geq 1/2$. In words, the majority-chosen value of y must not be too far from the median (too small if we start with $y \leq y_{med}$ as assumed here, or too large if we had rather started with $y \geq y_{med}$). If y is far enough from y_{med} , then the decisive voter is the one with the median location, θ_{med} , with all the voters with $\theta < \theta_{med}$ preferring a larger (resp., lower) value of x than $x(y, \theta_{med})$ if $y \leq y_{med}$ (resp., if $y > y_{med}$) and all voters with $\theta > \theta_{med}$ preferring a lower (resp. larger) value of x if $y \leq y_{med}$ (resp., if $y > y_{med}$).

This shows that the identity of the decisive voter(s) in the second stage changes continuously with the choice made in the first stage. In terms of policy, this implies that

$$x_m(y) = \begin{cases} (v')^{-1} \left(\frac{1}{\Psi(y - \theta_{med})} \right) & \text{if } y \leq y^*, \\ (v')^{-1} \left(\frac{1}{\Psi(\delta(y))} \right) & \text{if } y^* \leq y \leq y^{**}, \\ (v')^{-1} \left(\frac{1}{\Psi(y - \theta_{med})} \right) & \text{if } y \geq y^{**}, \end{cases}$$

where (with an abuse of notation) y^* is the unique solution to the equation $F(2y) = 1/2$, y^{**} is the unique solution to the equation $F(2y - 1) = 1/2$ and $\delta(y)$ is given by (9).

Figure 3 depicts the case where F is uniform. Panel (a) shows that $\delta(y)$ is defined only when y is at most distant of $1/4$ from the median value of y , and is constant when it exists. If y is lower than $1/4$ or larger than $3/4$, the decisive voter in the choice of x is $1/2$, as shown in panel (b). For intermediate values of y , there are actually two types of decisive voters (panel

b), both distant of $1/4$ from y (panel (a)). Panel (c) shows the majority-chosen value of x for any given $y, x_m(y)$: it first increases with y (since the decisive voter remains the same, while his distance from the chosen y decreases), then it is constant with y (even though the identity of the decisive voters changes with y , they all remain at the same distance from the chosen y), and finally decreases with y (as the distance between the decisive voter, located at $1/2$, and y increases).

Insert Figure 3 about here

The previous analysis is summarized in part i) of Proposition 6. Moving backward to the first stage voting over y , we assume that the utility function of a citizen of type θ , which is given by $W(y, \theta) = v(x_m(y))\Psi(y - \theta) - x_m(y)$, is concave in y for all θ . Part ii) of Proposition 6 (proved in Appendix 4) shows that individuals have no incentive to vote for $y = \theta$ in the first stage, but that strategic considerations related to the second-stage choice of x drive them to vote for a value of x that differs in a systematic way from θ .

Proposition 6 *Given the utility function (7), voting first over y and then over x ,*

i) in the second stage, the decisive voter type changes continuously with the choice made in the first stage;

ii) in the first stage, a voter of type $\theta < y^$ (with $F(2y^*) = 1/2$) always votes for a value of y larger than his peak θ while a voter of type $\theta > y^{**}$ (where $F(2y^{**} - 1) = 1/2$) always votes for a value of y smaller than his peak θ . Voters of type $y^* \leq \theta \leq y^{**}$ always vote for a value of y larger (resp., smaller) than their peak θ if $\delta(\theta)$ decreases (resp., increases) with θ . The sign of the derivative of $\delta(\theta)$ with respect to θ only depends upon the distribution function F .*

The intuition for part ii) runs as follows. Individuals know that, if they obtain their “naive” most-preferred location $y = \theta$ in the first-stage, the majority chosen public good level x will be much lower than their most-favored level, because they will be the ones with the largest willingness to pay for the public good. A small departure from $y = \theta$ then has a second-order direct cost (because, although less appealing, the location remains close to their first-best

choice) but a first-order gain, provided that this departure leads to a larger amount of public good in the second stage. A voter whose peak is to the left of y^* anticipates that a first-stage choice close to their peak will result in the median voter θ_{med} being decisive in the second stage. A value of y slightly larger than θ will then induce a larger second-stage value of x , as it increases the willingness to pay for the public good of the θ_{med} individual (since it decreases the distance between the first-stage location choice and his most-preferred location). A similar reasoning explains why individuals located to the right of y^{**} always prefer a value of y that is smaller than their first-best choice θ . Individuals with intermediate preferences ($y^* \leq \theta \leq y^{**}$) anticipate that voters located at a distance $\delta(y)$ from y will be decisive in the second stage. They then bias their first-stage choice in order to decrease this distance, so that the decisive voter increases his most-preferred public good amount. We show in Appendix 4 that the distance $\delta(y)$ is a function of the distribution function F only.

From Proposition 6 ii), we gather that the first-stage, most-preferred values of x need not be monotone in θ (once strategic considerations are taken into account), so that the individual with the median type θ_{med} need not be the decisive voter. A more precise assessment of the identity of the first-stage median voter would necessitate the introduction of functional forms for the utility function Ψ and for the distribution function F . Observe that, in the special case where F is uniform, the distance $\delta(y)$ is a constant (see Figure 3) so that individuals located between $1/4$ and $3/4$ have no incentive to distort their first-period choice and vote for $y = \theta$. The decisive individual in the first stage is then θ_{med} , and the first-stage choice of location is one half. In that special case, the KS equilibrium is also the ST equilibrium for both voting sequences.

3 Two-Dimensional Types

In this section, we move to the situation where the type of a voter is two-dimensional. The statistical distribution of types $\theta = (\theta_1, \theta_2)$ among the voters is now described by a continuous (i.e. absolutely continuous with respect to the Lebesgue measure on \mathfrak{R}^2) cumulative distribution

function F whose support is (a subset of) \mathbb{R}^2 ; we denote by f the corresponding density. The utility of a citizen of type θ for policy (x, y) remains denoted by $V(\theta, x, y)$, which is twice continuously differentiable and concave in (x, y) .

We skip the analysis of the simultaneous voting setting as, in contrast to the one-dimensional case, it is very similar to what is done in theoretical political science (Banks and Austen-Smith (1999), Ordeshook (1986)). Instead, we focus on the analysis of the sets of Kramer-Shepsle and Stackelberg equilibria. A new phenomenon appears. In contrast to the one-dimensional type setting where the Kramer-Shepsle equilibrium was unique under some mild monotonicity assumption (see Proposition 3), in the two-dimensional type setting, there may exist several KS equilibria.

Consider the KS equilibrium, and more precisely the vote over y for any given value of x . Since the utility V is concave with respect to y , the payoff of a citizen of type θ is then maximized for a choice $y(x, \theta)$ such that

$$\frac{\partial V(\theta, x, y(x, \theta))}{\partial y} = 0. \quad (10)$$

We are able to extend Lemma 1 to the bidimensional type setting, provided that the marginal single crossing assumption introduced in the previous section holds for both dimensions of types.

Assumption 4 (Marginal Single-Crossing (bidimensional types)) *We assume that*

$$\frac{\partial^2 V(\theta, x, y)}{\partial \theta_i \partial x} \geq 0 \text{ and } \frac{\partial^2 V(\theta, x, y)}{\partial \theta_i \partial y} \geq 0 \text{ for } i = 1, 2$$

and for all $(x, y) \in X$ and $\theta \in \mathbb{R}^2$.

Lemma 3 *Under Assumption 4, the most-preferred value of x (respectively, of y) is increasing in θ_i , for any given y (resp., of x):*

$$\frac{\partial x(y, \theta)}{\partial \theta_i} \geq 0 \text{ and } \frac{\partial y(x, \theta)}{\partial \theta_i} \geq 0 \text{ for } i = 1, 2.$$

Proof: From the implicit function theorem, we deduce that :

$$\frac{\partial y(x, \theta)}{\partial \theta_i} = -\frac{\frac{\partial^2 V(\theta, x, y(x, \theta))}{\partial \theta_i \partial y}}{\frac{\partial^2 V(\theta, x, y(x, \theta))}{\partial y^2}} \text{ for } i = 1, 2.$$

Since V is concave with respect to y , we deduce that the sign of $\frac{\partial y(x, \theta)}{\partial \theta_i}$ is the same as the sign of $\frac{\partial^2 V(\theta, x, y(x, \theta))}{\partial \theta_i \partial y}$. We proceed similarly to prove that $\frac{\partial x(y, \theta)}{\partial \theta_i} \geq 0$.

