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“Adaptive estimation in the linear random coefficients model when regressors have limited variation”

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ADAPTIVE ESTIMATION IN THE LINEAR RANDOM COEFFICIENTS MODEL WHEN REGRESSORS HAVE LIMITED VARIATION

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Abstract. We consider a linear model where the coefficients - intercept and slopes - are random and independent from regressors which support is a proper subset. When the slopes do not have heavy tails, the joint density of the random coefficients is identified. Lower bounds on the supremum risk for the estimation of the density are derived for this model and a related white noise model. We present an estimator, its rates of convergence, and a data-driven rule which delivers adaptive estimators. The corresponding R package is RandomCoefficients.

1. Introduction

For a random variable α and random vectors $\boldsymbol{X}$ and $\boldsymbol{\beta}$ of dimension p, the linear random coefficients model is

$$Y = \alpha + \boldsymbol{\beta}^\top \boldsymbol{X}, \tag{1}$$

$$(\alpha, \boldsymbol{\beta}^\top) \text{ and } \boldsymbol{X} \text{ are independent.} \tag{2}$$

The researcher has at her disposal n observations $(Y_i, \boldsymbol{X}_i^\top)_{i=1}^n$ of $(Y, \boldsymbol{X}^\top)$ but does not observe the realizations $(\alpha_i, \boldsymbol{\beta}_i^\top)_{i=1}^n$ of $(\alpha, \boldsymbol{\beta}^\top)$. α subsumes the intercept and error term and the vector of slope coefficients $\boldsymbol{\beta}$ is heterogeneous (*i.e.*, varies across i). For example, a researcher interested in the effect of class size on pupils' achievements might want to allow some pupils to be more sensitive than others to a decrease in the size and to estimate the density of the effect. $(\alpha, \boldsymbol{\beta}^\top)$ correspond to multidimensional unobserved heterogeneity and $\boldsymbol{X}$ to observed heterogeneity. Restricting unobserved heterogeneity to a scalar, as when only α is random, can have undesirable implications such as monotonicity in the literature on policy evaluation (see [24]). Parametric assumptions are often made by convenience and can drive the results (see [29]). For this reason, this paper considers a nonparametric setup. Model (1) is also a type of linear model with homegeneous slopes and heteroscedasticity, hence the averages of the coefficients are easy to obtain. However, the law of coefficients, their quantiles, prediction

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intervals for Y for $\boldsymbol{X}=\boldsymbol{x}$ as in [3], welfare measures, treatment and counterfactual effects, which depend on the distribution of the coefficients can be of great interest.

Estimation of the density of random coefficients $f_{\alpha,\boldsymbol{\beta}}$ when the support of $\boldsymbol{X}$ is $\mathbb{R}^p$ and $\boldsymbol{X}$ has heavy enough tails has been studied in [4, 31]. These papers notice that the inverse problem is related to a tomography problem (see, *e.g.*, [11, 12]) involving the Radon transform. Assuming the support of $\boldsymbol{X}$ is $\mathbb{R}^p$ amounts to assuming that the law of angles has full support, moreover a lower bound on the density of $\boldsymbol{X}$ is assumed so that the law of the angles is nondegenerate. When $p=1$ this is implied by densities of $\boldsymbol{X}$ which follow a Cauchy distribution. The corresponding tomography problem has a nonuniform and estimable density of angles and the dimension can be larger than in tomography due to more than one regressor. More general specifications of random coefficients model are important in econometrics (see, *e.g.*, [25, 30] and references therein) and there has been recent interest in nonparametric tests (see [10, 19]).

This paper considers the case where the support of $\boldsymbol{X}$ is a proper (*i.e.*, strict) subset. This is a much more useful and realistic framework for the random coefficients model. When $p=1$, this is related to limited angle tomography (see, *e.g.*, [20, 32]). There, one has measurements over a subset of angles and the unknown density has support in the unit disk. This is too restrictive for a density of random coefficients and implies that α has compact support, ruling out usual parametric assumptions on error terms. Due to (2), the conditional characteristic function of Y given $\boldsymbol{X}=\boldsymbol{x}$ at t is the Fourier transform of $f_{\alpha,\boldsymbol{\beta}}$ at $(t, t\boldsymbol{x}^\top)^\top$. Hence, the family of conditional characteristic functions indexed by $\boldsymbol{x}$ in the support of $\boldsymbol{X}$ gives access to the Fourier transform of $f_{\alpha,\boldsymbol{\beta}}$ on a double cone of axis $(1,0,\dots,0)\in\mathbb{R}^{p+1}$ and apex 0. When $\alpha=0$, $\mathbb{S}_{\boldsymbol{\beta}}$ is compact, and $\mathcal{X}\subseteq\mathbb{S}_{\boldsymbol{X}}$ is an arbitrary compact set of nonempty interior, this is the problem of out-of-band extrapolation or super-resolution (see, *e.g.*, [5] sections 11.4 and 11.5). Because we allow α to be nonzero, we generalize this approach. Estimation of $f_{\alpha,\boldsymbol{\beta}}$ is a statistical inverse problem for which the deterministic problem is the inversion of a truncated Fourier transform (see, *e.g.*, [2] and the references therein). The companion paper [23] presents conditions on the law of $\left(\alpha,\boldsymbol{\beta}^\top\right)^\top$ and the support of $\boldsymbol{X}$ that imply nonparametric identification. It considers weak conditions on α which could have infinite absolute moments and the marginals of $\boldsymbol{\beta}$ could have heavy tails. In this paper, we obtain rates of convergence when the marginals of $\boldsymbol{\beta}$ do not have heavy tails but can have noncompact support.

A related approach is extrapolation. It is used in [41] to perform deconvolution of compactly supported densities while allowing the Fourier transform of the error density to vanish on a set of positive measure. In this paper, the relevant operator is viewed as a composition of two operators based on partial Fourier transforms. One involves a truncated Fourier transform and we make use of properties of the singular value decomposition rather than extrapolation.

Similar to [26, 33], we study optimality in the minimax sense. We obtain lower bounds under weak to strong integrability in the first argument for this and a white noise model. We present an estimator involving: series based estimation of the partial Fourier transform of the density with respect to the first variable, interpolation around zero, and inversion of the partial Fourier transform. We give rates of convergence and use a Goldenshluger-Lepski type method to obtain data-driven estimators. We consider estimation of $f_{\boldsymbol{\beta}}$ in Appendix B.5. We present a numerical method to compute the estimator which is implemented in the R package RandomCoefficients.

2. Notations

$\mathbb{N}$ and $\mathbb{N}_0$ stand for the positive and nonnegative integers, $(\cdot)_+$ for $\max(\cdot,0)$, $a\wedge b$ (resp. $a\vee b$) for the minimum (resp. maximum) between a and b, and $\mathbb{1}\{\cdot\}$ for the indicator function. Bold letters are used for vectors. For all $r\in\mathbb{R}$, $\underline{\boldsymbol{r}}$ is the vector, which dimension will be clear from the text, where each entry is r. The iterated logarithms are $\ln_0(t)=t$ and, for $j\geq 1$ and t large enough, $\ln_j(t)=\ln(\ln_{j-1}(t))$. $|\cdot|_q$ for $q\in[1,\infty]$ stands for the ℓ_q norm of a vector. For all $\boldsymbol{\beta}\in\mathbb{C}^d$, $(f_m)_{m\in\mathbb{N}_0}$ functions with values in $\mathbb{C}$, and $\boldsymbol{m}\in\mathbb{N}_0^d$, denote by $\boldsymbol{\beta}^{\boldsymbol{m}}=\prod_{k=1}^d\boldsymbol{\beta}_k^{\boldsymbol{m}_k}$, $|\boldsymbol{\beta}|^{\boldsymbol{m}}=\prod_{k=1}^d|\boldsymbol{\beta}_k|^{\boldsymbol{m}_k}$, and $f_{\boldsymbol{m}}=\prod_{k=1}^d f_{\boldsymbol{m}_k}$. For a differentiable function f of real variables, $f^{(\boldsymbol{m})}$ denotes $\prod_{j=1}^d\frac{\partial^{\boldsymbol{m}_j}}{\partial x_j^{\boldsymbol{m}_j}}f$ and $\mathrm{supp}(f)$ its support. $C^\infty\left(\mathbb{R}^d\right)$ is the space of infinitely differentiable functions. The inverse of a mapping f, when it exists, is denoted by f^I. We denote the interior of $\mathcal{S}\subseteq\mathbb{R}^d$ by $\overset{\circ}{\mathcal{S}}$ and its closure by $\overline{\mathcal{S}}$. When $\mathcal{S}$ is measurable and μ a function from $\mathcal{S}$ to $[0,\infty]$, $L^2(\mu)$ is the space of complex-valued square integrable functions equipped with $\langle f,g\rangle_{L^2(\mu)}=\int_{\mathcal{S}}f(\boldsymbol{x})\overline{g}(\boldsymbol{x})\mu(\boldsymbol{x})d\boldsymbol{x}$. This is denoted by $L^2(\mathcal{S})$ when $\mu=1$. When $W_{\mathcal{S}}=\mathbb{1}\{\mathcal{S}\}+\infty\ \mathbb{1}\{\mathcal{S}^c\}$, we have $L^2\left(W_{\mathcal{S}}\right)=\left\{f\in L^2\left(\mathbb{R}^d\right):\ \mathrm{supp}(f)\subseteq\mathcal{S}\right\}$ and $\langle f,g\rangle_{L^2(W_{\mathcal{S}})}=\int_{\mathcal{S}}f(\boldsymbol{x})\overline{g}(\boldsymbol{x})d\boldsymbol{x}$. Denote by $\mathcal{D}$ the set of densities, by $\Pi: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ such that $\Pi f(\boldsymbol{x})=f(-\boldsymbol{x})$, and by $\otimes$ the product of functions (*e.g.*, $W^{\otimes d}(\boldsymbol{b})=\prod_{j=1}^d W(\boldsymbol{b}_j)$) or measures. The Fourier transform of $f\in L^1\left(\mathbb{R}^d\right)$ is $\mathcal{F}[f](\boldsymbol{x})=\int_{\mathbb{R}^d}e^{i\boldsymbol{b}^\top\boldsymbol{x}}f(\boldsymbol{b})d\boldsymbol{b}$ and $\mathcal{F}[f]$ is also the Fourier transform in $L^2\left(\mathbb{R}^d\right)$. For all $c>0$, denote the Paley-Wiener space by $PW(c):=\left\{f\in L^2(\mathbb{R}):\mathrm{supp}\left(\mathcal{F}[f]\right)\subseteq[-c,c]\right\}$, by $\mathcal{P}_c$ the projector from $L^2(\mathbb{R})$ to $PW(c)$ ($\mathcal{P}_c[f]=\mathcal{F}^I\left[\mathbb{1}\{[-c,c]\}\mathcal{F}[f]\right]$), and, for all $c\neq 0$, by

$$\begin{array}{ccccc}\mathcal{F}_c: & L^2\left(W^{\otimes d}\right) & \to & L^2\left([-1,1]^d\right) \\ & f & \to & \mathcal{F}[f](c\ \cdot)\end{array}\quad\text{and}\quad\begin{array}{cccc}\mathcal{C}_c: & L^2\left(\mathbb{R}^d\right) & \to & L^2\left(\mathbb{R}^d\right) \\ & f & \to & |c|^d f(c\ \cdot).\end{array}\tag{3}$$

Abusing notations, we sometimes use $\mathcal{F}_c[f]$ for the function in $L^2(\mathbb{R})$. $\mathcal{E}xt[f]$ assigns the value 0 outside $[-1,1]^d$ and $\mathcal{F}_{1\mathrm{st}}[f](t,\cdot)$ is the partial Fourier transform of f with respect to the first variable. For a random vector $\boldsymbol{X}$, $\mathbb{P}_{\boldsymbol{X}}$ is its law, $f_{\boldsymbol{X}}$ its density, $f_{\boldsymbol{X}|\mathcal{X}}$ the truncated density of $\boldsymbol{X}$ given $\boldsymbol{X}\in\mathcal{X}$, $\mathbb{S}_{\boldsymbol{X}}$ its support, and $f_{Y|\boldsymbol{X}=\boldsymbol{x}}$ the conditional density. For a sequence of random variables $(X_{n_0,n})_{(n_0,n)\in\mathbb{N}_0^2}$, $X_{n_0,n}=\underset{\mathcal{U}}{O_p}(1)$ means that, for all $\epsilon>0$, there exists M such that $\mathbb{P}(|X_{n_0,n}|\geq M)\leq\epsilon$ for all $(n_0,n)\in\mathbb{N}_0^2$ such that $\mathcal{U}$ holds. In the absence of constraint, we drop the notation $\mathcal{U}$. With a single index the $O_p(1)$ notation requires a bound holding for all value of the index (the usual notation if the random variables are bounded in probability).

3. Preliminaries

Assumption 1. (H1.1) $f_{\boldsymbol{X}}$ and $f_{\alpha,\boldsymbol{\beta}}$ exist;

(H1.2) $f_{\alpha,\boldsymbol{\beta}}\in L^2\left(w\otimes W^{\otimes p}\right)$, where $w\geq 1$ and W is even, nondecreasing on $[0,\infty)$, such that $W(0)>0$ and $\lim_{x\to\infty}W(x)=\infty$, $\sum_{k\in\mathbb{N}}M_k^{-1/k}=\infty$ with $M_k=\left(\int_{\mathbb{R}}b^{2k}W^{-1}(b)db\right)^{1/2}$;

(H1.3) There exists $x_0>0$ and $\mathcal{X}=[-x_0,x_0]^p\subseteq\mathbb{S}_{\boldsymbol{X}}$ and we have at our disposal i.i.d $(Y_i,\boldsymbol{X}_i)_{i=1}^n$ and an estimator $\widehat{f}_{\boldsymbol{X}|\mathcal{X}}$ based on $\mathcal{G}_{n_0}=(\boldsymbol{X}_i)_{i=-n_0+1}^0$ independent of $(Y_i,\boldsymbol{X}_i)_{i=1}^n$;

(H1.4) $\mathcal{E}$ is a set of densities on $\mathcal{X}$ such that, for $c_{\boldsymbol{X}}, C_{\boldsymbol{X}} \in (0,\infty)$, for all $f \in \mathcal{E}$, $\|f\|_{L^\infty(\mathcal{X})} \le C_{\boldsymbol{X}}$ and $\|1/f\|_{L^\infty(\mathcal{X})} \le c_{\boldsymbol{X}}$, and, for $(v(n_0,\mathcal{E}))_{n_0\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$ which tends to 0, we have

$$\frac{1}{v(n_0,\mathcal{E})} \sup_{f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}} \left\| \widehat{f}_{\boldsymbol{X}|\mathcal{X}} - f_{\boldsymbol{X}|\mathcal{X}} \right\|^2_{L^\infty(\mathcal{X})} = O_p(1).$$

We maintain this assumption for all results presenting upper bounds. When $w=1$, $\mathbb{E}\left[\alpha^k\right]$, for $k\in\mathbb{N}$, might not exist. Due to Theorem 3.14 in [18], if there exist $R>0$, $(a_j)_{j\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$, and $(p_j)_{j\in\mathbb{N}_0} \in (-\infty,1]^{\mathbb{N}_0}$ equal to 0 for j large enough, such that

$$W(x) \ge \exp\left(\frac{x^2}{\prod_{j=0}^{\infty} \log_j^{p_j}(a_j|x|)}\right) \mathbb{1}\{|x|\ge R\} \quad \left(e.g.,\ W(x) = \exp\left(\frac{|x|}{a_0\log(a_1|x|)}\right)\mathbb{1}\{|x|\ge R\}\right),$$

for all $x\in\mathbb{R}$, then $\sum_{m\in\mathbb{N}} 1/\left\|x\mapsto x^{2m}/W(x)\right\|^{1/(2m)}_{L^\infty(\mathbb{R})} = \infty$ which implies (H1.2). Marginal distributions can have an infinite moment generating function hence be heavy-tailed and their Fourier transforms belong to a quasi-analytic class but not be analytic. Now on, we use $W_{[-R,R]}$ or $\cosh(\cdot/R)$ for W. This rules out heavy tails and nonanalytic Fourier transforms. When $W=W_{[-R,R]}$, integrability in $\boldsymbol{b}$ amounts to $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R,R]^p$, but other W allow for non compact $\mathbb{S}_{\boldsymbol{\beta}}$. Though with a different scalar product, we have $L^2(\cosh(b\cdot)) = L^2\left(e^{b|\cdot|}\right)$ and (see Theorem IX.13 in [45]), for $a>0$, $\left\{f\in L^2(\mathbb{R}):\ \forall b<a,\ f\in L^2\left(e^{b|\cdot|}\right)\right\}$ is the set of square-integrable functions which Fourier transform have an analytic continuation on $\{z\in\mathbb{C}:\ |\mathrm{Im}(z)|<a/2\}$. In particular the Laplace transform is finite near 0. Equivalently, if f is a density, it does not have heavy-tails. The condition $\mathcal{X}=[-x_0,x_0]^p \subseteq \mathbb{S}_{\boldsymbol{X}}$ in (H1.4) is not restrictive because we can write (1) as $Y=\alpha+\boldsymbol{\beta}^\top\underline{\boldsymbol{x}}+\boldsymbol{\beta}^\top(\boldsymbol{X}-\underline{\boldsymbol{x}})$, take $\underline{\boldsymbol{x}}\in\mathbb{R}^p$ and x_0 such that $\mathcal{X}\subseteq\mathbb{S}_{\boldsymbol{X}-\underline{\boldsymbol{x}}}$, and there is a one-to-one mapping between $f_{\alpha+\boldsymbol{\beta}^\top\underline{\boldsymbol{x}},\boldsymbol{\beta}}$ and $f_{\alpha,\boldsymbol{\beta}}$. We assume (H1.4) because the estimator involves estimators of $f_{\boldsymbol{X}|\mathcal{X}}$ in denominators. Alternative solutions exist when $p=1$ (see, *e.g.*, [36]) only. Assuming the availability of an estimator of $f_{\boldsymbol{X}|\mathcal{X}}$ using the preliminary sample $\mathcal{G}_{n_0}$ is common in the deconvolution literature (see, *e.g.*, [15]). By using estimators of $f_{\boldsymbol{X}|\mathcal{X}}$ for a well chosen $\mathcal{X}$ rather than of $f_{\boldsymbol{X}}$, the assumption that $\left\|f_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})}\le C_{\boldsymbol{X}}$ and $\left\|1/f_{\boldsymbol{X}|\mathcal{X}}\right\|_{L^\infty(\mathcal{X})}\le c_{\boldsymbol{X}}$ in (H1.4) becomes very mild. This is feasible because of (2).

3.1. **Inverse problem in Hilbert spaces.** Estimation of $f_{\alpha,\boldsymbol{\beta}}$ is a statistical ill-posed inverse problem. The operator depends on w and W. Now on, the functions w and W are those of (H1.2). We have, for all $t\in\mathbb{R}$ and $\boldsymbol{u}\in[-1,1]^p$, $\mathcal{K}f_{\alpha,\boldsymbol{\beta}}(t,\boldsymbol{u}) = \mathcal{F}\left[f_{Y|\boldsymbol{X}=x_0\boldsymbol{u}}\right](t)\,|tx_0|^{p/2}$, where

$$\begin{array}{rccc} \mathcal{K}: & L^2\left(w\otimes W^{\otimes p}\right) & \to & L^2(\mathbb{R}\times[-1,1]^p) \\ & f & \to & (t,\boldsymbol{u}) \mapsto \mathcal{F}[f](t,tx_0\boldsymbol{u})\,|tx_0|^{p/2}. \end{array} \tag{4}$$

Proposition 1. $L^2\left(w\otimes W^{\otimes p}\right)$ is continuously embedded into $L^2(\mathbb{R}^{p+1})$. Moreover, $\mathcal{K}$ is injective and continuous, and not compact if $w=1$.

The case $w=1$ corresponds to mild integrability assumptions in the first variable when the SVD of $\mathcal{K}$ does not exist. This makes it difficult to prove rates of convergence even for estimators which do not rely explicitly on the SVD such as the Tikhonov and Landweber method (Gerchberg algorithm in out-of-band extrapolation, see, *e.g.*, [5]). Rather than work with $\mathcal{K}$ directly, we use that $\mathcal{K}$ is the composition of operators which are easier to analyze

$$\text{for } t\in\mathbb{R},\ \mathcal{K}[f](t,\star)=\mathcal{F}_{tx_0}\left[\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)\right](\star)\left|tx_0\right|^{p/2}\ \text{ in } L^2([-1,1]^p). \tag{5}$$

For all $f\in L^2\left(w\otimes W^{\otimes p}\right)$, W either $W_{[-R,R]}$ or $\cosh(\cdot/R)$, and $t\in\mathbb{R}$, $\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)$ belongs to $L^2(W^{\otimes p})$ and, for $c\neq 0$, $\mathcal{F}_c:\ L^2(W^{\otimes p})\to L^2([-1,1]^p)$ admits a SVD, where both orthonormal systems are complete. This is a tensor product of the SVD when $p=1$ that we denote by $\left(\sigma_m^{W,c},\varphi_m^{W,c},g_m^{W,c}\right)_{m\in\mathbb{N}_0}$, where $\left(\sigma_m^{W,c}\right)_{m\in\mathbb{N}_0}\in(0,\infty)^{\mathbb{N}_0}$ is in decreasing order repeated according to multiplicity, $\left(\varphi_m^{W,c}\right)_{m\in\mathbb{N}_0}$ and $\left(g_m^{W,c}\right)_{m\in\mathbb{N}_0}$ are orthonormal systems of, respectively, $L^2(W)$ and $L^2([-1,1])$. This holds for the following reason. Because $\mathcal{F}_c=\mathcal{F}\mathcal{C}_{c^{-1}}=|c|^{-1}\mathcal{C}_c\mathcal{F}$, $\Pi\mathcal{F}_c=\mathcal{F}_c\Pi$, $\mathcal{F}_c^*=W^{-1}\Pi\mathcal{F}_c\mathcal{E}xt$, and W is even, we obtain $\mathcal{F}_c^*=\Pi\left(W^{-1}\mathcal{F}_c\mathcal{E}xt\right)$ and $\mathcal{F}_c\mathcal{F}_c^*=\Pi\mathcal{F}_c\left(W^{-1}\mathcal{F}_c\mathcal{E}xt\right)=(2\pi/|c|)\mathcal{F}^I\left(\mathcal{C}_{c^{-1}}\left(W^{-1}\mathcal{C}_c\mathcal{F}\mathcal{E}xt\right)\right)=2\pi\mathcal{F}^I\left(\mathcal{C}_{c^{-1}}\left(W^{-1}\right)\mathcal{F}\mathcal{E}xt\right)$. The operator $\mathcal{Q}_c^W=(|c|/(2\pi))\mathcal{F}_c\mathcal{F}_c^*$ is a compact positive definite self-adjoint operator (see [44] and [49] for the two choices of W). Its eigenvalues in decreasing order repeated according to multiplicity are denoted by $\left(\rho_m^{W,c}\right)_{m\in\mathbb{N}_0}$ and a basis of eigenfunctions by $\left(g_m^{W,c}\right)_{m\in\mathbb{N}_0}$. The other elements of the SVD are $\sigma_m^{W,c}=\sqrt{2\pi\rho_m^{W,c}/|c|}$ and $\varphi_m^{W,c}=\mathcal{F}_c^*g_m^{W,c}/\sigma_m^{W,c}$.

Proposition 2. For all $c\neq 0$, $\left(\varphi_m^{W,c}\right)_{m\in\mathbb{N}_0}$ is a basis of $L^2(W)$.

The singular vectors $\left(g_m^{W_{[-1,1]},c}\right)_{m\in\mathbb{N}_0}$ are the Prolate Spheroidal Wave Functions (hereafter PSWF, see, *e.g.*, [44]). They can be extended as entire functions in $L^2(\mathbb{R})$ and form a complete orthogonal system of $PW(c)$ for which we use the same notation. They are useful to carry interpolation and extrapolation (see, *e.g.*, [40]) with Hilbertian techniques. In this paper, for all $t\neq 0$, $\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)$ plays the role of the Fourier transform in the definition of $PW(c)$. The weight $\cosh(\cdot/R)$ allows for larger classes than $PW(c)$ and noncompact $\mathbb{S}_{\boldsymbol{\beta}}$. This is useful even if $\mathbb{S}_{\boldsymbol{\beta}}$ is compact when the researcher does not know a superset containing $\mathbb{S}_{\boldsymbol{\beta}}$. The useful results on the corresponding SVD and a numerical algorithm to compute it are given in [22].

3.2. **Sets of smooth and integrable functions.** Define, for all $(\phi(t))_{t\geq 0}$ and $(\omega_m)_{m\in\mathbb{N}_0}$ increasing, $\phi(0)=\omega_0=1$, $l,M>0$, $q\in\{1,\infty\}$, $t\in\mathbb{R}$, $\boldsymbol{m}\in\mathbb{N}_0^p$, $k\in\mathbb{N}_0$, and $c(t):=tx_0$,

$$\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M):=\left\{f:\sum_{k\in\mathbb{N}_0}\int_{\mathbb{R}}\phi^2(|t|)\theta_{q,k}^2(t)dt\bigvee\sum_{k\in\mathbb{N}_0}\omega_k^2\|\theta_{q,k}\|_{L^2(\mathbb{R})}^2\leq 2\pi l^2,\ \|f\|_{L^2(w\otimes W^{\otimes p})}\leq M\right\}$$

and $\mathcal{H}_{w,W}^{q,\phi,\omega}(l)$ when we replace $\|f\|_{L^2(w\otimes W^{\otimes p})}\leq M$ by $\|f\|_{L^2(w\otimes W^{\otimes p})}<\infty$, where

$$b_{\boldsymbol{m}}(t):=\left\langle\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot),\varphi_{\boldsymbol{m}}^{W,c(t)}\right\rangle_{L^2(W^{\otimes p})},\ \theta_{q,k}(t):=\left(\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q=k}|b_{\boldsymbol{m}}(t)|^2\right)^{1/2}. \tag{6}$$

The first inequality in the definition of $\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)$ defines the notion of smoothness for functions in $L^2\left(1\otimes W^{\otimes p}\right)$ analyzed in this paper. It involves a maximum of two terms, thus two inequalities: the first corresponds to smoothness in the first variable and the second to smoothness in the other variables. The additional inequality imposes integrability in the first variable. The asymmetry in the treatment of the first and remaining variables is due to the fact that, in the statistical problem, only the random slopes are multiplied by regressors which have limited variation and we make integrability assumptions in the first variable which are as mild as possible. The use of the Fourier transform to express smoothness in the first variable is classical. For the remaining variables, we choose a framework that allows for both functions with compact and noncompact support and work with the bases $\left(\varphi_{\boldsymbol{m}}^{W,c(t)}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ for $t\neq 0$. For functions with compact support, it is possible to use Fourier series and we make a comparison in Section B.4. The use of different bases for different values of t is motivated by (5). Though the spaces are chosen for mathematical convenience, we analyze all types of smoothness. The smoothness being unknown anyway, we provide an adaptive estimator. We analyze two values of q and show that the choice of the ℓ_q norm matters for the rates of convergence for supersmooth functions.

Remark 1. The next model is related to (1) under Assumption 1 when $f_{\boldsymbol{X}}$ is known:

$$dZ(t)=\mathcal{K}\left[f\right](t,\cdot)dt+\frac{\sigma}{\sqrt{n}}dG(t),\quad t\in\mathbb{R}, \tag{7}$$

where f plays the role of $f_{\alpha,\boldsymbol{\beta}}$, $\sigma>0$ is known, and $(G(t))_{t\in\mathbb{R}}$ is a complex two-sided cylindrical Gaussian process on $L^2([-1,1]^p)$. This means, for Φ Hilbert-Schmidt from $L^2([-1,1]^p)$ to a separable Hilbert space H, $(\Phi G(t))_{t\in\mathbb{R}}$ is a Gaussian process in H of covariance $\Phi\Phi^*$ (see [17]). Taking $\Phi G(t)=\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}\Phi\left[g_{\boldsymbol{m}}^{W,c(t)}\right]B_{\boldsymbol{m}}(t)$, where $B_{\boldsymbol{m}}(t)=B_{\boldsymbol{m}}^{\mathfrak{R}}(t)+iB_{\boldsymbol{m}}^{\mathfrak{I}}(t)$, $(B_{\boldsymbol{m}}^{\mathfrak{R}}(t))_{t\in\mathbb{R}}$ and $\left(B_{\boldsymbol{m}}^{\mathfrak{I}}(t)\right)_{t\in\mathbb{R}}$ are independent two-sided Brownian motions, the system of independent equations

$$Z_{\boldsymbol{m}}(t):=\int_0^t\sigma_{\boldsymbol{m}}^{W,c(s)}b_{\boldsymbol{m}}(s)ds+\frac{\sigma}{\sqrt{n}}B_{\boldsymbol{m}}(t),\quad t\in\mathbb{R}, \tag{8}$$

where, $Z_{\boldsymbol{m}}(t):=\left\langle Z(t),g_{\boldsymbol{m}}^{W,c(t)}\right\rangle_{L^2([-1,1]^p)}$ and $\boldsymbol{m}\in\mathbb{N}_0^p$, is equivalent to (7). Because $\sigma_{\boldsymbol{m}}^{W,c(s)}$ is small when $|\boldsymbol{m}|_q$ is large or s is small (see Lemma B.4), the estimator of Section 4.1 truncates large values of $|\boldsymbol{m}|_q$ and does not rely on small values of $|s|$ but uses interpolation.

Remark 2. [32] considers a Gaussian sequence model corresponding to (7), $\mathcal{K}$ is the Radon transform, $p=1$, G is a two-sided cylindrical Wiener process, and $L^2\left(w\otimes W\right)$ is a weighted L^2 space of functions with support in the unit disk of $\mathbb{R}^2$ for which $\mathcal{K}$ has a SVD with a known rate of decay of the singular values.

3.3. **Interpolation.** Define, for all $\underline{a},\epsilon>0$, the operator

$$\mathcal{I}_{\underline{a},\epsilon}\left[f\right]:=\sum_{m\in\mathbb{N}_0}\frac{\rho_m^{W_{[-1,1]},\underline{a}\epsilon}}{\left(1-\rho_m^{W_{[-1,1]},\underline{a}}\right)\epsilon}\left\langle f,\mathcal{C}_{1/\epsilon}\left[g_m^{W_{[-1,1]},\underline{a}\epsilon}\right]\right\rangle_{L^2(\mathbb{R}\setminus(-\epsilon,\epsilon))}\mathcal{C}_{1/\epsilon}\left[g_m^{W_{[-1,1]},\underline{a}\epsilon}\right] \tag{9}$$

on $L^2(\mathbb{R})$ with domain $PW(\underline{a})$. For all $f\in L^2(\mathbb{R})$, $\mathcal{I}_{\underline{a},\epsilon}\left[f\right]$ is a distribution.

Proposition 3. For all $\underline{a}, \epsilon > 0$, we have $\mathcal{I}_{\underline{a},\epsilon}\left(L^2(\mathbb{R})\right) \subseteq L^2([-\epsilon,\epsilon])$ and, for all $g \in PW(\underline{a})$, $\mathcal{I}_{\underline{a},\epsilon}[g] = g$ in $L^2(\mathbb{R})$ and, for $C(\underline{a},\epsilon) = 4\underline{a}\epsilon / \left(\pi\left(1-\rho_0^{W_{[-1,1]},\underline{a}\epsilon}\right)^2\right)$ and all $f, h \in L^2(\mathbb{R})$,

$$\left\|f - \mathcal{I}_{\underline{a},\epsilon}[h]\right\|^2_{L^2([-\epsilon,\epsilon])} \le 2(1 + C(\underline{a},\epsilon)) \left\|f - \mathcal{P}_{\underline{a}}[f]\right\|^2_{L^2(\mathbb{R})} + 2C(\underline{a},\epsilon) \left\|f - h\right\|^2_{L^2(\mathbb{R}\setminus(-\epsilon,\epsilon))}. \tag{10}$$

If $f \in PW(\underline{a})$, $\mathcal{I}_{\underline{a},\epsilon}[f]$ only relies on $f\mathbb{1}\{\mathbb{R}\setminus(-\epsilon,\epsilon)\}$ and $\mathcal{I}_{\underline{a},\epsilon}[f] = f$ on $\mathbb{R}\setminus(-\epsilon,\epsilon)$, so (9) provides an analytic formula to carry interpolation on $[-\epsilon,\epsilon]$ of functions in $PW(\underline{a})$. Else, (10) provides an upper bound on the error made by approximating f by $\mathcal{I}_{\underline{a},\epsilon}[h]$ on $[-\epsilon,\epsilon]$ when h approximates f outside $[-\epsilon,\epsilon]$. We use interpolation when the variance of an initial estimator $\widehat{f}^0$ of f is large due to its values near 0 but $\left\|f - \widehat{f}^0\right\|^2_{L^2(\mathbb{R}\setminus(-\epsilon,\epsilon))}$ is small and work with

$$\forall t \in \mathbb{R},\ \widehat{f}(t) = \widehat{f}^0(t)\mathbb{1}\{|t| \ge \epsilon\} + \mathcal{I}_{\underline{a},\epsilon}\left[\widehat{f}^0\right](t)\mathbb{1}\{|t| < \epsilon\},$$

in which case, (10) yields

$$\left\|f - \widehat{f}\right\|^2_{L^2(\mathbb{R})} \le (1 + 2C(\underline{a},\epsilon)) \left\|f - \widehat{f}^0\right\|^2_{L^2(\mathbb{R}\setminus(-\epsilon,\epsilon))} + 2(1 + C(\underline{a},\epsilon)) \left\|f - \mathcal{P}_{\underline{a}}[f]\right\|^2_{L^2(\mathbb{R})}. \tag{11}$$

When $\operatorname{supp}(\mathcal{F}[f])$ is compact, $\underline{a}$ is taken such that $\operatorname{supp}(\mathcal{F}[f]) \subseteq [-\underline{a},\underline{a}]$. Else, $\underline{a}$ goes to infinity so the second term in (11) goes to 0. ϵ is taken such that $\underline{a}\epsilon$ is constant because, due to (3.87) in [44], $\lim_{\underline{a}\epsilon\to\infty} C(\underline{a},\epsilon) = \infty$ and (10) and (11) become useless. Then $C(\underline{a},\epsilon)$ is constant and we set $C = 2(1 + C(\underline{a},\epsilon))$. When $\underline{a}\epsilon = 1$, we get $\rho_0^{W_{[-1,1]},\underline{a}\epsilon} \approx 0.3019$ and $C \approx 7.2279$.

3.4. **Risk.** The risk of an estimator $\widehat{f}_{\alpha,\boldsymbol{\beta}}$ is the mean integrated squared error (MISE)

$$\mathcal{R}^W_{n_0}\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) := \mathbb{E}\left[\left\|\widehat{f}_{\alpha,\boldsymbol{\beta}} - f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(1\otimes W^{\otimes p})}\middle|\mathcal{G}_{n_0}\right].$$

When $W = W_{[-R,R]}$ and $\operatorname{supp}\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}\right) \subseteq \mathbb{R}\times[-R,R]^p$, it is $\mathbb{E}\left[\left\|\widehat{f}_{\alpha,\boldsymbol{\beta}} - f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^{p+1})}\middle|\mathcal{G}_{n_0}\right]$, else,

$$\mathbb{E}\left[\left\|\widehat{f}_{\alpha,\boldsymbol{\beta}} - f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^{p+1})}\middle|\mathcal{G}_{n_0}\right] \le \left\|W^{-1}\right\|^p_{L^\infty(\mathbb{R})} \mathcal{R}^W_{n_0}\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right). \tag{12}$$

We consider a risk conditional on $\mathcal{G}_{n_0}$ for simplicity of the treatment of the random regressors with unknown law. We adopt the minimax approach and consider the supremum risk. The lower bounds involve a function r (for rate) and take the form

$$\exists \nu > 0:\ \underline{\lim}_{n\to\infty} \inf_{\widehat{f}_{\alpha,\boldsymbol{\beta}}} \sup_{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l)\cap\mathcal{D}} \mathbb{E}\left[\left\|\widehat{f}_{\alpha,\boldsymbol{\beta}} - f_{\alpha,\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^{p+1})}\right] \ge \nu r(n). \tag{13}$$

When we replace $f_{\alpha,\boldsymbol{\beta}}$ by f, $\widehat{f}_{\alpha,\boldsymbol{\beta}}$ by $\widehat{f}$, and consider model (8), we refer to (13'); when we also replace $\mathcal{H}^{q,\phi,\omega}_{w,W}(l)$ by $\mathcal{H}^{q,\phi,\omega}_{w,W}(l)\cap\mathcal{S}_U$, we refer to (13"), where $\mathcal{S}_U$ is the set of functions in $\mathcal{H}^{q,\phi,\omega}_{w,W}(l)$ such that $t \mapsto \mathcal{F}_{1\mathrm{st}}[f](t,\cdot)$ is not arbitrarily concentrated close to 0: for all $\boldsymbol{m} \in \mathbb{N}_0^p$, $\sup\{|x|,\ x \in \operatorname{supp}(b_{\boldsymbol{m}})\} \ge U$.

4. Estimation

The sets of densities in the supremum risk and of estimators in this section depend on $q \in \{1, \infty\}$. The rates of convergence depend on q via $k_q := 1 + (p-1)\mathbb{1}\{q = \infty\}$.

