Isoperimetric stability of boundary barycenters in the plane

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Abstract

Consider an open domain D on the plane, whose isoperimetric deficit is smaller than 1. This note shows that the difference between the barycenter of D and the barycenter of its boundary is bounded above by a constant times the isoperimetric deficit to the power 1/4. This power can be improved to 1/2, when D is furthermore assumed to be a convex domain, in any Euclidean space of dimension larger than 2.

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1 Introduction

Consider a plane simple closed (or Jordan) curve C of length $L < +\infty$, bounding an open domain D of area A. The usual isoperimetric inequality asserts that

$$L^2 \geqslant 4\pi A \tag{1}$$

and that the equality is attained if and only if D is a disk.

The field of isoperimetric stability investigates what can be said about D when (1) is close to an equality, under an appropriate renormalisation. Recently there has been a lot of progress in this direction, see for instance the lecture notes of Fusco [5] and the references therein. Define $\rho \coloneqq \sqrt{A/\pi}$ and **the barycenter** $b(D)$ **of** D by

$$b(D) \coloneqq \frac{1}{A}\int_D x\,dx$$

There are several ways to measure how far D is from $B(b(D),\rho)$, the disk centered at $b(D)$ of radius ρ, when the **isoperimetric deficit**

$$d(D) \coloneqq L^2 - 4\pi A \tag{2}$$

is small. Here, we are interested in the difference between $b(D)$ and **the barycenter** $b(C)$ **of the boundary** C, defined by

$$b(C) \coloneqq \frac{1}{L}\int_C x\,\sigma(dx) \tag{3}$$

where σ is the one-dimensional Hausdorff measure (so that in particular $\sigma(C) = L$).

Of course when $d(D) = 0$, we have $b(C) = b(D) = b(B(b(D),\rho))$. It seems that the isoperimetric stability of the boundary barycenter has not been studied before. Our primary motivation comes from an illustrative example on the plane in [3], which investigates certain domain-valued stochastic evolutions associated by duality with elliptic diffusions on manifolds. Nevertheless, we found the isoperimetric stability of the boundary barycenter interesting in itself, as it contributes to a sharp understanding of the well-balancedness of almost minimizers of the isoperimetric inequality. Furthermore it shares some features with the strong form of isoperimetric stability recently developed by Fusco and Julin [6]. Here is the bound we needed in [3], it is the main result of this note:

Theorem 1 *There exists a constant $c > 0$ such that for any domain D with $d(D) \leqslant A/\pi$, we have*

$$\|b(D) - b(C)\| \leqslant cA^{1/4}d^{1/4}(D)$$

As observed by the referee, this estimate is clearly far from being optimal, since the l.h.s. can be zero with the r.h.s. being arbitrarily large.

Due to the invariance by translations and homotheties of the bound of Theorem 1, it is sufficient to show it when $\rho = 1$ and $b(D) = 0$. More precisely, translating by $-b(D)$ and applying the homothety of ratio $\sqrt{\pi/A}$, the above bound is equivalent to

$$\|b(C)\| \leqslant cd^{1/4}(D) \tag{4}$$

for any domain D with $d(D) \leqslant 1$ and whose barycenter is 0.

Due to Propositions 3 and 4 below, we are wondering if the exponent $1/4$ in (4) could not replaced by $1/2$ (or equivalently, replace $A^{1/4}d^{1/4}(D)$ by $\sqrt{d(D)}$ in Theorem 1). It would suffice to improve Lemma 9 below accordingly to obtain this conjecture.

We have not been very precise about the regularity assumption on the domain D, it should be such that the Bonnesen inequality [1] holds, as it is presented e.g. in the book of Burago and Zalgaller [2]. In particular, the above result is true if the boundary C of the open set D is piecewise $\mathcal{C}^1$. Probably it can be extended to the framework of sets of finite perimeter, as defined in the lectures of Fusco [5]. Then one has to be more careful with the definition of the boundary barycenter in (3): C has to be replaced by the *reduced boundary* $\partial^* D$ and σ by the *total variation measure of the distributional derivative of the indicator function of* D, see Fusco [5].