How do we relate the most-preferred value of y to the bidimensional voter's type? Graphically, we can draw in the two-dimensional space (θ_1, θ_2) the y -isopoly curves describing, for any given value of y and of x , the one-dimensional set of voters for whom y is the most-preferred policy given x . This curve is described implicitly by the equation :

$$y(x, \theta) = y. \tag{11}$$

Applying the implicit function theorem to equation (11) and making use of the marginal single crossing assumption (Assumption 4), we obtain that this curve is decreasing, and that moving in the north-eastern direction leads to larger most-preferred values of y . A typical curve is represented on Figure 4 below.

Insert Figure 4 here.

Making use of (11), we obtain that there exists (at most) one value of θ_2 such that, for any θ_1 , x and y , type (θ_1, θ_2) voters most-prefer y when the other dimension is exogenously set at x . We denote this value of θ_2 by the function $\phi_2(\theta_1, x; y)$. Since the utility functions $V(\theta, x, y)$ are single-peaked with respect to y for all x and all θ , we deduce from the median voter theorem that, for any x , there exists a majority voting equilibrium corresponding to the median value of the y -isopoly curve $y(x, \theta)$. This majority outcome, denoted by $y_m(x)$, is the unique solution to the equation

$$\int_{-\infty}^{+\infty} d\theta_1 \int_{-\infty}^{\phi_2(\theta_1, x; y)} f(\theta_1, \theta_2) d\theta_2 = \frac{1}{2}. \tag{12}$$

Turning to the choice of x for any given y , we obtain from (11) that there is (at most) one value of θ_2 such that, for any θ_1 , x and y , type (θ_1, θ_2) voters most-prefer x when the other dimension is exogenously set at y . We denote this value of θ_2 by the function $\psi_2(\theta_1, y; x)$. We deduce similarly that, for any y , there exists a majority voting equilibrium corresponding to the median value of the x -isopoly curve $x(y, \theta)$. This majority outcome, denoted by $x_m(y)$, is the unique solution to the equation

$$\int_{-\infty}^{+\infty} d\theta_1 \int_{-\infty}^{\psi_2(\theta_1, y; x)} f(\theta_1, \theta_2) d\theta_2 = \frac{1}{2}.$$

Since the functions $x_m(y)$ and $y_m(x)$ are continuous and X is convex, we deduce from Brouwer's fixed point theorem the existence of a Kramer-Shepsle's equilibrium. We have then proved the following:

Proposition 7 *In the bidimensional majority voting setting with a bidimensional type space, under Assumption 4, there always exists at least one KS equilibrium.*

The main difference between Propositions 7 and 3 is twofold. First, we have not proved the unicity of the KS equilibrium when the type space is bidimensional. Second, there is no reason for the KS equilibrium to coincide with the policy pair most-preferred by an individual of "median type".

The following proposition (proved in Appendix 5) sheds more light on the characteristics of the functions $x_m(y)$ and $y_m(x)$.

Proposition 8 *Under Assumptions 3 and 4, we have that $\frac{dy_m(x)}{dx} \geq 0$ and that $\frac{dx_m(y)}{dy} \geq 0$ — i.e., the policy variables x and y are strategic complements.*

There is little we can say at this level of generality about the existence or characteristics of the Stackelberg equilibria, and their relationship with the KS equilibria.¹² In order to illustrate

¹²We study in the last Appendix the determinants of the concavity of the function $U(\theta, x) \equiv V(\theta, x, y_m(x))$, i.e., the utility function used by voters in the first stage of the game where they vote first over x and then over y . This appendix shows the complexity of the task of assessing the concavity of $U(\cdot)$, since it includes several partial derivatives of order three.

our model, we now turn to the framework used most often in the formal political science literature, namely the spatial model where voters are assumed to have quadratic preferences (Banks and Austen-Smith (1999), Ordeshook (1986)). This will also allow us to show how multiple KS equilibria can arise.

The utility function of a voter of type (θ_1, θ_2) is defined here as follows:

$$\begin{aligned} V(\theta, x, y) &= - \begin{pmatrix} x - \theta_1 & y - \theta_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x - \theta_1 \\ y - \theta_2 \end{pmatrix} \\ &= -a_{11}(x - \theta_1)^2 - a_{12}(x - \theta_1)(y - \theta_2) \\ &\quad - a_{21}(x - \theta_1)(y - \theta_2) - a_{22}(y - \theta_2)^2, \end{aligned} \tag{13}$$

where $A \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is a symmetric positive definite matrix –i.e., $a_{12} = a_{21}$, $a_{11} > 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$. In words, voters differ only in the (bidimensional) location of their most-preferred policy, but share the same shape of indifference curves (as represented by the matrix A). That situation is easy to handle. Through a linear change of variables, we move from the current variables θ_1, θ_2, x and y to new variables θ'_1, θ'_2, x' and y' via a matrix P in order to change the matrix A into a diagonal matrix A' . In the new space of types, the utility functions are separable and we obtain therefore that there is a unique KS equilibrium which moreover coincides with the Stackelberg equilibrium.¹³

Uniqueness follows here from the specific choice of the two-dimensional type space: a type is the two-dimensional vector of ideal policies. Other parametric specifications of the type space in the spatial model of politics may display different features from the one described above, namely uniqueness¹⁴ and coincidence between KS and Stackelberg. This is illustrated in the following example where the voters are heterogeneous with respect to both the location of their

¹³Under some more stringent conditions, like for instance the radial symmetry of the density function f , the KS/Stackelberg equilibrium is also a Condorcet equilibrium when voting simultaneously over the two policy dimensions. This was observed by Tullock (1967) in the specific case of a uniform density function.

¹⁴Shepsle (1979) constructs Euclidean patterns leading to a multiplicity of KS equilibria. His examples are straightforward and do not apply here as the set of people admitted to cast a vote varies depending on the policy dimension being considered.

most-preferred policy and the shape of their indifference curves (i.e., the direction and intensity of the correlation between the two policy dimensions).

Let us consider the case depicted in Figure 5 below, where 5 voters are identified by their ideal policies, located at the points a, b, c, d, e , respectively.

Insert Figure 5 about here

We retain for voters a, d and e the simplest configuration of circular level curves around their ideal points.¹⁵ The indifference curves of individuals b and c are instead represented by two ellipses centered around their ideal points, for which we choose different shapes.¹⁶ We depict in Figure 5a the lines $y(\theta, x)$ (obviously, $y(\theta, x)$ is a horizontal line through point θ for voters $\theta = a, d, e$), as well as the locuses $y_m(x)$ in bold. We proceed similarly in Figure 5b, showing the lines $x(\theta, y)$ together with $x_m(y)$. Observe that Assumption 3 is satisfied so that, by Proposition 8, we have that $dy_m(x)/dx \geq 0$ and $dx_m(y)/dy \geq 0$. We report both $x_m(y)$ and $y_m(x)$ on Figure 6, and we obtain 3 KS equilibria: points c and d , but also a third point k^* that does not correspond to any voter's most-preferred location!

Insert Figure 6 about here

As for Stackelberg equilibria, observe first that both $U(\theta, x) = V(\theta, x, y_m(x))$ and $W(\theta, y) = V(\theta, y, x_m(y))$ are concave and single-peaked for all five voters. Moreover, location c constitutes the unique Stackelberg equilibrium, whatever the sequence of votes.

This example then shows that i) we may have multiple KS equilibria, ii) KS equilibria need not be Stackelberg equilibria and iii) KS equilibria need not correspond to any voter's most-preferred policy.

¹⁵We do not represent level curves for these voters to avoid cluttering the figure further.