4.1. **Estimator considered.** For all $q \in \{1, \infty\}$, $0 < \epsilon < 1 < T$ and $N : \mathbb{R} \to \mathbb{N}_0$ such that $N(t) = N(\epsilon)$ for $t \in [-\epsilon, \epsilon]$ and $N(t) = N(T)$ for $|t| > T$, a regularized inverse is obtained by:

(S.1) for all $t \neq 0$, obtain a preliminary approximation of $F_1(t, \cdot) := \mathcal{F}_{1\text{st}}[f_{\alpha,\boldsymbol{\beta}}](t, \cdot)$

$$F_1^{q,N,T,0}(t, \cdot) := \mathbb{1}\{|t| \leq T\} \sum_{|\boldsymbol{m}|_q \leq N(t)} \frac{c_{\boldsymbol{m}}(t)}{\sigma_{\boldsymbol{m}}^{W,tx_0}} \varphi_{\boldsymbol{m}}^{W,tx_0}, \ c_{\boldsymbol{m}}(t) := \left\langle \mathcal{F}\left[f_{Y|\boldsymbol{X}=x_0\cdot}\right](t), g_{\boldsymbol{m}}^{W,tx_0} \right\rangle_{L^2([-1,1]^p)},$$

(S.2) for all $t \in [-\epsilon, \epsilon]$, $F_1^{q,N,T,\epsilon}(t, \cdot) := F_1^{q,N,T,0}(t, \cdot)\mathbb{1}\{|t| \geq \epsilon\} + \mathcal{I}_{\underline{a},\epsilon}\left[F_1^{q,N,T,0}(\star, \cdot)\right](t)\mathbb{1}\{|t| < \epsilon\}$,

(S.3) $f_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}(\cdot_1, \cdot_2) := \mathcal{F}_{1\text{st}}^I\left[F_1^{q,N,T,\epsilon}(\star, \cdot_2)\right](\cdot_1)$.

To deal with the statistical problem, we carry (S.1)-(S.3) replacing $c_{\boldsymbol{m}}$ by the estimator

$$\widehat{c}_{\boldsymbol{m}}(t) := \frac{1}{n} \sum_{j=1}^{n} \frac{e^{itY_j}}{x_0^p \widehat{f}_{\boldsymbol{X}|\mathcal{X}}^{\delta}(\boldsymbol{X}_j)} \overline{g_{\boldsymbol{m}}^{W,tx_0}}\left(\frac{\boldsymbol{X}_j}{x_0}\right) \mathbb{1}\left\{\boldsymbol{X}_j \in \mathcal{X}\right\}, \tag{14}$$

where $\widehat{f}_{\boldsymbol{X}|\mathcal{X}}^{\delta}(\boldsymbol{X}_j) := \widehat{f}_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X}_j) \vee \sqrt{\delta(n_0)}$ and $\delta(n_0)$ is a trimming factor converging to zero with n_0. This yields the estimators $\widehat{F}_1^{q,N,T,0}$, $\widehat{F}_1^{q,N,T,\epsilon}$, and $\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}$. We use $\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}\right)_+$ as a final estimator of $f_{\alpha,\boldsymbol{\beta}}$ which always has a smaller risk than $\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}$ (see [25, 48]). We use $n_e = n \wedge (\delta(n_0)/v(n_0, \mathcal{E}))$ for the sample size required for an ideal estimator where $f_{\boldsymbol{X}|\mathcal{X}}$ is known to achieve the rate of the plug-in estimator. The upper bounds below take the form

$$\frac{1}{r(n_e)} \sup_{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l,M) \cap \mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) = O_p(1). \tag{15}$$

When we use instead the restriction $f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l) \cap \mathcal{D}$, we refer to (15').

4.2. **Logarithmic rates when ω is a power.** The first result below involves, for all $t, u > 0$ and $R, x_0 > 0$, the inverse $Q_{t,u}$ of $x \in (0, \infty) \mapsto x\ln(1 \vee (7e(2x+1)/(2Rx_0t))) + u\ln(2x+1)$ which is such that, for all $x, u \in (0, \infty)$, $t \in (0, \infty) \mapsto Q_{t,u}(x)$ is increasing.

Theorem 1. Let $q \in \{1, \infty\}$, $\phi = 1 \vee |\cdot|^s$, $(\omega_k)_{k \in \mathbb{N}_0} = (k^\sigma)_{k \in \mathbb{N}_0}$, $w = 1 \vee |\cdot|$, $l, M, s, R > 0$, $\sigma > 1/2$, $N(t) = \lceil \overline{N}(t) \rceil$ for $\epsilon \leq |t| \leq T$, and $\underline{a} = 1/\epsilon$. (15) holds with $r(n_e) = (\ln(n_e)/\ln_2(n_e))^{-2\sigma}$ in the following cases

(T1.1) $W = W_{[-R,R]}$, $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R, R]^p$, $T = n_e^{1/(2(p+1))}$, $\epsilon = (\ln(n_e)/\ln_2(n_e))^{-2\sigma}$, and $\overline{N}(t) = Q_{|t|,(2\sigma+p)/4}(\ln(n_e)/(8k_q))$,

(T1.2) $W = \cosh(\cdot/R)$, $T = n_e^{1/(2(p+1+k_q-\mathbb{1}\{q=\infty\}))}$, $\epsilon = \ln(n_e)^{-2\sigma}$, and

$$\overline{N}(t) = \frac{\ln(n_e)}{2}\left(\frac{\mathbb{1}\left\{|t| > \frac{\pi}{4Rx_0}\right\}}{2\sigma + p - k_q + \pi k_q \frac{(2+p\mathbb{1}\{q=1\})}{2Rx_0|t|}} + \frac{\mathbb{1}\left\{|t| \leq \frac{\pi}{4Rx_0}\right\}}{2\sigma + p - k_q + 2k_q\ln(7e^2/(4Rx_0|t|))}\right).$$

Theorem 2. Let $q \in \{1, \infty\}$, $\phi = 1 \vee |\cdot|^s$, $(\omega_k)_{k\in\mathbb{N}_0} = (k^\sigma)_{k\in\mathbb{N}_0}$, $w = 1 \vee |\cdot|$, and $0 < l, s, R < \infty$.

(T2.1) Let $W = W_{[-R,R]}$, $\sigma > 2 + k_q/2$, assume that $f_{\boldsymbol{X}}$ is known, $\mathbb{S}_{\boldsymbol{X}} = \mathcal{X}$, and $\|f_{\boldsymbol{X}}\|_{L^\infty(\mathcal{X})} < \infty$, (13) holds with $r(n) = (\ln(n)/\ln_2(n))^{-2\sigma}$.

(T2.2) In model (8) with $W = \cosh(\cdot/R)$ and $\sigma > 1/2$, (13') holds with $r(n) = (\ln(n/\ln(n)))^{-2\sigma}$.

Theorem 2 shows the rate in (T1.1) is optimal when $f_{\boldsymbol{X}}$ is known and $\mathbb{S}_{\boldsymbol{X}} = \mathcal{X}$. It is the same rate as in [41] for deconvolution with a known characteristic function of the noise on a bounded interval when the density of the signal has compact support, though for different smoothness. The rate in (T1.2) is for when $\mathbb{S}_{\boldsymbol{\beta}}$ can be noncompact but $\int\int f_{\alpha,\boldsymbol{\beta}}(a,\boldsymbol{b})^2 w(a) W^{\otimes p}(\boldsymbol{b}) da d\boldsymbol{b} < \infty$. Similarly, the discussion after Theorem 2 in [41] considers densities with non compact support but with a pointwise bound outside $[-1,1]$. By (12) and (T2.2), we obtain a lower bound on $\mathcal{R}^W_{n_0}\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)$ for densities with unbounded support. The rates in Theorem 1 are independent of p as common for severely ill-posed problems (see [14, 22]).

4.3. **Polynomial and nearly parametric rates when ω is exponential.** Here $Q_{t,u}$ is the inverse of the increasing function $x \in (0,\infty) \mapsto x\ln(1 \vee (7e(x+1)/(2Rx_0 t))) + ux\ln(x+1)$.

Theorem 3. Let $q \in \{1, \infty\}$ and $s, \kappa, \gamma, l, M, R, \underline{a} > 0$, $r, \rho \geq 1$.

(T3.1) When $W = W_{[-R,R]}$, $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R,R]^p$, $\phi(\cdot) = 1 \vee |\cdot|^s$, $(\omega_k)_{k\in\mathbb{N}_0} = \left(e^{\kappa(k\ln(1+k))^r}\right)_{k\in\mathbb{N}_0}$, $\kappa > k_q$, $N(t) = \lfloor \overline{N}(t) \rfloor$ for $\epsilon \leq |t| \leq T$, $\overline{N}(t) = Q_{|t|,\kappa/k_q}(\ln(n_e)/(2k_q))$, and $T = 7ee^{\kappa N(\epsilon)\ln(N(\epsilon)+1)/s}/(2Rx_0)$, we have

(T3.1.1) for $r = \rho = 1$, $w = e^{\gamma(|\cdot|\ln(1\vee|\cdot|))^\rho}$, $\gamma > 1/(4k_q)$, $s > \kappa/(2k_q)$, $\epsilon\underline{a} = 7e/(2Rx_0)$, and $\epsilon = 7e\ln_2(n)/(2\ln(n)Rx_0)$, (15) holds with $r(n_e) = n_e^{-\kappa/(\kappa+2k_q)}\ln(n_e)^{2\kappa+2p+2}$

(T3.1.2) for $r = 1$, $w = W_{[-\underline{a},\underline{a}]}$, $\mathbb{S}_\alpha \subseteq [-\underline{a},\underline{a}]$, $s > \kappa/k_q$, and $\epsilon = 7e/(2Rx_0)$, (15') holds with $r(n_e) = n_e^{-\kappa/(\kappa+k_q)}\ln(n_e)^{2\kappa+2p+2}$.

(T3.2) When $W = W_{[-R,R]}$, $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R,R]^p$, for $r > 1$, $(\omega_k)_{k\in\mathbb{N}_0} = \left(e^{\kappa(k\ln(1+k))^r}\right)_{k\in\mathbb{N}_0}$, $w = e^{\gamma(|\cdot|\ln(1\vee|\cdot|))^\rho}$, $\rho \geq r$, $\gamma > \kappa$, $\phi(\cdot) = e^{\nu|\cdot|}$, $\underline{a} = 1/\epsilon$, $\nu > 0$ large enough (to satisfy (B.69)), $N(t) = \lceil \overline{N} \rceil$ for $\epsilon \leq |t| \leq T$, $\epsilon = 7e/\left(2Rx_0\left(1+\overline{N}\right)\right)$, $T = \exp\left(2\overline{N}\ln\left(1+\overline{N}\right)\right)$,

$$\overline{N} = Q_{\infty,1}\left(\left(\frac{1}{2\kappa}\left(\ln(n_e) - \sum_{i=0}^{k} d_i \ln(n_e)^{(i+1)/r-i}\right)\right)^{1/r}\right),$$

where $k/(k+1) \leq 1/r \leq (k+1)/(k+2)$, $d_0 = 4/(2\kappa)^{1/r}$ and $(d_i)_{i=1}^k$ are such that

$$d_i := \frac{4(-1)^k}{(2\kappa)^{1/r}} \sum_{j=1}^{i} \frac{(1/r)\dots(1/r-j+1)}{j!} \sum_{p_1+\dots+p_j=i} d_{p_1-1}\dots d_{p_j-1},$$

and $\varphi(\cdot) = \exp\left(\sum_{i=0}^k d_i \ln(\cdot)^{(i+1)/r-i}\right)/\ln(\cdot)^{4p/r}$, (15) holds with $r(n_e) = (\varphi(n_e)\, n_e)^{-1}$.

(T3.3) When $W = \cosh(\cdot/R)$, $\phi(\cdot) = 1 \vee |\cdot|^s$, $(\omega_k)_{k\in\mathbb{N}_0} = (e^{\kappa k})_{k\in\mathbb{N}_0}$, $\kappa > k_q(\pi(s/(p+1)+1)/4-1)$ and $Rx_0 > \pi\kappa/(2(p+1))$, $w = W_{[-\underline{a},\underline{a}]}$, $\mathbb{S}_\alpha \subseteq [-\underline{a},\underline{a}]$, $\epsilon = \pi/(4Rx_0)$, $N(t) = \lfloor \overline{N}(t) \rfloor$ for

$\epsilon \le |t| \le T$, $\overline{N}(t) = \ln(n_e)/(2\kappa + \pi k_q/(2Rx_0|t|))$, and $T = e^{\kappa \overline{N}(\epsilon)/s}/(Rx_0)$, (15') holds with $r(n_e) = n_e^{-\kappa/(\kappa+k_q)} \ln(n_e)^{2p+2-p\mathbb{1}\{q=\infty\}}$.

In (T3.3), we relax the assumption that $\mathbb{S}_{\boldsymbol{\beta}}$ is compact maintained in (T3.1). The results of Theorem 3 are related to those for "2exp-severely ill-posed problems" (see [13] and [47] which obtains the same polynomial rates up to logarithmic factor as in (T3.1.2) when $1/v(n_0, \mathcal{E}) \ge n$ and $p = 1$). When $1/v(n_0, \mathcal{E}) \ge n$, the rate in (T3.1.2) matches the lower bound in model (8).

Theorem 4. Let $q \in \{1, \infty\}$ and consider model (8) with $\phi(\cdot) = 1 \vee |\cdot|^s$, $w(\cdot) = 1 \vee |\cdot|$, $s, \kappa, l, R > 0$. (13") holds with $r(n) = n^{-\kappa/(\kappa+k_q)}$ when either

(T4.1) $W = W_{[-R,R]}$, $U \ge 4/(eRx_0)$, and $(\omega_k)_{k\in\mathbb{N}_0} = \left(e^{\kappa k \ln(1+k)}\right)_{k\in\mathbb{N}_0}$,

(T4.2) $W = \cosh(\cdot/R)$, $U \ge 2/(eRx_0)$, and $(\omega_k)_{k\in\mathbb{N}_0} = \left(e^{\kappa k}\right)_{k\in\mathbb{N}_0}$.

4.4. **Data-driven estimator.** We use a variant of the Goldenshluger-Lepski method (see [28]) proposed by [39]. Let $\epsilon, \zeta_0 > 0$, $K_{\max} := \lfloor \zeta_0 \log(n)/\log(2) \rfloor$, $T_{\max} := 2^{K_{\max}}$, $\mathcal{T}_n := \left\{2^k : \ k = 1, \dots, K_{\max}\right\}$, and, for $N \in \mathbb{N}_0^{\mathbb{R}}$, $T \in \mathbb{N}_0$, $t \ne 0$, and $q \in \{1, \infty\}$,

$$B_1(t, N) := \max_{N \le N' \le N^W_{\max,q}(t)} \left(\sum_{N \le |\boldsymbol{m}|_q \le N'} \left(\frac{|\widehat{c}_{\boldsymbol{m}}(t)|}{\sigma^{W,tx_0}_{\boldsymbol{m}}} \right)^2 - \Sigma\left(t, N'\right) \right)_+,$$

$$B_2(T, N) := \max_{T' \in \mathcal{T}_n, T' \ge T} \left(\int_{T \le |t| \le T'} \sum_{|\boldsymbol{m}|_q \le N(t)} \left(\frac{|\widehat{c}_{\boldsymbol{m}}(t)|}{\sigma^{W,tx_0}_{\boldsymbol{m}}} \right)^2 - \Sigma(t, N(t))\, dt \right)_+,$$

$$\Sigma(t, N) := \frac{52(1 + 2((2\log(n)) \vee 3))c_{\boldsymbol{X}}}{3n} \left(\frac{|t|\, x_0}{2\pi} \right)^p \nu^W_q(N, tx_0);$$

(N.1) when $W = \cosh(\cdot/R)$,

$$\begin{aligned}\nu^W_q(N, t) = &\frac{(N+p-1)^{p-1}}{(p-1)!} \frac{2^{p+1} eR|t|}{\pi} \exp\left(\frac{\pi(N+p)}{2R|t|} \right) \mathbb{1}\left\{|t| > \frac{\pi}{4R}\right\} \mathbb{1}\{q = 1\} \\ &+ \left(\frac{4eR|t|}{\pi} \right)^p \exp\left(\frac{\pi p(N+1)}{2R|t|} \right) \mathbb{1}\left\{|t| > \frac{\pi}{4R}\right\} \mathbb{1}\{q = \infty\} \\ &+ \left(\frac{e\pi}{2} \right)^{2p} 2^{k_q} \exp\left(2k_q \ln\left(\frac{7e^2}{4R|t|} \right) N \right) \mathbb{1}\left\{|t| \le \frac{\pi}{4R}\right\},\end{aligned}$$

$$N^W_{\max,q}(t) = \left\lfloor \frac{2R|t|\ln(n)}{\pi k_q} - p \right\rfloor \mathbb{1}\left\{|t| > \frac{\pi}{4R}\right\} + \left\lfloor \frac{\ln(n)}{2k_q \ln\left(7e^2/(4R|t|)\right)} \right\rfloor \mathbb{1}\left\{|t| \le \frac{\pi}{4R}\right\};$$

(N.2) when $W = W_{[-R,R]}$ and $\mathcal{W}$ is the inverse of $x \in [0, \infty) \mapsto xe^x$,

$$\nu^W_q(N, t) = (N+1)^{k_q} \left[\frac{(N+p-1)^{p-1}\mathbb{1}\{q = 1\}}{(p-1)!} + 2^p \mathbb{1}\{q = \infty\} \right] \left(1 \bigvee \frac{7e(N+1)}{2R|t|} \right)^{2Nk_q},$$

$$N^W_{\max,q}(t) = \left\lfloor \frac{\ln(n)}{2k_q} \left(1 \vee \mathcal{W}\left(\frac{7e}{R|t|} \frac{\ln(n)}{4k_q} \right) \right)^{-1} \right\rfloor.$$

$\widehat{N}$ and $\widehat{T}$ are defined, using $c_1 \geq 211/208$ ($c_1 > 1$ to handle the estimation of $f_{\boldsymbol{X}|\mathcal{X}}$), as

$$\forall t \in \mathbb{R} \setminus (-\epsilon, \epsilon), \quad \widehat{N}(t) \in \underset{0 \leq N \leq N^W_{\max,q}(t)}{\operatorname{argmin}} \left(B_1(t, N) + c_1 \Sigma(t, N) \right), \tag{16}$$

$$\widehat{T} \in \underset{T \in \mathcal{T}_n}{\operatorname{argmin}} \left(B_2\left(T, \widehat{N}\right) + \int_{\epsilon \leq |t| \leq T} \Sigma\left(t, \widehat{N}(t)\right) dt \right). \tag{17}$$

Let us present the heuristic when $\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}} = f_{\boldsymbol{X}|\mathcal{X}}$ (hence we simply write $\mathcal{R}^W$). Denote by $N \vee N' : t \mapsto N(t) \vee N'(t)$. The Plancherel identity, (11), and Lemma B.1 yield

$$\frac{2\pi}{C} \mathcal{R}^W \left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}} \right) \leq \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} \mathbb{E}\left[\left\| \widehat{F}^{q,N,T,0}_1(t, \cdot) - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t, \cdot) \right\|^2_{L^2(W^{\otimes p})} \right] dt + \frac{2\pi M^2}{w(\underline{a})}.$$

By (A.24), the first term on the right-hand side can be written as

$$\begin{aligned}
&\int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} \left\| \left(F^{q,N,T,0}_1 - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right] \right)(t, \cdot) \right\|^2_{L^2(W^{\otimes p})} + \mathbb{E}\left[\left\| \left(\widehat{F}^{q,N,T,0}_1 - F^{q,N,T,0}_1 \right)(t, \cdot) \right\|^2_{L^2(W^{\otimes p})} \right] dt \\
&\leq \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} \left(\sup_{N'} \left\| \left(F^{q,N \vee N',T,0}_1 - F^{q,N,T,0}_1 \right)(t, \cdot) \right\|^2_{L^2(W^{\otimes p})} + \Sigma(t, N) \mathbb{1}\{|t| \leq T\} \right) dt.
\end{aligned} \tag{18}$$

Proposition 2 yields

$$B_1(t, N) = \max_{N \leq N' \leq N^W_{\max,q}(t)} \left(\left\| \left(\widehat{F}^{q,N \vee N',T,0}_1 - \widehat{F}^{q,N,T,0}_1 \right)(t, \cdot) \right\|^2_{L^2(W^{\otimes p})} - \Sigma\left(t, N'\right) \right)_+,$$

which, by concentration of measure, on an event of probability close to 1, is close to

$$\max_{N \leq N' \leq N^W_{\max,q}(t)} \left(\mathbb{E}\left[\left\| \left(\widehat{F}^{q,N \vee N',T,0}_1 - \widehat{F}^{q,N,T,0}_1 \right)(t, \cdot) \right\|^2_{L^2(W^{\otimes p})} \right] - \Sigma\left(t, N'\right) \right)_+. \tag{19}$$

By $\mathbb{E}\left[|\widehat{c}_{\boldsymbol{m}}(t)|^2 - |c_{\boldsymbol{m}}(t)|^2 \right] = \mathbb{E}\left[|\widehat{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t)|^2 \right]$ (see Lemma A.2) we can rewrite the expectation so that the term in parentheses in (19) becomes

$$\left\| \left(F^{q,N \vee N',T,0}_1 - F^{q,N,T,0}_1 \right)(t, \cdot) \right\|^2_{L^2(W^{\otimes p})} + \sum_{N \leq |\boldsymbol{m}|_q \leq N \vee N'} \frac{\mathbb{E}\left[|\widehat{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t)|^2 \right]}{\left(\sigma^{W,tx_0}_{\boldsymbol{m}} \right)^2} - \Sigma\left(t, N'\right)$$

and, by (A.23)-(A.24), is less than $\left\| \left(F^{q,N \vee N',T,0}_1 - F^{q,N,T,0}_1 \right)(t, \cdot) \right\|^2_{L^2(W^{\otimes p})}$. Hence (16) amounts to minimizing an estimator of the integrand in (18). Similarly, (17) amounts to minimizing an estimator of (18). Indeed, on an event of probability close to 1, $B_2(T, N)$ is close to

$$\max_{T' \in \mathcal{T}_n, T' \geq T} \left(\int_{T \leq |t| \leq T'} \mathbb{E}\left[\left\| \left(\widehat{F}^{q,N,T \vee T',0}_1 - \widehat{F}^{q,N,T,0}_1 \right)(t, \cdot) \right\|^2_{L^2(W^{\otimes p})} \right] - \Sigma\left(t, N(t)\right) dt \right)_+, \tag{20}$$

the term in parentheses in (20) is equal to

$$\int_{T\le|t|\le T'}\left\|\left(F_1^{q,N,T\vee T',0}-F_1^{q,N,T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}+\sum_{|\boldsymbol{m}|_q\le N}\frac{\mathbb{E}\left[\left|\widehat{c}_{\boldsymbol{m}}(t)-c_{\boldsymbol{m}}(t)\right|^2\right]}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}-\Sigma(t,N(t))dt$$

and, by (A.23)-(A.24), is less than $\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\left\|\left(F_1^{q,N,T\vee T',0}-F_1^{q,N,T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}dt$. The upper bounds take the form

$$\frac{1}{r(n)}\sup_{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)=\underset{v(n_0,\mathcal{E})/\delta(n_0)\le n^{-(2+\zeta)},\ n_e\ge e}{O_p}(1),\tag{21}$$

and we refer to (21') when we use instead the restriction $f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l)\cap\mathcal{D}$.

Theorem 5. Take $0<l,M,s,R,\underline{a}<\infty$, $H\in\mathbb{N}$, $q\in\{1,\infty\}$, $\zeta>1/12$, $\phi(\cdot)=1\vee|\cdot|^s$.

(T5.1) When $(\omega_k)_{k\in\mathbb{N}_0}=(k^\sigma)_{k\in\mathbb{N}_0}$, $\sigma>p/2$, for all σ_0 such that $\sigma_0>\sigma$, $s>1$, $w(\cdot)=1\vee|\cdot|$, and $\epsilon=(\ln_2(n)/\ln(n))^{-2\sigma_0}$, (21) holds with $r(n)=(\ln(n)/\ln_2(n))^{-2\sigma}$ when either
- (T5.1.1) $W=W_{[-R,R]}$, $\mathbb{S}_{\boldsymbol{\beta}}\subseteq[-R,R]^p$, and $\zeta_0=1/(6p)$,
- (T5.1.2) $W=\cosh(\cdot/R)$, $\epsilon=(\ln_2(n))^{-2\sigma_0}$, and $\zeta_0=1/(10p)$.

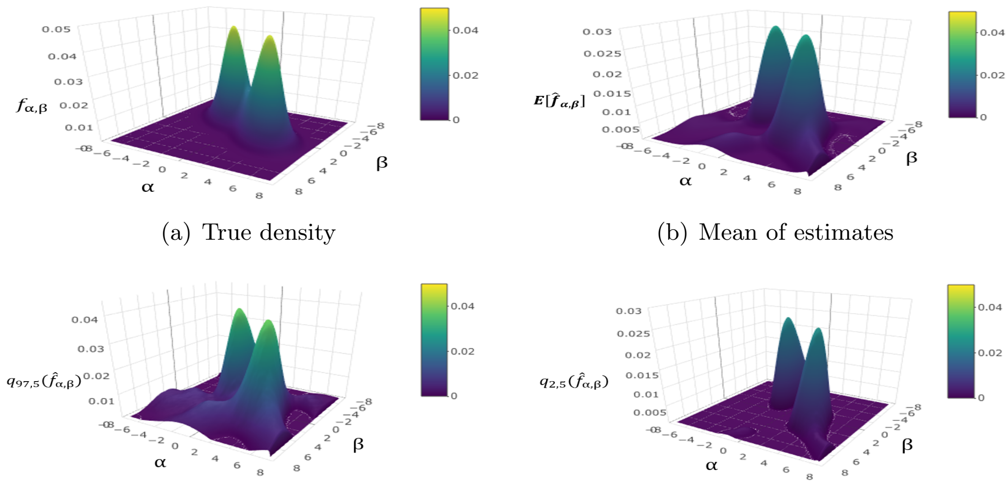
(T5.2) When $W=W_{[-R,R]}$, $\mathbb{S}_{\boldsymbol{\beta}}\subseteq[-R,R]^p$, $(\omega_k)_{k\in\mathbb{N}_0}=(e^{\kappa k\ln(1+k)})_{k\in\mathbb{N}_0}$, $w=W_{[-\underline{a},\underline{a}]}$, $\mathbb{S}_\alpha\subseteq[-\underline{a},\underline{a}]$, $\epsilon=7e/(2Rx_0)$, $\kappa>k_q$, $s>3p$, and $\zeta_0=1/(6p)$, (21') holds with $r(n)=n^{-\kappa/(\kappa+k_q)}\ln(n)^{2\kappa+2p+3}$.

(T5.3) When $W=\cosh(\cdot/R)$, $(\omega_k)_{k\in\mathbb{N}_0}=(e^{\kappa k})_{k\in\mathbb{N}_0}$, $\kappa>k_q(\pi(s/(p+1)+1)/4-1)$, $Rx_0>\pi\kappa/(2(p+1))$, $w=W_{[-\underline{a},\underline{a}]}$, $\mathbb{S}_\alpha\subseteq[-\underline{a},\underline{a}]$, $\epsilon=\pi/(4Rx_0)$, $s>5p\kappa/(\kappa+k_q)$, and $\zeta_0=1/(10p)$, (21') holds with $r(n)=n^{-\kappa/(\kappa+k_q)}\ln(n)^{2p+3-p\mathbb{1}\{q=\infty\}}$.

The results in Theorem 5 are for $v(n_0,\mathcal{E})/\delta(n_0)\le n^{-(2+\zeta)}$ with $\zeta>1/12$, in which case $n_e=n$. Theorem 2 and (T5.1) (a) show that $\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}}$ is adaptive. The rate in (T5.2) matches, up to a logarithmic factor, the lower bound in Theorem 4 (1) for model (8). For the other cases, the risk is different for the lower bounds and the upper bounds in Theorem 5, but using (12) we obtain the same rate up to logarithmic factors for the risk involving the weight W.

5. Simulations

Let $p=1$, $q=\infty$, and $(\alpha,\beta)^\top=\xi_1D+\xi_2(1-D)$ with $\mathbb{P}(D=1)=\mathbb{P}(D=0)=0.5$. The law of X is a truncated normal based on a normal of mean 0 and variance 2.5 and truncated to $\mathcal{X}$ with $x_0=1.5$. The laws of ξ_1 and ξ_2 are either: (Case 1) truncated normals based on normals with means $\mu_1=\begin{pmatrix}-2\\3\end{pmatrix}$ and $\mu_2=\begin{pmatrix}3\\0\end{pmatrix}$, same covariance $\begin{pmatrix}2&1\\1&2\end{pmatrix}$, and truncated to $[-6,6]^{p+1}$ or (Case 2) not truncated. Table 1 compares $\mathbb{E}\left[\left\|\widehat{f}^{\infty,\widehat{N},\widehat{T},\epsilon}_{\alpha,\beta}-f_{\alpha,\beta}\right\|^2_{L^2([-7.5,7.5]^2)}\right]$ and the risk of the oracle $\min_{T\in\mathcal{T}_n,N\in\mathcal{N}_{n,H}}\mathbb{E}\left[\left\|\widehat{f}^{\infty,N,T,\epsilon}_{\alpha,\beta}-f_{\alpha,\beta}\right\|^2_{L^2([-7.5,7.5]^2)}\right]$ for cases 1 and 2. The Monte-Carlo use 1000 simulations. Figure 1 (resp. Figure 2) displays summaries of the law



(a) True density (b) Mean of estimates

(c) 97.5% quantile of estimates (d) 2.5% quantile of estimates

FIGURE 1. Case 1, $W = W_{[-7.5,7.5]}$

	$W = W_{[-7.5,7.5]}$, Case 1			$W = \cosh(\cdot/7.5)$, Case 2		
	$n = 300$	$n = 500$	$n = 1000$	$n = 300$	$n = 500$	$n = 1000$
MISE (data-driven)	0.092	0.086	0.083	0.089	0.087	0.085
MISE (oracle)	0.091	0.086	0.082	0.088	0.087	0.085

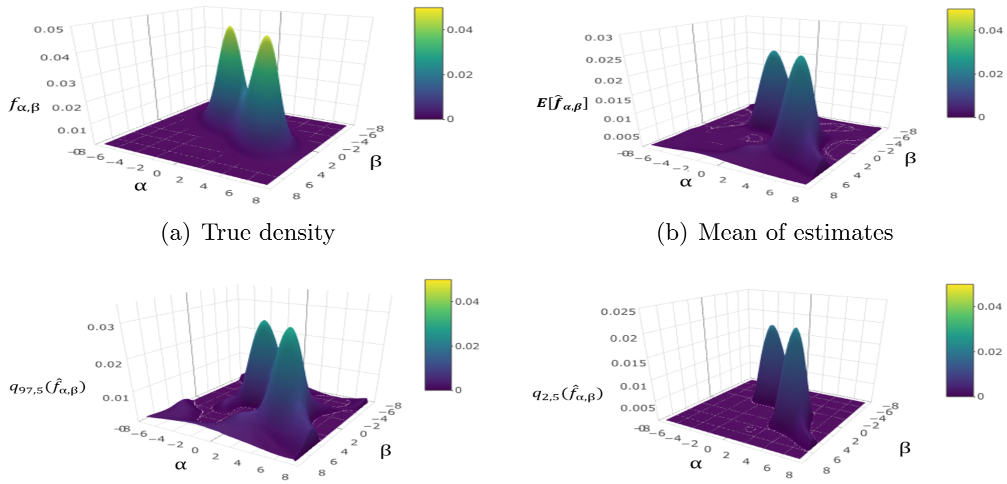
TABLE 1. Risk

of the estimator for $W = W_{[-7.5,7.5]}$ (resp. $W = \cosh(\cdot/7.5)$) in Case 1 (resp. Case 2) and $n = 1000$. $\widehat{f}_{X|X\in\mathcal{X}}$ is obtained with the same data. The estimator requires the SVD of $\mathcal{F}_c$. By Proposition B.1, we have $g_m^{W(\cdot/R),c} = g_m^{W,Rc}$ for all $m \in \mathbb{N}_0$. When $W = W_{[-1,1]}$, the first coefficients of the decomposition on the Legendre polynomials are obtained by solving for the eigenvectors of two tridiagonal symmetric Toeplitz matrices (see Section 2.6 in [44]). When $W = \cosh$, we refer to Section 7 in [22]. We use $\mathcal{F}_c^*\left(g_m^{W,Rc}\right) = \sigma_m^{W,Rc}\varphi_m^{W,Rc}$ and that $\varphi_m^{W,Rc}$ has norm 1 to get the rest of the SVD. The Fourier inverse is obtained by fast Fourier transform.

APPENDIX - PROOFS

$\mathfrak{R}$ and $\mathfrak{I}$ denote the real and imaginary parts. We denote, for all $m \in \mathbb{N}_0$, by ψ_m^c the function $g_m^{W_{[-1,1]},c}$ and $\mu_m^c = i^m \sigma_m^{W_{[-1,1]},c}$. Because $\psi_m^c = \mathcal{F}_c(\mathcal{E}xt[\psi_m^c])/\mu_m^c$ in $L^2([-1,1])$, ψ_m^c can be extended as an entire function which we denote with the same notation. Using the injectivity of $\mathcal{F}_c$ (see the proof of Proposition 1), we have $\varphi_m^{W_{[-1,1]},c} = i^{-m}\mathcal{E}xt[\psi_m^c]$. We make use of

$$\sup_{t\geq 1} \frac{\ln(t)^a}{t^b} = \left(\frac{a}{eb}\right)^a, \ a, b > 0. \tag{A.1}$$



(a) True density (b) Mean of estimates

(c) 97.5% quantile of estimates (d) 2.5% quantile of estimates

FIGURE 2. Case 2, $W = \cosh(\cdot/7.5)$

All expectations are conditional on $\mathcal{G}_{n_0}$ when $f_{\boldsymbol{X}|\mathcal{X}}$ is unknown and we rely on $\mathcal{G}_{n_0}$ to estimate it. We remove the conditioning in the notations for simplicity.

A.1. Proofs of Proposition 1, 2 and 3.