It could be tempting to extend Theorem 1 to the Euclidean spaces $\mathbb{R}^n$ of dimension $n \geqslant 3$. This is not possible, since the result is then wrong, as shown by the following example:

Example 2 Consider the case $n = 3$ and the set $D = B \cup F$, with B the unit open ball centered at 0 and

$$F \quad := \quad \{(x, y, z) \in \mathbb{R}^3 \,:\, x \geqslant x_0 \text{ and } \sqrt{y^2 + z^2} < f(x)\}$$

where $x_0 \in (0, 1)$, $f \,:\, [x_0, +\infty) \to \mathbb{R}_+$ is a decreasing function with $f(x_0) = \sqrt{1 - x_0^2}$ and$f(x) > \sqrt{1 - x^2}$ for all $x > x_0$. Here are the contributions of F to:

- the volume of D: $\pi \int_{x_0}^{+\infty} f^2(u)\, du$
- the area surface of D: $2\pi \int_{x_0}^{+\infty} f(u)\, du$
- the (unnormalized) barycenter of D: $\left(\pi \int_{x_0}^{+\infty} u f^2(u)\, du\right) (1, 0, 0)^{\mathrm{t}}$
- the (unnormalized) barycenter of ∂D: $\left(2\pi \int_{x_0}^{+\infty} u f(u)\, du\right) (1, 0, 0)^{\mathrm{t}}$

Let be given $\alpha > 0$ and consider the function g:

$$\forall\ u > 0, \qquad g(u) \quad := \quad u^{-\alpha}$$

For $v > 1$, consider as function f the function g shifted by v: $x_0 > 0$ is the solution of $x_0^2 + g^2(v + x_0) = 1$ and for any $u \geqslant x_0$, we take $f(u) := g(v + u)$. Since we have

$$\begin{aligned} \int_1^{+\infty} g^2(u)\, du \quad &< \quad +\infty \\ \int_1^{+\infty} g(u)\, du \quad &< \quad +\infty \\ \int_1^{+\infty} u g^2(u)\, du \quad &< \quad +\infty \\ \int_1^{+\infty} u g(u)\, du \quad &= \quad +\infty \end{aligned}$$

for any $\alpha \in (1, 2]$, we get a counter-example to Theorem 1 by letting v go to $+\infty$.

Similar considerations with $\alpha \in (1/2, 1]$ enable to see why the Bonnesen inequality [1], recalled below in Theorem 5, is no longer valid in $\mathbb{R}^3$. It is replaced by an upper bound on the *Fraenkel asymmetry index* in Fusco, Maggi and Pratelli [4]. The above construction also highlights the necessity of a restrictive assumption in Proposition 3 below.

These observations can easily be extended to the Euclidean spaces $\mathbb{R}^n$ of dimension $n \geqslant 3$.

$\square$

To avoid the pathologies of the previous example, one may want to work in the framework of compact Riemannian manifolds of dimension $n \geqslant 2$. Then consider the subsets D with a fixed volume and a fixed renormalized Fréchet mean $b(D)$ (replacing the notion of barycenter, in general $b(D)$ will not be unique and one may have to consider their whole set). Assume that among such D, there is a minimizer B for the $(n-1)$-Hausdorff measure of the boundary. There is no reason in general for the renormalized Fréchet mean $b(\partial B)$ to coincide with $b(B)$. But, under bounds on the total diameter and on the curvature, one could try to evaluate the difference between $b(\partial D)$ and $b(\partial B)$ in terms of the isoperimetric deficit of D. This investigation is clearly out of the scope of the present note.

Nevertheless, in the restricted framework of nearly spherical sets, there is an extension (even an improvement) of Theorem 1 to Euclidean spaces of dimension $n \geqslant 2$. An open set D from $\mathbb{R}^n$ is said to be **standard** if its volume is equal to the volume of the unit ball B and if its barycenter $b(D)$ is equal to 0. The standard set D is said to be **nearly spherical** if there exists a mapping u on the unitary sphere $S \coloneqq \partial B$ centered at 0 such that

$$C \ \coloneqq \ \partial D \ = \ \{(1+u(x))x \ : \ x \in S\}$$

Define the barycenter of C as in (3):

$$b(C) \ \coloneqq \ \frac{1}{\sigma(C)}\int_C x\,\sigma(dx)$$

where σ is the $(n-1)$-dimensional Hausdorff measure. The modified isoperimetric deficit is the non-negative quantity given by

$$\widetilde{d}(D) \ \coloneqq \ \sigma(C) - \sigma(S)$$

When $n = 2$, this quantity is similar to the isoperimetric deficit $d(D)$ defined in (2), at least when D is standard with $d(D) \in [0, 1]$, in which case we have

$$\frac{d(D)}{4\pi + 1} \ \leqslant \ \widetilde{d}(D) \ \leqslant \ \frac{d(D)}{2\pi} \tag{5}$$