¹⁶The equations of the ellipses represented in the figure are $(x - 4)^2 - 2(x - 4)(y - 3) + 2(y - 3)^2 = 1$ for the small ellipse around b , ($= 15$ for the bigger one) and $4(x - 5.4)^2 - 3(x - 5.4)(y - 4) + (y - 4)^2 = 9$ for the level curve of individual c .

4 Conclusions

Majority voting over a multidimensional policy space leads in general to negative results, requiring very stringent conditions for the existence of an equilibrium outcome when voting simultaneously over all dimensions. Such results have induced political economy scholars to introduce specific and restrictive assumptions on individual preferences, on the distribution of individuals' types across the population and on the voting rule, often based on a sequential scheme.

Our paper takes one step back: it assumes utility functions and a distribution of types as general as possible, and it focuses on two specific alternatives to simultaneous majority voting. Our analysis of Kramer-Shepsle and Stackelberg equilibria leads to promising results. We show that it is possible to conclude about the existence of these equilibria starting from simple single-crossing conditions widely used in the literature. Under the same weak assumptions, we compare the characteristics of the solutions issued by the two voting procedures under exam, emphasizing the relevance of the median type preferred policy. We also study the uniqueness of equilibrium solutions, showing that multiple Kramer-Shepsle equilibria become plausible when the domain of individual preferences is richer. While developing our analysis in a general setting, we also study thoroughly an environment modelled in the political economy literature exploring issues such as the quantity and the location of public goods in modern democracies, the connection with the size of the nations and the stability of national borders to secession threats.

Both additional theoretical advances and further applications could enrich and complete our main findings. Along the first line, it would be interesting to study a model where the set of alternatives consists in a finite hypercube and where voter preferences are orderings. Along the second research line, we recommend a systematic comparison of KS and ST equilibria in the main models studied in the applied political economy literature, in the spirit of De Donder, Le Breton and Peluso (2009).

5 References

- Alesina, A., Baqir, R. and W. Easterly. (1999) "Public Goods and Ethnic Divisions", *Quarterly Journal of Economics*, 114, 1243-1284.
- Alesina, A., Baqir, R. and C. Hoxby. (2004) "Political Jurisdictions in Heterogeneous Communities", *Journal of Political Economy*, 112, 348-396.
- Banks, J. and D. Austen-Smith (1999), *Positive Political Theory : Collective Preference*, Vol 1, Michigan University Press, Ann-Arbor.
- Banks, J. and J.Duggan (2004) "Existence of Nash Equilibria on Convex Sets", University of Rochester, unpublished manuscript.
- Banks, J., Duggan, J. and M. Le Breton (2006) "Social Choice and Electoral Competition in the General Spatial Model", *Journal of Economic Theory*, 126, 194-234.
- Bernheim, B.D. and S.N. Slavov (2009) "A Solution Concept for Majority Rule in Dynamic Settings", *Review of Economic Studies* 76, 33-62.
- Cremer H., De Donder P. and F. Gahvari (2004), "Taxes, Budgetary Rule and Majority Voting", *Public Choice*, 119: 335-358.
- Cremer H., De Donder, P., Maldonado, D. and P. Pestieau (2007) "Voting over the Size and Type of Social Security when some Individuals are Myopic", *Journal of Public Economics*, 91-10: 2041-2061.
- Davis, O.A., DeGroot, M.H. and M.J. Hinich (1972) "Social Preference Orderings and Majority Rule", *Econometrica*, 40, 147-157.
- De Donder P. and J. Hindriks (1998) "The Political Economy of Targeting", *Public Choice*, 95 (1-2), 177-200.
- De Donder, P., Le Breton, M. and E. Peluso (2009) "Sequential Majority Choice Over a Two-Dimensional Policy Choice", CEPR Working Paper 7223.
- Diba, B. and Feldman, A.M. (1984). "Utility functions for public outputs and majority Voting". *Journal of Public Economics*, 25, 235-243.
- Duggan, J. (2001) "Non-Cooperative Games Among Groups", University of Rochester, un-

published manuscript.

Etro, F. (2006) "Political Geography", *Public Choice*, 16, 517-535.

Gans, J. and M. Smart (1996) "Majority Voting with Single-Crossing Preferences", *Journal of Public Economics*, 59, 219-238.

Gregorini, F. (2009), "Political geography and income inequalities", Working Paper 152, Dipartimento di Economia Politica, Università di Milano - Bicocca.

Groseclose, T. (2007) "Black's Median Voter Theorem in 'One and a half' Dimensions", *Social Choice and Welfare*, 28, 321-335.

Haimanko, O., Le Breton, M. and S. Weber. (2005) "Transfers in a Polarized Country : Bridging the Gap between Efficiency and Stability", *Journal of Public Economics*, 89, 1277-1303.

Kramer, G.H. (1972) "Sophisticated Voting over Multidimensional Choice Spaces", *Journal of Mathematical Sociology*, 2, 165-180.

McKelvey, R., Ordeshook, P. and P. Ungar. (1980) "Conditions for Voting Equilibria in Continuous Voter Distributions", *Siam Journal of Applied Mathematics*, 39, 161-168.

Myerson, R. B. (1996) "Fundamentals of social choice theory." Discussion Paper 1162, Center for Mathematical Studies in Economics and Management Science, Northwestern University.

Nechyba, T.(1997) "Local Property and State Income Taxes: The Role of Interjurisdictional Competition and Collusion", *The Journal of Political Economy*, 105, 351-384.

Ordeshook, P. (1986) *Game Theory and Political Theory*, Cambridge University Press, Cambridge.

Perroni, C. and K. Scharf (2001) "Tiebout with Politics: Capital Tax Competition and Constitutional Choices", *Review of Economic Studies*, 68, 133-154.

Plott, C. (1967) "A Notion of Equilibrium and its Possibility under Majority Rule", *American Economic Review*, 57, 787-806.

Rothstein, P. (1990) "Order Restricted Preferences and Majority Rule", *Social Choice and Welfare*, 7, 331-342.

Sadanand, A. B. and J. M. Williamson (1991). "Equilibrium in a Stock Market Economy with Shareholder Voting" *International Economic Review*, 32, 1-35.

Shepsle, K.A. (1979) "Institutional Arrangements and Equilibrium in Multidimensional Voting Models", *American Journal of Political Science*, 23, 27-59.

Tullock, G. (1967) "The General Irrelevance of the General Impossibility Theorem", *Quarterly Journal of Economics*, 81, 256-270.

Appendix 1: Proof of Lemma 1

i) Since the utility V is concave with respect to x , for any given value of y (resp. x), the payoff of a citizen of type θ is maximized for a choice $x(y, \theta)$ (resp. $y(x, \theta)$) such that:

$$\frac{\partial V(\theta, x(y, \theta), y)}{\partial x} = 0 \text{ (resp. } \frac{\partial V(\theta, x, y(x, \theta))}{\partial y} = 0).$$

From the implicit function theorem, we deduce:

$$\frac{\partial x(y, \theta)}{\partial \theta} = -\frac{\frac{\partial^2 V(\theta, x(y, \theta), y)}{\partial \theta \partial x}}{\frac{\partial^2 V(\theta, x(y, \theta), y)}{\partial x^2}} \text{ and } \frac{\partial y(x, \theta)}{\partial \theta} = -\frac{\frac{\partial^2 V(\theta, x, y(x, \theta))}{\partial \theta \partial y}}{\frac{\partial^2 V(\theta, x, y(x, \theta))}{\partial y^2}}.$$

From the concavity of V with respect to x and y and Assumption 1, we get:

$$\frac{\partial x(y, \theta)}{\partial \theta} \geq 0 \text{ and } \frac{\partial y(x, \theta)}{\partial \theta} \geq 0.$$

ii) For any θ , the utility function $V(\theta, x, y)$ is single-peaked with respect to x and y . Then, from the median voter theorem, we know that for any y (resp. x), there exists a majority equilibrium corresponding to the median value of $x(y, \theta)$ (resp. $y(x, \theta)$).