Proof of Proposition 1. The first assertion comes from the fact that W is nondecreasing on $[0,\infty)$ and $W(0)>0$. For the rest, we use that, for every $h \in L^2(W^{\otimes p})$, if we do not restrict the argument in the definition of $\mathcal{F}_c[h]$ to $[-1,1]^p$, $\mathcal{F}_c[h]$ can be defined as a function in $L^2(\mathbb{R}^p)$. In what follows, for simplicity, we use $\mathcal{F}_c[h]$ for both the function in $L^2([-1,1]^p)$ and in $L^2(\mathbb{R}^p)$. Let us now show that, for all $c \neq 0$, $\mathcal{F}_c$ defined in (3) is injective. Take $h \in L^2\left(W^{\otimes p}\right) \subseteq L^2(\mathbb{R}^p)$ such that $\mathcal{F}_c[h] = 0$ in $L^2\left([-1,1]^p\right)$. When W^{-1} vanishes at one point, h is compactly supported, thus, by the Paley-Wiener theorem its Fourier transform can be extended as an entire function which restriction to $\mathbb{R}^p$ belongs to $L^2\left(\mathbb{R}^p\right)$. Because the Fourier transform vanishes on a subset with nonempty interior, then $\mathcal{F}[h]=0$ on $\mathbb{R}^p$, thus $h=0$ in $L^2\left(\mathbb{R}^p\right)$. Now, consider the case where $W^{-1}(x)>0$ for all $x \in \mathbb{R}$. $\mathcal{F}_c[h]$ belongs to $C^\infty(\mathbb{R}^p)$ by the Lebesgue dominated convergence theorem because, for all $(\boldsymbol{k},\boldsymbol{u}) \in \mathbb{N}_0^p \times \mathbb{R}^p$, $\int_{\mathbb{R}^p}\left|c^{|\boldsymbol{k}|_1}\boldsymbol{b}^{\boldsymbol{k}}e^{ic\boldsymbol{b}^T\boldsymbol{u}}h(\boldsymbol{b})\right|d\boldsymbol{b} \le c^{|\boldsymbol{k}|_1}\|h\|_{L^2(W^{\otimes p})}\prod_{j=1}^p M_{\boldsymbol{k}_j}$ and, for all $(\boldsymbol{k},\boldsymbol{u}) \in \mathbb{N}_0^p \times \mathbb{R}^p$, $\left|\mathcal{F}_c[h]^{(\boldsymbol{k})}(\boldsymbol{u})\right| \le c^{|\boldsymbol{k}|_1}\|h\|_{L^2(W^{\otimes p})}\prod_{j=1}^p M_{\boldsymbol{k}_j}$. Theorem B.1 in [18] and the fact that, by the Cauchy-Schwarz inequality, for all $j \in \{1,\dots,p\}$, $\boldsymbol{k} \in \mathbb{N}_0^p$, $M_{\boldsymbol{k}_j} \le M_{\boldsymbol{k}_j-1}M_{\boldsymbol{k}_j+1}$ yield that $\mathcal{F}_c[h]$ is zero on $\mathbb{R}^p$. Thus, $\mathcal{F}[h]$ and h are zero a.e.
We now show that $\mathcal{K}$ is injective. Take $f \in L^2\left(w \otimes W^{\otimes p}\right)$ such that $\mathcal{K}[f]=0$. By the Plancherel identity and the fact that $w \ge 1$, we have

$$\int_{\mathbb{R}^{p+1}} \left|\mathcal{F}_{1\text{st}}\left[f\right](t,\boldsymbol{b})\right|^2 W^{\otimes p}(\boldsymbol{b})dtd\boldsymbol{b} \le 2\pi \int_{\mathbb{R}^{p+1}} \left|f(a,\boldsymbol{b})\right|^2 w(a)W^{\otimes p}(\boldsymbol{b})dad\boldsymbol{b} < \infty$$

thus, there exists $\Omega_1 \subseteq \mathbb{R}$ of Lebesgue measure 1, such that, for all $t \in \Omega_1$, $\boldsymbol{b} \mapsto \mathcal{F}_{1\mathrm{st}}[f](t,\boldsymbol{b}) \in L^2(W^{\otimes p})$. Hence, by the above, for all $t \in \Omega_1$ and $c \in \mathbb{R}$, $\boldsymbol{u} \mapsto \mathcal{F}_c[\mathcal{F}_{1\mathrm{st}}[f](t,\cdot)](\boldsymbol{u})$ is continuous. Also, because $\|\mathcal{K}[f]\|_{L^2(\mathbb{R}\times[-1,1]^p)} = 0$, there exists $\Omega_2 \subseteq \mathbb{R}$ of Lebesgue measure 1, such that, for all $t \in \Omega_2$, $\|\mathcal{K}[f](t,\cdot)\|_{L^2([-1,1]^p)} = 0$. As a result, using (5), we have , for all $(t,\boldsymbol{u}) \in \Omega_1 \cap \Omega_2 \times [-1,1]^p$, $\mathcal{K}[f](t,\boldsymbol{u}) = 0$. Using again (5) and the injectivity of $\mathcal{F}_c$ for all $c \neq 0$, we obtain that for all $t \in (\Omega_1 \cap \Omega_2) \setminus \{0\}$, $\mathcal{F}_{1\mathrm{st}}[f](t,\cdot) = 0$ in $L^2(W^{\otimes p})$, thus $\mathcal{F}_{1\mathrm{st}}[f](\star,\cdot) = 0$ in $L^2(1 \otimes W^{\otimes p})$ and $f = 0$ in $L^2(1 \otimes W^{\otimes p})$, hence in $L^2(w \otimes W^{\otimes p})$.
We show that $\mathcal{K}$ is continuous at 0. Let $f \in L^2(w \otimes W^{\otimes p})$. By the change of variables, the Plancherel identity, and the lower bounds on the weights, we have

$$\|\mathcal{K}[f]\|^2_{L^2(\mathbb{R}\times[-1,1]^p)} \le \int_{\mathbb{R}^{p+1}} |\mathcal{F}[f](t,\boldsymbol{v})|^2 (t,\boldsymbol{v}) dt d\boldsymbol{v} \le \left(\frac{2\pi}{W(0)}\right)^p \|f\|^2_{L^2(w\otimes W^{\otimes p})}.$$

Let $w = 1$. We exhibit a bounded sequence $(f_k)_{k\in\mathbb{N}_0}$ in $L^2(1 \otimes W^{\otimes p})$ for which there does not exist a convergent subsequence of $(\mathcal{K}[f_k])_{k\in\mathbb{N}_0}$. Take v_0 such that $\mathrm{supp}(v_0) \subset [1,2]$, $\|v_0\|_{L^2(\mathbb{R})} = 1$ and, for all $k \in \mathbb{N}_0$ and $(a,\boldsymbol{b}^\top)^\top \in \mathbb{R}^{p+1}$, $v_k(\cdot) = 2^{-k/2} v_0(2^{-k}\cdot)$ and $f_k(a,\boldsymbol{b}) = \mathcal{F}^I\left[v_k(\cdot)\varphi_{\underline{\mathbf{0}}}^{W,x_0\cdot}(\boldsymbol{b})\right](a)$. $(f_k)_{k\in\mathbb{N}_0}$ is bounded by the Plancherel identity and

$$\|f_k\|^2_{L^2(1\otimes W^{\otimes p})} = \frac{1}{2\pi}\int_{\mathbb{R}} v_k(t)^2 \int_{\mathbb{R}^p} \left|\varphi_{\underline{\mathbf{0}}}^{W,tx_0}(\boldsymbol{b})\right|^2 W^{\otimes p}(\boldsymbol{b}) dt d\boldsymbol{b} \le \frac{1}{2\pi}.$$

Using $\mathcal{K}[f_k](\cdot,*) = \sigma_{\underline{\mathbf{0}}}^{W,x_0\cdot} v_k(\cdot) g_{\underline{\mathbf{0}}}^{W,x_0\cdot}(*) |x_0\cdot|^{p/2}$ and $c \in (0,\infty) \mapsto \rho_0^{W,c}$ is nondecreasing (by Lemma 1 in [22] which holds for all W which satisfy (H1.2)), and

$$\|\mathcal{K}[f_j] - \mathcal{K}[f_k]\|^2_{L^2(\mathbb{R}\times[-1,1]^p)} \ge \rho_{\underline{\mathbf{0}}}^{W,2^j x_0}(2\pi)^p \int_{\mathbb{R}} \left(v_j(t)^2 + v_k(t)^2\right) dt \ge 2(2\pi)^p \rho_{\underline{\mathbf{0}}}^{W,x_0} > 0.$$

for all $j \in \mathbb{N}_0$, $\|v_j\|_{L^2(\mathbb{R})} = 1$, we obtain, for all $(j,k) \in \mathbb{N}_0^2$, $j < k$, so $\mathcal{K}$ is not compact. □

Proof of Proposition 2. This holds by Theorem 15.16 in [37] and the injectivity of $\mathcal{F}_c$. □

Proof of Proposition 3. Take $f \in L^2(\mathbb{R})$ and start by showing that $\mathcal{I}_{\underline{a},\epsilon}[f] \in L^2([-\epsilon,\epsilon])$. The terms $1 - \rho_m^{W_{[-1,1]},\underline{a}\epsilon}$ in the denominator of (9) are nonzero because $\left(\rho_m^{W_{[-1,1]},\underline{a}\epsilon}\right)_{m\in\mathbb{N}_0}$ is nonincreasing and $\rho_0^{W_{[-1,1]},\underline{a}\epsilon} < 1$ (see (3.49) in [44]). Using that $\left(g_m^{W_{[-1,1]},\underline{a}\epsilon}(\cdot/\epsilon)/\sqrt{\epsilon}\right)_{m\in\mathbb{N}_0}$ is a basis of $L^2([-\epsilon,\epsilon])$, that $\left(\rho_m^{W_{[-1,1]},\underline{a}\epsilon}\right)_{m\in\mathbb{N}_0}$ is nonincreasing, and the Cauchy-Schwarz inequality for the first display, using that $\sum_{m\in\mathbb{N}_0} \rho_m^{W_{[-1,1]},\underline{a}\epsilon} = 2\underline{a}\epsilon/\pi$ (see (3.55) in [44]) and $\left\|g_m^{W_{[-1,1]},\underline{a}\epsilon}\right\|^2_{L^2(\mathbb{R})} = 1/\rho_m^{W_{[-1,1]},\underline{a}\epsilon}$ (see (3) in [8]) for the second inequality, we obtain

$$\sum_{m\in\mathbb{N}_0}\left(\frac{\rho_m^{W_{[-1,1]},\underline{a}\epsilon}}{\left(1-\rho_m^{W_{[-1,1]},\underline{a}\epsilon}\right)\epsilon}\right)^2 \left|\left\langle f, g_m^{W_{[-1,1]},\underline{a}\epsilon}\left(\frac{\star}{\epsilon}\right)\right\rangle_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}\right|^2 \left\|g_m^{W_{[-1,1]},\underline{a}}\left(\frac{\cdot}{\epsilon}\right)\right\|^2_{L^2([-\epsilon,\epsilon])}$$

$$\text{(A.2)}\qquad \leq \frac{\|f\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}}{\left(1-\rho_0^{W_{[-1,1]},\underline{a}\epsilon}\right)^2}\sum_{m\in\mathbb{N}_0}\left(\rho_m^{W_{[-1,1]},\underline{a}\epsilon}\right)^2\left\|g_m^{W_{[-1,1]},\underline{a}\epsilon}\right\|^2_{L^2(\mathbb{R})}\leq \frac{2\underline{a}\epsilon\,\|f\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}}{\pi\left(1-\rho_0^{W_{[-1,1]},\underline{a}\epsilon}\right)^2}.$$

Let us now show the second statement. Take $\epsilon>0$ and $g\in PW(\underline{a})$. Let $(\alpha_m)_{m\in\mathbb{N}}$ be the sequence of coefficients of $g(\epsilon\cdot)\in PW(\underline{a}\epsilon)$ on the complete orthogonal system $\left(g_m^{W_{[-1,1]},\underline{a}\epsilon}\right)_{m\in\mathbb{N}_0}$. Because $\left(g_m^{W_{[-1,1]},\underline{a}\epsilon}\right)_{m\in\mathbb{N}_0}$ is a basis of $L^2([-1,1])$, we have $\sum_{m\in\mathbb{N}_0}\alpha_m g_m^{W_{[-1,1]},\underline{a}\epsilon}=g(\epsilon\cdot)\mathbb{1}\{|\cdot|\geq 1\}+\sum_{m\in\mathbb{N}_0}\alpha_m g_m^{W_{[-1,1]},\underline{a}\epsilon}\mathbb{1}\{|\cdot|\leq 1\}$. We identify the coefficients by taking the Hermitian product in $L^2(\mathbb{R})$ with $g_m^{W_{[-1,1]},\underline{a}\epsilon}$ and obtain $\mathcal{I}_{\underline{a},\epsilon}[g]=g$ in $L^2(\mathbb{R})$ and, for all $f,h\in L^2(\mathbb{R})$,

$$\text{(A.3)}\qquad \left\|f-\mathcal{I}_{\underline{a},\epsilon}\left[h\right]\right\|^2_{L^2([-\epsilon,\epsilon])}\leq 2\left(\left\|f-\mathcal{P}_{\underline{a}}[f]\right\|^2_{L^2([-\epsilon,\epsilon])}+\left\|\mathcal{I}_{\underline{a},\epsilon}\left[\mathcal{P}_{\underline{a}}\left[f\right]-h\right]\right\|^2_{L^2([-\epsilon,\epsilon])}\right).$$

Replacing f by $\mathcal{P}_{\underline{a}}\left[f\right]-h$ in (A.2) yields

$$\text{(A.4)}\qquad \left\|\mathcal{I}_{\underline{a},\epsilon}\left[\mathcal{P}_{\underline{a}}\left[f\right]-h\right]\right\|^2_{L^2([-\epsilon,\epsilon])}\leq\frac{C(\underline{a},\epsilon)}{2}\left\|\mathcal{P}_{\underline{a}}\left[f\right]-h\right\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}.$$

Using (A.3) and (A.4) for the first display, $\mathcal{P}_{\underline{a}}\left[f\right]-h=\left(\mathcal{P}_{\underline{a}}[f]-f\right)+(f-h)$ and the Jensen inequality for the second display, we obtain

$$\begin{aligned}\left\|f-\mathcal{I}_{\underline{a},\epsilon}\left[h\right]\right\|^2_{L^2([-\epsilon,\epsilon])}&\leq 2\left\|f-\mathcal{P}_{\underline{a}}[f]\right\|^2_{L^2([-\epsilon,\epsilon])}+C(\underline{a},\epsilon)\left\|\mathcal{P}_{\underline{a}}[f]-h\right\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}\\&\leq 2(1+C(\underline{a},\epsilon))\left\|f-\mathcal{P}_{\underline{a}}[f]\right\|^2_{L^2(\mathbb{R})}+2C(\underline{a},\epsilon)\left\|f-h\right\|^2_{L^2(\mathbb{R}\setminus[-\epsilon,\epsilon])}.\quad\square\end{aligned}$$

A.2. **Lower bounds.** We denote by $\mathbb{P}_{j,n}$ the law of the data implied by $f_{j,n}$ and use

$$\inf_{\widehat{f}}\ \sup_{f\in\mathcal{H}}\mathbb{E}\left[\left\|\widehat{f}-f\right\|^2_{L^2(\mathbb{R}^{p+1})}\right]\geq\inf_{\widehat{f}}\ \max_{f_{j,n}\in\mathcal{H},\ j\in\{1,2\}}\mathbb{E}\left[\left\|\widehat{f}-f_{j,n}\right\|^2_{L^2(\mathbb{R}^{p+1})}\right]$$

and the next lemma (see Theorem 2.2, (2.5), and (2.9) in [48]).

Lemma A.1. If there exists $\xi<\sqrt{2}$ such that

(i) $\forall j\in\{1,2\},\ f_{j,n}\in\mathcal{H}$,
(ii) $\|f_{1,n}-f_{2,n}\|^2_{L^2(\mathbb{R}^{p+1})}\geq 4h_n^2>0$,
(iii) $\chi_2(\mathbb{P}_{2,n},\mathbb{P}_{1,n})\leq\xi^2$ or $K(\mathbb{P}_{2,n},\mathbb{P}_{1,n})\leq\xi^2$,

then we have

$$\inf_{\widehat{f}}\ \max_{f_{j,n}\in\mathcal{H},\ j\in\{1,2\}}\mathbb{E}\left[\left\|\widehat{f}-f_{j,n}\right\|^2_{L^2(\mathbb{R}^{p+1})}\right]\geq e^{-\xi^2}\bigvee\left(2-\xi\sqrt{2}\right).$$

Proof of (T2.1). For $j=1,2$, $f_{j,n}$ is a possible $f_{\alpha,\boldsymbol{\beta}}$, $\left(b^j_{\boldsymbol{m}}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ the sequence of its coefficients (see (6)), and $\mathbb{P}_{j,n}$ have marginal $f_{\boldsymbol{X}|\mathcal{X}}$. Steps 1-3 show (i)-(iii) in Lemma A.1 are satisfied with

$$\text{(A.5)}\qquad f_{1,n}:=f_0\text{ and }f_{2,n}:=f_0+\gamma_n H_N,\ \ f_0(a,\boldsymbol{b}):=\frac{1}{\pi\tau\left(1+(a/\tau)^2\right)}\frac{\mathbb{1}\{|\boldsymbol{b}|_\infty\leq R\}}{(2R)^{p/2}},$$

$$\text{(A.6)}\qquad \forall(a,\boldsymbol{b})\in\mathbb{R}^{p+1},\ H_N(a,\boldsymbol{b}):=\mathcal{F}^I_{1\text{st}}\left[\left(\frac{c(|\cdot|)}{2\pi}\right)^{p/2}\lambda(\cdot)\psi^{Rc(\cdot)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right)\right](a)\mathbb{1}\{|\boldsymbol{b}|_\infty\le R\},$$

$$\text{(A.7)}\qquad \forall U/2\le|t|\le U,\ \lambda(t):=\exp\left(1-\frac{1}{1-16\left(|t|-3U/4\right)^2/U^2}\right),\ \text{else }\lambda(t):=0,$$

$$\widetilde{\boldsymbol{N}}(1):=\left(N,\underline{\boldsymbol{H_1(U)}}^\top\right)^\top,\widetilde{\boldsymbol{N}}(\infty):=\underline{\boldsymbol{N}}\in\mathbb{N}^p,\ H_1(U)=\lceil H(Rc(U))\rceil,$$

or H defined in Section B.1.2, $R>0$, n large enough, N (odd), γ_n, τ, and U chosen in Step 4.
Step 1.1. We prove that $f_{1,n}$ and $f_{2,n}$ are nonnegative when $N\ge H_1(U)$ and γ_n satisfies

$$\text{(A.8)}\qquad \frac{\gamma_nU}{(1+p/2)}\left(\frac{Ux_0}{2\pi}\right)^{p/2}\left(N+\frac12\right)^{k_q/2}\Xi_q(U)\le\frac{1}{(2R)^{p/2}\tau(1+(1/\tau)^2)},$$

$$\text{(A.9)}\qquad \tau^2\le\frac{2\tau}{2^{p/2}\gamma_nUC_8(Rx_0,p,U)N^{k_q/2+2}}-1,$$

where $\Xi_q(U)=(H_1(U)+1/2)^{(p-1)/2}+\left(1-(H_1(U)+1/2)^{(p-1)/2}\right)\mathbb{1}\{q=\infty\}$ and $C_8(Rx_0,p,U)$ is defined in Lemma B.10. Let $N\ge H_1(U)$ and $(a,\boldsymbol{b})\in\mathbb{R}\times[-R,R]^p$. We show that (A.8) and (A.9) yield $f_0(a,\boldsymbol{b})\ge|\gamma_nH_N(a,\boldsymbol{b})|$ which ensures that $f_{2,n}(\cdot)$ is nonnegative. By the discussion before Lemma B.6, $N\ge Rc(U)$ and, by the third assertion in Lemma B.7, we obtain

$$\begin{aligned}|\gamma_nH_N(a,\boldsymbol{b})|\le&\frac{\gamma_n}{2\pi}\left(\frac{x_0}{2\pi}\right)^{p/2}\left(N+\frac12\right)^{k_q/2}\Xi_q(U)\int_{\mathbb{R}}|t|^{p/2}\lambda(t)dt\\ \le&\frac{\gamma_nU}{\pi(1+p/2)}\left(\frac{Ux_0}{2\pi}\right)^{p/2}\left(N+\frac12\right)^{k_q/2}\Xi_q(U)\ \left(\text{because }\|\lambda\|_{L^\infty(\mathbb{R})}\le1\right).\end{aligned}$$

This and (A.8) yield the result when $|a|<1$. Because $t\mapsto\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{b}/R)$ is analytic (see [21] page 320), $t\mapsto(c(|t|)/(2\pi))^{p/2}\lambda(t)\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}(\boldsymbol{b}/R)\in C^\infty(\mathbb{R})$ and its derivatives are square integrable because their support is compact. By integration by parts, we obtain, when $a\neq0$,

$$|H_N(a,\boldsymbol{b})|\le\frac{1}{\pi a^2R^{p/2}}\int_{U/2}^U\left|\frac{\partial^2}{\partial t^2}\left(\left(\frac{Rc(t)}{2\pi}\right)^{p/2}\lambda(t)\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right)\mathbb{1}\{|\boldsymbol{b}|_\infty\le R\}\right)\right|dt.$$

The result when $|a|\ge1$ is obtained by Lemma B.10, which yields

$$\text{(A.10)}\qquad \forall(a,\boldsymbol{b})\in\mathbb{R}^{p+1},\ |H_N(a,\boldsymbol{b})|\le\frac{UC_8(Rx_0,p,U)}{2\pi a^2R^{p/2}}N^{k_q/2+2}$$

and (A.9), which yields, for all $|a|\ge1$, $\gamma_nUC_8(Rx_0,p,U)N^{k_q/2+2}/(2a^2)\le1/\left(2^{p/2}\tau(1+(a/\tau)^2)\right)$.
$f_{1,n}=f_0$ has integral 1 and so has $f_{2,n}$ by Fubini's theorem and that ψ^c_N is odd when N is odd.
Step 1.2. We prove $f_{1,n},f_{2,n}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l)$. Clearly $f_{1,n}$ and $f_{2,n}$, because, by the conclusion of Step 1.1, for all $(a,\boldsymbol{b})\in\mathbb{R}^{p+1}$, $f_{2,n}(a,\boldsymbol{b})^2\le4f_{1,n}(a,\boldsymbol{b})^2$, belong to $L^2\left(w\otimes W^{\otimes p}_{[-R,R]}\right)$. Let us

show that $f_{2,n}$, hence $f_{1,n}$ which is $f_{2,n}$ when $\gamma_n=0$, satisfy the first condition in $\mathcal{H}^{q,\phi,\omega}_{w,W}(l)$ if

$$2\left(\frac{\Gamma(2s+1)}{(2\tau)^{2s+1}}+\frac{1}{2\tau}+\gamma_n^2\left(\frac{Rx_0}{2\pi}\right)^p\frac{\left(1\vee U^{2s}\right)U^{p+1}}{p+1}\right)\le \pi l^2 \tag{A.11}$$

$$\frac{C_{12}(Rx_0,\sigma,p)}{\tau k_q^{2\sigma}}+\frac{2U\left(\mathbb{1}\{q=\infty\}+p^{2\sigma}\mathbb{1}\{q=1\}\right)\gamma_n^2N^{2\sigma}}{p+1}\left(\frac{URx_0}{2\pi}\right)^p\le \pi l^2. \tag{A.12}$$

Let $\boldsymbol{m}\in\mathbb{N}_0^p$ and $c^P_{\boldsymbol{m}}(t):=\left\langle 2^{-p/2},\psi^{Rc(t)}_{\boldsymbol{m}}\right\rangle_{L^2([-1,1]^p)}$. By Proposition B.1 (iii), change of variables, for all $t\in\mathbb{R}$, $\mathcal{F}_{1\mathrm{st}}\left[f_0(\cdot,\star)\right](t)=e^{-|t|\tau}\mathbb{1}\{|\star|_\infty\le R\}/(2R)^{p/2}$, we have

$$b^2_{\boldsymbol{m}}(t)=i^{-|\boldsymbol{m}|_1}\left(e^{-\tau|t|}c^P_{\boldsymbol{m}}(t)+\gamma_n\mathbb{1}\{\boldsymbol{m}=\widetilde{\boldsymbol{N}}(q)\}\left(\frac{Rc(|t|)}{2\pi}\right)^{p/2}\lambda(t)\right). \tag{A.13}$$

Because $\left(\psi^{Rc(t)}_{\boldsymbol{m}}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ is an orthonormal basis, we have

$$\forall t\neq 0,\ \sum_{\boldsymbol{m}\in\mathbb{N}_0^p}\left|b^2_{\boldsymbol{m}}(t)\right|^2\le 2\left(e^{-2\tau|t|}+\gamma_n^2\left(\frac{Rc(|t|)}{2\pi}\right)^p\lambda(t)^2\right). \tag{A.14}$$

The first part of the first condition in $\mathcal{H}^{q,\phi,\omega}_{w,W}(l)$ holds by (A.11) and because, by (A.14),

$$\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}\int_{\mathbb{R}}\left(1\vee t^{2s}\right)\left|b^2_{\boldsymbol{m}}(t)\right|^2dt\le 4\left(\int_0^\infty\frac{1+t^{2s}}{e^{2\tau t}}dt+\gamma_n^2\left(\frac{Rx_0}{2\pi}\right)^p\int_{U/2}^U\left(1\vee t^{2s}\right)t^p\lambda^2(t)dt\right).$$

The second part of the first condition holds by (A.12) and because, by (A.13) and Lemma B.11, for all $\tau\ge\left(3e^{\sigma+p/2-1/4}Rx_0/8\right)\vee(1/2)$ and $N\ge H_1(U)$,

$$\begin{aligned}
&\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}|\boldsymbol{m}|_q^{2\sigma}\int_{\mathbb{R}}\left|b^2_{\boldsymbol{m}}(t)\right|^2dt\\
&\le 2\left(\int_{\mathbb{R}}e^{-2\tau|t|}\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}|\boldsymbol{m}|_q^{2\sigma}\left(c^P_{\boldsymbol{m}}(t)\right)^2dt+\gamma_n^2\left(\frac{Rx_0}{2\pi}\right)^p\left|\widetilde{\boldsymbol{N}}(q)\right|_q^{2\sigma}\int_{\mathbb{R}}\lambda^2(t)\,|t|^p\,dt\right)\\
&\le 2\left(\frac{C_{12}(Rx_0,\sigma,p)}{\tau k_q^{2\sigma}}+\frac{2U\left(\mathbb{1}\{q=\infty\}+p^{2\sigma}\mathbb{1}\{q=1\}\right)\gamma_n^2N^{2\sigma}}{(p+1)\left(2\pi/(URx_0)\right)^p}\right),
\end{aligned}$$

Step 2. (ii) holds with $4h_n^2=\gamma_n^2\left(Rx_0/(2\pi)\right)^p\int_{U/2}^U t^p\lambda(t)^2dt/\pi$ because

$$\|f_{1,n}-f_{2,n}\|^2_{L^2(\mathbb{R}^{p+1})}=\frac{\gamma_n^2}{\pi}\left(\frac{Rx_0}{2\pi}\right)^p\int_{U/2}^U t^p\lambda(t)^2dt. \tag{A.15}$$

Step 3. By (ii) page 97 in [48], we have $\chi_2(\mathbb{P}_{2,n},\mathbb{P}_{1,n})=\left(1+\chi_2\left(\mathbb{P}_2,\mathbb{P}_1\right)\right)^n-1$ so

$$\chi_2(\mathbb{P}_{2,n},\mathbb{P}_{1,n})=n\int_0^{\chi_2(\mathbb{P}_2,\mathbb{P}_1)}(1+u)^{n-1}du\le n\chi_2\left(\mathbb{P}_2,\mathbb{P}_1\right)\exp\left((n-1)\chi_2\left(\mathbb{P}_2,\mathbb{P}_1\right)\right).$$

Thus, if $\chi_2(\mathbb{P}_2,\mathbb{P}_1)\le 1/n$, we have $\chi_2(\mathbb{P}_{2,n},\mathbb{P}_{1,n})\le en\chi_2(\mathbb{P}_2,\mathbb{P}_1)$. Moreover, we have

$$\chi_2(\mathbb{P}_2,\mathbb{P}_1)\le\int_{\mathbb{S}_{\boldsymbol{X}}}\int_{\mathbb{R}}\frac{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{x})\left(f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x})-f^2_{Y|\boldsymbol{X}}(y|\boldsymbol{x})\right)^2}{f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x})}d\boldsymbol{x}dy.$$

Because, under f_0, α and $\boldsymbol{\beta}$ are independent for the first equation and using $f^1_{\boldsymbol{\beta}^\top\boldsymbol{x}}(\cdot)=\int_{\mathbb{R}^{p-1}}f^1_{\boldsymbol{\beta}_1\boldsymbol{x}_1}(\cdot-\sum_{j=1}^{p-1}\boldsymbol{w_j})\prod_{j=1}^{p-1}f^1_{\boldsymbol{\beta}_j\boldsymbol{x}_j}(\boldsymbol{w}_j)d\boldsymbol{w}$ for the second, we have, for all $(y,\boldsymbol{x})\in\mathbb{R}\times\mathbb{S}_{\boldsymbol{X}}$,

$$\begin{aligned}f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x})&=\int_{\mathbb{R}}f^1_\alpha(y-v)\,f^1_{\boldsymbol{\beta}^\top\boldsymbol{x}}(v)\,dv\\&=\frac{\prod_{j=1}^p 1/|\boldsymbol{x}_j|}{\pi\tau(2R)^{p/2}}\int_{\mathbb{R}^p}\frac{\prod_{k=1}^p\mathbb{1}\{|\boldsymbol{u}_k|\le|\boldsymbol{x}_k|R\}}{\left|\left(y-\sum_{k=1}^p\boldsymbol{u}_k\right)/\tau\right|^2+1}d\boldsymbol{u}\ge\frac{(2R)^{p/2}}{\pi\tau}\inf_{|u|\le|\boldsymbol{x}|_1R}\frac{1}{(|y-u|/\tau)^2+1}.\end{aligned}$$

This yields, using $\mathbb{S}_{\boldsymbol{X}}=[-x_0,x_0]^p$,

$$\begin{aligned}\chi_2(\mathbb{P}_2,\mathbb{P}_1)&\le\frac{\pi\tau C_{\boldsymbol{X}}}{(2R)^{p/2}}\int_{[-x_0,x_0]^p}\int_{\mathbb{R}}\left(\frac{2y^2}{\tau^2}+\frac{2(|\boldsymbol{x}|_1R)^2}{\tau^2}+1\right)\left(f^1_{Y|\boldsymbol{X}}(y|\boldsymbol{x})-f^2_{Y|\boldsymbol{X}}(y|\boldsymbol{x})\right)^2d\boldsymbol{x}dy\\&\le\frac{(2\pi)^2C_{\boldsymbol{X}}}{\tau(2R)^{p/2}}\int_{[-x_0,x_0]^p}\int_{\mathbb{R}}\left(|\partial_t\mathcal{F}[f_{2,n}-f_{1,n}](t,t\boldsymbol{x})|^2+\left((x_0pR)^2+\frac{\tau^2}{2}\right)|\mathcal{F}[f_{2,n}-f_{1,n}](t,t\boldsymbol{x})|^2\right)d\boldsymbol{x}dt\\&\le\frac{(2\pi)^2C_{\boldsymbol{X}}x_0^p\gamma_n^2}{\tau(2R)^{p/2}}\int_{[-1,1]^p}\int_{\mathbb{R}}\left(|\partial_t\mathcal{F}[H_N](t,tx_0\boldsymbol{x})|^2+\left((x_0pR)^2+\frac{\tau^2}{2}\right)|\mathcal{F}[H_N](t,tx_0\boldsymbol{x})|^2\right)d\boldsymbol{x}dt.\end{aligned}$$

By lemmas B.4 and B.12, we have, for all U such that $4/(eRc(U))\ge1$ and $N\ge H_1(U)$,

$$\chi_2(\mathbb{P}_2,\mathbb{P})\le C_{18}(H,U,x_0,R,\tau)\gamma_n^2N^2\exp\left(-2Nk_q\ln\left(\frac{4N}{eRc(U)}\right)\right),\tag{A.16}$$

$$C_{18}(H,U,x_0,R,\tau):=\frac{(2\pi)^2(x_0Rc(U))^pC_{\boldsymbol{X}}e^{3p}\left(C_{17}(H,U)+UR^p((x_0pR)^2+\tau^2/2)/H_1(U)^2\right)}{9^p\tau(2R)^{p/2}(\exp\left(2(p-1)H_1(U)\ln\left(4H_1(U)/(eRc(U))\right)\right)\mathbb{1}\{q=1\}+\mathbb{1}\{q=\infty\})}.$$

As a result, (iii) is satisfied if

$$C_{18}(H,U,x_0,R,\tau)e\gamma_n^2N^2\exp\left(-2Nk_q\ln\left(\frac{4N}{eRc(U)}\right)\right)n\le\xi^2.\tag{A.17}$$

Step 4. We chose the parameters as follows. Let $U:=4/(Rx_0e)$, $\tau\ge1$ such that

$$\tau\ge\left(\frac{2}{\pi l^2}\left(\frac{\Gamma(2s+1)}{2^{2s}}+1\right)\right)^{1/(2s+1)}\bigvee\frac{2C_{12}(Rx_0,\sigma,p)}{\pi l^2k_q^{2\sigma}}\bigvee\frac{3Rx_0e^{\sigma+p/2-1/4}}{8}\bigvee\frac{1}{2},$$

where Q_1 and Q_2 are such that $\overline{N}^\sigma/(N+1/2)^{k_q/2}\le Q_1$ and $\overline{N}^\sigma/N^{2+k_q/2}\le Q_2$ (possible because $\sigma>2+k_q/2$), $N:=\lceil\overline{N}\rceil$, where $\overline{N}:=3(\ln(n)/\ln_2(n))/(4k_q)$, $\gamma_n:=C_\Gamma/\overline{N}^\sigma$, $C_\Gamma:=M_1\wedge\sqrt{M_2}\wedge\sqrt{M_3}$, and

$$M_1:=\frac{1}{\tau(1+(1/\tau)^2)2^{p/2}U}\left(\left(\frac{Ux_0}{2\pi}\right)^{-p/2}\frac{Q_1(1+p/2)}{\Xi_q(U)R^{p/2}}\bigwedge\frac{2Q_2}{C_8(Rx_0,p,U)}\right),$$

$$M_2 := \frac{\pi l^2 (p+1) U}{4} \left(\frac{2\pi}{URx_0}\right)^p \left(\frac{1}{1 \vee U^{2s}} \bigwedge \frac{1}{\mathbb{1}\{q=\infty\} + p^{2\sigma}\mathbb{1}\{q=1\}}\right),$$

$M_3 := \xi^2 / \left(C_{18}(H, U, x_0, R, \tau) e\right)$. Hence, (A.8)-(A.9) and (A.11)-(A.12) hold. This yields, for all $\overline{N} \geq H_1(c(U))$ (satisfied for n large enough),

$$\begin{aligned} n\gamma_n^2 N^2 \exp\left(-2k_q N \ln(N)\right) &\leq \frac{2C_\Gamma^2 n \overline{N}^2}{\overline{N}^{2\sigma}} \exp\left(-2k_q \overline{N} \ln(\overline{N})\right) \\ &\leq \frac{2C_\Gamma^2}{\sqrt{n}} \exp\left(\frac{3}{2} \frac{\ln(n)\ln(4k_q \ln_2(n)/3)}{\ln_2(n)} - 2(\sigma-1)\ln\left(\frac{3\ln(n)}{4k_q\ln_2(n)}\right)\right) \end{aligned}$$

hence $\lim_{n\to\infty} n\gamma_n^2 \exp\left(-2k_q N \ln(N)\right) N^2 = 0$. Thus, (A.17) holds for n large enough.

For this choice of γ_n, we have $h_n^2 = \overline{N}^{-2\sigma} C_\Gamma^2 \left(Rx_0/(2\pi)\right)^p \int_{U/2}^U t^p \lambda(t)^2 dt/(4\pi)$. □

Proof of (T2.2). Equip $L^2(\mathbb{R})\times L^2(\mathbb{R})$ with $\langle \boldsymbol{g}, \boldsymbol{h}\rangle^2_{L^2(\mathbb{R})\times L^2(\mathbb{R})} = \langle \boldsymbol{g}_1, \boldsymbol{h}_1\rangle^2_{L^2(\mathbb{R})} + \langle \boldsymbol{g}_2, \boldsymbol{h}_2\rangle^2_{L^2(\mathbb{R})}$. It is a separable Hilbert space. Denote by $\mathbb{P}^{\boldsymbol{m}}_{j,n}$ the law of $\left(\Re\left(Z^j_{\boldsymbol{m}}(t)\right), \Im\left(Z^j_{\boldsymbol{m}}(t)\right)\right)_{t\in\mathbb{R}}$ in $L^2(\mathbb{R})\times L^2(\mathbb{R})$ and by $\mathbb{P}_{j,n}$ the law on the space $\ell^2\left(L^2(\mathbb{R})\times L^2(\mathbb{R})\right)$ of square summable sequences with values in $L^2(\mathbb{R})\times L^2(\mathbb{R})$ of $\left(Z^j_{\boldsymbol{m}}(t)\right)_{\boldsymbol{m}\in\mathbb{N}_0^p,\ t\in\mathbb{R}} = \left(\Re\left(Z^j_{\boldsymbol{m}}(t)\right), \Im\left(Z^j_{\boldsymbol{m}}(t)\right)\right)_{\boldsymbol{m}\in\mathbb{N}_0^p,\ t\in\mathbb{R}}$ defined using $f_{j,n}$, hence $\left(b^j_{\boldsymbol{m}}(t)\right)_{\boldsymbol{m}\in\mathbb{N}_0^p,\ t\in\mathbb{R}}$, for $j=1,2$. Take $f_{1,n}=0$ and $f_{2,n}$ like (A.5) replacing $\widetilde{\boldsymbol{N}}(1)$ by $\widetilde{\boldsymbol{N}}(1) := (N, \underline{\boldsymbol{0}}^\top)^\top \in \mathbb{N}_0^p$, where N is odd and N and γ_n are chosen in Step 4. Using (A.13), this yields, for all $\boldsymbol{m} \in \mathbb{N}_0^p$, $b^2_{\boldsymbol{m}}(t) = \gamma_n \mathbb{1}\{\boldsymbol{m} = \widetilde{\boldsymbol{N}}(q)\}\left(Rc(|t|)/2\pi\right)^{p/2}\lambda(t)$. By independence, we have, for $j=1,2$, $\mathbb{P}_{j,n} = \bigotimes_{\boldsymbol{m}\in\mathbb{N}_0^p} \mathbb{P}^{\boldsymbol{m}}_{j,n}$ and

$$K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) = \int_{L^2(\mathbb{R})\times L^2(\mathbb{R})} \ln\left(\frac{d\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{2,n}}{d\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{1,n}}(y)\right) d\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{2,n}(y).$$

Step 1. Using (A.10), $f_{2,n} \in L^2\left(w \otimes \cosh(\cdot/R)^{\otimes p}\right)$ and, like (A.11)-(A.12), $f_{2,n} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l)$ if

$$\left(\frac{URx_0}{2\pi}\right)^p \left(2\gamma_n^2 \frac{\left(1\vee U^{2s}\right)U}{p+1} \bigvee \frac{2U\left(\mathbb{1}\{q=\infty\} + p^{2\sigma}\mathbb{1}\{q=1\}\right)\gamma_n^2 N^{2\sigma}}{p+1}\right) \leq \pi l^2. \tag{A.18}$$

Step 2. It is the same as for (T2.1).