Indeed, we have, in one hand,

$$\begin{aligned}
\widetilde{d}(D) &= L - 2\pi \\
&= \frac{L^2 - 4\pi^2}{L + 2\pi} \\
&= \frac{L^2 - 4\pi A}{L + 2\pi} \\
&\leqslant \frac{d(D)}{2\pi}
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
d(D) &= L^2 - 4\pi \\
&= (L + 2\pi)(L - 2\pi) \\
&\leqslant (\sqrt{d(D) + 4\pi^2} + 2\pi)(L - 2\pi) \\
&\leqslant (\sqrt{d(D)} + 2\pi + 2\pi)(L - 2\pi) \\
&\leqslant (1 + 4\pi)\widetilde{d}(D)
\end{aligned}$$

The interest of the (modified) isoperimetric deficit is:

Proposition 3 *There exist two constants $\epsilon(n) > 0$ and $c(n) > 0$ depending only on n, such that for any standard nearly spherical set D with $\|u\|_{W^{1,\infty}(S)} \leqslant \epsilon(n)$, we have*

$$\|b(C)\| \ \leqslant \ c(n)\sqrt{\widetilde{d}(D)}$$

Proof

This is an immediate consequence of Theorem 3.1 from Fusco [5], which finds two constants $\epsilon_1(n) > 0$ and $c_1(n) > 0$ depending only on n, such that for any standard nearly spherical set D with $\|u\|_{W^{1,\infty}(S)} \leqslant \epsilon_1(n)$, we have

$$\|u\|_{W^{1,2}(S)} \leqslant c_1(n)\sqrt{\widetilde{d}(D)}$$

Up to replacing $\epsilon_1(n)$ by $\epsilon(n) \coloneqq (1/2) \wedge \epsilon_1(n)$, we can assume that the mapping $\psi : S \ni y \mapsto (1 + u(y))y \in \partial D$ is one-to-one. It enables to use the change of variable formula to get

$$\int_C x\,\sigma(dx) = \int_S \psi(y)\,\mathrm{Jac}[\psi](y)\sigma(dy)$$

where $\mathrm{Jac}[\psi](y)$ stands for the Jacobian of ψ at $y \in S$. From the form of ψ, we deduce there exists a constant $c_2(n) > 0$, a function $w : S \to \mathbb{R}$ and a vector field v on S such that

$$\forall\ y \in S, \qquad \left\{ \begin{array}{rcl} \mathrm{Jac}[\psi](y) &=& 1 + w(y)u(y) + \langle v, \nabla_S u\rangle (y) \\ |w(y)| &\leqslant& c_2(n)\, \|u\|^{n-1}_{W^{1,\infty}(S)} \\ \|v(y)\| &\leqslant& c_2(n)\, \|u\|^{n-1}_{W^{1,\infty}(S)} \end{array} \right.$$

It follows that there exists a constant $c_2(n) > 0$ depending only on n such that as soon as $\|u\|_{W^{1,\infty}(S)} \leqslant \epsilon(n)$, we have

$$\forall\ y \in S, \qquad \|y - \psi(y)\mathrm{Jac}[\psi](y)\| \leqslant c_3(n)(|u(y)| + \|\nabla_S u(y)\|)$$

Thus we get that

$$\begin{aligned} \left\|\int_C x\,\sigma(dx)\right\| &= \left\|\int_S \psi(y)\mathrm{Jac}[\psi](y)\sigma(dy) - \int_S y\sigma(dy)\right\| \\ &\leqslant c_3(n)\int_S |u(y)| + \|\nabla_S u(y)\|\ \sigma(dy) \\ &\leqslant c_3(n)\sqrt{\sigma(S)}\, \|u\|_{W^{1,2}(S)} \end{aligned}$$

where Cauchy-Schwarz' inequality was used in the last bound. It remains to write that

$$\begin{aligned} \|b(C)\| &= \left\|\frac{1}{\sigma(C)}\int_C x\,\sigma(dx)\right\| \\ &\leqslant \frac{c_3(n)}{\sqrt{\sigma(S)}}\, \|u\|_{W^{1,2}(S)} \\ &\leqslant \frac{c_1(n)c_3(n)}{\sqrt{\sigma(S)}}\sqrt{\widetilde{d}(D)} \end{aligned}$$

to get the announced result with $c(n) \coloneqq c_1(n)c_3(n)/\sqrt{\sigma(S)}$.