Appendix 2: Proof of Proposition 1

From Assumption 1, we deduce that a necessary condition for the status quo (x^*, y^*) to be preferred to any other feasible policy bundle by a majority of voters is

$$\frac{\partial V(\theta_{med}, x^*, y^*)}{\partial x} = 0 \text{ and } \frac{\partial V(\theta_{med}, x^*, y^*)}{\partial y} = 0,$$

which implies, if V is further assumed to be strictly concave, that (x^*, y^*) is the unique ideal point of the median type voter.

Appendix 3: Majority choice of x in section 2.3

The most-preferred value of x decreases from $\bar{x} \equiv v'^{-1}(\frac{1}{\Psi(0)})$ to $\underline{x} \equiv v'^{-1}(\frac{1}{\text{Min}(\Psi(y_{med}), \Psi(y_{med}-1))})$ as θ moves away from θ_{med} . Without loss of generality, suppose that $\Psi(y_{med}) \leq \Psi(y_{med}-1)$.

The proportion $B(x)$ of voters with an ideal peak below the fixed level x is given by :

$$B(x) = \begin{cases} F\left(y_{med} - \Psi^{-1}\left(\frac{1}{v'(x)}\right)\right) & \text{if } x \leq x^*, \\ F\left(y_{med} - \Psi^{-1}\left(\frac{1}{v'(x)}\right)\right) + \left[1 - F\left(y_{med} + \Psi^{-1}\left(\frac{1}{v'(x)}\right)\right)\right] & \text{if } x \geq x^*, \end{cases}$$

where x^* is the unique solution to the equation

$$v'(x) = \frac{1}{\Psi(y_{med} - 1)}.$$

When F is symmetric, $y_{med} = \frac{1}{2}$, $x^* = \underline{x}$ and B is a cumulative distribution function on $[\underline{x}, \bar{x}]$ defined as follows:

$$B(x) = 2F\left(\frac{1}{2} - \Psi^{-1}\left(\frac{1}{v'(x)}\right)\right).$$

Then, the majority choice x_{med} is the unique solution x to the equation :

$$F\left(\frac{1}{2} - \Psi^{-1}\left(\frac{1}{v'(x)}\right)\right) = \frac{1}{4}.$$

For instance, when F is uniform, x_{med} is the peak of a voter located at a distance from the median equal to $\frac{1}{4}$.

Appendix 4: Proof of Proposition 6 ii)

The first-order condition for y of an individual θ is given by

$$v'(x_m(y))\Psi(y - \theta)\frac{dx_m(y)}{dy} + v(x_m(y))\Psi'(y - \theta) - \frac{dx_m(y)}{dy} = 0.$$

Our objective is to assess under which circumstances the value of y that maximizes $W(y, \theta)$ differs from θ (which is the “true peak” of the utility function—i.e., the value of y that maximizes $V(\theta, x, y)$ for any given value of x). To this effect, we evaluate the derivative of $W(y, \theta)$ at $y = \theta$ to obtain

$$\frac{\partial W(y, \theta)}{\partial y} \Big|_{y=\theta} = \frac{dx_m(\theta)}{dy} [v'(x_m(\theta))\Psi(0) - 1] + v(x_m(\theta))\Psi'(0).$$

The function $x_m(y)$ is characterized by the equality $v'(x_m(y))\Psi(d) - 1 = 0$, where $d = \delta(\theta)$ if $y^* \leq \theta \leq y^{**}$ and $d = \theta - \theta_{med}$ if $\theta < y^*$ or $\theta > y^{**}$. Therefore, the above derivative is equal

to

$$\begin{aligned}\frac{\partial W(y, \theta)}{\partial y} \Big|_{y=\theta} &= v'(x_m(\theta)) [\Psi(0) - \Psi(d)] \frac{dx_m(\theta)}{dy} + v(x_m(\theta)) \Psi'(0) \\ &= v'(x_m(\theta)) [\Psi(0) - \Psi(d)] \frac{dx_m(\theta)}{dy},\end{aligned}$$

as $\Psi'(0) = 0$. Since $\Psi(0) - \Psi(d) > 0$, the sign of the derivative at $y = \theta$ is the same as the sign of $dx_m(\theta)/dy$. If $\theta < y^*$, $x_m(\theta) = x(\theta, \theta_{med})$ so that $dx_m(\theta)/dy > 0$. If $\theta > y^{**}$, $x_m(\theta) = x(\theta, \theta_{med})$ so that $dx_m(\theta)/dy > 0$. If $y^* \leq \theta \leq y^{**}$, $x_m(\theta) = x(\theta, \theta \pm \delta(\theta))$ and we obtain that

$$\frac{dx_m(y)}{dy} = -\frac{v'(x_m(y))\Psi'(\delta(y))}{v''(x_m(y))\Psi(\delta(y))} \frac{d\delta(y)}{dy}.$$

Since $\Psi'(\delta) < 0$, the sign of $dx_m(y)/dy$ is the opposite of the sign of $d\delta(y)/dy$. From the definition of $\delta(y)$ and the implicit function theorem, we obtain that

$$\frac{d\delta(y)}{dy} = \frac{f(y - \delta(y)) - f(y + \delta(y))}{f(y + \delta(y)) + f(y - \delta(y))},$$

so that the sign of $d\delta(y)/dy$ depends exclusively upon the shape of the density function f .

Appendix 5: Proof of Proposition 8

Differentiating (12) with respect to x , we get

$$\int_{-\infty}^{+\infty} \left[\frac{\partial \phi_2(\theta_1, x; y_m(x))}{\partial x} + \frac{\partial \phi_2(\theta_1, x; y_m(x))}{\partial y} \frac{dy_m(x)}{dx} \right] f(\theta_1, \phi_2(\theta_1, x; y)) d\theta_1 = 0,$$

which implies that

$$\frac{dy_m(x)}{dx} = -\frac{\int_{-\infty}^{+\infty} \frac{\partial \phi_2(\theta_1, x; y_m(x))}{\partial x} f(\theta_1, \phi_2(\theta_1, x; y)) d\theta_1}{\int_{-\infty}^{+\infty} \frac{\partial \phi_2(\theta_1, x; y_m(x))}{\partial y} f(\theta_1, \phi_2(\theta_1, x; y)) d\theta_1}.$$

On the other hand, differentiating

$$\frac{\partial V(\theta_1, \phi_2(\theta_1, x; y), x, y)}{\partial y} = 0$$

with respect to x and y leads to

$$\frac{\partial \phi_2(\theta_1, x; y)}{\partial x} = -\frac{\frac{\partial^2 V(\theta_1, \phi_2(\theta_1, x; y), x, y)}{\partial y \partial x}}{\frac{\partial^2 V(\theta_1, \phi_2(\theta_1, x; y), x, y)}{\partial y \partial \theta_2}}, \quad (14)$$

and

$$\frac{\partial \phi_2(\theta_1, x; y)}{\partial y} = -\frac{\frac{\partial^2 V(\theta_1, \phi_2(\theta_1, x; y), x, y)}{\partial^2 y}}{\frac{\partial^2 V(\theta_1, \phi_2(\theta_1, x; y), x, y)}{\partial y \partial \theta_2}}. \quad (15)$$

We deduce that

$$\frac{dy_m(x)}{dx} = -\frac{\int_{-\infty}^{+\infty} \frac{\frac{\partial^2 V(\theta_1, \phi_2(\theta_1, x; y_m(x)), x, y_m(x))}{\partial y \partial x}}{\frac{\partial^2 V(\theta_1, \phi_2(\theta_1, x; y_m(x)), x, y_m(x))}{\partial y \partial \theta_2}} f(\theta_1, \phi_2(\theta_1, x; y_m(x))) d\theta_1}{\int_{-\infty}^{+\infty} \frac{\frac{\partial^2 V(\theta_1, \phi_2(\theta_1, x; y_m(x)), x, y_m(x))}{\partial^2 y}}{\frac{\partial^2 V(\theta_1, \phi_2(\theta_1, x; y_m(x)), x, y_m(x))}{\partial y \partial \theta_2}} f(\theta_1, \phi_2(\theta_1, x; y_m(x))) d\theta_1}. \quad (16)$$

Given the concavity of V with respect to y and Assumption 4, the sign of $\frac{dy_m(x)}{dx}$ is the same as the sign of $\frac{\partial^2 V(\theta_1, \theta_2, x, y)}{\partial y \partial x}$, which is non negative by Assumption 3.