Step 3. Let $\xi < \sqrt{2}$. Denote by $G^W_{\widetilde{\boldsymbol{N}}(q)} : s \mapsto \left(\Re\left(\sigma^{W,c(s)}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(s)\right), \Im\left(\sigma^{W,c(s)}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(s)\right)\right)^\top$. We start by proving, for all $y \in L^2(\mathbb{R})\times L^2(\mathbb{R})$ and $\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{1,n}$ *a.s.*,

$$\frac{d\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{2,n}}{d\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{1,n}}(y) = \exp\left(\left\langle y, \frac{\sqrt{n}}{\sigma}\mathcal{L}\left[G^W_{\widetilde{\boldsymbol{N}}(q)}\right]\right\rangle_{\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{1,n}} - \frac{1}{2}\left|\frac{\sqrt{n}}{\sigma}\mathcal{L}\left[G^W_{\widetilde{\boldsymbol{N}}(q)}\right]\right|^2_{\mathbb{P}^{\widetilde{\boldsymbol{N}}(q)}_{1,n}}\right),$$

where $\langle , \rangle_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}$ is the scalar product on $H_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}$, which is the image of

$$\mathcal{L}: \begin{array}{rcl} L^2(\mathbb{R}) \times L^2(\mathbb{R}) & \to & L^2(\mathbb{R}) \times L^2(\mathbb{R}) \\ \boldsymbol{h} & \mapsto & \dfrac{\sigma}{\sqrt{n}} \begin{pmatrix} \int_0^{\cdot} \boldsymbol{h}_1(s)ds \\ \int_0^{\cdot} \boldsymbol{h}_2(s)ds \end{pmatrix} \end{array},$$

with the norm of the image structure (*i.e.*, $\|\boldsymbol{f}\|^2_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} = \|\boldsymbol{h}_1\|_2^2 + \|\boldsymbol{h}_2\|_2^2$, where $\boldsymbol{f} = \mathcal{L}[\boldsymbol{h}]$), defined, letting $\boldsymbol{f_j} = \left(\int_0^{\cdot} \boldsymbol{h_j}_1(s)ds, \int_0^{\cdot} \boldsymbol{h_j}_2(s)ds\right)^\top$, $\boldsymbol{h_j} \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$, $j \in \{1, 2\}$ (the functions $\boldsymbol{h_j}$ are unique *a.e.* because they are the derivatives of $\boldsymbol{f_j}$ in the sense of distributions), as

$$\langle \boldsymbol{f_1}, \boldsymbol{f_2} \rangle^2_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} = \frac{n}{\sigma^2} \left(\langle \boldsymbol{h_1}_1, \boldsymbol{h_2}_1 \rangle^2_{L^2(\mathbb{R})} + \langle \boldsymbol{h_1}_2, \boldsymbol{h_2}_2 \rangle^2_{L^2(\mathbb{R})} \right)$$

and using (2.12) page 41 in [17] when one function belongs to $H_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}$ and for $\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}$ *a.e.* other function in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Indeed, the reproducing kernel Hilbert space $H_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}}$ of $\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}$ on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ is the image of $\mathcal{Q}^{1/2}$ with the scalar product of the image structure and where $\mathcal{Q}$ is its covariance operator. Using Corollary B.3 in [17], that $\mathcal{Q} = \mathcal{L}\mathcal{L}^*$ and, by the Cameron-Martin formula (Proposition 2.26 in [17]), we obtain

$$K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) = \mathbb{E}\left[\left\langle Z^2_{\widetilde{\boldsymbol{N}}(q)}, \frac{\sqrt{n}}{\sigma} \mathcal{L}\left[G^W_{\widetilde{\boldsymbol{N}}(q)} \right] \right\rangle_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} \right] - \frac{n}{2\sigma^2} \int_{\mathbb{R}} \left| \sigma^{W,c(s)}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(s) \right|^2 ds.$$

Because

$$\left\langle Z^2_{\widetilde{\boldsymbol{N}}(q)}, \frac{\sqrt{n}}{\sigma} \mathcal{L}\left[G^W_{\widetilde{\boldsymbol{N}}(q)} \right] \right\rangle_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} = \left| \frac{\sqrt{n}}{\sigma} \mathcal{L}\left[G^W_{\widetilde{\boldsymbol{N}}(q)} \right] \right|^2_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}} + \left\langle \begin{pmatrix} B^{\mathfrak{R}}_{\widetilde{\boldsymbol{N}}(q)} \\ B^{\mathfrak{I}}_{\widetilde{\boldsymbol{N}}(q)} \end{pmatrix}, \mathcal{L}\left[G^W_{\widetilde{\boldsymbol{N}}(q)} \right] \right\rangle_{\mathbb{P}_{1,n}^{\widetilde{\boldsymbol{N}}(q)}},$$

and the second term in the right-hand side is a limit in quadratic mean of mean zero Gaussian random variables, hence has mean zero (see the arguments page 41 in [17]), we have

$$K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) = \frac{n}{2\sigma^2} \int_{\mathbb{R}} \left| \sigma^{W,c(t)}_{\widetilde{\boldsymbol{N}}(q)} b^2_{\widetilde{\boldsymbol{N}}(q)}(t) \right|^2 dt. \tag{A.19}$$

By Proposition B.1 (ii) and $\left(\sigma^{\cosh,c}_{\widetilde{\boldsymbol{N}}(q)}\right)^2 (c/(2\pi))^p = \rho^{\cosh,c}_{\widetilde{\boldsymbol{N}}(q)}$ for all $c \neq 0$, we obtain

$$K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) = \frac{\gamma_n^2 n R^p}{2\sigma^2} \int_{\mathbb{R}} \left(\sigma^{\cosh,Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\right)^2 \left(\frac{Rc(|t|)}{2\pi}\right)^p \lambda(t)^2 dt = \frac{\gamma_n^2 n R^p}{2\sigma^2} \int_{\mathbb{R}} \rho^{\cosh,Rc(t)}_{\widetilde{\boldsymbol{N}}(q)} \lambda(t)^2 dt.$$

Using Theorem 3 in [22], we have, for all $U/2 \leq |t| \leq U$ and $2/(Rx_0U) \geq 1$,

$$\rho^{\cosh,Rc(t)}_{\widetilde{\boldsymbol{N}}(q)} \leq \left(\frac{2Rx_0Ue}{\pi(1 - (Rx_0U/2)^2)} \right)^p \exp\left(-2k_q N \log\left(\frac{2}{Rx_0U} \right) \right)$$

Thus (iii) is satisfied if

$$\left(\frac{2Rx_0Ue}{\pi(1 - (Rx_0U/2)^2)} \right)^p \frac{U\gamma_n^2 n}{2\sigma^2} \exp\left(-2k_q N \log\left(\frac{2}{Rx_0U} \right) \right) \leq \xi^2. \tag{A.20}$$

Step 4. Let $N = \lceil \overline{N} \rceil$, where $\overline{N} := \ln(n/\ln(n))/2k_q$, $\gamma_n := C_{\Gamma,q,1}/\overline{N}^{\sigma}$, $U := 2/(Rx_0 e)$, and

$$C_{\Gamma,q,1}^2 := \left(\frac{(p+1)\,(2\pi)^p\,\pi l^2}{2U(URx_0)^p} \left[\frac{1}{1 \vee U^{2s}} \bigwedge \frac{1}{\mathbb{1}\{q=\infty\} + p^{2\sigma}\mathbb{1}\{q=1\}} \right] \right) \bigwedge \left(\frac{2(\sigma\xi)^2}{U} \left(\frac{\pi(1-e^{-2})}{2R^2 x_0 U e} \right)^p \right)$$

hence (A.18) is satisfied and $f_{1,n}$ and $f_{2,n}$ belong to $\mathcal{H}_W^{\omega,\phi}(l)$. Moreover, (B.74) is satisfied as

$$\gamma_n^2 n \exp\left(-2k_q N\right) \le C_{\Gamma,q,1}^2 n \exp(-\ln(n) + \ln_2(n) - 2\sigma\ln_2(n) + 2\sigma\ln_3(n)) \le C_{\Gamma,q,1}^2 \;\; (\text{using } \sigma > 1/2).$$

For such γ_n, we have $h_n^2 = C_{\Gamma,q,1}^2 \ln(n/\ln(n))^{2\sigma} \left(Rx_0/(2\pi)\right)^p \int_{U/2}^{U} t^p \lambda(t)^2 dt/(4\pi)$. □

The proof of Theorem 4 is similar to (T2.1) so it is postponed to Section B.2.

A.3. **Upper bounds.** We use the notations

$$Z_{n_0} := \sup_{f_{\boldsymbol{X}|\mathcal{X}} \in \mathcal{E}} \left\| 1 - \frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}} \right\|_{L^\infty(\mathcal{X})}^2 ; \;\; \forall \boldsymbol{m} \in \mathbb{N}_0^p,\; \widetilde{c}_{\boldsymbol{m}}(t) := \frac{1}{n} \sum_{j=1}^n \frac{e^{itY_j} \overline{g_{\boldsymbol{m}}^{W,tx_0}} (\boldsymbol{X}_j/x_0)\, \mathbb{1}\{\boldsymbol{X}_j \in \mathcal{X}\}}{x_0^p f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X}_j)}.$$

Lemma A.2. For all $\boldsymbol{m} \in \mathbb{N}_0^p$, we have $\mathbb{E}\left[\widetilde{c}_{\boldsymbol{m}}(t)\right] = c_{\boldsymbol{m}}(t)$ and $\mathbb{E}\left[\left| \widetilde{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t) \right|^2 \right] \le c_{\boldsymbol{X}}/(n x_0^p)$.

Proof. The first assertion comes from

$$\begin{aligned}
\mathbb{E}\left[\widetilde{c}_{\boldsymbol{m}}(t)\right] &= \frac{1}{x_0^p} \mathbb{E}\left[\frac{e^{itY}}{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X})} \overline{g_{\boldsymbol{m}}^{W,tx_0}} \left(\frac{\boldsymbol{X}}{x_0} \right) \mathbb{1}\{\boldsymbol{X} \in \mathcal{X}\} \right] \;\; \text{(the observations are i.i.d.)} \\
&= \frac{1}{x_0^p} \int_{\mathcal{X}} \mathbb{E}\left[e^{it\alpha + it\boldsymbol{\beta}^\top \boldsymbol{x}} \right] \overline{g_{\boldsymbol{m}}^{W,tx_0}} \left(\frac{\boldsymbol{x}}{x_0} \right) d\boldsymbol{x} \quad \text{(by (2))}.
\end{aligned}$$

Similarly, the second assertion follows from

$$\begin{aligned}
\mathbb{E}\left[\left| \widetilde{c}_{\boldsymbol{m}}(t) - c_{\boldsymbol{m}}(t) \right|^2 \right] &\le \frac{1}{n x_0^{2p}} \mathbb{E}\left[\left| \frac{e^{itY}}{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{X})} \overline{g_{\boldsymbol{m}}^{W,tx_0}} \left(\frac{\boldsymbol{X}}{x_0} \right) \right|^2 \middle| \boldsymbol{X} \in \mathcal{X} \right] \\
&\le \frac{1}{n x_0^{2p}} \int_{\mathcal{X}} \frac{1}{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{x})} \left| \overline{g_{\boldsymbol{m}}^{W,tx_0}} \left(\frac{\boldsymbol{x}}{x_0} \right) \right|^2 d\boldsymbol{x} \le \frac{c_{\boldsymbol{X}}}{n x_0^p} \int_{[-1,1]^p} \left| \overline{g_{\boldsymbol{m}}^{W,tx_0}} (\boldsymbol{u}) \right|^2 d\boldsymbol{u}. \quad \square
\end{aligned}$$

Proofs of theorems 1, 3. Let $K_1 := \left\| \mathbb{1}\{|\cdot| \ge \epsilon\} \left(\widehat{F}_1^{q,N,T,0} - \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right] \right)(\cdot,\star) \right\|_{L^2(1 \otimes W^{\otimes p})}^2$, $K_2 := \left\| \mathbb{1}\{|\cdot| < \epsilon\} \left(\mathcal{I}_{\underline{a},\epsilon}\left[\widehat{F}_1^{q,N,T,0} \right] - \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right] \right)(\cdot,\star) \right\|_{L^2(1 \otimes W^{\otimes p})}^2$, and $f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)$. The Plancherel and Chasles identities yield $\left\| \widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon} - f_{\alpha,\boldsymbol{\beta}} \right\|_{L^2(1 \otimes W^{\otimes p})}^2 \le (K_1 + K_2)/(2\pi)$.

Consider K_2. For a.e. $\boldsymbol{b}$, $a \mapsto f_{\alpha,\boldsymbol{\beta}}(a,\boldsymbol{b}) \in L_w^2(\mathbb{R})$ and for those $\boldsymbol{b}$ we have $\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\cdot,\boldsymbol{b}) \in L^2(\mathbb{R})$. Using (10) for the first display and Lemma B.1 for the second, we obtain

$$\begin{aligned}
K_2 \le& \int_{\mathbb{R}^p} 2(1 + C(\underline{a},\epsilon)) \left\| \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\cdot,\boldsymbol{b}) - \mathcal{P}_{\underline{a}}\left[\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\star,\boldsymbol{b}) \right](\cdot) \right\|_{L^2(\mathbb{R})}^2 W^{\otimes p}(\boldsymbol{b}) d\boldsymbol{b} \\
&+ \int_{\mathbb{R}^p} 2C(\underline{a},\epsilon) \left\| \mathbb{1}\{|\cdot| \ge \epsilon\} \left(\widehat{F}_1^{q,N,T,0} - \mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right] \right)(\cdot,\boldsymbol{b}) \right\|_{L^2(\mathbb{R})}^2 W^{\otimes p}(\boldsymbol{b}) d\boldsymbol{b}
\end{aligned}$$

$$\le \frac{4\pi(1+C(\underline{a},\epsilon))}{w(\underline{a})}\int_{\mathbb{R}^p}\|f_{\alpha,\boldsymbol{\beta}}(\cdot,\boldsymbol{b})\|^2_{L^2(w)}W^{\otimes p}(\boldsymbol{b})d\boldsymbol{b}+2C(\underline{a},\epsilon)K_1$$

$$\le(1+C(\underline{a},\epsilon))\frac{4\pi M^2}{w(\underline{a})}+2C(\underline{a},\epsilon)K_1, \tag{A.21}$$

hence

$$K_1+K_2\le(1+2C(\underline{a},\epsilon))K_1+(1+C(\underline{a},\epsilon))\frac{4\pi M^2}{w(\underline{a})}. \tag{A.22}$$

Then, by the Jensen inequality, we have $K_1\le 4\sum_{j=1}^4\|R_j\|^2_{L^2(1\otimes W^{\otimes p})}$, where

$R_1(t,\boldsymbol{b}):=\mathbb{1}\{\epsilon\le|t|\}\left(\widetilde{F}^{q,N,T,0}-F^{q,N,T,0}\right)(t,\boldsymbol{b})$, $R_2(t,\boldsymbol{b}):=\mathbb{1}\{\epsilon\le|t|\}\left(\widehat{F}^{q,N,T,0}-\widetilde{F}^{q,N,T,0}\right)(t,\boldsymbol{b})$,

$R_3(t,\boldsymbol{b}):=\mathbb{1}\{\epsilon\le|t|\}\left(F^{q,N,T,0}-F^{q,\infty,T,0}\right)(t,\boldsymbol{b})$, $R_4(t,\boldsymbol{b}):=\mathbb{1}\{\epsilon\le|t|\}\left(F^{q,\infty,T,0}-\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\boldsymbol{b})$,

$\widetilde{F}_1^{q,N,T,0}$ is defined like $\widehat{F}_1^{q,N,T,0}$ replacing $\widehat{c}_{\boldsymbol{m}}(t)$ by $\widetilde{c}_{\boldsymbol{m}}(t)$ (c.f. Lemma A.2), $\widetilde{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}$ is defined like $\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}$ replacing $\widehat{F}_1^{q,N,T,0}$ by $\widetilde{F}_1^{q,N,T,0}$.

Term R_1. Using Proposition 2 for the first display, Lemma A.2 for the second, and Lemma B.2 for the third, we have

$$\mathbb{E}\left[\|R_1\|^2_{L^2(1\otimes W^{\otimes p})}\right]\le\int_{\mathbb{R}}\mathbb{1}\{\epsilon\le|t|\le T\}\sum_{|\boldsymbol{m}|_q\le N(t)}\frac{\mathbb{E}\left[|\widetilde{c}_{\boldsymbol{m}}(t)-c_{\boldsymbol{m}}(t)|^2\right]}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2}dt \tag{A.23}$$

$$\le\frac{c_{\boldsymbol{X}}}{(2\pi)^p n}\int_{\mathbb{R}}\mathbb{1}\{\epsilon\le|t|\le T\}\sum_{|\boldsymbol{m}|_q\le N(t)}\frac{|t|^p}{\rho_{\boldsymbol{m}}^{W,tx_0}}dt$$

$$\le\frac{c_{\boldsymbol{X}}}{(2\pi)^p n}\int_{\mathbb{R}}\mathbb{1}\{\epsilon\le|t|\le T\}\,|t|^p\,\nu_q^W(N(t),tx_0)dt. \tag{A.24}$$

Term R_2. Denoting by $\Delta_f(x):=\left(1/\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}-1/f_{\boldsymbol{X}|\mathcal{X}}\right)(x)$ and

$$\left(\widetilde{\omega}^{q,W,c}_{N(t)}\right)^2:=\sup_{|\boldsymbol{m}|_q\le N(t)}\frac{1}{\rho_{\boldsymbol{m}}^{W,c}}, \tag{A.25}$$

we have

$$\|R_2\|^2_{L^2(1\otimes W^{\otimes p})}=\int_{\mathbb{R}}\int_{\mathbb{R}^p}\mathbb{1}\{\epsilon\le|t|\le T\}\left|\widehat{F}_1^{q,N,T,0}(t,\boldsymbol{b})-\widetilde{F_1}^{q,N,T,0}(t,\boldsymbol{b})\right|^2W^{\otimes p}(\boldsymbol{b})dtd\boldsymbol{b}$$

$$=\int_{\epsilon\le|t|\le T}\int_{\mathbb{R}^p}\left|\sum_{|\boldsymbol{m}|_q\le N(t)}\frac{\varphi_{\boldsymbol{m}}^{W,tx_0}(\boldsymbol{b})}{\sigma_{\boldsymbol{m}}^{W,tx_0}}\sum_{j=1}^n\frac{e^{itY_j}}{nx_0^p}\Delta_f(\boldsymbol{X}_j)\overline{g_{\boldsymbol{m}}^{W,tx_0}\left(\frac{\boldsymbol{X}_j}{x_0}\right)}\mathbb{1}\{\boldsymbol{X}_j\in\mathcal{X}\}\right|^2W^{\otimes p}(\boldsymbol{b})dtd\boldsymbol{b}$$

$$\le\int_{\epsilon\le|t|\le T}\frac{|tx_0|^p\left(\widetilde{\omega}^{q,W,tx_0}_{N(t)}\right)^2}{(2\pi)^p}\sum_{|\boldsymbol{m}|_q\le N(t)}\left|\sum_{j=1}^n\frac{e^{itY_j}}{nx_0^p}\Delta_f(\boldsymbol{X}_j)\overline{g_{\boldsymbol{m}}^{W,tx_0}\left(\frac{\boldsymbol{X}_j}{x_0}\right)}\mathbb{1}\{\boldsymbol{X}_j\in\mathcal{X}\}\right|^2dt$$

$$= \int_{\epsilon\le|t|\le T} \frac{|tx_0|^p\left(\widetilde{\omega}_{N(t)}^{q,W,tx_0}\right)^2}{(2\pi)^p}\left\|S_0^N(\cdot,t)\right\|_{L^2([-1,1]^p)}^2 dt,$$

where $S_0^N(\cdot,t) := \sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q\le N} g_{\boldsymbol{m}}^{W,tx_0}\Delta_{\boldsymbol{m}}(t)$, $\Delta_{\boldsymbol{m}}(t) := (1/n)\sum_{j=1}^n Z_j^{\boldsymbol{m},t}$, and, for all $j = 1,\dots,n$, $Z_j^{\boldsymbol{m},t} := (e^{itY_j}/x_0^p)\Delta_f(\boldsymbol{X}_j)\overline{g_{\boldsymbol{m}}^{W,tx_0}(\boldsymbol{X}_j/x_0)}\mathbb{1}\{\boldsymbol{X}_j\in\mathcal{X}\}$.
We have $\mathbb{E}\left[\left\|S_0^N(\cdot,t)\right\|_{L^2([-1,1]^p)}^2\right] = \mathbb{E}\left[\left\|S_1^N(\cdot,t)\right\|_{L^2([-1,1]^p)}^2\right] + \mathbb{E}\left[\left\|S_2^N(\cdot,t)\right\|_{L^2([-1,1]^p)}^2\right]$, where $S_1^N(\cdot,t) := \sum_{|\boldsymbol{m}|_q\le N(t)} g_{\boldsymbol{m}}^{W,tx_0}\mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)\right]$, $S_2^N(\cdot,t) := \sum_{|\boldsymbol{m}|_q\le N(t)} g_{\boldsymbol{m}}^{W,tx_0}\left(\Delta_{\boldsymbol{m}}(t) - \mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)\right]\right)$,

$$\begin{aligned}
\left\|S_1^N(\cdot,t)\right\|_{L^2([-1,1]^p)}^2 &= \left\|\sum_{|\boldsymbol{m}|_q\le N(t)} g_{\boldsymbol{m}}^{W,tx_0}\left\langle \mathcal{F}\left[f_{Y|\boldsymbol{X}=x_0\cdot}\right](t)\left(\frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}}(x_0\cdot)-1\right), g_{\boldsymbol{m}}^{W,tx_0}\right\rangle_{L^2([-1,1]^p)}\right\|_{L^2([-1,1]^p)}^2\\
&\le \left\|\mathcal{F}\left[f_{Y|\boldsymbol{X}=x_0\cdot}\right](t)\left(\frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}}(x_0\cdot)-1\right)\right\|_{L^2([-1,1]^p)}^2\\
&\le Z_{n_0}\left\|\mathcal{F}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,tx_0\cdot)\right\|_{L^2([-1,1]^p)}^2 \le Z_{n_0}\left(\frac{2\pi}{|t|x_0}\right)^p\left\|\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)\right\|_{L^2(\mathbb{R}^p)}^2, \qquad \text{(A.26)}
\end{aligned}$$

and, by independence of $Z_j^{\boldsymbol{m},t}$ for $j = 1,\dots,n$,

$$\begin{aligned}
\mathbb{E}\left[\left\|S_2^N(\cdot,t)\right\|_{L^2([-1,1]^p)}^2\right] &= \sum_{|\boldsymbol{m}|_q\le N(t)}\mathbb{E}\left[\left|\Delta_{\boldsymbol{m}}(t)-\mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)\right]\right|^2\right]\\
&\le \sum_{|\boldsymbol{m}|_q\le N(t)}\frac{1}{n}\mathbb{E}\left[\left|Z_j^{\boldsymbol{m},t}-\mathbb{E}\left[Z_j^{\boldsymbol{m},t}\right]\right|^2\right]\\
&\le \sum_{|\boldsymbol{m}|_q\le N(t)}\frac{Z_{n_0}}{nx_0^{2p}}\int_{\mathcal{X}}\frac{1}{f_{\boldsymbol{X}|\mathcal{X}}(\boldsymbol{x})}\left|\overline{g_{\boldsymbol{m}}^{W,tx_0}}\left(\frac{\boldsymbol{x}}{x_0}\right)\right|^2 d\boldsymbol{x} \le \frac{(N(t)+1)^p c_{\boldsymbol{X}} Z_{n_0}}{nx_0^p}, \qquad \text{(A.27)}
\end{aligned}$$

where $\sum_{|\boldsymbol{m}|_q\le N} 1 = \binom{N+p}{p}\mathbb{1}\{q=1\} + (N+1)^p\mathbb{1}\{q=\infty\} \le (N+1)^p$.
Collecting (A.26) and (A.27) with $L^2(t) := (2\pi)^p\left\|\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)\right\|_{L^2(\mathbb{R}^p)}^2$, we obtain

$$\mathbb{E}\left[\left\|R_2\right\|_{L^2(1\otimes W^{\otimes p})}^2\right] \le \frac{Z_{n_0}}{(2\pi)^p}\int_{\epsilon\le|t|\le T}\left(L^2(t) + \frac{c_{\boldsymbol{X}}(N(t)+1)^p|t|^p}{n}\right)\left(\widetilde{\omega}_{N(t)}^{q,W,tx_0}\right)^2 dt. \qquad \text{(A.28)}$$

Term R_3. By Lemma B.3 and Proposition 2 for the first inequality and $f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)$ for the second, we obtain

$$\left\|R_3\right\|_{L^2(1\otimes W^{\otimes p})}^2 \le \int_{\mathbb{R}}\sum_{k>N(t)}\sum_{|\boldsymbol{m}|_q=k}\left|b_{\boldsymbol{m}}(t)\right|^2 dt \le \int_{\mathbb{R}}\frac{\omega_k^2}{\omega_{N(t)}^2}\sum_{k>N(t)}\theta_{q,k}^2(t) \le \sup_{t\in\mathbb{R}}\frac{2\pi l^2}{\omega_{N(t)}^2}. \qquad \text{(A.29)}$$

Term R_4**.** We obtain, by Proposition 2,

$$\|R_4\|^2_{L^2(1\otimes W^{\otimes p})} \le \sum_{k\in\mathbb{N}_0}\int_{|t|\ge T}\sum_{|\boldsymbol{m}|_q=k}|b_{\boldsymbol{m}}(t)|^2\,dt \le \sum_{k\in\mathbb{N}_0}\int_{\mathbb{R}}\frac{\phi^2(|t|)}{\phi^2(T)}\theta^2_{q,k}(t)dt \le \frac{2\pi l^2}{\phi^2(T)}. \tag{A.30}$$

Thus, we have

$$\begin{aligned}
&\sup_{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}} \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)\\
&\le C\int_{\epsilon\le|t|\le T}\frac{2}{\pi(2\pi)^p}\left(\frac{c_{\boldsymbol{X}}|t|^p}{n}\nu^W_q(N(t),tx_0)+Z_{n_0}\left(L^2(t)+\frac{c_{\boldsymbol{X}}(1+N(t))^p|t|^p}{n}\right)\left(\widetilde{\omega}^{q,W,tx_0}_{N(t)}\right)^2\right)dt\\
&\quad + C\left(4l^2\left(\sup_{t\in\mathbb{R}}\frac{1}{\omega^2_{N(t)}}+\frac{1}{\phi(T)^2}\right)+\frac{M^2}{w(\underline{a})}\right).
\end{aligned} \tag{A.31}$$

The remaining of the proof is in Section B.2 particularising (A.31) to the different smoothness.

A.4. **Data-driven choice of the parameters.** Denote by $\mathcal{N}_n$ the set of functions $N\in\mathbb{N}_0^{\mathbb{R}}$ such that, for all $t\in\mathbb{R}\setminus(-\epsilon,\epsilon)$, $N(t)\in\{0,\dots,N^W_{\max,q}(t)\}$. For all $t\in\mathbb{R}$ and $N\in\mathbb{N}_0$, let

$$\mathcal{R}_{0,q}(N,t) := \mathbb{E}\left[\left\|\left(\widehat{F}^{q,N,T,0}_1-\mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}\right].$$

The upper bounds that we derive depend on the parameters of the class $\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)$. For all $t\in[-T,T]\setminus[-\epsilon,\epsilon]$ and $N\in\mathbb{N}_0$, by convexity of $x\mapsto x^2$, we have

$$\mathcal{R}_{0,q}(N,t)\le\mathbb{E}\left[\Xi(t,N)\right]+3\left(\mathbb{E}\left[S_1(t,N)\right]+\mathbb{E}\left[S_2(t,N)\right]+\mathbb{E}\left[S_3(t,N)\right]\right), \tag{A.32}$$

$\Delta_{\boldsymbol{m}}(t):=\widehat{c}_{\boldsymbol{m}}(t)-\widetilde{c}_{\boldsymbol{m}}(t)$, $\widetilde{\Delta}_{\boldsymbol{m}}(t):=\widetilde{c}_{\boldsymbol{m}}(t)-c_{\boldsymbol{m}}(t)$,

$$\Xi(t,N):=\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q>N}\left|\frac{c_{\boldsymbol{m}}(t)}{\sigma^{W,tx_0}_{\boldsymbol{m}}}\right|^2,\quad S_1(t,N):=\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q\le N}\left|\frac{\mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)\right]}{\sigma^{W,tx_0}_{\boldsymbol{m}}}\right|^2,$$

$$S_2(t,N):=\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q\le N}\left|\frac{\Delta_{\boldsymbol{m}}(t)-\mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)\right]}{\sigma^{W,tx_0}_{\boldsymbol{m}}}\right|^2,$$

$$S_3(t,N):=\int_{\mathbb{R}^p}\left|\widetilde{F}^{q,N,T,0}_1(t,\boldsymbol{b})-F^{q,N,T,0}_1(t,\boldsymbol{b})\right|^2W^{\otimes p}(\boldsymbol{b})d\boldsymbol{b}=\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q\le N}\left|\frac{\widetilde{\Delta}_{\boldsymbol{m}}(t)}{\sigma^{W,tx_0}_{\boldsymbol{m}}}\right|^2.$$

Lemma A.3. Let $q\in\{1,\infty\}$, for all $t\in[-T,T]\setminus(-\epsilon,\epsilon)$ with $0<\epsilon<1<T<T_{\max}=2^{K_{\max}}$, $\widehat{N}(t)$ chosen from (16), and $N\in\{0,\dots,N^W_{\max,q}(t)\}$, the following inequalities hold

$$\mathbb{E}\left[S_1\left(t,\widehat{N}(t)\right)\right]\le Z_{n_0}\left\|\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)\right\|^2_{L^2(\mathbb{R}^p)}|tx_0|^p\mathbb{E}\left[\nu^W_q\left(\widehat{N}(t),tx_0\right)\right], \tag{A.33}$$

$$\mathbb{E}\left[S_2\left(t,\widehat{N}(t)\right)\right] \le Z_{n_0}\left(\frac{|tx_0|}{2\pi}\right)^p \frac{c_{\boldsymbol{X}}\nu_q^W(N_{\max,q}^W(t),tx_0)}{nx_0^p}, \tag{A.34}$$

$$\mathbb{E}\left[\left(S_3(t,N)-\frac{\Sigma(t,N)}{2(2+c_0)}\right)_+\right] \le \frac{48c_{\boldsymbol{X}}\,|t|^p\,\nu_q^W(N,tx_0)}{(2\pi)^p n}\Psi_{0,n}(t), \tag{A.35}$$

where $p_n := (2\ln(n))\vee 3$, $K_1 := x_0^p\sqrt{2c_{\boldsymbol{X}}}/42$, $H_W(t)$ is defined in Proposition B.2, and

$$\Psi_{0,n}(t) := x_0^p \exp\left(-\frac{p_n}{6}\right) + \frac{294c_{\boldsymbol{X}}^2K_n^2(t)}{(2\pi x_0)^p n}\exp\left(-\frac{K_1\sqrt{p_n n}}{K_n(t)}\right),\ K_n(t) := H_W(t)\left(N_{\max,q}^W(t)+\frac{1}{2}\right)^p.$$

Proof. Let $t\in[-T,T]\setminus(-\epsilon,\epsilon)$ and $N\in\{0,\dots,N_{\max,q}^W(t)\}$.
Proof of (A.33). Let $\boldsymbol{m}\in\mathbb{N}_0^p$, we have, like in (A.26),

$$\begin{aligned}|\mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)\right]|^2 &= \left|\int_{[-1,1]^p}\mathcal{F}\left[f_{Y|\boldsymbol{X}=x_0\boldsymbol{u}}\right](t)\left(\frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}}(x_0\boldsymbol{u})-1\right)g_{\boldsymbol{m}}^{W,tx_0}(\boldsymbol{u})d\boldsymbol{u}\right|^2\\ &\le \left\|\mathcal{F}\left[f_{Y|\boldsymbol{X}=x_0\cdot}\right](t)\left(\frac{f_{\boldsymbol{X}|\mathcal{X}}}{\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}}(x_0\cdot)-1\right)\right\|^2_{L^2([-1,1]^p)} \le Z_{n_0}(2\pi)^p\left\|\mathcal{F}_{1\text{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)\right\|^2_{L^2(\mathbb{R}^p)}.\end{aligned}$$

(A.33) now follows because, by Lemma B.2,

$$\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q\le N}\frac{1}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2} \le \frac{|tx_0|^p}{(2\pi)^p}\nu_q^W(N,tx_0). \tag{A.36}$$

Proof of (A.34)**.** Using (A.36) and (A.27) for the second display, we have

$$\mathbb{E}\left[S_2\left(t,\widehat{N}(t)\right)\right] \le \sum_{\substack{|\boldsymbol{m}|_q\le N_{\max,q}^W(t)\\ \boldsymbol{m}\in\mathbb{N}_0^p}}\frac{\mathbb{E}\left[|\Delta_{\boldsymbol{m}}(t)-\mathbb{E}\left[\Delta_{\boldsymbol{m}}(t)\right]|^2\right]}{\left(\sigma_{\boldsymbol{m}}^{W,tx_0}\right)^2} \le \frac{c_{\boldsymbol{X}}Z_{n_0}|t|^p\nu_q^W\left(N_{\max,q}^W(t),tx_0\right)}{(2\pi)^p n}.$$

Proof of (A.35)**.** We have

$$S_3(t,N) = \int_{\mathbb{R}^p}\left|\widetilde{F}_1^{q,N,T,0}(t,\boldsymbol{b})-F_1^{q,N,T,0}(t,\boldsymbol{b})\right|^2W^{\otimes p}(\boldsymbol{b})d\boldsymbol{b} = \sup_{u\in\mathcal{U}}\left|\nu_n^t(u)\right|^2, \tag{A.37}$$

where

$$\nu_n^t(u) := \left\langle\widetilde{F}_1^{q,N,T,0}(t,\cdot)-F_1^{q,N,T,0}(t,\cdot),u(\cdot)\right\rangle_{L^2(W^{\otimes p})} = \frac{1}{n}\sum_{j=1}^n\left(f_u^t(Y_j,X_j)-\mathbb{E}\left[f_u^t(Y_j,X_j)\right]\right),$$

$$f_u^t(\star,\cdot) := \int_{\mathbb{R}^p}\sum_{|\boldsymbol{m}|_q\le N}\frac{e^{it\star}}{x_0^pf_{\boldsymbol{X}|\mathcal{X}}(\cdot)}\overline{g_{\boldsymbol{m}}^{W,tx_0}}\left(\frac{\cdot}{x_0}\right)\frac{\varphi_{\boldsymbol{m}}^{W,tx_0}(\boldsymbol{b})}{\sigma_{\boldsymbol{m}}^{W,tx_0}}\overline{u}(\boldsymbol{b})W^{\otimes p}(\boldsymbol{b})d\boldsymbol{b},$$

and $\mathcal{U}$ is a countable dense class of functions of $\left\{u:\ \|u\|_{L^2(W^{\otimes p})}=1\right\}$. We now check the conditions of the Talagrand inequality given in Lemma B.16.