■

The situation of convex sets is even simpler:

Proposition 4 *There exist two constants $\delta(n) > 0$ and $C(n) > 0$ depending only on n, such that any standard convex set D from $\mathbb{R}^n$ with $\widetilde{d}(D) \leqslant \delta(n)$ satisfies*

$$\|b(C)\| \leqslant C(n)\sqrt{\widetilde{d}(D)}$$

Proof

From Lemma 3.3 from Fusco [5], we deduce that there exists a constant $\delta(n) > 0$ such that any standard convex set D from $\mathbb{R}^n$ with $\widetilde{d}(D) \leqslant \delta(n)$ is nearly spherical with $\|u\|_{W^{1,\infty}(S)} \leqslant \epsilon(n)$. Proposition 3 then shows that it is sufficient to take $C(n) \coloneqq c(n)$ to insure the validity of the above statement.

■

2 Proof of Theorem 1

In all this section, the set D will be as in the beginning of the introduction.

The arguments will be based on two results of the literature. The first one is quite old and is due to Bonnesen [1] (see also Theorem 1.3.1 of Burago and Zalgaller [2]):

Theorem 5 *Let r and R be the radii of the incircle and the circumcircle of D. We have*

$$\pi^2(R-r)^2 \leqslant d(D)$$

This result is not sufficient to deduce Theorem 1, since one can construct a set D whose boundary is included into the centered annulus of radii $1-\epsilon$ and $1+\epsilon$, with small $\epsilon > 0$, with a lot of folds in one direction, so that $b(C)$ drifts in this direction, without $b(D)$ moving a lot.

Thus we need a second result, due quite recently to Fusco and Julin [6]. Let us recall their oscillation index $\beta(D)$, while referring to their paper for its motivation. To simplify the notation, assume that $\rho = 1$, i.e. $A = \pi$. Consider

$$\beta(D) \coloneqq \min_{y\in\mathbb{R}^2} \sqrt{\int_C \left\| \nu_C(x) - \frac{x-y}{\|x-y\|}\right\|^2 \sigma(dx)} \tag{6}$$

where $\nu_C(x)$ is the exterior unitary normal of C at x, under our assumption it is defined σ-a.s. on C (Fusco and Julin [6] defined it more generally for the sets of finite parameter, with the caution recalled after the statement of Theorem 1). Fusco and Julin [6] obtained the (multi-dimensional version of the) following result

Theorem 6 *Under the assumption $A = \pi$, there exists a constant $\widetilde{\gamma} > 0$ such that*

$$\beta(D) \leqslant \widetilde{\gamma}\sqrt{\widetilde{d}(D)}$$

Recalling the upper bound of (5) (which does note require $d(D) \leqslant 1$), we deduce that if $A = \pi$,

$$\beta(D) \leqslant \gamma\sqrt{d(D)} \tag{7}$$

with $\gamma \coloneqq \widetilde{\gamma}/\sqrt{2\pi}$.

With these ingredients at hand, we now come to the proof of Theorem 1. As already mentioned, it is sufficient to consider a standard set D with $d(D) \leqslant 1$, for which the wanted bound reduces to (4) with a universal constant $c > 0$.

Let us denote by o and O the respective centers of the incircle and the circumcircle of D. We begin by showing that o, O and 0 are quite close when the isoperimetric deficit is small.

Lemma 7 *As soon as D is a standard set with $d(D) \leqslant 1$, we have*

$$\max\{\|o\|, \|O\|\ \|O-o\|\} < 3\sqrt{d(D)}$$

Proof

Consider two numbers $0 < r' < R'$ and two points $o', O' \in \mathbb{R}^2$. If we want the inclusion of $B(o', r')$ into $B(O', R')$, we must have $\|O' - o'\| \leqslant R' - r'$. Indeed, the equality in the previous bound (which is also its worse case) corresponds to the situation where $B(o', r')$ and $B(O', R')$ are tangential at a point p which is at the intersection of $B(o', r')$ with $B(O', R')$. Then the three points p, O' and o' are on the same line and we have $r + \|O' - o'\| + R = 2R$, namely $\|O' - o'\| = R' - r'$. Since $B(o, r) \subset D \subset B(O, R)$, we deduce that $\|O - o\| \leqslant R - r \leqslant \sqrt{d(D)}/\pi$, according to Theorem 5.