The same observations apply of course to the derivative of x_m with respect to y .

Appendix 6: Concavity of the first-period utility function $U(\theta, x)$

In this appendix, we examine the primitive conditions for the sign of $\frac{\partial^2 U(\theta, x)}{\partial x^2}$ when individuals vote first over x and then over y .

- One-dimensional types

The second-order derivative $\partial^2 U(\theta, x) / \partial x^2$ of the reduced utility function of a citizen of type θ is:

$$\begin{aligned} & \frac{\partial^2 V(\theta, x, y_m(x))}{\partial x^2} + 2 \frac{\partial^2 V(\theta, x, y_m(x))}{\partial x \partial y} \frac{dy_m(x)}{dx} + \\ & \frac{\partial^2 V(\theta, x, y_m(x))}{\partial y^2} \frac{dy_m(x)}{dx} + \frac{\partial V(\theta, x, y_m(x))}{\partial y} \frac{d^2 y_m(x)}{dx^2}. \end{aligned} \quad (17)$$

From this, we see that a key element to guarantee the existence of a ST equilibrium is the sign of the second-order derivative $d^2 y_m(x) / dx^2$. From the differentiation of (4), we obtain

$$\begin{aligned} \frac{d^2 y_m(x)}{dx^2} = & -\frac{\left[\frac{\partial^3 V(\theta_{med}, x, y_m(x))}{\partial x^2 \partial y} + \frac{\partial^3 V(\theta_{med}, x, y_m(x))}{\partial x \partial y^2} \frac{dy_m(x)}{dx} \right] \frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial y^2}}{\left(\frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial y^2} \right)^2} \\ & + \frac{\left[\frac{\partial^3 V(\theta_{med}, x, y_m(x))}{\partial x \partial y^2} + \frac{\partial^3 V(\theta_{med}, x, y_m(x))}{\partial y^3} \frac{dy_m(x)}{dx} \right] \frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial x \partial y}}{\left(\frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial y^2} \right)^2}, \end{aligned}$$

and therefore, by using (4) again

$$\frac{d^2 y_m(x)}{dx^2} = \frac{\left[\frac{\partial^3 V(\theta_{med}, x, y_m(x))}{\partial x^2 \partial y} + \frac{\partial^3 V(\theta_{med}, x, y_m(x))}{\partial x \partial y^2} \frac{\frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial x \partial y}}{\frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial y^2}} \right] \frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial y^2}}{\left(\frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial y^2} \right)^2} - \frac{\left[\frac{\partial^3 V(\theta_{med}, x, y_m(x))}{\partial x \partial y^2} + \frac{\partial^3 V(\theta_{med}, x, y_m(x))}{\partial y^3} \frac{\frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial x \partial y}}{\frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial y^2}} \right] \frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial x \partial y}}{\left(\frac{\partial^2 V(\theta_{med}, x, y_m(x))}{\partial y^2} \right)^2}$$

We then substitute this expression of $d^2 y_m(x)/dx^2$ into (17). As we can see, checking that $\partial^2 U(\theta, x)/\partial x^2 \leq 0$ is quite tricky as it involves the sign of many high-order partial derivatives, including third-order cross derivatives. No general principle can be provided and it is necessary to proceed to this computation for each specific environment.

-Two-dimensional types

We have that

$$\frac{\partial U(\theta, x)}{\partial x} = \frac{\partial V(\theta, x, y_m(x))}{\partial x} + \frac{\partial V(\theta, x, y_m(x))}{\partial y} \frac{dy_m(x)}{dx},$$

which implies in turn

$$\begin{aligned} \frac{\partial^2 U(\theta, x)}{\partial x^2} &= \frac{\partial^2 V(\theta, x, y_m(x))}{\partial x^2} + 2 \frac{\partial^2 V(\theta, x, y_m(x))}{\partial x \partial y} \frac{dy_m(x)}{dx} \\ &\quad + \frac{\partial^2 V(\theta, x, y_m(x))}{\partial y^2} \left(\frac{dy_m(x)}{dx} \right)^2 \\ &\quad + \frac{\partial V(\theta, x, y_m(x))}{\partial y} \frac{d^2 y_m(x)}{dx^2}. \end{aligned}$$

While the sign of the first three terms can be deduced from our assumptions, the sign of the last term depends upon more information. The value of the second-order derivative $\frac{d^2 y_m(x)}{dx^2}$ is obtained by differentiation of (16):

$$\frac{d^2 y_m(x)}{dx^2} = \frac{\int_{-\infty}^{+\infty} [\Delta(\theta_1, \theta_2, x) H(\theta_1, \theta_2, x) - S(\theta_1, \theta_2, x) \Gamma(\theta_1, \theta_2, x)] f(\theta_1, \phi_2(\theta_1, x; y_m(x))) d\theta_1}{\left(\int_{-\infty}^{+\infty} H(\theta_1, \theta_2, x) f(\theta_1, \phi_2(\theta_1, x; y_m(x))) d\theta_1 \right)^2},$$

where

$$S(\theta_1, \theta_2, x) \equiv \frac{\frac{\partial^2 V(\theta_1, \phi_2(\theta_1, x; y_m(x)), x, y_m(x))}{\partial y \partial x}}{\frac{\partial^2 V(\theta_1, \phi_2(\theta_1, x; y_m(x)), x, y_m(x))}{\partial y \partial \theta_2}},$$

$$H(\theta_1, \theta_2, x) \equiv \frac{\frac{\partial^2 V(\theta_1, \phi_2(\theta_1, x; y_m(x)), x, y_m(x))}{\partial^2 y}}{\frac{\partial^2 V(\theta_1, \phi_2(\theta_1, x; y_m(x)), x, y_m(x))}{\partial y \partial \theta_2}},$$

$$\Delta(\theta_1, \theta_2, x) \equiv \frac{\partial S(\theta_1, \theta_2, x)}{\partial x} f(\theta_1, \phi_2(\theta_1, x; y_m(x))) + S(\theta_1, \theta_2, x) \frac{\partial f(\theta_1, \phi_2(\theta_1, x; y_m(x)))}{\partial \theta_2} \left(\frac{\partial \phi_2(\theta_1, x; y)}{\partial x} + \frac{\partial \phi_2(\theta_1, x; y)}{\partial y} \frac{dy_m(x)}{dx} \right),$$

and

$$\Gamma(\theta_1, \theta_2, x) \equiv \frac{\partial H(\theta_1, \theta_2, x)}{\partial x} f(\theta_1, \phi_2(\theta_1, x; y_m(x))) + H(\theta_1, \theta_2, x) \frac{\partial f(\theta_1, \phi_2(\theta_1, x; y_m(x)))}{\partial \theta_2} \left(\frac{\partial \phi_2(\theta_1, x; y)}{\partial x} + \frac{\partial \phi_2(\theta_1, x; y)}{\partial y} \frac{dy_m(x)}{dx} \right).$$

To conclude, it is of course necessary to evaluate $\Delta(\theta_1, \theta_2, x)$ and $\Gamma(\theta_1, \theta_2, x)$. Using (14), (15) and (16), we can replace $\frac{\partial \phi_2(\theta_1, x; y)}{\partial x}$, $\frac{\partial \phi_2(\theta_1, x; y)}{\partial y}$ and $\frac{dy_m(x)}{dx}$ by expressions depending upon the primitives. The sign of $\frac{\partial S(\theta_1, \theta_2, x)}{\partial x}$ and $\frac{\partial H(\theta_1, \theta_2, x)}{\partial x}$ depends however on third-order partial derivatives. Note, finally, that the sign of the first derivative of the density function f also appears in the expression of $\frac{d^2 y_m(x)}{dx^2}$.