Condition (B.79). For all $u \in \mathcal{U}$, Proposition B.2, the Cauchy-Schwarz inequality, and setting $M(t,N) := K_n(t)\,|t|^{p/2}\,c_{\boldsymbol{X}}\left(\nu_q^W(N,tx_0)\right)^{1/2}/(2\pi x_0)^{p/2}$ yield

$$\left\|f_u^t\right\|_{L^\infty(\mathbb{R}\times\mathcal{X})} \le \left\|\left(\int_{\mathbb{R}^p}\sum_{|\boldsymbol{m}|_q\le N}\frac{|t|^p\left|\overline{g_{\boldsymbol{m}}^{W,tx_0}}(\cdot/x_0)\right|^2}{(2\pi)^p f_{\boldsymbol{X}|\mathcal{X}}^2(\cdot)\rho_{\boldsymbol{m}}^{W,tx_0}x_0^p}\left|\varphi_{\boldsymbol{m}}^{W,tx_0}(\boldsymbol{b})\right|^2 W^{\otimes p}(\boldsymbol{b})d\boldsymbol{b}\right)^{1/2}\right\|_{L^\infty(\mathcal{X})}\|u\|_{L^2(W^{\otimes p})}$$

$$\le \frac{H_W(t)\left(N_{\max,q}^W(t)+1/2\right)^p|t|^{p/2}c_{\boldsymbol{X}}}{(2\pi x_0)^{p/2}}\left(\sum_{|\boldsymbol{m}|_q\le N}\frac{1}{\rho_{\boldsymbol{m}}^{W,tx_0}}\right)^{1/2} \le M(t,N).$$

Condition (B.80). Using (A.24) and $c_0 = 1/6$, we have

$$\mathbb{E}\left[\sup_{u\in\mathcal{U}}\left|\nu_n^t(u)\right|^2\right] \le \mathbb{E}\left[\sup_{u\in\mathcal{U}}\left\|\widetilde{F}_1^{q,N,T,0}(t,\cdot)-F_1^{q,N,T,0}(t,\cdot)\right\|_{L^2(W^{\otimes p})}^2\|u\|_{L^2(W^{\otimes p})}^2\right]$$

$$\le \frac{\Sigma(N,t)}{8(2+c_0)(1+2p_n)} = \frac{c_{\boldsymbol{X}}\,|tx_0|^p}{n(2\pi)^p}\nu_q^W(N,tx_0) =: H^2.$$

Condition (B.81). We have, using the Cauchy-Schwarz inequality and that $\left(\varphi_{\boldsymbol{m}}^{W,tx_0}\right)_{\boldsymbol{m}\in\mathbb{N}_0^p}$ is an orthonormal basis of $L^2\left(W^{\otimes p}\right)$ for the second display and Lemma B.2 for the third display

$$\mathrm{Var}\left(\Re(f_u^t(Y_j,X_j))\right)\vee\mathrm{Var}\left(\Im(f_u^t(Y_j,X_j))\right) \le \int_{\mathbb{R}\times\mathcal{X}}\left|f_u^t(y,\boldsymbol{x})\right|^2 f_{Y,\boldsymbol{X}}(y,\boldsymbol{x})dyd\boldsymbol{x}$$

$$\le c_{\boldsymbol{X}}\sum_{|\boldsymbol{m}|_q\le N}\frac{|tx_0|^p}{(2\pi)^p\rho_{\boldsymbol{m}}^{W,tx_0}}\int_{[-1,1]^p}\left|g_{\boldsymbol{m}}^{W,tx_0}(\boldsymbol{x})\right|^2 d\boldsymbol{x}$$

$$\le \frac{c_{\boldsymbol{X}}\,|tx_0|^p}{(2\pi)^p}\nu_q^W(N,tx_0).$$

The result follows because, by Lemma B.16 with $\eta = p_n$ and $\Lambda(p_n)\ge 1$,

$$\mathbb{E}\left[\left(\sup_{u\in\mathcal{U}}\left|\nu_n^t(u)\right|^2-\frac{\Sigma(t,N)}{2(2+c_0)}\right)_+\right] \le \frac{48\nu_q^W(N,tx_0)c_{\boldsymbol{X}}\,|t|^p}{(2\pi)^p n}\Psi_{0,n}(t). \qquad \square$$

Denote by

$$\widetilde{B}\left(\widehat{N}\right) := \mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\le|t|\le T'}\left(\sum_{|\boldsymbol{m}|_q\le\widehat{N}(t)}\left(\frac{\left|\widetilde{\Delta}_{\boldsymbol{m}}(t)\right|}{\sigma_{\boldsymbol{m}}^{W,tx_0}}\right)^2-\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+dt\right].$$

Lemma A.4. Let $\|f_{\alpha,\boldsymbol{\beta}}\|_{L^2(1\otimes W^{\otimes p})}\le M$, $M,\epsilon>0$, $q\in\{1,\infty\}$. For all $T\in\mathcal{T}_n$, we have

$$\mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,\widehat{N},\widehat{T},\epsilon},f_{\alpha,\boldsymbol{\beta}}\right) \le \frac{C(5+2c_0)}{2\pi}\left(1+\frac{2}{c_0}\right)\mathbb{E}\left[\left\|\mathbb{1}\{|\cdot|\ge\epsilon\}\left(\widehat{F}_1^{q,\widehat{N},T,0}-\mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(\cdot,\star)\right\|_{L^2(1\otimes W^{\otimes p})}^2\right]$$

$$+\frac{CM^2}{w(\underline{a})}+C_{19}\Pi(n,Z_{n_0},T_{\max},N^W_{\max,q})+\frac{C(2+c_0)}{\pi}\int_{\epsilon\le|t|\le T}\mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\right]dt,$$

where $C_{19}:=2(2+c_0)^2C/\pi$, $c_0:=1/6$,

$$\Pi(n,Z_{n_0},T_{\max},N^W_{\max,q}):=Z_{n_0}\int_{\epsilon\le|t|\le T_{\max}}\Psi_n(t)dt+\Pi_1(n,T_{\max},N_{\max,q})$$

$$\Pi_1(n,T_{\max},N^W_{\max,q}):=\frac{96(1+2c_0)c_{\boldsymbol{X}}K_{\max}}{(2\pi)^p n}\int_\epsilon^{T_{\max}}N^W_{\max,q}(t)t^p\nu_q^W(N^W_{\max,q}(t),tx_0)\Psi_{0,n}(t)dt,$$

$$\Psi_n(\star):=\left(2+\frac{1}{c_0}\right)|\star x_0|^p\left(\frac{c_{\boldsymbol{X}}\nu_q^W(N^W_{\max,q}(\star),\star x_0)}{n(2\pi x_0)^p}+\|\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\star,\cdot)\|^2_{L^2(\mathbb{R}^p)}\mathbb{E}\left[\nu_q^W\left(\widehat{N}(\star),\star x_0\right)\right]\right).$$

Proof. Let $T\in\mathcal{T}_n$. Consider the term $\widehat{F}_1^{q,\widehat{N},\widehat{T},0}-\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]=\sum_{j=1}^3R_j^T$, where $R_1^T:=\widehat{F}_1^{q,\widehat{N},\widehat{T},0}-\widehat{F}_1^{q,\widehat{N},T\vee\widehat{T},0}$, $R_2^T:=\widehat{F}_1^{q,\widehat{N},T\vee\widehat{T},0}-\widehat{F}_1^{q,\widehat{N},T,0}$, and $R_3^T:=\widehat{F}_1^{q,\widehat{N},T,0}-\mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]$. Using Proposition 3 and Lemma B.1 for the first inequality and using that for all $c_0>0$ and $a,b\in R$, $ab\le a^2/(2c_0)+b^2c_0/2$ for the second display, we have

$$\begin{aligned}\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)&\le\frac{C}{2\pi}\mathbb{E}\left[\left\|\mathbb{1}\{|\cdot|\ge\epsilon\}\left(\widehat{F}_1^{q,\widehat{N},\widehat{T},0}-\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(\cdot,\star)\right\|^2_{L^2(1\otimes W^{\otimes p})}\right]+\frac{CM^2}{w(\underline{a})}\\&\le\frac{C(2+c_0)}{2\pi}\sum_{j=1}^2\mathbb{E}\left[\|R_j^T\|^2_{L^2(1\otimes W^{\otimes p})}\right]+\frac{CM^2}{w(\underline{a})}\\&\quad+\frac{C\left(1+2/c_0\right)}{2\pi}\mathbb{E}\left[\|\mathbb{1}\{|\cdot|\ge\epsilon\}R_3^T(\cdot,\star)\|^2_{L^2(1\otimes W^{\otimes p})}\right].\end{aligned}$$

Denote by $\Sigma_2(T,N):=\int_{\epsilon\le|t|\le T}\Sigma(t,N(t))dt$. Because

$$B_2\left(T,\widehat{N}\right)=\max_{T'\in\mathcal{T}_n}\left(\int_{\epsilon\le|t|}\left\|\left(\widehat{F}_1^{q,\widehat{N},T\vee T',0}-\widehat{F}_1^{q,\widehat{N},T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}-\Sigma\left(t,\widehat{N}(t)\right)dt\right)_+,$$

we have

$$\begin{aligned}\mathbb{E}\left[\left\|R_1^T\right\|^2_{L^2(1\otimes W^{\otimes p})}\right]&\le\mathbb{E}\left[\max_{T'\in\mathcal{T}_n}\int_{\epsilon\le|t|}\left(\left\|R_1^{T'}(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}-\mathbb{1}\{|t|\le T'\}\Sigma\left(t,\widehat{N}(t)\right)\right)_+dt\right]\\&\quad+\mathbb{E}\left[\int_{\epsilon\le|t|\le T}\Sigma\left(t,\widehat{N}(t)\right)dt\right]\le\mathbb{E}\left[B_2\left(\widehat{T},\widehat{N}\right)\right]+\mathbb{E}\left[\Sigma_2\left(T,\widehat{N}\right)\right]\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[\left\|R_2^T\right\|^2_{L^2(W^{\otimes p})}\right]&\le\mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\le|t|}\left(\left\|R_2^{T'}(t,\cdot)\right\|^2_{L^2(1\otimes W^{\otimes p})}-\mathbb{1}\{|t|\le T'\}\Sigma\left(t,\widehat{N}(t)\right)\right)_+dt\right]\\&\quad+\mathbb{E}\left[\Sigma_2\left(\widehat{T},\widehat{N}\right)\right]\le\mathbb{E}\left[B_2\left(T,\widehat{N}\right)\right]+\mathbb{E}\left[\Sigma_2\left(\widehat{T},\widehat{N}\right)\right].\end{aligned}$$

Thus, using the definition of $\widehat{T}$ we have

$$\mathbb{E}\left[\left\|\mathbb{1}\{|\cdot|\geq\epsilon\}R_3^{\widehat{T}}(\cdot,\star)\right\|^2_{L^2(1\otimes W^{\otimes p})}\right]$$
$$\leq 2(2+c_0)\left(\mathbb{E}\left[B_2\left(T,\widehat{N}\right)\right]+\mathbb{E}\left[\Sigma_2\left(T,\widehat{N}\right)\right]\right)+\left(1+\frac{2}{c_0}\right)\mathbb{E}\left[\left\|\mathbb{1}\{|\cdot|\geq\epsilon\}R_3^{T}(\cdot,\star)\right\|^2_{L^2(1\otimes W^{\otimes p})}\right].$$

Consider $\mathbb{E}\left[B_2\left(T,\widehat{N}\right)\right]$ and let $T'\in\mathcal{T}_n$. Using $K_1:=\widehat{F}_1^{q,\widehat{N},T\vee T',0}-F_1^{q,\widehat{N},T\vee T',0}$, $K_2:=F_1^{q,\widehat{N},T,0}-\widehat{F}_1^{q,\widehat{N},T,0}$, and $K_3:=F_1^{q,\widehat{N},T\vee T',0}-F_1^{q,\widehat{N},T,0}$, $B_2\left(T,\widehat{N}\right)$ is smaller than

$$\int_{\epsilon\leq|t|}\left((2+c_0)\sum_{j=1}^2\|K_j(t,\cdot)\|^2_{L^2(W^{\otimes p})}+\left(1+\frac{2}{c_0}\right)\|K_3(t,\cdot)\|^2_{L^2(W^{\otimes p})}-\mathbb{1}\{|t|\leq T'\}\Sigma\left(t,\widehat{N}(t)\right)\right)_+dt.$$

Using that $F_1^{q,\infty,\infty,0}=\mathcal{F}_{1\text{st}}\left[f_{\alpha,\beta}\right]$, we have, for all $t\in\mathbb{R}\setminus(-\epsilon,\epsilon)$,

$$\|K_3(t,\cdot)\|^2_{L^2(W^{\otimes p})}=\mathbb{1}\{T\leq|t|\leq T\vee T'\}\sum_{0\leq|\boldsymbol{m}|_q\leq\widehat{N}}\left|\frac{c_{\boldsymbol{m}}(t)}{\sigma_{\boldsymbol{m}}^{W,tx_0}}\right|^2\leq\left\|\left(F_1^{q,\widehat{N},T,0}-\mathcal{F}_{1\text{st}}\left[f_{\alpha,\beta}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}$$

hence

$$B_2\left(T,\widehat{N}\right)\leq\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|}\left(2(2+c_0)\left\|\left(\widehat{F}_1^{q,\widehat{N},T',0}-F_1^{q,\widehat{N},T',0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}-\mathbb{1}\{|t|\leq T'\}\Sigma\left(t,\widehat{N}(t)\right)\right)_+dt$$
$$+\left(1+\frac{2}{c_0}\right)\int_{\epsilon\leq|t|}\left\|R_3^T(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}dt.$$

Finally, we have

$$\mathbb{E}\left[\left\|\mathbb{1}\{|\cdot|\geq\epsilon\}R_3^{\widehat{T}}(\cdot,\star)\right\|^2_{L^2(1\otimes W^{\otimes p})}\right]$$
$$\leq 4(2+c_0)^2\mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|}\left(\left\|\left(\widehat{F}_1^{q,\widehat{N},T',0}-F_1^{q,\widehat{N},T',0}\right)(t,\star)\right\|^2_{L^2(W^{\otimes p})}-\mathbb{1}\{|t|\leq T'\}\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+\right]$$
$$+2(2+c_0)\mathbb{E}\left[\Sigma_2\left(T,\widehat{N}\right)\right]+\left(1+\frac{2}{c_0}\right)(5+2c_0)\mathbb{E}\left[\left\|\mathbb{1}\{|\cdot|\geq\epsilon\}R_3^{T}(\cdot,\star)\right\|^2_{L^2(1\otimes W^{\otimes p})}\right].$$

By the Young inequality in the second display and Lemma A.3 for the third, we have

$$\mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|}\left(\left\|\left(\widehat{F}_1^{q,\widehat{N},T',0}-F_1^{q,\widehat{N},T',0}\right)(t,\star)\right\|^2_{L^2(W^{\otimes p})}-\mathbb{1}\{|t|\leq T'\}\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+dt\right]$$
$$=\mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|\leq T'}\left(\sum_{|\boldsymbol{m}|_q\leq\widehat{N}(t)}\left(\frac{|\widehat{c}_{\boldsymbol{m}}(t)-c_{\boldsymbol{m}}(t)|}{\sigma_{\boldsymbol{m}}^{W,tx_0}}\right)^2-\frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+dt\right]$$

$$\leq (1+2c_0)\widetilde{B}\left(\widehat{N}\right) + \left(2+\frac{1}{c_0}\right)\mathbb{E}\left[\sup_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|\leq T'} S_1\left(\widehat{N}(t),t\right)+S_2\left(\widehat{N}(t),t\right)dt\right]$$
$$\leq (1+2c_0)\widetilde{B}\left(\widehat{N}\right) + Z_{n_0}\int_{\epsilon\leq|t|\leq T_{\max}}\Psi_n(t)dt.$$

We now focus on the first term of the last inequality. Using (A.35) for the third display,

$$\widetilde{B}\left(\widehat{N}\right) \leq \mathbb{E}\left[\sum_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|\leq T'}\left(\sum_{|\boldsymbol{m}|_q\leq\widehat{N}(t)}\left(\frac{\left|\widetilde{\Delta}_{\boldsymbol{m}}(t)\right|}{\sigma_{\boldsymbol{m}}^{W,tx_0}}\right)^2 - \frac{\Sigma\left(t,\widehat{N}(t)\right)}{2(2+c_0)}\right)_+ dt\right]$$
$$\leq \sum_{T'\in\mathcal{T}_n}\int_{\epsilon\leq|t|\leq T'}\sum_{0\leq N\leq N^W_{\max,q}(t)}\mathbb{E}\left[\left(\sum_{|\boldsymbol{m}|_q\leq N(t)}\left(\frac{\left|\widetilde{\Delta}_{\boldsymbol{m}}(t)\right|}{\sigma_{\boldsymbol{m}}^{W,tx_0}}\right)^2 - \frac{\Sigma(t,N)}{2(2+c_0)}\right)_+\right]dt$$
$$\leq \sum_{T'\in\mathcal{T}_n}\int_{\epsilon}^{T'}\sum_{0\leq N\leq N^W_{\max,q}(t)}\frac{96c_{\boldsymbol{X}}t^p\nu_q^W(N,tx_0)}{(2\pi)^p n}\Psi_{0,n}(t)dt$$
$$\leq \frac{96c_{\boldsymbol{X}}K_{\max}}{(2\pi)^p n}\int_{\epsilon}^{T_{\max}} N^W_{\max,q}(t)t^p\nu_q^W(N^W_{\max,q}(t),tx_0)\Psi_{0,n}(t)dt. \qquad \square$$

Lemma A.5. For $W_{[-R,R]}$ and $\cosh(\cdot/R)$, $q\in\{1,\infty\}$, and all $T\in\mathcal{T}_n$ and $N\in\mathcal{N}_n$, we have

$$\int_{\epsilon\leq|t|\leq T}\mathcal{R}_{0,q}\left(\widehat{N}(t),t\right)dt + C_{c_0}\int_{\epsilon\leq|t|\leq T}\mathbb{E}\left[\Sigma\left(T,\widehat{N}(t)\right)\right]dt$$
$$\leq 4(2+c_0)^2\Pi(n,Z_{n_0},T_{\max},N_{\max,q}) + C_{c_0,1}\int_{\epsilon\leq|t|\leq T}\left(\mathcal{R}_{0,q}\left(N(t),t\right)+C_{c_0}c_1\mathbb{E}\left[\Sigma(t,N(t))\right]\right)dt,$$

$C_{c_0}:=2(2+c_0)/((5+2c_0)(1+2/c_0))$, and $C_{c_0,1}:=(5+2c_0)(1+2/c_0)$.

The proof of Lemma A.5 is similar to the one of Lemma A.4, hence postponed to Section B.2.

Proof of Theorem 5. Take $\mathcal{E},\eta>0$, and $c_0:=1/6$. Let $(n,n_0)\in\mathbb{N}^2$ such that $v(n_0,\mathcal{E})/\delta(n_0)\leq n^{-(2+\zeta)}$. By Lemma B.14, there exists $M_{1,\mathcal{E},\eta}$ such that, for all $n_0\in\mathbb{N}$, $\mathbb{P}\left(E\left(\mathcal{G}_{n_0},\mathcal{E}\right)\right)\geq 1-\eta$, where $E\left(\mathcal{G}_{n_0},\mathcal{E}\right):=\{Z_{n_0}\leq M_{1,\mathcal{E},\eta}v(n_0,\mathcal{E})/\delta(n_0)\}$. We work on this event.
Proof of (T5.1). Let $W=W_{[-R,R]}$, $q=1$, $T\in\mathcal{T}_n$, and $N\in\mathcal{N}_n$. The other cases can be treated similarly. Use $C_{c_0,2}:=CC_{c_0,1}/(2\pi)$ and $C_{c_0,3}:=C(2+c_0)/\pi$. Using Lemma A.4 and (A.22) for the first display and $C_{c_0}=C_{c_0,3}/C_{c_0,2}$ for the second yield

$$\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\leq C_{c_0,2}\mathbb{E}\left[\left\|\mathbb{1}\{|\cdot|\geq\epsilon\}\left(\widehat{F}_1^{q,\widehat{N},T,0}-\mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(\cdot,\star)\right\|^2_{L^2(1\otimes W^{\otimes p})}\right]$$
$$+C_{c_0,3}\int_{\epsilon\leq|t|\leq T}\mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\right]dt + C_{19}\Pi(n,Z_{n_0},T_{\max},N_{\max,q}) + \frac{CM^2}{w(\underline{a})}$$

$$
\begin{aligned}
&\leq C_{c_0,2}\left(\int_{\epsilon\leq|t|\leq T}\mathcal{R}_{0,q}\left(\widehat{N}(t),t\right)+C_{c_0}\mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\right]dt\right)+C_{19}\Pi(n,Z_{n_0},T_{\max},N_{\max,q})+\frac{CM^2}{w(\underline{a})}\\
&\leq C_{c_0,2}\int_{\epsilon\leq|t|\leq T}\left(\frac{2\pi C_{c_0,2}}{C}\mathcal{R}_{0,q}\left(N(t),t\right)+2(2+c_0)c_1\mathbb{E}\left[\Sigma(t,N(t))\right]\right)dt\\
&\quad+\left(C_{19}+4(2+c_0)^2C_{c_0,2}\right)\Pi(n,Z_{n_0},T_{\max},N_{\max,q})+\frac{CM^2}{w(\underline{a})}\quad\text{(by Lemma A.5)}.
\end{aligned}
$$

By Lemma (B.15), $v(n_0,\mathcal{E})/\delta(n_0)\leq n^{-(2+\zeta)}$, the definition of Σ, and (B.41)-(B.43), we obtain

$$
\begin{aligned}
\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\leq&\frac{CM^2}{w(\underline{a})}+\frac{C_{25,\mathcal{E},\eta}}{n}+\frac{2\pi C_{c_0,2}^2}{C}\left(4l^2\sup_{t\in\mathbb{R}}\frac{1}{N(t)^{2\sigma}}+\frac{4l^2}{(1\vee T)^{2s}}\right)\\
&+\frac{2\pi C_{c_0,2}^2}{C}\int_{\epsilon\leq|t|\leq T}\widetilde{\Delta}_{2,q}\left(t,N(t),n,\frac{M_{1,\mathcal{E},\eta}v(n_0,\mathcal{E})}{\delta(n_0)}\right)dt,
\end{aligned}
\tag{A.38}
$$

$$
\begin{aligned}
\widetilde{\Delta}_{2,q}(t,N,n,z):=&\Delta_{3,q}(t,N)\left(1+\frac{120(1+2p_n)C(2+c_0)c_1}{\pi C_{c_0,2}}\right)\frac{2p^{p-1}c_{\boldsymbol{X}}N(t)^p\,|t|^p}{(2\pi)^p(p-1)!n}\\
&+\frac{z\Delta_{3,q}(t,N)}{\pi^p}\left(L(t)^2+\frac{c_{\boldsymbol{X}}(N(t)+1)^p\,|t|^p}{n}\right),\\
C_{25,\mathcal{E},\eta}:=&\left(C_{19}+4(2+c_0)^2C_{c_0,2}\right)\left(C_{22,\mathcal{E},\eta}\left(\frac{p-1}{e(\zeta-\zeta_0)}\right)^{p-1}(1\vee l^2)+C_{24}\right)
\end{aligned}
$$

and $\Delta_{3,q}(t,N)$ is given in (B.42). Let $T^*:=2^{k^*}$, where $k^*:=\lfloor\ln(n)/(\ln(2)6s(p+1))\rfloor$ hence $n_e^{1/(6s(p+1))}/2\leq T^*\leq n_e^{1/(6s(p+1))}$, and $N^*(t):=\lceil\overline{N}^*(t)\rceil$, where $\overline{N}^*(t):=Q_{|t|,(2\sigma+p+1)/4}\left(\ln(n_e)/8\right)$ and $(2\sigma+p+1)/4$ replaces $(2\sigma+p)/4$ in the definition of $\overline{N}$ in (T1.1). We have $N^*(t)\leq N^W_{\max,q}(t)$ for all $t\in\mathbb{R}\setminus(-\epsilon,\epsilon)$, thus $N^*\in\mathcal{N}_n$. We also have $n_e^{1/(6s(p+1))}\leq T_{\max}=n^{\zeta_0}$ for all $s>1$, thus $T^*\in\mathcal{T}_n$. We have

$$
\begin{aligned}
&\sup_{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\\
&\leq\frac{CM^2}{(\ln(n)/\ln_2(n))^{2\sigma_0}}+\frac{C_{25,\mathcal{E},\eta}}{n}+\frac{2\pi C_{c_0,2}^2}{C}\int_{\epsilon\leq|t|\leq T^*}\widetilde{\Delta}_{2,q}\left(t,N^*(t),n,\frac{M_{1,\mathcal{E},\eta}v(n_0,\mathcal{E})}{\delta(n_0)}\right)dt\\
&\quad+\frac{2\pi C_{c_0,2}^2}{C}\left(4l^2\sup_{t\in\mathbb{R}}\frac{1}{N^*(t)^{2\sigma}}+\frac{4l^2}{(1\vee T^*)^{2s}}\right).
\end{aligned}
$$

Adapting the constants in the proof of (T1.1) to account for the new value of T, we obtain

$$
\begin{aligned}
&\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,\widehat{N},\widehat{T},\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\left(\frac{\ln(n)}{\ln_2(n)}\right)^{2\sigma}\leq M_{13,\mathcal{E},\eta},\\
&M_{13,\mathcal{E},\eta}:=2\pi C_{c_0,2}^2\left(\frac{1}{2\tau_{4,1}}\bigwedge\frac{1}{2\tau_{3,1}}\right)^{-2\sigma}\left(4l^2\left(1+2\left(\frac{6\sigma(p+1)}{e}\right)^{2\sigma}\right)+M'_{4,\mathcal{E},\eta}+\frac{M^2}{(4k_1\tau_{5,1})^{2\sigma}}\right)
\end{aligned}
$$

$$M'_{4,\mathcal{E},\eta} := \left(1 + \frac{120(1+2p_n)C(2+c_0)c_1}{\pi C_{c_0,2}}\right)\frac{8}{\pi}\frac{\tau_2 c_{\boldsymbol{X}} p^p}{(2\pi)^p(p+1)!} + 4M_{1,\mathcal{E},\eta}\left(2^p l^2 + \frac{c_{\boldsymbol{X}}}{\pi^{p+1}(p+1)}\left(\frac{p}{e}+1\right)^p\right).$$

The other smoothness classes are treated in Section B.2.

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SUPPLEMENTAL APPENDIX

Appendix B.1. Harmonic analysis

B.1.1. **Preliminaries.** P_m is the Legendre polynomial of degree m with $\|P_m\|_{L^2([-1,1])} = 1$.

Lemma B.1. For all $f \in L^2_w(\mathbb{R})$, w even, nondecreasing on $[0,\infty)$, and $w(0), R > 0$, we have $\left\|\mathcal{P}_R\left[\mathcal{F}\left[f\right]\right] - \mathcal{F}\left[f\right]\right\|^2_{L^2(\mathbb{R})} \le (2\pi/w(R))\|f\|^2_{L^2(w)}$.

Proof. The result uses the Plancherel identity and

$$\left\|\mathcal{P}_R\left[\mathcal{F}\left[f\right]\right] - \mathcal{F}\left[f\right]\right\|^2_{L^2(\mathbb{R})} = 2\pi \int_{\mathbb{R}} \mathbb{1}\{|a| > R\} \left|f\left(a\right)\right|^2 da \le \frac{2\pi}{w(R)} \int_{\mathbb{R}} \left|f\left(a\right)\right|^2 w(a) da. \quad \square$$

Proposition B.1. For all weighting function W, $c \in \mathbb{R}$, $R > 0$, and $m \in \mathbb{N}_0$, we have

(i) $g_m^{W(\cdot/R),c} = g_m^{W,Rc}$ in $L^2([-1,1])$,

(ii) $\sigma_m^{W(\cdot/R),c} = \sigma_m^{W,Rc}\sqrt{R}$,

(iii) $\varphi_m^{W(\cdot/R),c} = \varphi_m^{W,Rc}\left(\star/R\right)/\sqrt{R}$ *a.e.*

Proof. (i) follows from $\mathcal{Q}_c^{W(\cdot/R)} = \mathcal{Q}_{Rc}^{W}$ and (ii) from $\sigma_m^{W(\cdot/R),c} = \sqrt{2\pi\rho_m^{W(\cdot/R),c}/\left|c\right|} = \sqrt{2\pi\rho_m^{W,Rc}/\left|c\right|}$ (by the argument yielding (i)). Now, using (i) in the first display and (ii) in the last display, we have, for a.e. $t \in \mathbb{R}$,

$$\begin{aligned}
\sigma_m^{W,Rc}\varphi_m^{W,Rc}\left(\frac{t}{R}\right) &= \mathcal{F}^*_{Rc}\left[g_m^{W(\cdot/R),c}\right]\left(\frac{t}{R}\right) \quad \left(\text{where } \mathcal{F}^*_{Rc}:\ L^2([-1,1]) \to L^2(W)\right)\\
&= \mathcal{F}^*_{c}\left[g_m^{W(\cdot/R),c}\right](t) \quad \left(\text{where } \mathcal{F}^*_{c}:\ L^2([-1,1]) \to L^2(W(\cdot/R))\right)\\
&= \sigma_m^{W(\cdot/R),c}\varphi_m^{W(\cdot/R),c}(t) = \sigma_m^{W,Rc}\sqrt{R}\varphi_m^{W(\cdot/R),c}(t),
\end{aligned}$$

hence (iii) when we divide by $\sigma_m^{W,Rc}$ which is nonzero. $\square$

Proposition B.2. For all $\boldsymbol{m} \in \mathbb{N}_0^p$, $R > 0$, $W = W_{[-R,R]}$ or $W = \cosh(\cdot/R)$, $t \ne 0$, we have $\left\|g_{\boldsymbol{m}}^{W,tx_0}\right\|_{L^\infty([-1,1]^p)} \le H_W(t)\prod_{j=1}^p \sqrt{m_j + 1/2}$, where $H_{W_{[-R,R]}}(t) = H_0^p\left(1 + (|t|x_0)^2\right)^p$, $H_0 = 2(1 + 1/\sqrt{3})$, $H_{\cosh(\cdot/R)}(t) = H_1^p(1 \vee (|t|x_0)^4)^p$, $H_1 > 0$.

Proof. When $W = W_{[-R,R]}$, this is (66) in [9] else this is Corollary 1 in [22]. $\square$

Lemma B.2. For all $q \in \{1,\infty\}$, $t \ne 0$, $R > 0$, $N \in \mathbb{N}_0$, in cases (N.1) and (N.2) of Section 4.4, we have $\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_q \le N} 1/\rho_{\boldsymbol{m}}^{W,t} \le \nu_q^W(N,t)$.

Proof. Let $R > 0$. We use repeatedly, for all $x > 0$ and $N \in \mathbb{N}_0$,

$$\sum_{k\le N} \exp\left(kx\right) \le \frac{\exp\left((N+1/2)x\right)}{2\sinh\left(x/2\right)} \le \frac{\exp\left((N+1/2)x\right)}{x} \quad \left(\text{because } \sinh(|x|) \ge |x|\right), \tag{B.1}$$

$$\le \frac{\exp\left(Nx\right)}{1 - \exp(-x)}, \tag{B.2}$$

the cardinal of $\{\boldsymbol{m} \in \mathbb{N}_0^p : \ |\boldsymbol{m}|_1 = k\}$ is $\binom{k+p-1}{p-1}$, and $(k+p-1)!/k! \le (k+p-1)^{p-1}$, and for all $m \in \mathbb{N}_0$, $\rho_m^{\cosh,Rt} = \rho_m^{\cosh(\cdot/R),t}$ and $\rho_m^{W_{[-1,1]},Rt} = \rho_m^{W_{[-R,R]},t}$.
Start by case (N.1). Let $|t| > \pi/4$ and $q = 1$. By (8) in [22] (there is difference of normalisations for $\mathcal{Q}_t$ by a factor $1/(2\pi)$), we have, for all $m \in \mathbb{N}_0$,

$$\rho_m^{\cosh,t} \ge \frac{1}{2}\exp\left(-\frac{\pi(m+1)}{2\,|t|}\right). \tag{B.3}$$

The result is obtained from the above with (B.1) and

$$\begin{aligned}\sum_{|\boldsymbol{m}|_1 \le N} \frac{1}{\rho_{\boldsymbol{m}}^{\cosh,t}} &\le 2^p \sum_{k \le N} \sum_{|\boldsymbol{m}|_1 = k} \exp\left(\frac{\pi(|\boldsymbol{m}|_1 + p)}{2\,|t|}\right) \\ &\le \frac{2^{p+1}(N+p-1)^{p-1} e\,|t|}{\pi(p-1)!} \exp\left(\frac{\pi(N+p)}{2\,|t|}\right).\end{aligned} \tag{B.4}$$

Let $|t| \le \pi/4$ and $q = 1$. By Theorem 1 in [22], we have, for all $m \in \mathbb{N}_0$,

$$\rho_m^{\cosh,t} \ge \left(\frac{2}{e\pi}\right)^2 \exp\left(-2\ln\left(\frac{7e^2}{4|t|}\right) m\right). \tag{B.5}$$

Let $q = 1$. The result is obtained from the above with (B.2) and

$$\begin{aligned}\sum_{|\boldsymbol{m}|_1 \le N} \frac{1}{\rho_{\boldsymbol{m}}^{\cosh,t}} &\le \left(\frac{e\pi}{2}\right)^{2p} \sum_{k \le N} \sum_{|\boldsymbol{m}|_1 = k} \exp\left(2\ln\left(\frac{7e^2}{4|t|}\right)|\boldsymbol{m}|_1\right) \\ &\le \left(\frac{e\pi}{2}\right)^{2p} \frac{(N+p-1)^{p-1}}{(p-1)!} \exp\left(2\ln\left(\frac{7e^2}{4|t|}\right) N\right) \frac{1}{1 - \left(\pi/(14e^2)\right)^2}.\end{aligned} \tag{B.6}$$

The results for $q = \infty$ are obtained using (B.4) and (B.6) with $p = 1$ and

$$\sum_{\boldsymbol{m} \in \mathbb{N}_0^p:\ |\boldsymbol{m}|_\infty \le N} \frac{1}{\rho_{\boldsymbol{m}}^{\cosh,t}} \le \prod_{j=1}^p \left(\sum_{\boldsymbol{m}_j = 0}^N \frac{1}{\rho_{\boldsymbol{m}_j}^{\cosh,t}}\right). \tag{B.7}$$

Consider case (N.2). Let $t \ne 0$. Because $7e/\pi \ge 1$ and by Lemma B.5, we have, for all $m \in \mathbb{N}_0$,

$$\rho_m^{W_{[-1,1]},t} \ge \frac{1}{2}\left(\frac{2\,|t|}{7e(m+1)} \bigwedge 1\right)^{2m}. \tag{B.8}$$

When $q = 1$, the result follows from the following sequence of inequalities

$$\begin{aligned}\sum_{|\boldsymbol{m}|_1 \le N} \frac{1}{\rho_{\boldsymbol{m}}^{W_{[-1,1]},t}} &\le 2^p \sum_{k \le N} \sum_{|\boldsymbol{m}|_1 = k} \prod_{j=1}^p \exp\left(2\boldsymbol{m}_j \ln\left(\frac{7e(\boldsymbol{m}_j + 1)}{2\,|t|} \bigvee 1\right)\right) \\ &\le \frac{2^p (N+p-1)^{p-1}(N+1)}{(p-1)!} \exp\left(2N\ln\left(\frac{7e(N+1)}{2\,|t|} \bigvee 1\right)\right).\end{aligned}$$

When $q = \infty$, we obtain the result using the above with $p = 1$ and (B.7).

Lemma B.3. Let $f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l, M)$. For all $\boldsymbol{m} \in \mathbb{N}_0^p$, $t \ne 0$, we have $c_{\boldsymbol{m}}(t) = \sigma_{\boldsymbol{m}}^{W,tx_0} b_{\boldsymbol{m}}(t)$.

Proof. Let $\boldsymbol{m} \in \mathbb{N}_0^p$ and $t \neq 0$. We have $c_{\boldsymbol{m}}(t) = \int_{[-1,1]^p} \int_{\mathbb{R}^p} e^{itx_0 \boldsymbol{b}^\top \boldsymbol{u}} \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b}) \overline{g_{\boldsymbol{m}}^{W,tx_0}(\boldsymbol{u})} d\boldsymbol{u} d\boldsymbol{b}$, hence, when $W = \cosh(\cdot/R)$,

$$\begin{aligned} c_{\boldsymbol{m}}(t) &= \int_{\mathbb{R}^p} \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b}) \overline{\left(\left(W^{\otimes p}\right)^{-1}(\boldsymbol{b}) \int_{[-1,1]^p} e^{-itx_0 \boldsymbol{b}^\top \boldsymbol{u}} g_{\boldsymbol{m}}^{W,tx_0}(\boldsymbol{u})\, d\boldsymbol{u} \right)} W^{\otimes p}(\boldsymbol{b}) d\boldsymbol{b} \\ &= \sigma_{\boldsymbol{m}}^{W,tx_0} \int_{\mathbb{R}^p} \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b}) \overline{\varphi_{\boldsymbol{m}}^{W,tx_0}}(\boldsymbol{b})\, W^{\otimes p}(\boldsymbol{b})\, d\boldsymbol{b} = \sigma_{\boldsymbol{m}}^{W,tx_0} b_{\boldsymbol{m}}(t), \end{aligned}$$

while, when $W = W_{[-R,R]}$, because $\mathbb{S}_{\boldsymbol{\beta}} \subseteq [-R,R]^p$,

$$c_{\boldsymbol{m}}(t) = \int_{\mathbb{R}^p} \mathbb{1}\left\{|\boldsymbol{b}|_\infty \leq R\right\} \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\boldsymbol{b}) \overline{\mathcal{F}_{tx_0}^*\left[g_{\boldsymbol{m}}^{W_{[-R,R]},tx_0}\right](\boldsymbol{b})} d\boldsymbol{b} = \sigma_{\boldsymbol{m}}^{W_{[-R,R]},tx_0} b_{\boldsymbol{m}}(t). \qquad \square$$

B.1.2. **Properties of the PSWF and eigenvalues.**

Lemma B.4. For all $c \neq 0$ and $m \in \mathbb{N}_0$, we have $|\mu_m^c| \leq \sqrt{2\pi} e^{3/2} \left(e\,|c| / (4(m+1/2))\right)^m / 3$.

Proof. By (69) in [46], 6.1.18 in [1], (7) in [27], (1.3) in [42], and $\sup_{x \geq 0} (x+1)^{1/2}(x+1/2)^x/(x+3/2)^{x+1} \leq 2/3$, we obtain, for all $c \neq 0$ and $m \in \mathbb{N}_0$,

$$\begin{aligned} |\mu_m^c| &\leq \frac{\sqrt{\pi}\,|c|^m\,(m!)^2}{(2m)!\Gamma(m+3/2)} \\ &\leq \frac{\pi\,|c|^m}{4^m \Gamma\left(m+3/2\right)} \frac{\Gamma(m+1)}{\Gamma\left(m+1/2\right)} \\ &\leq \frac{\pi\,|c|^m}{4^m \Gamma(m+3/2)} (m+1)^{1/2} \\ &\leq \frac{\sqrt{\pi e^3}(e\,|c|)^m (m+1)^{1/2}}{4^m \sqrt{2}(m+3/2)^{m+1}} \leq \frac{\sqrt{2\pi} e^{3/2}}{3} \left(\frac{e\,|c|}{4(m+1/2)}\right)^m. \qquad \square \end{aligned}$$

Lemma B.5. For all $c \neq 0$ and $m \in \mathbb{N}_0$, we have

$$\rho_m^{W_{[-1,1]},c} \geq \frac{1}{2}\left(\mathbb{1}\left\{m \leq \frac{2\,|c|}{\pi} - 1\right\} + \frac{2c}{7e(m+1)} \mathbb{1}\left\{m > \frac{2\,|c|}{\pi} - 1\right\} \right)^{2m}.$$

Proof. When $m \geq 2\,|c|/\pi - 1$, the result follows from the fact that, by Proposition 5.1 in [6] and the Turán-Nazarov inequality (see [43] page 240), $\rho_m^{W_{[-1,1]},c} \geq \left(2c/\left(7e(m+1)\right)\right)^{2m}/2$. For all $m \leq 2\,|c|/\pi - 1$, the result follows from Remark 5.2 in [6] and that, for all $m \in \mathbb{N}_0$, $c \in (0,\infty) \mapsto \rho_m^c$ is nondecreasing (by the arguments in the proof of Lemma 1 in [22]). $\square$

In the next proofs, we use $\Pi(c) := 3c^2 \exp\left(2c^2/\sqrt{3}\right)/16$, $H(c) := \sqrt{2\Pi(c)} \vee 2$, $r(c) := \left(1 + 4c^2/3^{3/2}\right)\left(1 + 2c^2 3^{3/2}\right)$, if $N \geq H(c)$ then $N \geq c$ because, for all $c \geq 2$, $N \geq c\sqrt{3\exp(8/\sqrt{3})/16} > c$ and else $N \geq H(c) \geq 2 > c$, and $f(x) := |x|/(1-x^2)$, $g(x) := |x|/(1-x)^2$, $h(x) := |x|/(1-|x|)$, $c_f := 4/3$, $c_g := 4$, $c_h := 2$,

$$\forall x \in [-1/2, 1/2],\ f(x) \leq c_f\,|x|,\ g(x) \leq c_g\,|x|,\ h(x) \leq c_h\,|x|; \tag{B.9}$$

$$2\sum_{k\equiv N[2],\ 0<k<N} 2k+1 = N(N-1). \tag{B.10}$$

(B.10) is obtained because for all N even the sum is $2\sum_{p=1}^{N/2-1} 4p+1$ and else $2\sum_{p=0}^{(N-1)/2-1} 4p+3$.