Since the barycenter of D is 0, we have

$$\begin{aligned}
0 &= \int_D x\,dx \\
&= \int_{B(O,R)} x\,dx - \int_{B(O,R)\setminus D} x\,dx \\
&= \pi R^2 O - \int_{B(O,R)\setminus D} x\,dx
\end{aligned}$$

It follows that

$$\begin{aligned}
\pi R^2 \|O\| &= \left\| \int_{B(O,R)\setminus D} x\,dx \right\| \\
&\leqslant \int_{B(O,R)\setminus D} \|x\|\,dx \\
&\leqslant (\|O\| + R) \int_{B(O,R)\setminus B(o,r)} dx \\
&\leqslant (\|O\| + R)\pi(R^2 - r^2) \\
&\leqslant (\|O\| + R)\pi(R + r)\frac{\sqrt{d(D)}}{\pi} \\
&= 2(\|O\| + R)R\sqrt{d(D)}
\end{aligned}$$

We deduce that

$$(\pi R^2 - 2R\sqrt{d(D)})\,\|O\| \;\leqslant\; 2R^2\sqrt{d(D)}$$

Due to the assumption $d(D) \leqslant 1$ and from the fact that $R \geqslant 1$, we have $(\pi R^2 - 2R\sqrt{d(D)}) \geqslant (\pi - 2)R^2$, so that finally

$$\|O\| \;\leqslant\; \frac{2}{\pi - 2}\sqrt{d(D)}$$

The triangle inequality enables to conclude to the last inequality:

$$\begin{aligned}
\|o\| &\leqslant \|O - o\| + \|O\| \\
&\leqslant \left(\frac{1}{\pi} + \frac{2}{\pi - 2}\right)\sqrt{d(D)} \\
&< 3\sqrt{d(D)}
\end{aligned}$$

■

Our next step consists in checking that $\mathcal{M}$, **the set of minimizers in** (6), is also close to 0. It was remarked by Fusco and Julin [6], as a simple consequence of the divergence theorem, that such minimizers coincide with the points $y \in \mathbb{R}^2$ maximizing the mapping

$$U_D \,:\, \mathbb{R}^2 \ni y \;\mapsto\; \int_D \frac{1}{\|x - y\|}\,dx \tag{8}$$

It leads us to study the function f defined by

$$\mathbb{R}_+ \ni t \;\mapsto\; f(t) := \int_B \frac{1}{\|x - te_1\|}dx$$

where B is the unit disk centered at 0 and e_1 is the usual horizontal unit vector.

Lemma 8 *The mapping f is decreasing and as t goes to 0_+,*

$$f(t) - f(0) \quad \sim \quad \frac{\pi}{2} t^2$$

Proof

For any $t \geqslant 0$, we have

$$\begin{aligned} f(t) &= \int_{-1}^{1} dx_2 \int_{-\sqrt{1-x_2^2}}^{\sqrt{1-x_2^2}} \frac{1}{\sqrt{(x_1+t)^2 + x_2^2}} dx_1 \\ &= 2\int_0^1 g_{x_2}(t)\, dx_2 \end{aligned}$$

with for any $x_2 \in [0,1]$,

$$\forall\ t \geqslant 0, \qquad g_{x_2}(t) \quad := \quad \int_{-\sqrt{1-x_2^2}+t}^{\sqrt{1-x_2^2}+t} \frac{1}{\sqrt{x_1^2 + x_2^2}} dx_1$$

Differentiating with respect to $t \geqslant 0$, for fixed $x_2 \in (0,1)$, we get

$$\begin{aligned} & g'_{x_2}(t) \\ &= \frac{1}{\sqrt{(\sqrt{1-x_2^2}+t)^2 + x_2^2}} - \frac{1}{\sqrt{(-\sqrt{1-x_2^2}+t)^2 + x_2^2}} \\ &= \frac{1}{\sqrt{1+2\sqrt{1-x_2^2}t + t^2}} - \frac{1}{\sqrt{1-2\sqrt{1-x_2^2}t + t^2}} \\ &= \frac{1-2\sqrt{1-x_2^2}t + t^2 - (1+2\sqrt{1-x_2^2}t + t^2)}{\sqrt{1+2\sqrt{1-x_2^2}t+t^2}\sqrt{1-2\sqrt{1-x_2^2}t+t^2}\left(\sqrt{1+2\sqrt{1-x_2^2}t+t^2} + \sqrt{1-2\sqrt{1-x_2^2}t+t^2}\right)} \\ &= \frac{-4\sqrt{1-x_2^2}t}{\sqrt{1+2\sqrt{1-x_2^2}t+t^2}\sqrt{1-2\sqrt{1-x_2^2}t+t^2}\left(\sqrt{1+2\sqrt{1-x_2^2}t+t^2} + \sqrt{1-2\sqrt{1-x_2^2}t+t^2}\right)} \\ &< 0 \end{aligned}$$