Figure 1: Voters favoring direction $d=(d_x,d_y)$
General case

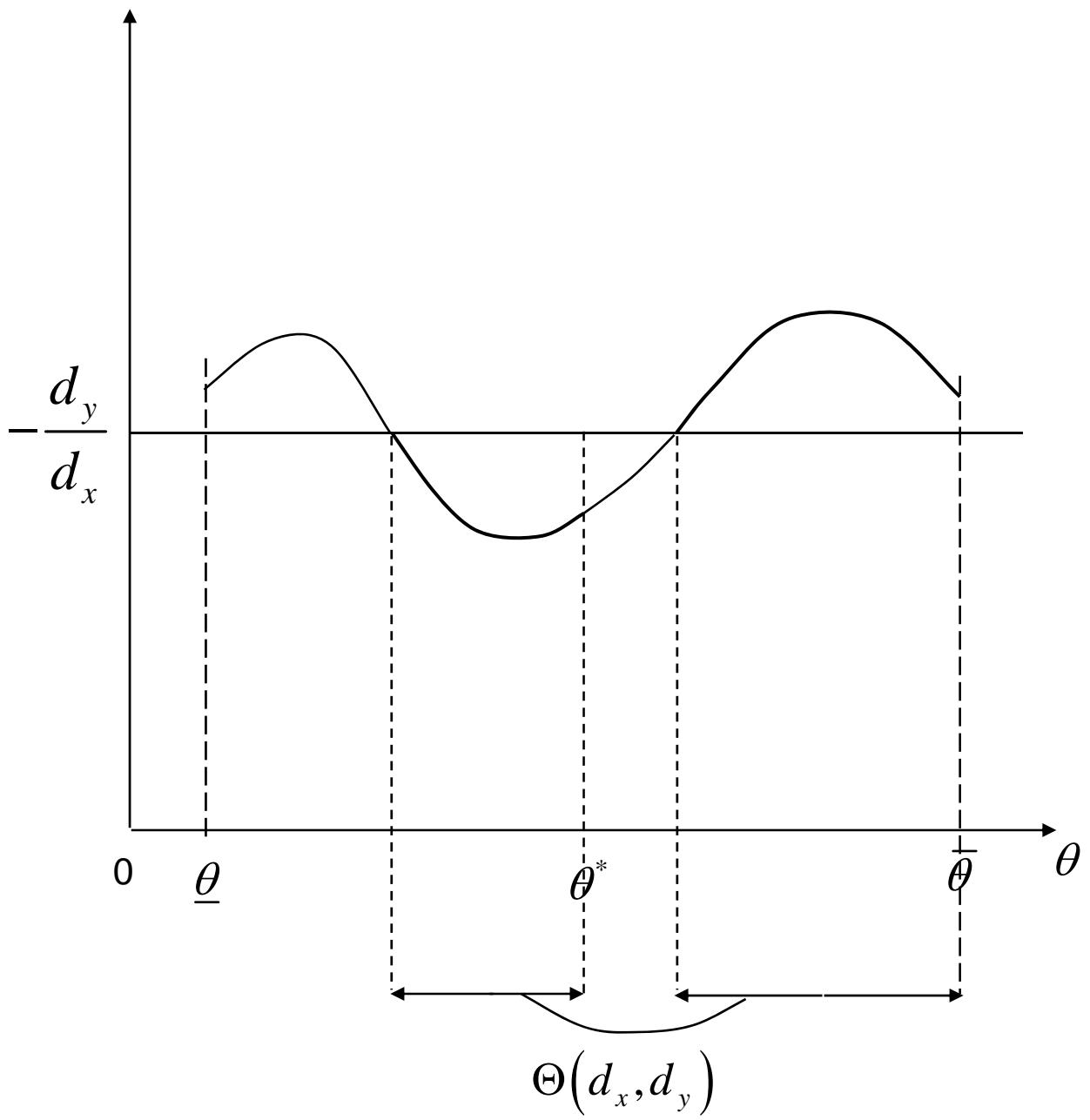


Figure 2 : Voters favoring direction $d=(d_x,d_y)$

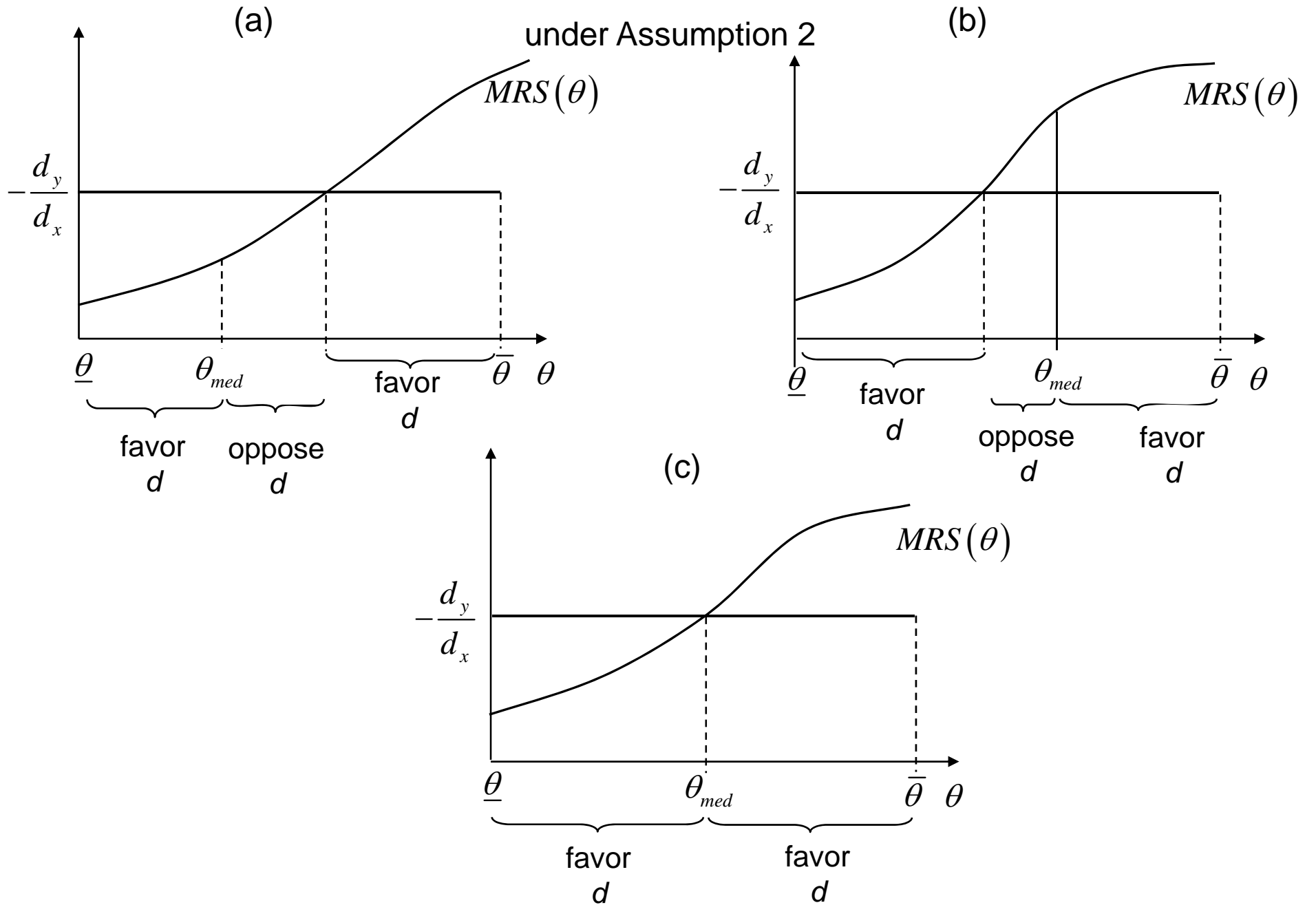
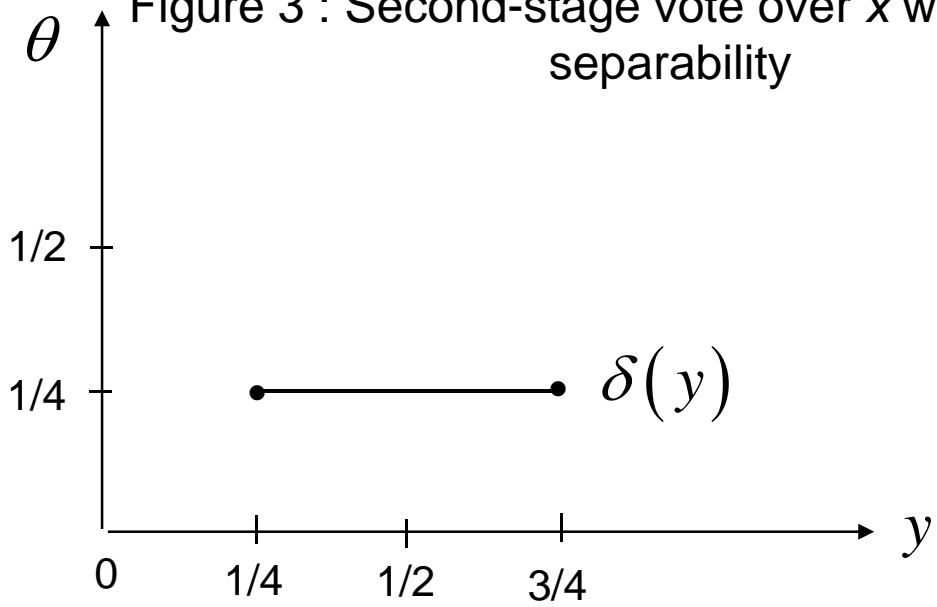
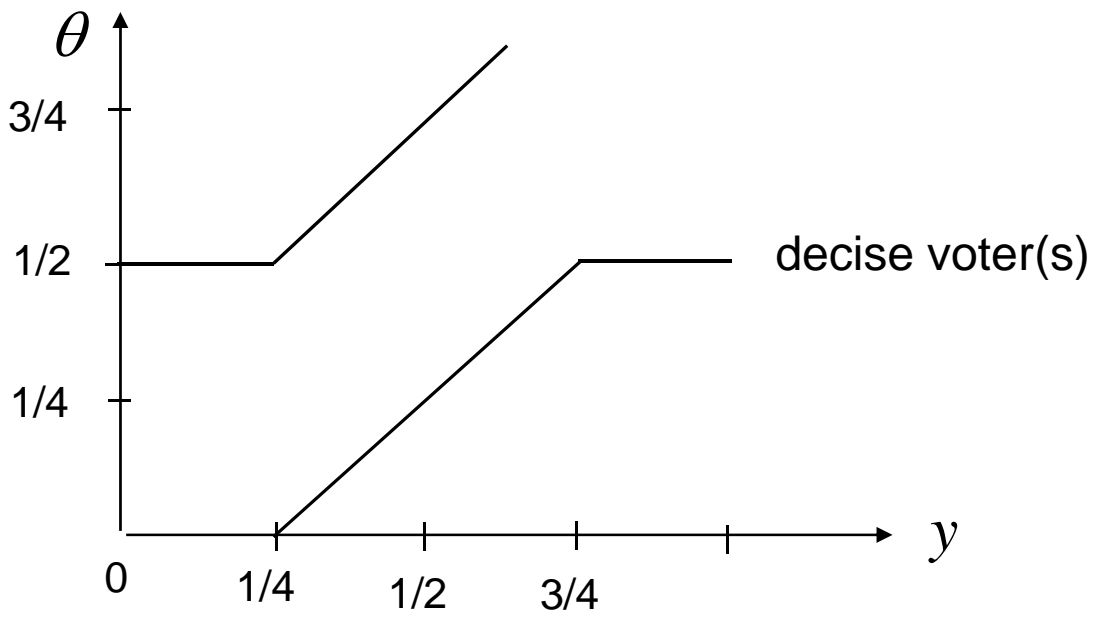