Lemma B.6. For all $c\neq 0$ and $m\geq 2$, we have $\left|\mu_m^c/\mu_{m-2}^c\right|\leq \Pi(c)/m^2$.

Proof. Let $c>0$ and $m\in\mathbb{N}_0$ (for $c<0$, we use $\mu_m^c=\overline{\mu_m^{-c}}$). By Theorem 8.1 in [44], we have

$$|\mu_m^c| = \frac{\sqrt{\pi}c^m(m!)^2}{(2m)!\Gamma(m+3/2)}e^{F_m(c)},\ F_m(c)=\int_0^c\left(\frac{2\left(\psi_m^t(1)\right)^2-1}{2t}-\frac{m}{t}\right)dt.$$

Moreover, by (65) in [9], for all $t>0$,

$$\left(\sqrt{m+\frac{1}{2}}-\frac{t^2}{\sqrt{3}\sqrt{m+1/2}}\right)^2\leq\left(\psi_m^t(1)\right)^2\leq\left(\sqrt{m+\frac{1}{2}}+\frac{t^2}{\sqrt{3}\sqrt{m+1/2}}\right)^2$$

which yields, if $m\geq 2$,

$$\begin{aligned}\left(\psi_m^t(1)\right)^2-\left(\psi_{m-2}^t(1)\right)^2 &\leq \left(\sqrt{m+\frac{1}{2}}+\frac{t^2}{\sqrt{3}\sqrt{m+1/2}}\right)^2-\left(\sqrt{(m-2)+\frac{1}{2}}-\frac{t^2}{\sqrt{3}\sqrt{(m-2)+1/2}}\right)^2\\ &= 2+\frac{4t^2}{\sqrt{3}}+\frac{t^4}{3}\left(\frac{1}{m+1/2}-\frac{1}{m-3/2}\right)\leq 2+\frac{4t^2}{\sqrt{3}}. \qquad \text{(B.11)}\end{aligned}$$

Using $\sup_{x\geq 2}x^3(x-1)/\left((x^2-1/4)(x-1/2)(x-3/2)\right)\leq 3$ and (B.11), for all $m\geq 2$,

$$\begin{aligned}\left|\frac{\mu_m^c}{\mu_{m-2}^c}\right| &= \frac{c^2}{16}\frac{m(m-1)}{(m^2-1/4)(m-1/2)(m-3/2)}\exp\left(F_m(c)-F_{m-2}(c)\right)\\ &\leq \frac{3c^2}{16m^2}\exp\left(\int_0^c\left(\frac{\left(\psi_m^t(1)\right)^2-\left(\psi_{m-2}^t(1)\right)^2}{t}-\frac{2}{t}\right)dt\right)\leq\frac{3c^2}{16m^2}\exp\left(\frac{2c^2}{\sqrt{3}}\right). \quad \square\end{aligned}$$

Lemma B.7. For all $c\neq 0$ and $k\in\mathbb{N}$, we have $\left(\psi_k^c(1)\right)^2\leq(k+1/2)\left(1+2c^2/3^{3/2}\right)^2$ and $\|\psi_k^c\|^2_{L^\infty([-1,1])}\leq(k+1/2)\left(1+4c^2/3^{3/2}\right)^2$. For all $c\neq 0$ and $k\geq c$, we have $\|\psi_k^c\|^2_{L^\infty([-1,1])}\leq k+1/2$. We also have $\|\psi_0^c\|^2_{L^\infty([-1,1])}\leq 2|c|/\pi$.

Proof. The first assertion follows from (65) in [9]. For the second, we use (66) in [9] in the first display, 22.14.7 and 22.2.10 in [1], hence $\|P_k\|_{L^\infty([-1,1])}\leq\sqrt{k+1/2}$, in the second inequality,

$$\begin{aligned}\|\psi_k^c\|_{L^\infty([-1,1])} &\leq \|P_k\|_{L^\infty([-1,1])}+\frac{c^2}{\sqrt{3(k+1/2)}}\left(1+\frac{\sqrt{3/2}}{\sqrt{k+1/2}}\right)\\ &\leq\sqrt{k+1/2}\left(1+\frac{c^2}{\sqrt{3}(k+1/2)}\left(1+\frac{\sqrt{3/2}}{\sqrt{k+1/2}}\right)\right)\leq\sqrt{k+1/2}\left(1+\frac{4c^2}{3^{3/2}}\right).\end{aligned}$$

The third uses (3.4) and (3.125) in [44]. We obtain the last by the proof of Proposition 1 in [35] which yields $\|\psi_0^c\|^2_{L^\infty([-1,1])}\leq 2/(\mu_0^c)^2$ and Lemma B.5. For all $c<0$, we use $\psi_m^{-c}=\psi_m^c$. $\square$

Lemma B.8. For all $c \neq 0$ and $N \geq H(c)$, we have

$$\left\| \frac{\partial \psi_N^c}{\partial c} \right\|_{L^\infty([-1,1])} \leq \frac{2c_f \left(C_1(c) + C_2(c)\right) C_3(c)\Pi(c)}{|c|} \sqrt{N},$$

$$C_1(c) := \frac{2H(c)+9}{(H(c)+2)^2},\ C_2(c) := \frac{2\,|c|}{\pi H(c)(H(c)-1)} + \frac{r(c)}{4},\ C_3(c) := \sqrt{1 + \frac{1}{2H(c)}}.$$

Proof. Take $c \neq 0$, $N \geq H(c)$, and $w \in [-1,1]$. Theorem 7.11 in [44] yields

$$\frac{\partial \psi_N^c}{\partial c}(w) = \frac{2\psi_N^c(1)}{|c|} \sum_{k \equiv N[2],\ k \neq N} \frac{\mu_N^c \mu_k^c}{\left(\mu_N^c\right)^2 - \left(\mu_k^c\right)^2} \psi_k^c(1)\psi_k^c(w). \tag{B.12}$$

Using $\mu_k^c/\mu_N^c \in \mathbb{R}$ if $k \equiv N[2]$ and Lemma B.7, we obtain

$$\left| \frac{\partial \psi_N^c}{\partial c}(w) \right| \leq \frac{\sqrt{4N+2}}{|c|} \mathcal{C}(f,N,c),$$

$$\mathcal{C}(f,N,c) := f\left(\frac{\mu_N^c}{\mu_0^c}\right) \frac{2|c|\mathbb{1}\{N \equiv 0[2]\}}{\pi} + \sum_{\substack{0<k<N \\ k\equiv N[2]}} f\left(\frac{\mu_N^c}{\mu_k^c}\right) r(c)\left(k+\frac{1}{2}\right) + \sum_{\substack{k>N \\ k \equiv N[2]}} f\left(\frac{\mu_k^c}{\mu_N^c}\right)\left(k+\frac{1}{2}\right).$$

Lemma B.6 yields, if $k \equiv N[2]$,

$$\left|\frac{\mu_N^c}{\mu_k^c}\right| \leq \left|\frac{\mu_N^c}{\mu_{N-2}^c}\right| \leq \frac{\Pi(c)}{N^2} \leq \frac{1}{2} \text{ if } k < N \text{ and } \left|\frac{\mu_k^c}{\mu_N^c}\right| \leq \left(\frac{\sqrt{\Pi(c)}}{N+2}\right)^{k-N} \leq \frac{1}{2} \text{ if } k > N. \tag{B.13}$$

Using (B.10), (B.9), (B.13), and $\sum_{k\in\mathbb{N}} k2^{-k} = 2$ in the third display, the result follows from

$$\begin{aligned}
\mathcal{C}(f,N,c) &\leq c_f \left(\left(\frac{2\,|c|}{\pi} + \frac{r(c)N(N-1)}{4} \right) \frac{\Pi(c)}{N^2} + \sum_{k\equiv N[2],\ k>N} \frac{k+1/2}{2^{(k-N)/2}} \left(\frac{\sqrt{2\Pi(c)}}{N+2}\right)^{k-N} \right) \\
&\leq c_f \Pi(c) \left(\frac{2\,|c|}{\pi H(c)(H(c)-1)} + \frac{r(c)}{4} + \frac{2}{(N+2)^2} \sum_{l\equiv 0[2],\ l\geq 2} \left(l+N+\frac{1}{2}\right)\frac{1}{2^{l/2}} \right) \\
&\leq c_f \Pi(c) \left(C_2(c) + \frac{2}{(N+2)^2}\left(N+\frac{9}{2}\right)\right) \leq c_f \Pi(c) \left(C_1(c)+C_2(c)\right). \qquad (\text{B.14})
\end{aligned}$$

$\square$

Lemma B.9. For all $c \neq 0$ and $N \geq H(c)$, we have

$$\left\| \frac{\partial^2 \psi_N^c}{\partial c^2} \right\|_{L^\infty([-1,1])} \leq \frac{\Pi(c)C_3(c)}{c^2} \left(C_4(c)N^{5/2} + C_5(c)N^{3/2} + C_6(c)\sqrt{N} + C_7(c) \right),$$

$$C_4(c) := c_g\left(C_2(c) - C_1(c)\right),\ C_7(c) := \frac{c_g}{(H(c)+2)^{1/2}}\left(85 + \frac{246}{H(c)+2}\right),$$

$$C_5(c) := 8\left(c_f\left(C_1(c)+C_2(c)\right)C_3(c)\right)^2 \Pi(c) + \left(c_g + 4c_f\right)C_2(c) + \left(8c_f - c_g\right)C_1(c) + 2c_g,$$

$$C_6(c) := 8c_h c_f (C_1(c)+C_2(c))^2 \Pi(c) + \left(C_1(c)+C_2(c)\right)\left(c^2 c_g + 4c_f\right) + 19c_g.$$

Proof. For all $c<0$, $\mu_m^c=\overline{\mu_m^{-c}}$ and $\psi_m^{-c}=\psi_m^c$, hence we only consider $c>0$. Using $c\in(0,\infty)\mapsto\psi_N^c(x)$ is analytic (see [21] page 320) and (7.99) in [44], we have by differentiating

$$\mu_N^c\psi_N^c(x)=\int_{-1}^1 e^{icxt}\psi_N^c(t)dt: \tag{B.15}$$

$$\mu_N^c\frac{\partial\psi_N^c}{\partial x}(x)=\int_{-1}^1 icte^{icxt}\psi_N^c(t)dt, \tag{B.16}$$

$$\mu_N^c\frac{\partial^2\psi_N^c}{\partial x^2}(x)=-\int_{-1}^1 (ct)^2e^{icxt}\psi_N^c(t)dt, \tag{B.17}$$

(B.18)
$$\left(\frac{\partial^2\mu_N^c}{\partial c^2}\psi_N^c+2\frac{\partial\mu_N^c}{\partial c}\frac{\partial\psi_N^c}{\partial c}+\mu_N^c\frac{\partial^2\psi_N^c}{\partial c^2}\right)(x)=\int_{-1}^1 e^{icxt}\left(\frac{\partial^2\psi_N^c}{\partial c^2}(t)+2ixt\frac{\partial\psi_N^c}{\partial c}(t)-(xt)^2\psi_N^c(t)\right)dt.$$

Multiplying (B.18) by $\psi_k^c(x)$, integrating, and using (B.15)-(B.17), we obtain, for all $k\neq N$,

$$\begin{aligned}
&2\frac{\partial\mu_N^c}{\partial c}\int_{-1}^1\frac{\partial\psi_N^c}{\partial c}(x)\psi_k^c(x)dx+\mu_N^c\int_{-1}^1\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx\\
&=\mu_k^c\int_{-1}^1\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx+2\frac{\mu_k^c}{c}\int_{-1}^1 x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx+\frac{\mu_k^c}{c^2}\int_{-1}^1 x^2\psi_N^c(x)\frac{\partial^2\psi_k^c}{\partial x^2}(x)dx.
\end{aligned}$$

Recombining and using that, for all $k\neq N$, $\mu_k^c\neq\mu_N^c$ (see (3.45) in [44]), we obtain

$$\begin{aligned}
&\int_{-1}^1\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx\\
&=\frac{1}{\mu_N^c-\mu_k^c}\left(2\frac{\mu_k^c}{c}\int_{-1}^1 x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx+\frac{\mu_k^c}{c^2}\int_{-1}^1 x^2\psi_N^c(x)\frac{\partial^2\psi_k^c}{\partial x^2}(x)dx-2\frac{\partial\mu_N^c}{\partial c}\int_{-1}^1\frac{\partial\psi_N^c}{\partial c}(x)\psi_k^c(x)dx\right).
\end{aligned}$$

This yields, for all $k\not\equiv N[2]$, using (B.12), (7.69)-(7.70), and Theorem 7.11 in [44], $\int_{-1}^1\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx=0$, while, for all $k\equiv N[2]$ and $k\neq N$, using (7.69)-(7.70), Theorem 7.11, (7.99) and the eigenvalues $(\chi_N^c)_{N\in\mathbb{N}_0}$ of the differential operator in (1.1) in [44],

$$\int_{-1}^1\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_k^c(x)dx=\frac{2}{c}\frac{\mu_k^c}{\mu_N^c-\mu_k^c}\int_{-1}^1 x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx+\Xi_{N,k},$$

$$\Xi_{N,k}:=\frac{\psi_N^c(1)\psi_k^c(1)}{c^2}\left(\frac{\mu_N^c\mu_k^c(\chi_k^c-\chi_N^c)}{\left(\mu_N^c-\mu_k^c\right)^2}-2\frac{\mu_N^c\mu_k^c}{\left(\mu_N^c\right)^2-\left(\mu_k^c\right)^2}\left(2+\frac{\mu_N^c\left(2\psi_N^c(1)^2-1\right)}{\mu_N^c-\mu_k^c}\right)\right).$$

Differentiating (7.114) in [44] in c yields $\int_{-1}^1\frac{\partial^2\psi_N^c}{\partial c^2}(x)\psi_N^c(x)dx=-\int_{-1}^1\left(\frac{\partial\psi_N^c}{\partial c}(x)\right)^2dx$. Also, by (B.13), for all $k\equiv N[2]$,

$$\frac{|\mu_N^c|}{|\mu_N^c-\mu_k^c|}\le 1\text{ if }k<N\text{ and else }\frac{|\mu_N^c|}{|\mu_N^c-\mu_k^c|}\le 2. \tag{B.19}$$

We obtain, using Lemma B.7 and $N \geq c$ for the first term,

$$\left\| \frac{\partial^2 \psi_N^c}{\partial c^2} \right\|_{L^\infty([-1,1])} \leq \sqrt{N + \frac{1}{2}} \int_{-1}^{1} \left(\frac{\partial \psi_N^c}{\partial c}(x) \right)^2 dx + \sum_{k \equiv N[2],\ k \neq N} |\Xi_{N,k}| \, \|\psi_k^c\|_{L^\infty([-1,1])}$$

$$+ \sum_{k \equiv N[2],\ k \neq N} \frac{2\,|\mu_k^c|}{c\,|\mu_N^c - \mu_k^c|} \left| \int_{-1}^{1} x \frac{\partial \psi_N^c}{\partial c}(x) \frac{\partial \psi_k^c}{\partial x}(x) dx \right| \|\psi_k^c\|_{L^\infty([-1,1])}. \tag{B.20}$$

For the first term on the right-hand side of (B.20), using Lemma B.8, we obtain

$$\sqrt{N + \frac{1}{2}} \int_{-1}^{1} \left(\frac{\partial \psi_N^c}{\partial c}(x) \right)^2 dx \leq 8 \left(c_f \left(C_1(c) + C_2(c) \right) C_3(c) \right)^2 C_3(c) \left(\frac{\Pi(c)}{c} \right)^2 N^{3/2}.$$

For the second term in (B.20), using that for all $k \equiv N[2]$, $\mu_N^c / \mu_k^c \in \mathbb{R}$ and (B.13) we obtain

$$|\Xi_{N,k}| \leq \frac{|\psi_N^c(1)|\,|\psi_k^c(1)|}{c^2} \left(g\left(\rho_k\right) \left(\chi_k^c - \chi_N^c \right) + 2 \left(2 + \frac{\left|2\psi_N^c(1)^2 - 1\right| |\mu_N^c|}{|\mu_N^c - \mu_k^c|} \right) f\left(\rho_k\right) \right),$$

where $\rho_k = \mu_N^c / \mu_k^c$ when $k < N$ and $\rho_k = \mu_k^c / \mu_N^c$ when $k > N$. Using $N \geq c$, (B.19), $|\chi_N^c - \chi_k^c| \leq |N - k|\,(k + N + 1) + c^2$ (see (13) in [8]), (B.9), and $\left|2\psi_N^c(1)^2 - 1\right| \leq 2N$ (by Lemma B.7) for the first inequality, $(N-k)(k+N+1) \leq N(N+1)$ for all $0 < k < N$, (B.13), and (B.10) for the second, $(k-N)(k+N+1) = k(k+1) - N^2 - N$ for the third, the computations in (B.14), $\sum_{k=1}^{\infty} k^2 2^{-k} = 6$ and $\sum_{k=1}^{\infty} k^3 2^{-k} = 26$, and Euclidean division for the fourth, yield

$$\sum_{k \equiv N[2],\ k \neq N} |\Xi_{N,k}| \, \|\psi_k^c\|_{L^\infty([-1,1])} \leq \frac{c_g \sqrt{4N+2}\mathbb{1}\{N \equiv 0[2]\}}{|c|\,\pi} \left| \frac{\mu_N^c}{\mu_0^c} \right| \left(N(N+1) + c^2 + \frac{4c_f}{c_g}(N+1) \right)$$

$$+ \frac{c_g\sqrt{4N+2}}{2c^2} \sum_{k \equiv N[2],\ 0<k<N} \left(k + \frac{1}{2} \right) r(c) \left| \frac{\mu_N^c}{\mu_k^c} \right| \left((N-k)(k+N+1) + c^2 + \frac{4c_f}{c_g}(N+1) \right)$$

$$+ \frac{c_g\sqrt{4N+2}}{2c^2} \sum_{k \equiv N[2],\ k>N} \left(k + \frac{1}{2} \right) \left| \frac{\mu_k^c}{\mu_N^c} \right| \left(|N-k|\,(k+N+1) + c^2 + \frac{4c_f}{c_g}(2N+1) \right)$$

$$\leq \frac{c_g\sqrt{4N+2}}{2c^2} \left(N(N+1) + c^2 + \frac{4c_f}{c_g}(N+1) \right) \left(\frac{2\,|c|}{\pi} + \frac{r(c)N(N-1)}{4} \right) \frac{\Pi(c)}{N^2}$$

$$+ \frac{c_g\sqrt{4N+2}}{2c^2} \sum_{k \equiv N[2],\ k>N} \frac{k+1/2}{2^{(k-N)/2}} \left(\frac{\sqrt{2\Pi(c)}}{N+2} \right)^{k-N} \left((k-N)(k+N+1) + c^2 + \frac{4c_f}{c_g}(2N+1) \right)$$

$$\leq \frac{c_g\sqrt{4N+2}\Pi(c)}{2c^2} \left(N(N+1) + c^2 + \frac{4c_f}{c_g}(N+1) \right) \left(\frac{2\,|c|}{\pi H(c)(H(c)-1)} + \frac{r(c)}{4} \right)$$

$$+ \frac{c_g\sqrt{4N+2}}{2c^2} \frac{2\Pi(c)}{(N+2)^2} \sum_{l \equiv 0[2],\ l \geq 2} \frac{l+N+1/2}{2^{l/2}} \left(c^2 + \frac{4c_f}{c_g}(2N+1) - N - N^2 \right)$$

$$+\frac{c_g\sqrt{4N+2}}{2c^2}\frac{2\Pi(c)}{(N+2)^2}\sum_{l\equiv 0[2],\ l\geq 2}\left(l+N+\frac{1}{2}\right)(l+N)(l+N+1)\frac{1}{2^{l/2}}$$

$$\leq \frac{c_g\sqrt{4N+2}\Pi(c)}{2c^2}\left[C_2(c)\left(N(N+1)+c^2+\frac{4c_f}{c_g}(N+1)\right)+C_1(c)\left(c^2+\frac{4c_f}{c_g}(2N+1)-N-N^2\right)\right.$$
$$\left.+2N+19+\frac{85}{N+2}+\frac{246}{(N+2)^2}\right]$$

$$\leq \frac{c_g\Pi(c)}{c^2}C_3(c)\left[N^{5/2}\left(C_2(c)-C_1(c)\right)+N^{3/2}\left(\left(1+\frac{4c_f}{c_g}\right)C_2(c)+\left(\frac{8c_f}{c_g}-1\right)C_1(c)+2\right)\right.$$
$$\left.+\sqrt{N}\left((C_1(c)+C_2(c))\left(c^2+\frac{4c_f}{c_g}\right)+19\right)+\frac{85}{(H(c)+2)^{1/2}}+\frac{246}{(H(c)+2)^{3/2}}\right].$$

For the third term in (B.20), using (B.12), the triangle inequality, and (7.74) in [44] for the first inequality and using $|\mu_m^c|/|\mu_m^c+\mu_k^c|\leq 1$ for the second, we obtain

$$\left|\int_{-1}^{1}x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx\right|\leq\frac{4\left|\psi_N^c(1)\right|\left|\psi_k^c(1)\right|}{|c|}\sum_{m\neq N,\ m\equiv N[2]}\frac{\left|\mu_N^c\right|\left|\mu_m^c\right|\left|\psi_m^c(1)\right|^2}{\left|\left(\mu_m^c\right)^2-\left(\mu_N^c\right)^2\right|}\frac{\left|\mu_m^c\right|}{\left|\mu_m^c+\mu_k^c\right|}$$
$$\leq\frac{4\left|\psi_N^c(1)\right|\left|\psi_k^c(1)\right|}{|c|}\mathcal{C}(f,N,c),$$

hence, using (B.14) for the first inequality and (B.9) and (B.14) replacing c_f by c_h for the third,

$$\sum_{k\equiv N[2],\ k\neq N}\frac{2\left|\mu_k^c\right|}{c\left|\mu_N^c-\mu_k^c\right|}\left|\int_{-1}^{1}x\frac{\partial\psi_N^c}{\partial c}(x)\frac{\partial\psi_k^c}{\partial x}(x)dx\right|\|\psi_k^c\|_{L^\infty([-1,1])}$$
$$\leq 4c_f\sqrt{4N+2}(C_1(c)+C_2(c))\frac{\Pi(c)}{c^2}\sum_{k\equiv N[2],\ k\neq N}\frac{\left|\mu_k^c\right|}{\left|\mu_N^c-\mu_k^c\right|}\left|\psi_k^c(1)\right|\|\psi_k^c\|_{L^\infty([-1,1])}$$
$$\leq 4c_f\sqrt{4N+2}(C_1(c)+C_2(c))\frac{\Pi(c)}{c^2}\mathcal{C}(h,N,c)$$
$$\leq 4c_hc_f\sqrt{4N+2}(C_1(c)+C_2(c))^2\frac{\Pi(c)^2}{c^2}\leq 8c_hc_fC_3(c)(C_1(c)+C_2(c))^2\frac{\Pi(c)^2}{c^2}\sqrt{N}.\quad\square$$

Lemma B.10. For all $N\geq H(Rc(U))$, $t\in\mathbb{R}$, and $U>0$, $H_1(U):=\lceil H(Rc(U))\rceil$, ϕ from (A.7), $\widetilde{\boldsymbol{N}}(1):=(N,\underline{\boldsymbol{H_1(U)}})\in\mathbb{N}^p$, $\widetilde{\boldsymbol{N}}(\infty):=\underline{\boldsymbol{N}}\in\mathbb{N}^p$, we have

$$\sup_{\boldsymbol{b}\in[-R,R]^p}\left|\frac{\partial^2}{\partial t^2}\left(\left(\frac{Rc(t)}{2\pi}\right)^{p/2}\phi(t)\psi_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}\left(\frac{\boldsymbol{b}}{R}\right)\right)\right|\leq\mathbb{1}\{U/2\leq|t|\leq U\}C_8(Rx_0,p,U)N^{k_q/2+2},$$

$$C_8(Rx_0,p,U):=\left(\frac{URx_0}{\pi}\right)^{p/2}C_3(c(U))^p\left((H_1(U))^{(p-1)/2}\mathbb{1}\{q=1\}+\mathbb{1}\{q=\infty\}\right)R_0(Rx_0,p,U),$$

$$
\begin{aligned}
R_0(Rx_0,p,U) &:= \left(\frac{p|p-2|}{U^2}+C_9(U)\frac{2p}{U}+C_{10}(U)\right)\frac{1}{H_1(U)^2}+\left(\frac{2p}{U}+2C_9(U)\right)\frac{pC_{16}(U)}{H_1(U)^2}\\
&\quad+\frac{p(p-1)C_{16}(U)}{H_1(U)^2}+pC_{11}(U),\\
C_9(U) &:= \sup_{t\in[U/2,U]}\left|\phi'(t)\right|,\ C_{10}(U):=\sup_{t\in[U/2,U]}\left|\phi''(t)\right|,\\
C_{11}(U) &:= \frac{(Rx_0)^2\,\Pi(Rc(U))}{(Rc(U))^2}\left(C_4(Rc(U))+\frac{C_5(Rc(U))}{H_1(U)}+\frac{C_6(Rc(U))}{H_1(U)^2}+\frac{C_7(Rc(U))}{H_1(U)^{5/2}}\right),\\
C_{16}(U) &:= 2c_fRx_0\left(C_1(Rc(U))+C_2(Rc(U))\right)C_3(Rc(U))\frac{\Pi(Rc(U))}{Rc(U)}.
\end{aligned}
$$

Proof. Let $q=1$. By $\operatorname{supp}(\phi)\subseteq[-U,-U/2]\cup[U/2,U]$ and symmetry, we only consider $t\in[U/2,U]$ and $\boldsymbol{b}\in[-R,R]^p$. For such $(t,\boldsymbol{b})$ and $c>0$ we have

$$
\begin{aligned}
R(t,\boldsymbol{b}) &:= \left|\frac{\partial^2}{\partial t^2}\left(\left(\frac{Rc(t)}{2\pi}\right)^{p/2}\phi(t)\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right)\right)\right|\\
&\le \left(\frac{Rx_0}{2\pi}\right)^{p/2}t^{p/2}\Bigg[\left(\frac{p\,|p-2|}{4t^2}\phi(t)+\frac{p}{t}\left|\phi'(t)\right|+\left|\phi''(t)\right|\right)\left|\psi^{Rc(t)}_{\widetilde{\boldsymbol{N}}(q)}\left(\frac{\boldsymbol{b}}{R}\right)\right|\\
&\quad+Rx_0\left(\frac{p}{t}\phi(t)+2|\phi'(t)|\right)\left|\left.\frac{\partial\psi^c_{\widetilde{\boldsymbol{N}}(q)}}{\partial c}\right|_{c=Rc(t)}\left(\frac{\boldsymbol{b}}{R}\right)\right|+(Rx_0)^2\,\phi(t)\left|\left.\frac{\partial^2\psi^c_{\widetilde{\boldsymbol{N}}(q)}}{\partial c^2}\right|_{c=Rc(t)}\left(\frac{\boldsymbol{b}}{R}\right)\right|\Bigg],
\end{aligned}
$$

$$
\frac{\partial\psi^c_{\widetilde{\boldsymbol{N}}(q)}}{\partial c}\left(\frac{\boldsymbol{b}}{R}\right)=\sum_{j=2}^p\psi^c_N\left(\frac{\boldsymbol{b}_1}{R}\right)\frac{\partial\psi^c_{H_1(U)}}{\partial c}\left(\frac{\boldsymbol{b}_j}{R}\right)\prod_{\substack{l=2\\ l\neq j}}^p\psi^c_{H_1(U)}\left(\frac{\boldsymbol{b}_l}{R}\right)+\frac{\partial\psi^c_N}{\partial c}\left(\frac{\boldsymbol{b}_1}{R}\right)\prod_{l=2}^p\psi^c_{H_1(U)}\left(\frac{\boldsymbol{b}_l}{R}\right),
$$

$$
\begin{aligned}
\frac{\partial^2\psi^c_{\widetilde{\boldsymbol{N}}(q)}}{\partial c^2}\left(\frac{\boldsymbol{b}}{R}\right) =&2\sum_{j=2}^p\frac{\partial\psi^c_{H_1(U)}}{\partial c}\left(\frac{\boldsymbol{b}_j}{R}\right)\frac{\partial\psi^c_N}{\partial c}\left(\frac{\boldsymbol{b}_1}{R}\right)\prod_{\substack{l=2\\ l\neq j}}^p\psi^c_{H_1(U)}\left(\frac{\boldsymbol{b}_l}{R}\right)\\
&+\sum_{k=2}^p\sum_{\substack{j=2\\ j\neq k}}^p\psi^c_N\left(\frac{\boldsymbol{b}_1}{R}\right)\frac{\partial\psi^c_{H_1(U)}}{\partial c}\left(\frac{\boldsymbol{b}_j}{R}\right)\frac{\partial\psi^c_{H_1(U)}}{\partial c}\left(\frac{\boldsymbol{b}_k}{R}\right)\prod_{\substack{l=2\\ l\neq j,l\neq k}}^p\psi^{c(t)}_{H_1(U)}\left(\frac{\boldsymbol{b}_l}{R}\right)\\
&+\frac{\partial^2\psi^c_N}{\partial c^2}\left(\frac{\boldsymbol{b}_1}{R}\right)\prod_{l=2}^p\psi^c_{H_1(U)}\left(\frac{\boldsymbol{b}_l}{R}\right)+\sum_{j=2}^p\psi^c_N\left(\frac{\boldsymbol{b}_l}{R}\right)\frac{\partial^2\psi^c_{H_1(U)}}{\partial c^2}\left(\frac{\boldsymbol{b}_j}{R}\right)\prod_{\substack{l=2\\ l\neq j}}^p\psi^c_{H_1(U)}\left(\frac{\boldsymbol{b}_l}{R}\right),
\end{aligned}
$$

using that, by the discussion before Lemma B.6, $N\ge Rc(U)$, the third assertion of Lemma B.7, and Lemma B.9, we obtain $R(t,\boldsymbol{b})\le(URx_0/\pi)^{p/2}\,C_3(c(U))^pN^{5/2}(H_1(U))^{(p-1)/2}R_0(Rx_0,p,U)$. The case $q=\infty$ is obtained by replacing $H_1(U)$ by N above. $\square$

Lemma B.11. For all $R, x_0 > 0$, $\sigma > k_q/2 + 2$, $q \in \{1, \infty\}$, $\tau \geq \left(3e^{\sigma+p/2-1/4}Rx_0/8\right) \vee (1/2)$, we have

$$\int_{\mathbb{R}} e^{-2\tau|t|} \left(\sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |\boldsymbol{m}|_q^{2\sigma} \left(c_{\boldsymbol{m}}^P(t)\right)^2 \right) dt \leq \frac{C_{12}(Rx_0,\sigma,p)}{\tau k_q^{2\sigma}},$$

$$C_{12}(Rx_0,\sigma,p) := \frac{2^{p-1}p\Gamma(2\sigma+p+1)}{2\sigma+p}\left(\frac{8}{3}\right)^{2\sigma+p} + \frac{\pi e^3 p 2^p\sqrt{3}}{9\tau}\frac{\Gamma(2\sigma+p+1/2)}{(2\sigma+p-1/2)^{2\sigma+p+1/2}}.$$

Proof. When $q = 1$, we use $|\boldsymbol{m}|_1 \leq p\,|\boldsymbol{m}|_\infty$. Let $q = \infty$, $R, x_0 > 0$, $\sigma > k_q/2 + 2$, $\tau \geq \left(3e^{\sigma+p/2-1/4}Rx_0/8\right) \vee (1/2)$. Because $P_0 = \mathbb{1}\{|\cdot|_\infty \leq 1\}/2^{p/2}$, for all $m \in \mathbb{N}_0$, $\left|\langle P_0, \psi_m^c\rangle_{L^2([-1,1])}\right| \leq 1$, and, for all $m > |c|$, $\left|\langle P_0, \psi_m^c\rangle_{L^2([-1,1])}\right| \leq |\mu_m^c|/\sqrt{2}$ (see Proposition 3 in [8]), we obtain, for all $t \neq 0$,

$$\sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |\boldsymbol{m}|_\infty^{2\sigma}\left(c_{\boldsymbol{m}}^P(t)\right)^2 \leq \sum_{|\boldsymbol{m}|_\infty \leq Rc(|t|)} |\boldsymbol{m}|_\infty^{2\sigma}\mathbb{1}\{Rc(|t|)\geq 1\} + \sum_{|\boldsymbol{m}|_\infty > Rc(|t|)} \frac{|\boldsymbol{m}|_\infty^{2\sigma}\left|\mu_{|\boldsymbol{m}|_\infty}^{Rc(t)}\right|^2}{2}. \tag{B.21}$$

Using (B.21), Lemma B.4 and $\sum_{|\boldsymbol{m}|_\infty=k} 1 \leq p(k+1)^{p-1}$ for the first inequality, $m+1 \leq 2m$ when $m \geq 1$ for the second, and $2m+1 \leq 3m$, $(Rc(t)+1)^{2\sigma+p} \leq (2Rc(t))^{2\sigma+p}$ when $m, Rc(t) \geq 1$, and (see (1.3) in [42])

$$\int_0^\infty e^{-2\tau t}t^{2m}dt = \frac{\Gamma(2m+1)}{(2\tau)^{2m+1}} \leq e^{-2m}e^{2m\ln(2m+1)}\frac{\sqrt{2m+1}}{(2\tau)^{2m+1}}$$

for the third, we have

$$\int_{\mathbb{R}} e^{-2\tau|t|}\left(\sum_{\boldsymbol{m}\in\mathbb{N}_0^p} |\boldsymbol{m}|_\infty^{2\sigma}\left(c_{\boldsymbol{m}}^P(t)\right)^2\right) dt$$

$$\leq \int_0^\infty 2pe^{-2\tau t}\left(\sum_{m\leq Rc(t)} (m+1)^{p-1}m^{2\sigma}\mathbb{1}\{Rc(t)\geq 1\} + \frac{\pi e^3}{9}\sum_{m>Rc(t)}(m+1)^{p-1}m^{2\sigma}\left(\frac{eRc(t)}{4m}\right)^{2m}\right)dt$$

$$\leq \int_0^\infty 2^p p e^{-2\tau t}\int_1^{Rc(t)+1} u^{2\sigma+p-1}du\mathbb{1}\{Rc(t)\geq 1\}dt + \frac{\pi e^3 p 2^p}{9}\sum_{m\geq 1} m^{2\sigma+p-1}\left(\frac{eRx_0}{4m}\right)^{2m}\int_0^\infty e^{-2\tau t}t^{2m}dt$$

$$\leq \frac{2^{2(\sigma+p)}p}{2\sigma+p}\int_{1/(Rx_0)}^\infty (Rc(t))^{2\sigma+p}e^{-2\tau t}dt + \frac{\pi e^3 p 2^p\sqrt{3}}{9\tau}\sum_{m\geq 1} m^{2\sigma+p-1/2}e^{2m\ln(3Rx_0/(8\tau))}$$

$$\leq \frac{2^{p-1}p}{2\sigma+p}\frac{\Gamma(2\sigma+p+1)(Rx_0)^{2\sigma+p}}{\tau^{2\sigma+p+1}} + \frac{\pi e^3 p 2^p\sqrt{3}}{9\tau}\sum_{m\geq 1}\frac{m^{2\sigma+p-1/2}}{e^{(2\sigma+p-1/2)m}}$$

$$\leq \frac{2^{p-1}p\Gamma(2\sigma+p+1)}{(2\sigma+p)\tau}\left(\frac{8}{3}\right)^{2\sigma+p} + \frac{\pi e^3 p 2^p\sqrt{3}}{9\tau}\int_0^\infty \frac{t^{2\sigma+p-1/2}}{e^{t(2\sigma+p-1/2)}}dt \leq \frac{C_{12}(Rx_0,\sigma,p)}{\tau k_q^{2\sigma}}. \quad \square$$