The last expression is bounded uniformly in $x_2 \in [0,1]$ and for t in a compact of $\mathbb{R}_+ \backslash \{1\}$. It follows that we can differentiate under the integral to get that for $t \geqslant 0$, $t \neq 1$,

$$\begin{aligned} f'(t) &= 2\int_0^1 g'_{x_2}(t)\, dx_2 \\ &< 0 \end{aligned}$$

This is sufficient to insure that f is decreasing on $\mathbb{R}_+$.

Furthermore the above computation shows that uniformly over $x_2 \in [0,1]$, we have as t goes to 0_+,

$$g'_{x_2}(t) \quad \sim \quad -2\sqrt{1-x_2^2}t$$

This implies that as t goes to 0_+,

$$f'(t) \ \sim\ -4t\int_0^1 \sqrt{1-x_2^2}\, dx_2 \ =\ -\pi t$$

and next the last assertion of the lemma.

■

Note that by homothety and rotation, we have for any $\varrho > 0$ and $y \in \mathbb{R}^2$,

$$\int_{B(0,\varrho)} \frac{1}{\|z-y\|}\,dz \;=\; \int_{B(0,1)} \frac{\varrho^2}{\|\varrho z-y\|}\,dz \;=\; \varrho\int_{B(0,1)} \frac{1}{\|z-y/\varrho\|}\,dz \;=\; \varrho f(\|y\|/\varrho) \tag{9}$$

In conjunction with the previous lemma, we deduce the following upper bound on the elements from $\mathcal{M}$:

Lemma 9 *There exists a constant $c > 0$ such that for any standard set D with $d(D) \leqslant 1$, we have*

$$\forall\; y \in \mathcal{M}, \qquad \|y\| \;\leqslant\; cd^{1/4}(D)$$

Proof

It is sufficient to show that there exists $\epsilon \in (0,1]$ such that for any standard set D satisfying $d(D) \leqslant \epsilon$, we have

$$\forall\; y \in \mathbb{R}^2, \qquad \|y\| \geqslant cd^{1/4}(D) \;\;\Rightarrow\;\; U_D(y) < U_D(0) \tag{10}$$

where U_D was defined in (8).

Note that

$$\begin{aligned} U_D(0) &\geqslant \int_{B(o,r)} \frac{1}{\|x\|}dx \\ &> \int_{B(0,r-3\sqrt{d(D)})} \frac{1}{\|x\|}dx \end{aligned}$$

since the bound $\|o\| < 3\sqrt{d(D)}$ from Lemma 7 implies that $B(0, r-3\sqrt{d(D)})$ is strictly included into $B(o,r)$. From (9) we deduce that

$$\begin{aligned} U_D(0) &> (r-3\sqrt{d(D)})_+ f(0) \\ &= (r-3\sqrt{d(D)})_+ 2\pi \int_0^1 s/s\,ds \\ &= 2\pi(r-3\sqrt{d(D)})_+ \end{aligned}$$

Recall that $r \leqslant 1 \leqslant R$, so from Theorem 5 we have that $r \geqslant 1-\sqrt{d(D)/\pi}$. It follows that ϵ can be chosen sufficiently small so that $r - 3\sqrt{d(D)} \geqslant 1-(3+1/\pi)\sqrt{d(D)} > 0$.