Figure 3 : Second-stage vote over x with one-sided separability

(a)



(b)



(c)

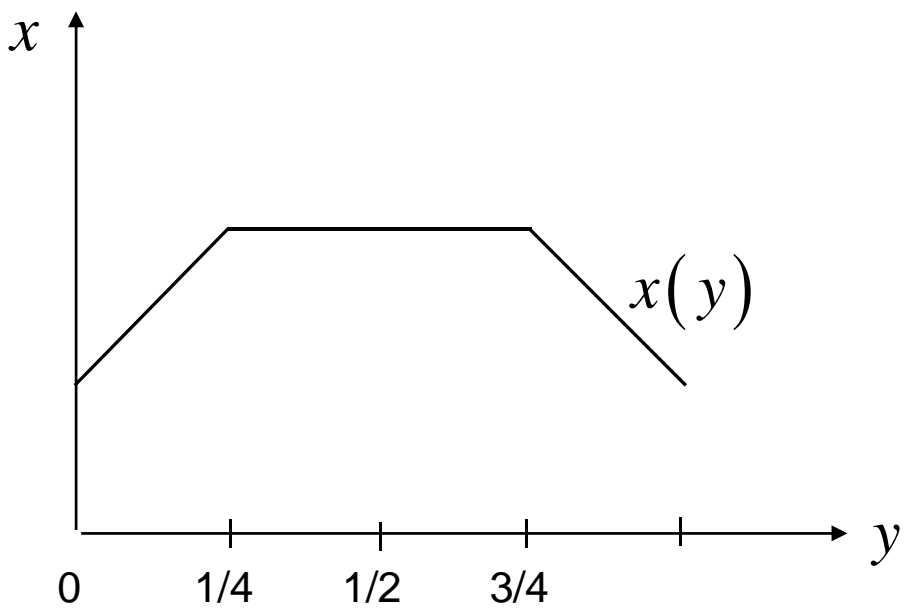
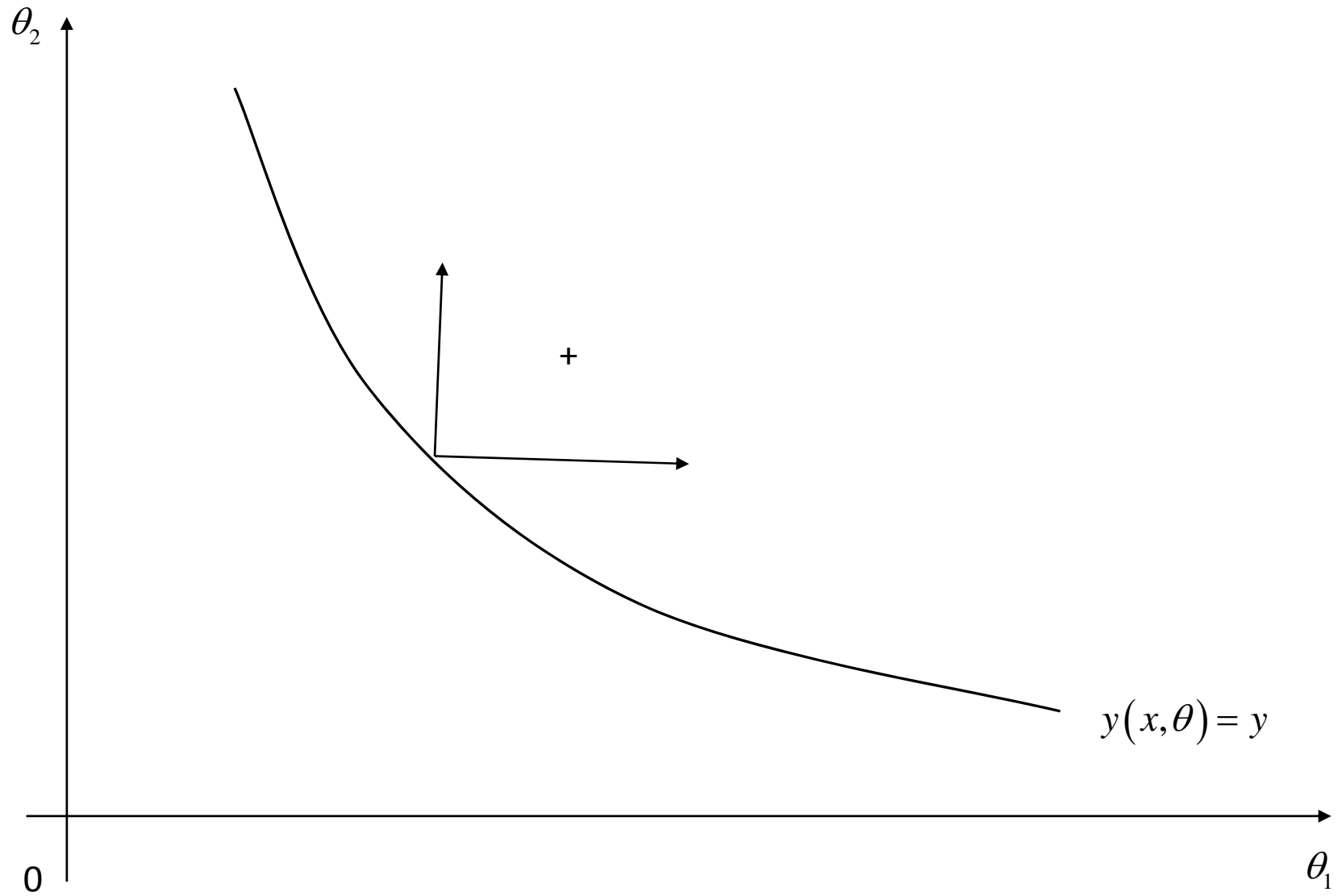
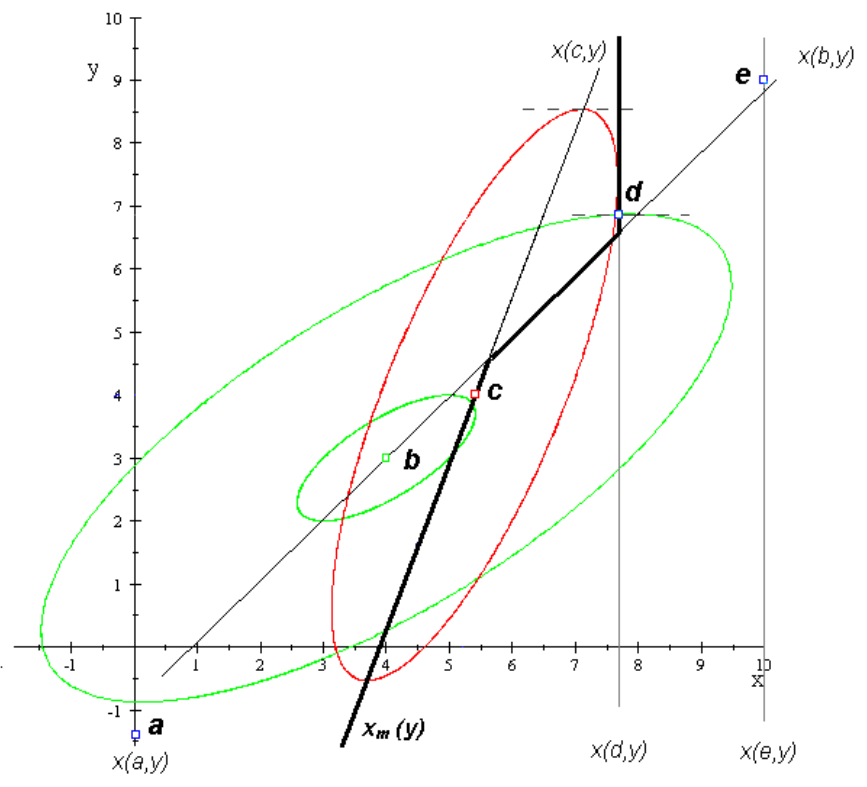
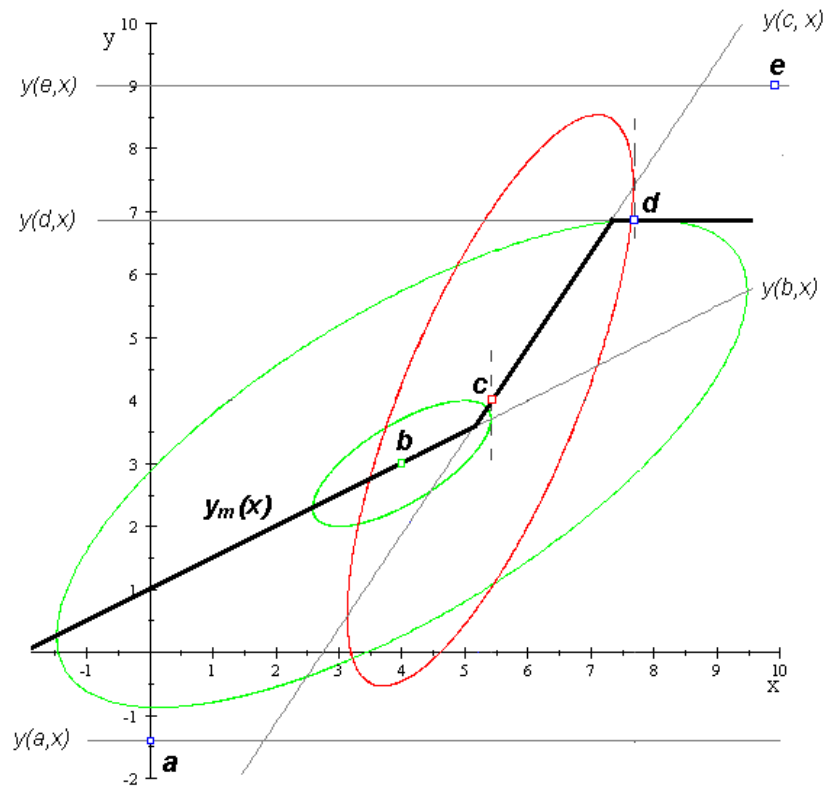