Lemma B.12. For all $N \geq H(Rc(U))$, $R, U > 0$, $q \in \{1, \infty\}$, $\widetilde{\boldsymbol{N}}(1) = (N, \underline{\boldsymbol{H_1(U)}}) \in \mathbb{N}^p$, $\widetilde{\boldsymbol{N}}(\infty) = \underline{\boldsymbol{N}} \in \mathbb{N}^p$, H and H_1 are defined in Lemma B.8, and H_N is defined in (A.6), we have

$$I_1 := \int_{[-1,1]^p} \int_{\mathbb{R}} \left|\partial_t \mathcal{F}\left[H_N\right](t, tx_0\boldsymbol{x})\right|^2 d\boldsymbol{x}dt \leq C_{17}(U) N^2 \rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(U)} \tag{B.22}$$

$$I_2 := \int_{[-1,1]^p} \int_{\mathbb{R}} \left|\mathcal{F}\left[H_N\right](t, tx_0\boldsymbol{x})\right|^2 d\boldsymbol{x}dt \leq R^p U \rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(U)}, \tag{B.23}$$

$$C_{15}(U) := \frac{25p^2}{8U}\left(1 + \frac{2(Rc(U))^2}{3^{3/2}}\right)^4 + \frac{UC_9(U)^2}{H_1(U)^2} + \frac{5pC_9(U)\ln(2)}{2H_1(U)}\left(1 + \frac{2(Rc(U))^2}{3^{3/2}}\right)^2,$$

$$C_{17}(U) := R^p\left(C_{15}(U) + \frac{2pUC_{16}(U)^2}{H_1(U)}\right).$$

Proof. Let $N \geq H(Rc(U))$. I_1 is bounded using that, for all $(t, \boldsymbol{x}) \in \mathbb{R} \times [-1,1]^p$,

$$\begin{aligned}
\mathcal{F}\left[H_N\right](t, tx_0\boldsymbol{x}) &= \left(\frac{c(|t|)}{2\pi}\right)^{p/2} \lambda(t) \mathcal{F}\left[\psi_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}\left(\frac{\cdot}{R}\right) \mathbb{1}\{|\cdot|_\infty \leq R\}\right](tx_0\boldsymbol{x}) \\
&= \left(\frac{c(|t|)}{2\pi}\right)^{p/2} \lambda(t) \mathcal{F}_{Rc(t)}\left[\psi_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}\right](\boldsymbol{x}) \\
&= R^{p/2} i^{|\widetilde{\boldsymbol{N}}(q)|_1} \lambda(t) \sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}} \psi_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}(\boldsymbol{x}) \quad \left(\text{because } \mu_m^{Rc(t)} = i^m \left(\frac{2\pi}{Rc(|t|)}\right)^{1/2} \sqrt{\rho_m^{Rc(t)}}\right),
\end{aligned} \tag{B.24}$$

which yields

$$\frac{I_1}{R^p} \leq \int_{\mathbb{R}} \int_{[-1,1]^p} \left(\left(\frac{\mathrm{d}\sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}}}{\mathrm{d}t} \lambda(t) + \lambda'(t) \sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}}\right) \psi_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}(\boldsymbol{x}) + \sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}} \lambda(t) \frac{\partial \psi_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}(\boldsymbol{x})}{\partial t}\right)^2 dt d\boldsymbol{x}.$$

Using (7.114) in [44], cross-products terms of the last inequality are zero hence we obtain

$$\begin{aligned}
\frac{I_1}{R^p} \leq& \int_{\mathbb{R}} \lambda(t)^2 \left(\frac{\mathrm{d}\sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}}}{\mathrm{d}t}\right)^2 dt + \int_{\mathbb{R}} \left(\left(\lambda'(t)\right)^2 \rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)} + 2\lambda(t)|\lambda'(t)| \sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}} \frac{\mathrm{d}\sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}}}{\mathrm{d}t}\right) dt \\
&+ \int_{\mathbb{R}} \left(\int_{[-1,1]^p} \left(\frac{\psi_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}(\boldsymbol{x})}{\partial t}\right)^2 d\boldsymbol{x}\right) \lambda(t)^2 \rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)} dt.
\end{aligned} \tag{B.25}$$

Then, using (7.100) in [44] for the second equality yields, for all $|t| \geq \epsilon$,

$$\frac{\mathrm{d}\sqrt{\rho_N^{Rc(t)}}}{\mathrm{d}t} = \frac{x_0 R}{2\sqrt{\rho_N^{Rc(t)}}} \left.\frac{\mathrm{d}\rho_N^c}{\mathrm{d}c}\right|_{c=Rc(t)} = \frac{\sqrt{\rho_N^{Rc(t)}}}{|t|} \left(\psi_N^{Rc(t)}(1)\right)^2,$$

hence, using the first and the last assertions of Lemma B.7, for all $U/2 \leq |t| \leq U$,

(B.26)
$$\frac{\mathrm{d}\sqrt{\rho_N^{Rc(t)}}}{\mathrm{d}t} \leq \frac{(N+1/2)\sqrt{\rho_N^{Rc(t)}}}{|t|}\left(1+\frac{2(Rc(U))^2}{3^{3/2}}\right)^2 \mathbb{1}\{N>0\} + \sqrt{\rho_0^{Rc(t)}}\left(\frac{2x_0R}{\pi}\right)\mathbb{1}\{N=0\}.$$

When $q=1$ and using $N \geq H_1(U) \geq 2$ and $(N+1/2) \leq 5N/4$ for all$N \geq 2$, we have

$$\frac{\mathrm{d}\sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}}}{\mathrm{d}t} = (p-1)\left(\sqrt{\rho_{H_1(U)}^{Rc(t)}}\right)^{p-2}\sqrt{\rho_N^{Rc(t)}}\left(\frac{\mathrm{d}\sqrt{\rho_{H_1(U)}^{Rc(t)}}}{\mathrm{d}t}\right) + \left(\sqrt{\rho_{H_1(U)}^{c(t)}}\right)^{p-1}\left(\frac{\mathrm{d}\sqrt{\rho_{H_1(U)}^{Rc(t)}}}{\mathrm{d}t}\right)$$

$$\leq \frac{5pN}{4\,|t|}\left(1+\frac{2(Rc(U))^2}{3^{3/2}}\right)^2\sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}}. \tag{B.27}$$

Because $N \geq H_1(U)$, (B.27) holds for $q=\infty$. Hence, using (B.26) and that for all $m \in \mathbb{N}_0$, $c \in (0,\infty) \mapsto \rho_m^{W_{[-1,1]},c}$ is nondecreasing (using an adaptation of Lemma 1 in [22]), we have

$$2\int_{U/2}^{U}\lambda(t)^2\left(\frac{\mathrm{d}\sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}}}{\mathrm{d}t}\right)^2 + \left(\lambda'(t)\right)^2\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)} + 2\left|\lambda'(t)\right|\sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}}\frac{\mathrm{d}\sqrt{\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}}}{\mathrm{d}t}dt$$

$$\leq \left(\frac{25p^2N^2}{8}\int_{U/2}^{U}\frac{dt}{t^2}\left(1+\frac{2(Rc(U))^2}{3^{3/2}}\right)^4 + UC_9(U)^2 + \frac{5pNC_9(U)}{2}\left(1+\frac{2(Rc(U))^2}{3^{3/2}}\right)^2\int_{U/2}^{U}\frac{dt}{t}\right)\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(U)}$$

$$\leq C_{15}(U)N^2\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(U)}.$$

Then, using (7.114) in [44] for the first inequality and $N \geq H_1(U)$ and Lemma B.8 for the second inequality, we have, for all $U/2 \leq |t| \leq U$,

$$\int_{[-1,1]^p}\left(\frac{\partial\psi_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}(\boldsymbol{z})}{\partial t}\right)^2 d\boldsymbol{z}$$

$$= (Rx_0)^2\left((p-1)\int_{[-1,1]}\left(\left.\frac{\partial\psi_{H_1(U)}^{c}(z)}{\partial c}\right|_{c=Rc(t)}\right)^2 dz + \int_{[-1,1]}\left(\left.\frac{\partial\psi_{N}^{c}(z)}{\partial c}\right|_{c=Rc(t)}\right)^2 dz\right)$$

$$\leq 2pC_{16}(U)^2N \quad (\text{using } N \geq H_1(U)).$$

Because $N \geq H_1(U)$, the same holds for $q=\infty$. This and (B.25), yield (B.22) for all $N \geq H_1(U)$.

By (B.24) and the fact that $c \in (0,\infty) \mapsto \rho_m^{W_{[-1,1]},c}$ is nondecreasing, (B.23) follows from

$$I_2 = \int_{\mathbb{R}}\int_{[-1,1]^p} R^p\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}\lambda(t)^2\left|\psi_{\widetilde{\boldsymbol{N}}(q)}^{Rc(t)}(\boldsymbol{z})\right|^2 dtd\boldsymbol{z} \leq \int_{\mathbb{R}} R^p\lambda(t)^2dt\rho_{\widetilde{\boldsymbol{N}}(q)}^{Rc(U)}. \qquad \square$$

Lemma B.13. Assume that $v(n_0,\mathcal{E}) \le n^{-(2+\zeta)}$ with $\zeta>0$. For the weights W of Section 4.4 and $q\in\{1,\infty\}$, there exists $C_{20}^{W,q}$ independent of n such that, for all $\epsilon\le|t|\le T_{\max}$,

$$\left(\frac{|t|\,x_0}{2\pi}\right)^p \nu_q^W\left(N_{\max,q},tx_0\right)v(n_0,\mathcal{E}) \le C_{20}^{W,q}\ln(n)^{p-1}\left(\frac{\mathbb{1}\{W=W_{[-R,R]}\}\ln(n)}{n^{1+\zeta-p\zeta_0}}+\frac{\mathbb{1}\{W=\cosh(\cdot/R)\}}{n^{1+\zeta-2p\zeta_0}}\right).$$

Proof. Let $\epsilon\le|t|\le T_{\max}\le n^{\zeta_0}$. Let $q=1$ and $W=W_{[-R,R]}$. Using $N_{\max,1}^{W_{[-R,R]}}(t)\le\ln(n)/2$, $\left((7e/(2\,|t|\,Rx_0))\,N_{\max,1}^W(t)\right)^{2pN_{\max,1}^W(t)}\le n$, (B.29), and the definition of $\nu_1^{W_{[-R,R]}}$, we have

$$\left(\frac{|t|\,x_0}{2\pi}\right)^p \nu_1^W\left(N_{\max,1}^W(t),tx_0\right) \le \left(\frac{x_0\,|t|}{2\pi}\right)^p\frac{2p^{p-1}(N_{\max,1}^W(t))^p}{(p-1)!}\left(\frac{7e(N_{\max,1}^W(t)+1)}{2\,|t|\,x_0}\bigvee 1\right)^{2N_{\max,1}^W(t)}$$

$$\le C_{20}^{W_{[-R,R]}}T_{\max}^p\ln(n)^pn\le C_{20}^{W_{[-R,R]}}\ln(n)^pn^{1+p\zeta_0}, \tag{B.28}$$

using $C_{20}^{W_{[-R,R]}}:=2p^{p-1}(x_0/2)^p/(p-1)!$, hence the result.
Let $q=1$ and $W=\cosh(\cdot/R)$. Using the definition of $N_{\max,1}^{\cosh(\cdot/R)}(t)$, we have

$$\nu_1^{\cosh(\cdot/R)}\left(N_{\max,1}^W(t),tx_0\right)\le\left(\frac{2^{2p}eR^p\ln(n)^{p-1}\,|t|^p}{\pi^p}+2\left(\frac{e\pi}{2}\right)^{2p}\right)n,$$

hence, using $C_{20}^{\cosh(\cdot/R)}:=(x_0/(2\pi))^p\left(p^{p-1}2^{2p}eR^p/\pi^p+2\left(e\pi/2\right)^{2p}\right)$ and $\ln(n)^{p-1}n^{p\zeta_0}\ge1$,

$$\left(\frac{|t|\,x_0}{2\pi}\right)^p\nu_1^W\left(N_{\max,q}^W(t),tx_0\right)\le C_{20}^{\cosh(\cdot/R)}\ln(n)^{p-1}T_{\max}^{2p}n\le C_{20}^{\cosh(\cdot/R)}\ln(n)^{p-1}n^{1+2p\zeta_0}.$$

Similar computations yield the results when $q=\infty$. □

Appendix B.2. Complements on the proofs of the main results

Lemma B.14. If $\widehat{f}_{\boldsymbol{X}|\mathcal{X}}$ satisfies (H1.4) then $Z_{n_0}=O_p\left(v(n_0,\mathcal{E})/\delta(n_0)\right)$.

Proof. For all n_0 sufficiently large so that $\sqrt{\delta(n_0)}c_{\boldsymbol{X}}\le1$, we have, for all $x\in\mathcal{X}$,

$$\begin{aligned}\left|\left(\widehat{f}^{\delta}_{\boldsymbol{X}|\mathcal{X}}-f_{\boldsymbol{X}|\mathcal{X}}\right)(x)\right| &\le\left|\left(\widehat{f}_{\boldsymbol{X}|\mathcal{X}}-f_{\boldsymbol{X}|\mathcal{X}}\right)(x)\right|\mathbb{1}\left\{\widehat{f}_{\boldsymbol{X}|\mathcal{X}}(x)\ge\sqrt{\delta(n_0)}\right\}\\ &\quad+\left|\sqrt{\delta(n_0)}-f_{\boldsymbol{X}|\mathcal{X}}(x)\right|\mathbb{1}\left\{\widehat{f}_{\boldsymbol{X}|\mathcal{X}}(x)-f_{\boldsymbol{X}|\mathcal{X}}(x)<\sqrt{\delta(n_0)}-f_{\boldsymbol{X}|\mathcal{X}}(x)\right\}\\ &\le\left|\left(\widehat{f}_{\boldsymbol{X}|\mathcal{X}}-f_{\boldsymbol{X}|\mathcal{X}}\right)(x)\right|,\end{aligned}$$

hence $\delta(n_0)Z_{n_0}\le\sup_{f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}\left\|\widehat{f}_{\boldsymbol{X}|\mathcal{X}}-f_{\boldsymbol{X}|\mathcal{X}}\right\|^2_{L^\infty(\mathcal{X})}$. We conclude by (H1.4). □

We complete the proofs of theorems 1 and 3, considering (A.31) in all smoothness cases. We use $\theta:=7e/(2Rx_0)$, take $\mathcal{E}$, $\eta>0$, and work on $E\left(\mathcal{G}_{n_0},\mathcal{E}\right)$ defined in the proof of Theorem 5. In the next proofs we use that for all $k,l\ge0$, $N\ge1$, and for $f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)$),

$$(N+l)^k\le((l+1)N)^k,\ \int_{\epsilon\le|t|\le T}L^2(t)dt\le(2\pi)^{p+1}l^2\ \left(L^2(t)\text{ defined above (A.28)}\right). \tag{B.29}$$

Proof of (T1.2). Let $q=1$ and $\theta_0=\pi/(4Rx_0)$. By (B.3) and (B.5), we obtain the upper bound on $\left(\widetilde{\omega}_{N(t)}^{1,W,tx_0}\right)^2$ (see (A.25) and (A.28)),

$$\left(\widetilde{\omega}_{N(t)}^{1,W,tx_0}\right)^2 \le \left(\frac{e\pi}{2}\right)^{2p} \exp\left(2\ln\left(\frac{7e^2}{4Rx_0|t|}\right)N(t)\right)\mathbb{1}\{|t|\le\theta_0\} + 2^p\exp\left(\frac{\pi(N(t)+p)}{2Rx_0\,|t|}\right)\mathbb{1}\{|t|>\theta_0\}.$$

Use, for all $z>0$, $N\in\mathbb{N}$, and $t\neq 0$, if $|t|>\theta_0$

$$\begin{aligned}\Delta_1(t,N,n,z) :=& \frac{4c_{\boldsymbol{X}}(N+p-1)^{p-1}\,|t|^{p+1}\,eRx_0}{\pi^{p+1}n(p-1)!}\exp\left(\frac{\pi(N+p)}{2\,|t|\,Rx_0}\right)\\ &+\frac{2z}{\pi^{p+1}}\left(L^2(t)+\frac{c_{\boldsymbol{X}}(N+1)^p\,|t|^p}{n}\right)\exp\left(\frac{\pi(N+p)}{2\,|t|\,Rx_0}\right),\end{aligned} \tag{B.30}$$

and, if $|t|\le\theta_0$,

$$\begin{aligned}\Delta_1(t,N,n,z) :=& \frac{c_{\boldsymbol{X}}(N+p-1)^{p-1}\,|t|^p}{n(p-1)!}\frac{2e^{2p}\pi^{p-1}}{8^p}\exp\left(2\ln\left(\frac{7e^2}{4Rx_0|t|}\right)N\right)\\ &+z\left(L^2(t)+\frac{c_{\boldsymbol{X}}(N+1)^p\,|t|^p}{n}\right)\frac{2e^{2p}\pi^{p-1}}{8^p}\exp\left(2\ln\left(\frac{7e^2}{4Rx_0|t|}\right)N\right).\end{aligned}$$

We have, using Δ_1 to collect the bounds on $\mathbb{E}\left[\|R_1\|^2_{L^2_{1\otimes W^{\otimes p}}}\right]$ and $\mathbb{E}\left[\|R_2\|^2_{L^2_{1\otimes W^{\otimes p}}}\right]$ in (A.31),

(B.31)
$$\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\le C\left(\int_{\epsilon\le|t|\le T}\Delta_1(t,N(t),n,Z_{n_0})dt+\sup_{t\in\mathbb{R}}\frac{4l^2}{N(t)^{2\sigma}}+\frac{4l^2}{(1\vee T)^{2s}}+M^2\epsilon\right).$$

Step 1. We check that $\int_{\epsilon\le|t|\le T}\Delta_1(t,N(t),n,Z_{n_0})N(t)^{2\sigma}dt\le M_{2,\mathcal{E},\eta,1}$, where

$$\begin{aligned}M_{2,\mathcal{E},\eta,1} :=& \frac{4c_{\boldsymbol{X}}p^{p-1}}{\pi(p-1)!}\left(\frac{\tau_1'(e^2\pi)^p}{8^p}\frac{1\vee\theta_0}{p+1}+\frac{2\tau_1 eRx_0}{\pi^p}\right)+2^{p+2}\pi^p M_{1,\mathcal{E},\eta}\left(\frac{\tau_1'(e^2\pi)^p}{8^p}\bigvee\frac{\tau_1}{\pi^p}\right)l^2\\ &+\left(\frac{\tau_1'(e^2\pi)^p}{8^p}\bigvee\frac{\tau_1}{\pi^p}\right)\frac{4M_{1,\mathcal{E},\eta}c_{\boldsymbol{X}}}{\pi(p+1)}\left(\frac{p}{e}+2\right)^p,\end{aligned}$$

$\tau_1:=\exp(2(p+1))+1$, and $\tau_1':=(7e^2/(4Rx_0))^2(8\sigma/e)^{4\sigma}+1$, from which we deduce

(B.32)
$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}}\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\le C\left(\sup_{\epsilon\le|t|\le T}\frac{M_{2,\mathcal{E},\eta,1}+4l^2}{N(t)^{2\sigma}}+\frac{4l^2}{(1\vee T)^{2s}}+M^2\epsilon\right).$$

Step 1.1. Let $|t|>\theta_0$. We have, using (B.29) with $k=p-1$,

$$\Delta_1(t,N(t),n,Z_{n_0})N(t)^{2\sigma}\le\left(\frac{4c_{\boldsymbol{X}}n_e p^{p-1}eRx_0}{\pi^{p+1}(p-1)!n}\right)\frac{|t|^{p+1}\,N(t)^{2\sigma+p-1}}{n_e}\exp\left(\frac{\pi(N(t)+p)}{2\,|t|\,Rx_0}\right)$$

$$+\left(\frac{2M_{1,\mathcal{E},\eta}v(n_0,\mathcal{E})n_e}{\pi^{p+1}}\right)L^2(t)\frac{N(t)^{2\sigma}}{n_e}\exp\left(\frac{\pi(N(t)+p)}{2\,|t|\,Rx_0}\right)$$

$$+\left(\frac{2c_{\boldsymbol{X}}(N(t)+1)^pM_{1,\mathcal{E},\eta}n_ev(n_0,\mathcal{E})}{n\pi^{p+1}}\right)\frac{|t|^p\,N(t)^{2\sigma}}{n_e}\exp\left(\frac{\pi(N(t)+p)}{2\,|t|\,Rx_0}\right). \tag{B.33}$$

Using that, when $N(t)>1$, $N(t)+p\le\overline{N}(t)(p+2)$ and (B.29) with $k=2\sigma+p-1$ for the first display, using for the second display that, for all $t>0$, $\ln(t)\le t-1$, and using that $n_e\ge1$ and the definition of $\overline{N}$ in the last inequality, we obtain, for all $|t|\ge\epsilon\vee\theta_0$,

$$\begin{aligned}&\frac{N(t)^{2\sigma+p-1}}{n_e}\exp\left(\frac{\pi(N(t)+p)}{2\,|t|\,Rx_0}\right)\\ &\le\frac{\mathbb{1}\{N(t)=1\}}{n_e}\exp\left(\frac{\pi(p+1)}{2\,|t|\,Rx_0}\right)+\frac{\mathbb{1}\{N(t)>1\}2^{2\sigma+p-1}}{n_e}\exp\left((2\sigma+p-1)\ln\left(\overline{N}(t)\right)+\frac{\pi(p+2)}{2\,|t|\,Rx_0}\overline{N}(t)\right)\\ &\le\frac{\mathbb{1}\{N(t)=1\}}{n_e}\exp\left(2(p+1)\right)+\frac{\mathbb{1}\{N(t)>1\}}{n_e}\exp\left(\left(2\sigma+p-1+\frac{\pi(p+2)}{2\,|t|\,Rx_0}\right)\overline{N}(t)\right)\le\frac{\tau_1}{n_e^{1/2}}.\end{aligned}$$

For the remaining terms we use

$$\frac{N(t)^{2\sigma}}{n_e}\exp\left(\frac{\pi(N(t)+p)}{2\,|t|\,Rx_0}\right)\le\frac{N(t)^{2\sigma+p-1}}{n_e}\exp\left(\frac{\pi(N(t)+p)}{2\,|t|\,Rx_0}\right)$$

which holds when $N(t)=0$ and $N(t)\ge1$.
Then, using that $n_e/n\le1$, $n_ev(n_0,\mathcal{E})/\delta(n_0)\le1$, we obtain

(B.34)

$$\begin{aligned}&\int_{\epsilon\vee\theta_0|t|\le T}\Delta_1(t,N(t),n,Z_{n_0})N(t)^{2\sigma}dt\\ &\le\frac{\tau_18c_{\boldsymbol{X}}p^{p-1}eRx_0T^{p+2}}{\pi^{p+1}(p-1)!(p+2)n_e^{1/2}}+\int_{\epsilon\vee\theta_0\le|t|\le T}\frac{2\tau_1M_{1,\mathcal{E},\eta}L^2(t)}{\pi^{p+1}}+\frac{\tau_14c_{\boldsymbol{X}}M_{1,\mathcal{E},\eta}}{\pi^{p+1}n_e^{1/2}}\frac{|t|^p\,(1+N(t))^p}{n}dt.\end{aligned}$$

Step 1.2. Let $|t|\le\theta_0$. We have, using (B.29) with $k=p-1$

$$\begin{aligned}&\Delta_1(t,N(t),n,Z_{n_0})N(t)^{2\sigma}\\ &\le\frac{2c_{\boldsymbol{X}}p^{p-1}n_e\,|t|^p\,e^{2p}\pi^p}{n(p-1)!8^p\pi}\frac{N(t)^{2\sigma+p+1}}{n_e}\exp\left(2\ln\left(\frac{7e^2}{4Rx_0|t|}\right)N(t)\right)\\ &+\frac{2M_{1,\mathcal{E},\eta}v(n_0,\mathcal{E})n_eL^2(t)(e^2\pi)^p}{8^p\pi}\frac{N(t)^{2\sigma}}{n_e}\exp\left(2\ln\left(\frac{7e^2}{4Rx_0|t|}\right)N(t)\right)\\ &+\frac{2c_{\boldsymbol{X}}M_{1,\mathcal{E},\eta}n_ev(n_0,\mathcal{E})(N(t)+1)^p(|t|\,e^2\pi)^p}{8^p\pi n}\frac{N(t)^{2\sigma}}{n_e}\exp\left(2\ln\left(\frac{7e^2}{4Rx_0|t|}\right)N(t)\right).\end{aligned}$$

Using (B.29) when $N(t)>1$ with $k=2\sigma+p-1$ and for all $t>0$, $\ln(t)\le t-1$ for the first display, and $n_e\ge1$ and the definition of $\overline{N}$ for the second, we have, for all $\epsilon\le|t|\le\epsilon\vee\theta_0$,

$$\frac{N(t)^{2\sigma+p-1}}{n_e}\exp\left(2\ln\left(\frac{7e^2}{4Rx_0|t|}\right)N(t)\right)$$

$$\leq \frac{\mathbb{1}\{N(t)=1\}}{n_e}\left(\frac{7e^2}{4Rx_0\epsilon}\right)^2+\frac{\mathbb{1}\{N(t)>1\}}{n_e}\exp\left(\left(2\sigma+p-1+2\ln\left(\frac{7e^2}{4Rx_0|t|}\right)\right)\overline{N}(t)\right)$$
$$\leq \frac{\mathbb{1}\{N(t)=1\}(7e^2)^2\ln(n_e)^{4\sigma}}{(4Rx_0)^2 n_e}+\frac{\mathbb{1}\{N(t)>1\}}{n_e^{1/2}}\leq\frac{\tau_1'}{n_e^{1/2}}\quad(\text{by (A.1)}).$$

For the remaining terms, we use

$$\frac{N(t)^{2\sigma}}{n_e}\exp\left(2\ln\left(\frac{7e^2}{4Rx_0|t|}\right)N(t)\right)\leq\frac{N(t)^{2\sigma+p-1}}{n_e}\exp\left(2\ln\left(\frac{7e^2}{4Rx_0|t|}\right)N(t)\right)$$

which holds when $N(t)=0$ and $N(t)\geq 1$. Using $n_e/n\leq 1$, $n_e v(n_0,\mathcal{E})/\delta(n_0)\leq 1$, we get

$$\int_{\epsilon\leq|t|\leq\epsilon\vee\theta_0}\Delta_1(t,N(t),n,Z_{n_0})N(t)^{2\sigma}dt\leq\int_{\epsilon\leq t\leq\epsilon\vee\theta_0}\frac{4\tau_1'c_{\boldsymbol{X}}p^{p-1}(e^2\pi)^{p+1}t^p}{(p-1)!8^p n_e^{1/2}}+\frac{2\tau_1'(e^2\pi)^pM_{1,\mathcal{E},\eta}L^2(t)}{8^p\pi}dt$$
$$+\frac{4\tau_1'M_{1,\mathcal{E},\eta}c_{\boldsymbol{X}}(e^2\pi)^p}{n_e^{1/2}8^p\pi}\int_{\epsilon\leq t\leq\epsilon\vee\theta_0}\frac{t^p(1+N(t))^p}{n}dt. \tag{B.35}$$

Step 1.3. Conclusion. We have

$$\int_\epsilon^T\frac{|t|^p\,(1+N(t))^p}{n}dt\leq\frac{T^{p+1}(\ln(n_e)+2)^p}{(p+1)n}\quad\left(\text{because } N(t)+1\leq\overline{N}(t)+2\leq\ln(n_e)+2\right)$$
$$\leq\frac{T^{p+1}}{p+1}\left(\frac{p}{e}+2\right)^p\ \ (\text{because } n\geq n_e \text{ and by (A.1)}), \tag{B.36}$$

hence, by (B.34) and (B.35), we obtain

$$\int_{\epsilon\leq|t|\leq T}\Delta_1(t,N(t),n,Z_{n_0})N(t)^{2\sigma}dt$$
$$\leq\frac{4c_{\boldsymbol{X}}p^{p-1}}{(p-1)!\pi}\left(\frac{\tau_1'(e^2\pi)^p}{8^p}\frac{1\vee\theta_0}{(p+1)n_e^{1/2}}+\frac{2\tau_1 eRx_0}{\pi^p}\frac{T^{p+2}}{n_e^{1/2}}\right)$$
$$+\frac{2M_{1,\mathcal{E},\eta}}{\pi}\left(\frac{\tau_1'(e^2\pi)^p}{8^p}\bigvee\frac{\tau_1}{\pi^p}\right)\int_{\epsilon\leq|t|\leq T}L^2(t)dt+\left(\frac{\tau_1'(e^2\pi)^p}{8^p}\bigvee\frac{\tau_1}{\pi^p}\right)\frac{4M_{1,\mathcal{E},\eta}c_{\boldsymbol{X}}T^{p+1}}{n_e^{1/2}(p+1)\pi}\left(\frac{p}{e}+2\right)^p.$$

Then, using (B.29), $\phi\geq 1$, and that $T^{p+2}=n_e^{1/2}$, we have the result of Step 1 hence (B.32).
Step 2. Let $n_e\geq n_1$, where $n_1:=\exp\left((2Rx_0/e^{2\sigma+p})^{1/2\sigma}\right)\vee e^e$. Using $\sup_{|t|\in[\epsilon,T]}N(t)^{-2\sigma}=N(\epsilon)^{-2\sigma}\leq\overline{N}(\epsilon)^{-2\sigma}$ and (B.32), we obtain

$$\sup_{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\leq\frac{C}{\overline{N}(\epsilon)^{2\sigma}}\left(M_{2,\mathcal{E},\eta,1}+4l^2+\frac{4l^2\overline{N}(\epsilon)^{2\sigma}}{(1\vee T)^{2s}}+M^2\epsilon\overline{N}(\epsilon)^{2\sigma}\right). \tag{B.37}$$

Because, $\epsilon = 1/\ln(n_e)^{2\sigma}$, we have $2\sigma + p - 1 + \ln\left(7e^2/(4Rx_0\epsilon)\right) \geq 1$ which yields

$$\epsilon\overline{N}(\epsilon)^{2\sigma} = (\ln(n_e))^{-2\sigma}\left(\frac{\ln(n_e)}{2\left(2\sigma+p-1+\ln\left(\ln(n_e)^{2\sigma}7e^2/(4Rx_0)\right)\right)}\right)^{2\sigma} \leq 1.$$

We also obtain, for all $n_e \geq n_1$,

$$\begin{aligned}\overline{N}(\epsilon)^{-2\sigma} &= \left(\frac{\ln(n_e)}{2(2\sigma+p-1)+4\sigma\ln_2(n_e)+2\ln(7e^2/(4Rx_0))}\right)^{-2\sigma}\\ &\leq \left(\frac{\ln(n_e)}{\ln_2(n_e)}\right)^{-2\sigma}\left(1\vee\left(2\left(4\sigma+p-1+\ln\left(\frac{7e^2}{4Rx_0}\right)\right)\right)^{2\sigma}\right).\end{aligned}$$

Thus, using, the definition of $\overline{N}(\epsilon)$, $T^{p+2} = n_e^{1/2}$, and $n_e \geq n_1$, we have

$$\frac{\overline{N}(\epsilon)^{2\sigma}}{(1\vee T)^{2s}} \leq \frac{1}{2^{2\sigma}}\frac{\ln(n_e)^{2\sigma}}{n_e^{s/(p+2)}} \leq \left(\frac{\sigma(p+2)}{se}\right)^{2\sigma} \quad \text{(by (A.1))}.$$

Finally, by (B.37), we obtain

$$\left(\frac{\ln(n_e)}{\ln_2(n_e)}\right)^{2\sigma}\sup_{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \leq M_{3,\mathcal{E},\eta,1}, \tag{B.38}$$

where

$$M_{3,\mathcal{E},\eta,1} := C\left(1\vee\left(2\left(4\sigma+p-1+\ln\left(\frac{7e^2}{4Rx_0}\right)\right)\right)^{2\sigma}\right)\left(M_{2,\mathcal{E},\eta,1}+4l^2+4l^2\left(\frac{\sigma(p+2)}{se}\right)^{2\sigma}+M^2\right).$$

Let $q = \infty$. Similarly, using that

$$\begin{aligned}\left(\widetilde{\omega}^{\infty,W,tx_0}_N\right)^2 &\leq \left(\frac{e\pi}{2}\right)^{2p}\exp\left(2p\ln\left(\frac{7e^2}{4Rx_0|t|}\right)N\right)\mathbb{1}\left\{|t|\leq\frac{\pi}{4Rx_0}\right\}\\ &\quad + 2^p\exp\left(\frac{\pi p(N+1)}{2Rx_0\,|t|}\right)\mathbb{1}\left\{|t|>\frac{\pi}{4Rx_0}\right\},\end{aligned}$$

we obtain (B.38) in the case $q = \infty$ with

$$M_{3,\mathcal{E},\eta,\infty} := C\left(1\vee\left(2\left(4\sigma+1+p\ln\left(\frac{7e^2}{4Rx_0}\right)\right)\right)\right)^{2\sigma}\left(M_{2,\mathcal{E},\eta,\infty}+4l^2+4l^2\left(\frac{2p\sigma}{se}\right)^{2\sigma}+M^2\right).$$

Proof of (T1.1). Let $q = 1$. Let $t \neq 0$, $z > 0$, and $N \in \mathbb{N}$. Using (B.8), we have

$$\left(\widetilde{\omega}^{1,W,tx_0}_N\right)^2 \leq 2^p\left(1\bigvee\frac{7e(N+1)}{2Rx_0\,|t|}\right)^{2N}. \tag{B.39}$$

and, using (A.28), we have

(B.40)

$$\mathbb{E}\left[\|R_2\|^2_{L^2(1\otimes W^{\otimes p})}\right] \leq \frac{Z_{n_0}}{\pi^p}\int_{\epsilon\leq|t|\leq T}\left(L(t)^2+\frac{c_{\boldsymbol{X}}(N+1)^p\,|t|^p}{n}\right)\left(1\bigvee\frac{7e(N+1)}{2Rx_0|t|}\right)^{2N}dt.$$

We collect the upper bounds on $\mathbb{E}\left[\|R_1\|_{L^2(1\otimes W^{\otimes p})}\right]$ and $\mathbb{E}\left[\|R_2\|_{L^2(1\otimes W^{\otimes p})}\right]$ and use

$$\Delta_{2,q}(t,N,n,z) := \frac{2\Delta_{3,q}(t,N)}{\pi^p}\left(\frac{p^{p-1}c_{\boldsymbol{X}}N^p|t|^p}{2^{p-1}(p-1)!n} + z\left(L(t)^2 + \frac{c_{\boldsymbol{X}}(N+1)^p|t|^p}{n}\right)\right), \tag{B.41}$$

$$\Delta_{3,q}(t,N) := \frac{2}{\pi}\left(1\bigvee\frac{7e(N+1)}{2Rx_0|t|}\right)^{2N}. \tag{B.42}$$

By Lemma B.2, (A.31), and (B.29) with $k=p-1$ we have, for all $N\in\mathbb{N}_0$,

(B.43)

$$\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \le C\left(\int_{\epsilon\le|t|\le T}\Delta_{2,q}(t,N(t),n,Z_{n_0})dt + \sup_{t\in\mathbb{R}}\frac{4l^2}{N(t)^{2\sigma}} + \frac{4l^2}{(1\vee T)^{2s}} + M^2\epsilon\right).$$

Step 1. We check, for all $n_e\ge e^e$, $\int_{\epsilon\le|t|\le T}\Delta_{2,q}(t,N(t),n,Z_{n_0})N(t)^{2\sigma}dt \le M_{4,\mathcal{E},\eta}$, where

$$M_{4,\mathcal{E},\eta,1} := \frac{4\tau_2}{\pi^{p+1}(Rx_0)^p}\left(\frac{c_{\boldsymbol{X}}p^p}{2^{p-1}(p+1)!} + \frac{M_{1,\mathcal{E},\eta}c_{\boldsymbol{X}}}{p+1}\left(\frac{p}{e}+1\right)^p\right) + 2^{p+2}\tau_2 M_{1,\mathcal{E},\eta}l^2$$

and $\tau_2 := 1+(8\sigma/e)^{4\sigma}(1\vee(7e/(2Rx_0))^2)$, from which we deduce

$$\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \le C\left(\sup_{t\in\mathbb{R}}N(t)^{-2\sigma}\left(M_{4,\mathcal{E},\eta}+4l^2\right) + \frac{4l^2}{(1\vee T)^{2s}} + M^2\epsilon\right). \tag{B.44}$$

Using that $n_e/n\le 1$, $n_e v(n_0,\mathcal{E})/\delta(n_0)\le 1$, and $N(t)\le 2\overline{N}(t)$ we obtain, for all $t>0$,

$$\begin{aligned}
&\left(1\bigvee\frac{7e(N(t)+1)}{2Rx_0t}\right)^{2N(t)}N(t)^{2\sigma+p}\\
&\le \mathbb{1}\{N(t)=1\}\left(1\bigvee\frac{7e}{Rx_0\epsilon}\right)^2\\
&\quad + \mathbb{1}\{N(t)>1\}\exp\left(4\left(\overline{N}(t)\ln\left(1\bigvee\frac{7e(2\overline{N}(t)+1)}{2Rx_0t}\right) + \frac{2\sigma+p}{4}\ln\left(2\overline{N}(t)+1\right)\right)\right)\\
&\le \mathbb{1}\{N(t)=1\}\left(1\bigvee\frac{7e}{Rx_0}\right)^2 n_e^{1/2}\left(\frac{\ln(n_e)^{4\sigma}}{n_e^{1/2}}\right) + \mathbb{1}\{N(t)>1\}n_e^{1/2}\ \left(\text{by definition of }\overline{N}(t)\right),\\
&\le \tau_2 n_e^{1/2}\ (\text{by (A.1)}).
\end{aligned}$$

This yields, for all $t\ne 0$,

$$\Delta_{2,q}(t,N(t),n,Z_{n_0})N(t)^{2\sigma} \le \frac{2}{\pi^{p+1}}\frac{\tau_2}{n_e^{1/2}}\left(\frac{c_{\boldsymbol{X}}p^{p-1}|t|^p}{2^{p-1}(p-1)!} + M_{1,\mathcal{E},\eta}\left(L(t)^2 + \frac{c_{\boldsymbol{X}}(N(t)+1)^p|t|^p}{n}\right)\right).$$

Then, using $n_e/n\le 1$, $n_e v(n_0,\mathcal{E})/\delta(n_0)\le 1$, and (B.36) we obtain

$$\int_{\epsilon\le|t|\le T}\Delta_2(t,N(t),n,Z_{n_0})N(t)^{2\sigma}dt$$

$$\leq \frac{2\tau_2}{\pi^{p+1}n_e^{1/2}}\left(\frac{4c_{\boldsymbol{X}}p^pT^{p+1}}{2^p(p+1)!}+\frac{2M_{1,\mathcal{E},\eta}}{\pi}\int_{\epsilon\leq|t|\leq T}L(t)^2dt+\frac{4c_{\boldsymbol{X}}M_{1,\mathcal{E},\eta}T^{p+1}}{(p+1)\pi}\left(\frac{p}{e}+1\right)^p\right).$$

Using (B.29) and $T^{p+1}=n_e^{1/2}$ we obtain the result of Step 1 hence (B.44).