Next let us find an upper bound on $U_D(y)$, for $y \in \mathbb{R}^2$ not too small. We have

$$\begin{aligned} U_D(y) &\leqslant \int_{B(O,R)} \frac{1}{\|x-y\|}dx \\ &= \int_{B(0,R)} \frac{1}{\|x+O-y\|}dx \\ &= Rf(\|y-O\|/R) \end{aligned}$$

where (9) was taken into account. Assume that for some constant $c_1 > 0$, $\|y\| \geqslant (c_1+3)d^{1/4}(D)$, so that we are insured of

$$\|y\| \;\geqslant\; c_1 d^{1/4}(D) + 3d^{1/2}(D) \;\geqslant\; c_1 d^{1/4}(D) + \|O\|$$

Then we deduce from Lemmas (8) and (7) that for $\epsilon > 0$ chosen small enough,

$$\begin{aligned} Rf(\|y-O\|/R) &\leqslant Rf(\|y\|/R - \|O\|/R) \\ &\leqslant Rf(0) - \frac{\pi}{4}R\,(\|y\|/R - \|O\|/R)^2 \\ &\leqslant 2\pi R - c_2\sqrt{d(D)}/R \end{aligned}$$

with $c_2 \coloneqq \pi c_1^2/4$. Note that $R \leqslant 1 + \sqrt{d(D)}/\pi$, so that (10) amounts to find c_2 large enough (i.e. c_1 large enough) so that

$$\forall\; d \in [0,\epsilon), \qquad 2\pi(1+\sqrt{d}/\pi) - c_2\sqrt{d}/(1+\sqrt{d}/\pi) \;\leqslant\; 2\pi(1-(3+1/\pi)\sqrt{d})$$

where $\epsilon \in (0,1]$ has been chosen above. An elementary computation shows that this is true with $c_2 \coloneqq 2(1+\pi)(3+2/\pi)$.

■

The end of the proof of Theorem 1 is immediate. Remark that by an application of the divergence theorem, we have $\int_C \nu_C(x)\,dx = 0$, so that for any standard set D,

$$\begin{aligned} \|b(C)\| &= \frac{1}{L}\left\|\int_C x\,\sigma(dx)\right\| \\ &= \frac{1}{L}\left\|\int_C x - \nu_C(x)\,\sigma(dx)\right\| \\ &\leqslant \frac{1}{L}\int_C \|x-\nu_C(x)\|\;\sigma(dx) \end{aligned}$$

Consider $y \in \mathcal{M}$ and write

$$\|\nu_C(x) - x\| \;\leqslant\; \left\|\nu_C(x) - \frac{x-y}{\|x-y\|}\right\| + \left\|\frac{x-y}{\|x-y\|} - (x-y)\right\| + \|y\|$$

The middle term of the r.h.s. can be treated as follows:

$$\begin{aligned} \left\|\frac{x-y}{\|x-y\|} - (x-y)\right\| &= \left|\frac{1}{\|x-y\|} - 1\right|\|x-y\| \\ &= |1-\|x-y\|| \\ &\leqslant \|y\| + |\|x\|-1| \end{aligned}$$

Concerning the last term, use Theorem 5 and Lemma 7 to see that for $x \in C$, if $d(D) \leqslant 1$, on one hand,

$$\begin{aligned} \|x\| &\leqslant \|x-O\| + \|O\| \\ &\leqslant R + 3\sqrt{d(D)} \\ &\leqslant 1 + (3+1/\pi)\sqrt{d(D)} \end{aligned}$$

and on the other hand,

$$\begin{aligned} \|x\| &\geqslant \|x-O\| - \|O\| \\ &\geqslant r - 3\sqrt{d(D)} \\ &\geqslant 1 - (3+1/\pi)\sqrt{d(D)} \end{aligned}$$

It follows that $|\|x\|-1| \leqslant (3+1/\pi)\sqrt{d(D)}$. Putting together the above considerations, we get

$$\begin{aligned} \|b(C)\| &\leqslant \frac{1}{L}\int_C \left\|\nu_C(x) - \frac{x-y}{\|x-y\|}\right\| + 2\|y\| + (3+1/\pi)\sqrt{d(D)}\,\sigma(dx) \\ &\leqslant \sqrt{\int_C \left\|\nu_C(x) - \frac{x-y}{\|x-y\|}\right\|^2 \frac{\sigma(dx)}{L}} + \|y\| + (3+1/\pi)\sqrt{d(D)} \\ &\leqslant \frac{\beta(D)}{\sqrt{2\pi}} + Cd^{1/4}(D) + (3+1/\pi)\sqrt{d(D)} \end{aligned}$$

where we used Lemma 9. From (7) and the fact that $d(D) \leqslant 1$, we conclude that

$$\|b(C)\| \quad \leqslant \quad \left(\frac{\gamma}{\sqrt{2\pi}} + C + 3 + \frac{1}{\pi}\right) d^{1/4}(D)$$

as wanted.

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