Figure 4 : Iso-y curve in the bidimensional type space





Panel 1 Figure 5 : An example with quadratic preferences **Panel 2**

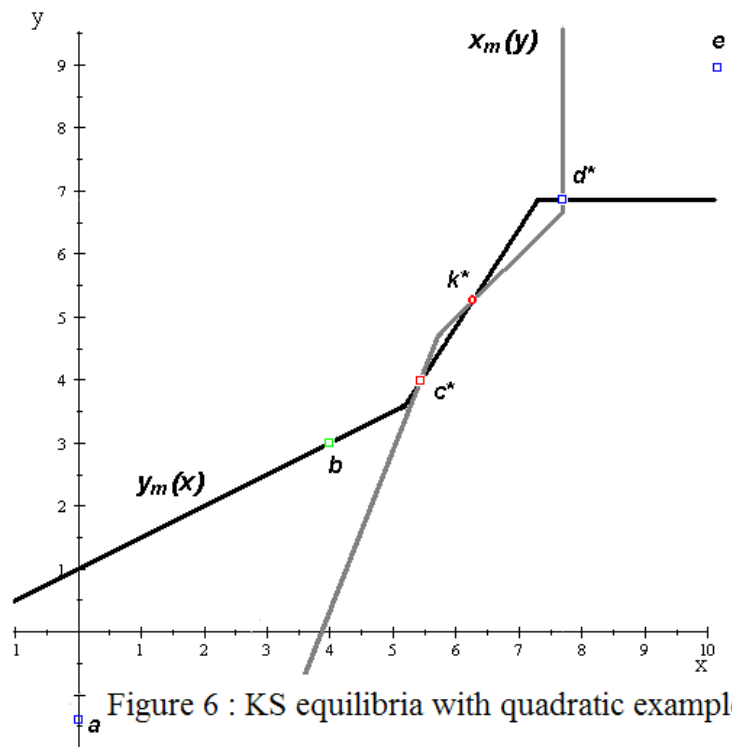


Figure 6 : KS equilibria with quadratic example