Step 2. Using $\sup_{|t|\in[\epsilon,T]}N(t)^{-2\sigma}=N(\epsilon)^{-2\sigma}\leq\overline{N}(\epsilon)^{-2\sigma}$ and (B.44), we obtain

$$\sup_{\substack{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D}\\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}}\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\leq C\overline{N}(\epsilon)^{-2\sigma}\left(M_{4,\mathcal{E},\eta}+4l^2+\frac{4l^2\overline{N}(\epsilon)^{2\sigma}}{(1\vee T)^{2s}}+M^2\epsilon\overline{N}(\epsilon)^{2\sigma}\right).$$

We first prove that

$$N(\epsilon)^{-2\sigma}\leq\left(\frac{1}{2}\left(\frac{1}{\tau_{4,1}}\bigwedge\frac{1}{\tau_{3,1}}\right)\frac{\ln(n_e)}{\ln_2(n_e)}\right)^{-2\sigma}, \tag{B.45}$$

$\tau_{3,q}:=4\sigma k_q\left(1+\left(\ln\left((7e/(2Rx_0))^{1/2\sigma}\right)\vee 1\right)\right)>0$, and $\tau_{4,q}:=\tau_{2,q}(1+\ln(1+\tau_{2,q}))$. Because $\epsilon=\left(\ln\left(n_e\right)/\ln_2\left(n_e\right)\right)^{-2\sigma}$,

$$\begin{aligned}
&\overline{N}(\epsilon)\ln\left(\frac{7e(2\overline{N}(\epsilon)+1)}{2Rx_0\epsilon}\right)+\frac{2\sigma+p}{4}\ln\left(2\overline{N}(\epsilon)+1\right)\\
&\leq\overline{N}(\epsilon)\ln\left(\frac{7e}{2Rx_0\epsilon}\right)+\left(\frac{2\sigma+p}{4}+\overline{N}(\epsilon)\right)\ln\left(2\overline{N}(\epsilon)+1\right)\\
&\leq 2\left(\left(2\sigma\overline{N}(\epsilon)\ln\left(\left(\frac{7e}{2Rx_0}\right)^{1/2\sigma}\frac{\ln\left(n_e\right)}{\ln_2\left(n_e\right)}\right)\right)\bigvee\left(1+\frac{2\sigma+p}{4}\right)(2\overline{N}(\epsilon)+1)\ln\left(2\overline{N}(\epsilon)+1\right)\right)
\end{aligned}$$

which yields $\overline{N}(\epsilon)\geq\widetilde{N}\wedge\widetilde{N}_1$ where $\widetilde{N}$ and $\widetilde{N}_1$ are defined using the equations

$$2\left(1+\frac{2\sigma+p}{4}\right)\left(2\widetilde{N}+1\right)\ln\left(2\widetilde{N}+1\right)=\frac{\ln(n_e)}{8k_q}=4\sigma\widetilde{N}_1\ln\left(\left(\frac{7e}{2Rx_0}\right)^{1/2\sigma}\frac{\ln\left(n_e\right)}{\ln_2\left(n_e\right)}\right).$$

Using $\tau_{2,q}:=4k_q(4+2\sigma+p)$, $\mathcal{W}$ the W-Lambert function (satisfying $e^{\mathcal{W}(z)}\mathcal{W}(z)=z$ for all $z>0$), and that $\mathcal{W}(x)\leq\ln(x+1)$ for all $x>0$ (see Theorem 2.3 in [34]), we have, for $n_e\geq e^e$,

$$2\widetilde{N}+1\geq\frac{\ln(n_e)}{\tau_{2,1}\mathcal{W}\left(\frac{\ln(n_e)}{\tau_{2,1}}\right)}\geq\frac{\ln(n_e)}{\tau_{2,1}\ln\left(\ln(n_e)+\tau_{2,1}\right)}\geq\frac{\ln(n_e)}{\tau_{2,1}\left(1+\ln(1+\tau_{2,1})\right)\ln_2(n_e)}. \tag{B.46}$$

Moreover, we have

$$\widetilde{N}_1\geq\frac{\ln(n_e)}{\ln_2\left(n_e\right)}\frac{1}{4\sigma k_q\left(1+\left(\ln\left((7e/(2Rx_0))^{1/2\sigma}/\ln_2\left(n_e\right)\right)\vee 1\right)/\ln_2\left(n_e\right)\right)}\geq\frac{\ln(n_e)}{\tau_{3,1}\ln_2\left(n_e\right)}.$$

Thus, we have

$$\left(1+\overline{N}(\epsilon)\right)^{-2\sigma}\leq\left(\frac{\ln(n_e)}{\tau_{4,1}\ln_2(n_e)}\bigwedge\frac{\ln(n_e)}{\tau_{3,1}\ln_2\left(n_e\right)}\right)^{-2\sigma}.$$

Then, using $\overline{N}(\epsilon) \le N(\epsilon)$, hence $2^{-2\sigma}(1 \vee N(\epsilon))^{-2\sigma} \le \left(1+\overline{N}(\epsilon)\right)^{-2\sigma}$, and $N(\epsilon) \ge 1$, we obtain (B.45). For $n_e \ge 1$, by (A.1), we have

$$\frac{N(\epsilon)^{2\sigma}}{(1 \vee T)^{2s}} \le \frac{\ln(n_e)^{2\sigma}}{n_e^{s/(p+1)}} \le \left(\frac{2\sigma(p+1)}{se}\right)^{2\sigma}.$$

Moreover, there exists $\tau_{5,1} > 0$ such that for all n_e large enough we have $\epsilon\overline{N}(\epsilon)^{2\sigma} \le 1/(4k_1\tau_{5,1})^{2\sigma}$. Indeed, using

$$\overline{N}(\epsilon)\ln\left(\frac{7e\left(2\overline{N}(\epsilon)+1\right)}{2Rx_0\epsilon}\right) + \frac{2\sigma+p}{4}\ln\left(2\overline{N}(\epsilon)+1\right) \ge \overline{N}(\epsilon)\ln\left(\frac{7e\overline{N}(\epsilon)}{Rx_0\epsilon}\right),$$

we have, similarly to (B.46) and using that, for all $x \ge e$, $\mathcal{W}(x) \ge \ln(x) - \ln_2(x)$ (see Corollary 2.4 in [34]), for all n_e such that $n_e \ge \exp(2ek_1(\ln(n_e)/\ln_2(n_e))^{-2\sigma}/\theta) \vee e^e$

$$\overline{N}(\epsilon) \le \frac{\ln(n_e)}{4k_1\mathcal{W}\left(\ln(n_e)\theta/(2k_1\epsilon)\right)} \le \frac{\ln(n_e)}{4k_1\ln_2(n_e)}\frac{1}{(1+\ln(\theta/(2k_1\epsilon))/\ln_2(n_e) - \ln_2(\ln(n_e)\theta/(2k_1\epsilon)/\ln_2(n_e))}.$$

Thus, we obtain

$$\left(\frac{\ln(n_e)}{\ln_2(n_e)}\right)^{2\sigma} \sup_{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}} \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)$$
$$\le C\left(\frac{1}{2\tau_{4,1}}\bigwedge\frac{1}{2\tau_{3,1}}\right)^{-2\sigma}\left(4l^2\left(1+\left(\frac{2\sigma}{se}\right)^{2\sigma}\right) + M_{4,\mathcal{E},\eta} + \frac{M^2}{(4k_1\tau_{5,1})^{2\sigma}}\right) \tag{B.47}$$

which yields the result. Similar computations yield the result when $q=\infty$ using $\left(\widetilde{\omega}^{\infty,W,tx_0}_N\right)^2 \le 2^p\left(1\bigvee 7e(N+1)/(2Rx_0|t|)\right)^{2pN}$. $\square$

Proof of (T3.1.1). Let $q=1$. Using (A.31), (B.41), and (B.42), we obtain

$$\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)$$
$$\le C\left(\int_{\epsilon\le|t|\le T}\Delta_{2,q}(t,N(t),n,Z_{n_0})dt + 4l^2\sup_{t\in\mathbb{R}} e^{-2\kappa(N(t)\ln(1+N(t)))^r} + \frac{4l^2}{(1\vee T)^{2s}} + \frac{M^2}{w(\underline{a})}\right). \tag{B.48}$$

By (B.41), $n_e/n \le 1$, $n_e v(n_0,\mathcal{E}))/\delta(n_0) \le 1$, and, for all $t \ne 0$,

$$\left(1\bigvee\frac{7e(1+N(t))}{2Rx_0|t|}\right)^{2N(t)} e^{2\kappa N(t)\ln(1+N(t))}$$
$$\le \exp\left(2\overline{N}(t)\ln\left(1\bigvee\frac{7e(1+\overline{N}(t))}{2Rx_0|t|}\right) + 2\kappa\overline{N}(t)\ln\left(1+\overline{N}(t)\right)\right) \le n_e \text{ (by definition of } \overline{N}(t)),$$

we have

$$\frac{\Delta_{2,q}(t,N(t),n,Z_{n_0})e^{2\kappa N(t)\ln(1+N(t))}}{(1+N(t))^p(((1+\overline{N}(T))\theta)\vee|t|)^{p+2}} \le \frac{2}{\pi(\theta\vee|t|)^{p+2}}\left(\frac{c_{\boldsymbol{X}}2p^{p-1}|t|^p}{(2\pi)^p(p-1)!} + \frac{M_{1,\mathcal{E},\eta}}{\pi^p}\left(L(t)^2 + \frac{c_{\boldsymbol{X}}|t|^p}{n}\right)\right).$$

Then, using $T \geq \theta \geq \epsilon$,

$$\int_{\epsilon}^{T} \frac{t^p}{(\theta \vee t)^{p+2}} dt \leq \int_{\epsilon}^{T} \frac{t^p}{(\theta \vee t)^{p+2}} dt \leq \frac{(p+2)}{\theta(p+1)} \text{ and } \int_{\epsilon \leq |t| \leq T} L^2(t) dt \leq (2\pi)^{p+1} l^2,$$

we obtain

$$\text{(B.49)} \qquad \int_{\epsilon \leq |t| \leq T} \frac{\Delta_{2,q}(t, N(t), n, Z_{n_0}) e^{2\kappa N(t) \ln(1+N(t))}}{(1+N(t))^p \left(\left(1+\overline{N}(T)\right)\theta\right) \vee |t|)^2} dt \leq M_{5,\mathcal{E},\eta},$$

$$M_{5,\mathcal{E},\eta} := \frac{4}{\theta\pi} \frac{c_{\boldsymbol{X}} p^p (p+2)}{(2\pi)^p (p+1)!} + 4M_{1,\mathcal{E},\eta} \left(2^p l^2 + \frac{c_{\boldsymbol{X}}(p+2)}{\pi^{p+1}\theta(p+1)}\right).$$

By (B.48) and (B.49), we obtain

$$\mathcal{R}_{n_0}^W \left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right)$$

(B.50)

$$\leq C \left(\sup_{|t| \in [\epsilon, T]} \frac{(1+N(t))^p \left(\left(\left(1+\overline{N}(T)\right)\theta\right) \vee |t|\right)^{p+2}}{e^{2\kappa(N(t)\ln(1+N(t)))}} \left(M_{5,\mathcal{E},\eta} + 4l^2\right) + \frac{4l^2}{(1 \vee T)^{2s}} + \frac{M^2}{w(\underline{a})}\right).$$

We now show that

$$\text{(B.51)} \qquad \sup_{|t| \in [\epsilon, T]} \frac{(1+N(t))^p \left(\left(\left(\overline{N}(T)+1\right)\theta\right) \vee |t|\right)^{p+2}}{4e^{2\kappa N(t)\ln(1+N(t))}} \leq \frac{e^{2\kappa}\theta^2 (1 \vee (\ln(n_e)/(2\kappa)))^{2\kappa+2p+2}}{4e^{2\kappa \overline{N}(\epsilon)\ln(\overline{N}(\epsilon)+1)}}.$$

then

(B.52)

$$e^{2\kappa \overline{N}(\epsilon) \ln\left(1+\overline{N}(\epsilon)\right)} \geq n_e^{\kappa/(\kappa+2k_q)} \left(1 \bigwedge e \exp\left(\frac{\ln\left(\ln_2(n_e)/(2(\kappa+2k_q)\ln(\ln(n_e)/\ln_2(n_e)))\right)}{\ln(\ln(n_e)/\ln_2(n_e))}\right)\right),$$

hence, by (B.50) and (B.51),

$$\frac{n_e^{\kappa/(\kappa+2k_q)}}{\ln(n_e)^{2\kappa+2p+2}} \sup_{\substack{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}_{w,W}^{q,\phi,\omega}(l,M) \cap \mathcal{D} \\ f_{\boldsymbol{X}}|_{\mathcal{X}} \in \mathcal{E}}} \mathcal{R}_{n_0}^W \left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,N,T,\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right)$$

$$\text{(B.53)} \quad \leq C \left(\frac{e^{2\kappa}\theta^2}{(2\kappa)^{2\kappa+2p+2}} \left(4l^2 + M_{6,\mathcal{E},\eta,q}\right) + \frac{e^{2\kappa \overline{N}(\epsilon) \ln(1+\overline{N}(\epsilon))}}{\ln(n_e)^{2\kappa+2p+2}} \left(\frac{4l^2}{(1 \vee T)^{2s}} + \frac{M^2}{e^{2\gamma \underline{a} \ln(1+\underline{a})}}\right)\right).$$

Proof of (B.51). Let $t \neq 0$, because $\overline{N}(t) - 1 \leq N(t) \leq \overline{N}(t)$, we obtain, for all $n_e \in \mathbb{N}$,

$$\begin{aligned}
\frac{\exp\left(2\kappa N(t) \ln(1+N(t))\right)}{(1+N(t))^p} &\geq \frac{\exp\left(2\kappa \overline{N}(t) \ln(1+N(t)) - 2\kappa \ln(1+\overline{N}(t))\right)}{(1+N(t))^p} \\
&\geq \frac{\exp\left(2\kappa \overline{N}(t) \ln\left(1+\overline{N}(t)\right)\right)}{(1+\overline{N}(t))^{2\kappa+p}} \exp\left(-2\kappa \overline{N}(t) \ln\left(1+\frac{1}{\overline{N}(t)}\right)\right) \\
&\geq \frac{\exp\left(2\kappa \overline{N}(t) \ln\left(1+\overline{N}(t)\right)\right)}{e^{2\kappa}(1+\overline{N}(t))^{2\kappa+p}} \text{ (because } \ln(1+x) \leq x)
\end{aligned}$$

$$\geq \frac{\exp\left(2\kappa\overline{N}(t)\ln\left(1+\overline{N}(t)\right)\right)}{\sup_{t\in[\epsilon,T]} e^{2\kappa}(1+\overline{N}(t))^{2\kappa+p}}.$$

Denoting by $g(t)=\left(\left(\left(1+\overline{N}(T)\right)\theta\right)\vee|t|\right)^{p+2}e^{-2\kappa(\overline{N}(t)\ln(1+\overline{N}(t)))}$, we have

$$\sup_{|t|\in[\epsilon,T]}\frac{(1+N(t))^p\left(\left(\left(1+\overline{N}(T)\right)\theta\right)\vee|t|\right)^{p+2}}{e^{2\kappa(N(t)\ln(1+N(t)))}}\leq\left(\sup_{t\in[\epsilon,T]}e^{2\kappa}(1+\overline{N}(t))^{2\kappa+p}\right)\sup_{|t|\in[\epsilon,T]}g(t)$$

$$=\left(\sup_{t\in[\epsilon,T]}e^{2\kappa}(1+\overline{N}(t))^{2\kappa+p}\right)g(\epsilon), \tag{B.54}$$

because we show below that $\|g\|_{L^\infty([\epsilon,T])}=g(\epsilon)$. Indeed, for all $x>0$, differentiating $Q_{t,u}^{-1}(Q_{t,u}(x))=x$ with respect to t yields

$$\frac{\partial Q_{t,u}^{-1}}{\partial t}(Q_{t,u}(x))+\frac{\partial Q_{t,u}^{-1}}{\partial x}(Q_{t,u}(x))\frac{\partial Q_{t,u}}{\partial t}(x)=0.$$

Hence, for all $t\neq 0$ such that $|t|<(1+\overline{N}(t))\theta$, we obtain

$$\overline{N}'(t)=\frac{\overline{N}(t)}{|t|\left((1+\kappa/k_q)\overline{N}(t)/(1+\overline{N}(t))+\ln((1+\overline{N}(t))^2\theta/|t|)\right)}>0, \tag{B.55}$$

while, for all $|t|\geq(1+\overline{N}(t))\theta$, $\overline{N}'(t)=0$. Thus, we have, for all $|t|\leq\left(1+\overline{N}(T)\right)\theta$,

$$g'(t)=-2\kappa\overline{N}'(t)\left(\ln(\overline{N}(t)+1)+\frac{\overline{N}(t)}{\overline{N}(t)+1}\right)\left(\left(1+\overline{N}(T)\right)\theta\right)^2e^{-2\kappa\overline{N}(t)\ln(\overline{N}(t)+1)},$$

which yields that g is decreasing on $|t|\leq(1+\overline{N}(T))\theta$, and increasing on $|t|\geq(1+\overline{N}(T))\theta$, $\overline{N}(\cdot)$ being independent of $|t|$. This yields, because g is positive,

$$\|g\|_{L^\infty([\epsilon,T])}=g(\epsilon)\vee g(T). \tag{B.56}$$

Moreover, we have

$$k_q\overline{N}(\epsilon)\ln\left(\frac{7e\left(\overline{N}(\epsilon)+1\right)}{2Rx_0\epsilon}\right)+\kappa\overline{N}(\epsilon)\ln\left(\overline{N}(\epsilon)+1\right)\geq(k_q+\kappa)\,\overline{N}(\epsilon)\ln\left(\overline{N}(\epsilon)+1\right) \tag{B.57}$$

which yields that $\overline{N}(\epsilon)\leq N_1$, where $N_1\ln(N_1+1)=\ln(n_e)/(2k_q(k_q+\kappa))$. Then, using $T\geq\left(1+\overline{N}(T)\right)\theta$ (by (B.62)) and the definition of $\overline{N}(T)$, we have

$$\kappa\overline{N}(T)\ln\left(\overline{N}(T)+1\right)=\frac{\ln(n_e)}{2} \tag{B.58}$$

and, with (B.57) and the definition of $\overline{N}(\epsilon)$,

$$\frac{\ln(n_e)}{2}\geq(k_q+\kappa)\,\overline{N}(\epsilon)\ln\left(\overline{N}(\epsilon)+1\right) \tag{B.59}$$

hence, by (B.58)-(B.59) and because $s > \kappa/k_q$,

$$-\frac{1}{s}+\frac{\overline{N}(T)\ln\left(\overline{N}(T)+1\right)}{\overline{N}(\epsilon)\ln\left(\overline{N}(\epsilon)+1\right)} \geq -\frac{1}{s}+\frac{k_q+\kappa}{\kappa} \geq 1. \tag{B.60}$$

Thus, we have

$$\begin{aligned} g(T) &= \exp\left(-2\kappa\overline{N}(\epsilon)\ln(\overline{N}(\epsilon)+1)\left[-\frac{1}{s}+\frac{\overline{N}(T)\ln(\overline{N}(T)+1)}{\overline{N}(\epsilon)\ln(\overline{N}(\epsilon)+1)}\right]\right) \\ &\leq \exp\left(-2\kappa\overline{N}(\epsilon)\ln(\overline{N}(\epsilon)+1)\right) \leq g(\epsilon), \end{aligned} \tag{B.61}$$

hence $\|g\|_{L^\infty([\epsilon,T])} = g(\epsilon)$.
Now, by definition of $\overline{N}$, for all $t \in [\epsilon, T]$, $\overline{N}(t) \leq \ln(n_e)/(2\kappa)$, hence

$$\sup_{t\in[\epsilon,T]} (1+\overline{N}(t))^{2\kappa+p} \leq \left(1 \bigvee \frac{\ln(n_e)}{2\kappa}\right)^{2\kappa+p},$$

by (B.54) this yields (B.51).
Proof of (B.52). Considering the cases $\ln(n_e)/\ln_2(n_e) \leq 1+N(\epsilon)$ and $\ln(n_e)/\ln_2(n_e) > 1+N(\epsilon)$, we have

$$\begin{aligned} & k_q\overline{N}(\epsilon)\ln\left(\frac{7e(\overline{N}(\epsilon)+1)}{2Rx_0\epsilon}\right)+\kappa\overline{N}(\epsilon)\ln\left(\overline{N}(\epsilon)+1\right) \\ & \leq \left((\kappa+2k_q)\overline{N}(\epsilon)\ln\left(\frac{\ln(n_e)}{\ln_2(n_e)}\right)\right) \bigvee \left((\kappa+2k_q)\,\overline{N}(\epsilon)\ln\left(\overline{N}(\epsilon)+1\right)\right) \end{aligned}$$

which yields $\overline{N}(\epsilon) \geq \widetilde{N}_3 \wedge \widetilde{N}_4$, where $\widetilde{N}_3$ and $\widetilde{N}_4$ are defined using the equations

$$(\kappa+2k_q)\widetilde{N}_3\ln\left(\frac{\ln(n_e)}{\ln_2(n_e)}\right)=\frac{\ln(n_e)}{2},\ \ (\kappa+2k_q)\widetilde{N}_4\ln\left(\widetilde{N}_4+1\right)=\frac{\ln(n_e)}{2}. \tag{B.62}$$

This yields (B.52) hence (B.53). Using $\underline{a}\epsilon = 7e/(2Rx_0)$, the definitions of ϵ and $\overline{N}(\epsilon)$ for the first display, and $\gamma > 1/(4k_q)$ for the second display, there exists $M_{6,\mathcal{E},\eta}$ such that

$$\begin{aligned} & \exp\left(-2\gamma\underline{a}\ln(1+\underline{a})+2\kappa\overline{N}(\epsilon)\ln\left(1+\overline{N}(\epsilon)\right)\right) \\ & \leq \exp\left(-\frac{\gamma 7e}{Rx_0\epsilon}\ln\left(\frac{7e}{2Rx_0\epsilon}\right)+\frac{\ln(n_e)}{2k_q}\right) \leq \exp\left(\left(\frac{1}{2k_q}-2\gamma\right)\ln(n_e)+2\gamma\frac{\ln(n_e)\ln_3(n_e)}{\ln_2(n_e)}\right) \leq M_{6,\mathcal{E},\eta}. \end{aligned}$$

Denoting by $M_{7,\mathcal{E},\eta} := C\left(4l^2+e^{2\kappa}\theta^2(2\kappa)^{-(2\kappa+2p+2)}\left(4l^2+M_{5,\mathcal{E},\eta}\right)+M^2M_{6,\mathcal{E},\eta}\right)$,

$$\frac{n_e^{\kappa/(\kappa+2)}}{\ln(n_e)^{2\kappa+2p+2}} \sup_{f_{\alpha,\beta}\in\mathcal{H}_{w,W}^{1,\phi,\omega}(l,M)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{1,N,T\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right) \leq M_{7,\mathcal{E},\eta}.$$

Similar computations are used to handle the case $q = \infty$.
Proof of (T3.1.2). Let $q = 1$. Proceeding like in the proof of (T3.1.11) to obtain (B.50) and

using that $w = W_{\mathcal{A}}$, we have,

$$\sup_{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}} \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)$$

(B.63)

$$\leq 2(1+C(\underline{a},\epsilon))\left(\sup_{\epsilon\leq|t|\leq T}\frac{(1+N(t))^p\left(\left(\left(\overline{N}(T)+1\right)\theta\right)\vee|t|\right)^{p+2}}{e^{2\kappa(N(t)\ln(1+N(t)))}}\left(M_{6,\mathcal{E},\eta,q}+4l^2\right)+\frac{4l^2}{(1\vee T)^{2s}}\right).$$

Then, using (B.56-B.61), that $\overline{N}(T)\ln(1+\overline{N}(T)) = \ln(n_e)/(2\kappa)$, that

$$\overline{N}(\epsilon)\ln\left(1+\overline{N}(\epsilon)\right) = \frac{\ln(n_e)}{2+2\kappa} - \frac{\overline{N}(\epsilon)\ln(\theta/\epsilon)}{1+2\kappa},$$

$\theta/\epsilon = 1$, and that $s > \kappa/k_q$, we obtain $\|g\|_{L^\infty([\epsilon,T])} = g(\epsilon)$. Thus, we have (B.51). Then, for $q = 1,\infty$, $M_{8,\mathcal{E},\eta,q} := 2(1+C(\underline{a},1/\theta))\left(4l^2 + e^{2\kappa}\theta^2(2\kappa)^{-(2\kappa+2p+2)}\left(M_{6,\mathcal{E},\eta,q}+4l^2\right)\right)$, by definition of $\overline{N}(\epsilon)$, and using that (B.63) we have, for all n_e,

$$\frac{n_e^{\kappa/(\kappa+1)}}{\ln(n_e)^{2\kappa+2p+2}}\sup_{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{1,\phi,\omega}_{w,W}(l)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}\mathcal{R}^W_{n_0,n}\left(\widehat{f}^{1,N,T\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)\leq M_{8,\mathcal{E},\eta,1}.$$

Proceeding similarly for the case $q=\infty$ yields

$$\frac{n_e^{\kappa/(\kappa+p)}}{\ln(n_e)^{2\kappa+2p+2}}\sup_{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{\infty,\phi,\omega}_{w,W}(M)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}\mathcal{R}^W_{n_0}\left(\widehat{f}^{\infty,N,T\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)\leq M_{8,\mathcal{E},\eta,\infty}.$$

Proof of (T3.2). Let $q=1$ and $\widetilde{n}_e := n_e/\ln(n_e)^{p/r}$. We follow the arguments in [38]. Using (B.48) and $\Phi_{n_e} := \sup_{t\in[\epsilon,T]}\exp\left(2N\ln\left(1\bigvee((1+N)\theta/t)\right)\right)(1\vee(Rx_0t))^2$, we have

$$\begin{aligned}
\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) &\leq C\left(\int_{\epsilon\leq|t|\leq T}\Delta_{2,q}(t,N,n,Z_{n_0})dt + 4l^2e^{-2\kappa(N\ln(1+N))^r}+\frac{4l^2}{\phi(T)}+\frac{M^2}{w(\underline{a})}\right)\\
&\leq C\left(\frac{\Phi_{n_e}}{\widetilde{n}_e}\frac{4l^2\widetilde{n}_e}{\Phi_{n_e}e^{2\kappa(N\ln(1+N))^r}}+\frac{4l^2}{e^{\nu T}}\right.\\
&\qquad\left.+\frac{\Phi_{n_e}}{\widetilde{n}_e}\int_{\epsilon\leq|t|\leq T}\frac{\Delta_{2,q}(t,N,n,Z_{n_0})\widetilde{n}_e}{(1\vee(Rx_0|t|))^2e^{2N\ln((1+N)\theta/|t|)}}dt+\frac{M^2}{e^{2\gamma(\underline{a}\ln(1+\underline{a}))^\rho}}\right).
\end{aligned}$$

Now, we study Φ_{n_e}. Because $g_1 : t\in(0,\infty)\mapsto\exp\left(2N\ln\left(1\bigvee(1+N)\theta/t\right)\right)(1\vee(Rx_0t))^2$ is nonincreasing on $\left[\epsilon,(1+\overline{N})\theta\right]$ and increasing on $\left[(1+\overline{N})\theta,\infty\right)$, we have $\|g_1\|_{L^\infty([\epsilon,T])} = g_1(\epsilon)\vee g_1(T)$. Using $N\leq\overline{N}+1$ and $1+N\leq 2\left(1+\overline{N}\right)$ for the first display and that $\theta/|t|\leq\theta/\epsilon = (1+\overline{N})$ and $\ln(1+t)\leq t$ for $t\geq 0$ for the second display, we have, for all $t\neq 0$,

$$\exp\left(2N\ln\left(1\bigvee\frac{(1+N)\theta}{|t|}\right)\right)\leq\exp\left(2\overline{N}\ln\left(1\bigvee\frac{2\left(2+\overline{N}\right)\theta}{|t|}\right)\right)\left(1\bigvee\frac{\theta\left(1+\overline{N}\right)}{|t|}\right)^2$$

$$\leq \exp\left(4\overline{N}\ln\left(\overline{N}+1\right)\right)\exp\left(\frac{2\overline{N}}{\overline{N}+1}\right)\left(1+\overline{N}\right)^3.$$

Thus, using $2\theta/\epsilon = \left(1+\overline{N}\right)$, $\overline{N} \leq (\ln(n_e)/(2\kappa))^{1/r}$, and $T^2 = \exp\left(4\overline{N}\ln\left(\overline{N}+1\right)\right)$, we have

$$\Phi_{n_e} \leq \exp\left(4\overline{N}\ln\left(\overline{N}+1\right)\right) 2e\left(1 \bigvee \frac{\ln(n_e)^{3p/r}}{(2\kappa)^{3p/r}}\right)(1\vee(Rx_0))^2. \tag{B.64}$$

Now, we show that there exists $\widetilde{D}$ such that

$$\exp\left(2N\ln\left(\frac{(N+1)\theta}{\epsilon}\right)\right)\exp\left(2\kappa(N\ln(N+1))^r\right) \geq \widetilde{D}n_e. \tag{B.65}$$

Define d_i by the equations $D_0 = D_1 = \cdots = D_k = 0$, where $D_0 = -d_0 + 4/(2\kappa)^{1/r}$,

$$\forall 1 \leq i \leq k+1,\ D_i = -d_i + \frac{4(-1)^k}{(2\kappa)^{1/r}}\sum_{j=1}^{i}\frac{1/r\dots(1/r-j+1)}{j!}\sum_{p_1+\cdots+p_j=i} d_{p_1-1}\dots d_{p_j-1} \tag{B.66}$$

and $d_{k+1}=0$. We use $u_{n_e} := \sum_{i=0}^k d_i \ln\left(n_e\right)^{(i+1)/r-(i+1)}$ and, for all $t\neq 0$,

$$g\left(n_e,d,r,t\right) := \ln(n_e) - \sum_{i=0}^{k} d_i\ln\left(n_e\right)^{(i+1)/r-i}.$$

Using $\theta/\epsilon = \overline{N}+1$, that $N \geq \overline{N}$, and using the definition of $\overline{N}$, we obtain

$$\begin{aligned}\exp\left(2\kappa(N\ln(N+1))^r + 2N\ln\left(\frac{(N+1)\,\theta}{\epsilon}\right)\right) &\geq \exp\left(2\kappa(\overline{N}\ln(\overline{N}+1))^r + 4\overline{N}\ln\left(\overline{N}+1\right)\right)\\ &\geq h(n_e,d,r),\end{aligned}$$

where

$$\begin{aligned}h(n_e,d,r) &:= \exp\left(g\left(n_e,d,r\right) + \frac{4}{(2\kappa)^{1/r}}g\left(n_e,d,r\right)^{1/r}\right)\\ &= n_e\exp\left(-\sum_{i=0}^{k} d_i\ln\left(n_e\right)^{(i+1)/r-i}\right)\exp\left(\frac{4}{(2\kappa)^{1/r}}\left(\ln\left(n_e\right)\right)^{1/r}\left(1-\nu(n_0,\mathcal{E},n)\right)\right),\\ \nu(n_0,\mathcal{E},n) &:= \frac{1}{r}u_{n_e} + \cdots + (-1)^k\frac{1/r\dots(1/r-k)}{(k+1)!}u_{n_e}^{k+1} + o\left(u_{n_e}^{k+1}\right).\end{aligned}$$

Thus, denoting by $\tau_4 := (k+1)r-(k+2)$, we have

$$h(n_e,d,r) = n_e\exp\left(1+\sum_{i=1}^{k+1} D_i\ln\left(n_e\right)^{(i+1)r-i} + o\left(\left(\ln\left(n_e\right)\right)^{\tau_4}\right)\right) \tag{B.67}$$

which yields (B.65) with $\widetilde{D}$ depending only on $(D_i)_{i=1}^{k+1}$ thus on r, κ, and p. We also have

$$\frac{\Delta_{2,q}(t,N(t),n,Z_{n_0})\widetilde{n}_e}{(1\vee|t|)^{p+2}e^{2N\ln(1\vee((N+1)\theta/|t|))}}$$

$$\leq \frac{2}{\pi(1\vee|t|)^{p+2}}\left(\frac{c_{\boldsymbol{X}}2p^{p-1}|t|^p}{(2\pi)^p(p-1)!}+\frac{M_{1,\mathcal{E},\eta}}{\pi^p}\left(L(t)^2+\frac{c_{\boldsymbol{X}}(N(t)+1)^p\,|t|^p}{n}\right)\right).$$

Hence, we obtain

$$\int_{\epsilon\leq|t|\leq T}\frac{\Delta_{2,q}(t,N(t),n,Z_{n_0})\widetilde{n}_e}{(1\vee|t|)^{p+2}e^{2N\ln(1\vee((N+1)\theta/|t|))}}dt\leq M_{9,\mathcal{E},\eta,q},$$

$$M_{9,\mathcal{E},\eta,1}:=\frac{4}{(2\pi)^p}\left(1+\frac{1}{p+1}\right)\left(\frac{c_{\boldsymbol{X}}2p^{p-1}}{(p-1)!}+2^pM_{1,\mathcal{E},\eta}\left(1+\frac{p}{e}\right)^pc_{\boldsymbol{X}}\right)+2^{p+2}M_{1,\mathcal{E},\eta}l^2.$$

Thus, we obtain

$$\begin{aligned}&\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}},f_{\alpha,\boldsymbol{\beta}}\right)\\&\leq\frac{Ce^{4\overline{N}\ln(1+\overline{N})}}{\widetilde{n}_e\ln(n_e)^{-3p/r}}\left(2e\left(1\bigvee\frac{1}{(2\kappa)^{3p/r}}\right)\left(\frac{4l^2}{\widetilde{D}}+M_{9,\mathcal{E},\eta,1}+M^2\right)+\frac{4l^2\ln(n_e)^{-3p/r}\widetilde{n}_e}{e^{\nu T}\exp\left(4\overline{N}\ln(1+\overline{N})\right)}\right).\end{aligned}$$

Then, using the definition of N for the first equality, the definition of $h(n_e,d,r)$, and (B.67) for the inequality yield that there exists $M_{10,\mathcal{E},\eta}$ such that

$$\begin{aligned}\frac{\widetilde{n}_e}{\exp\left(4\overline{N}\ln(1+\overline{N})\right)}&=\widetilde{n}_e\exp\left(-\frac{4}{(2\kappa)^{1/r}}g\left(n_e,d,r,t\right)^{1/r}\right)\\&\geq\frac{M_{10,\mathcal{E},\eta}}{\ln(n_e)^{p/r}}\exp\left(g\left(n_e,d,r,t\right)\right)=\frac{M_{10,\mathcal{E},\eta}n_e}{\ln(n_e)^{p/r}}\exp\left(-\Sigma(n_e)\right),\end{aligned}\tag{B.68}$$

where $\Sigma(n_e)=\sum_{i=0}^k d_i\ln\left(n_e\right)^{(i+1)/r-i}$. Then, because

$$\begin{aligned}\frac{\widetilde{n}_e}{\exp\left(4\overline{N}\ln(1+\overline{N})\right)}&\leq\frac{\exp\left((2\kappa(\overline{N}\ln(1+\overline{N}))^r\right)}{\widetilde{D}\ln(n_e)^{p/r}}\quad\text{(by (B.65))}\\&\leq\frac{n_e\exp\left(-\Sigma(n_e)\right)}{\widetilde{D}\ln(n_e)^{p/r}}\ \text{(by definition of }\overline{N}),\end{aligned}$$

there exists $\overline{D}$ depending only on $(d_i)_{i=1}^k$ such that $\exp\left(\Sigma(n_e)\right)\geq\exp\left(\overline{D}\ln(n_e)^{1/r}\right)$, $\sup_{n\geq e}\ln_2(n)-\overline{D}\ln(n)^{1/r}=r\left(\ln\left(r/\overline{D}\right)-1\right)$, we obtain, for all ν such that

$$\ln(\nu)\geq r\left(\ln\left(\frac{r}{\overline{D}}\right)-1\right)-\ln\left(\widetilde{D}\right),\tag{B.69}$$

$$\begin{aligned}\frac{\widetilde{n}_e}{e^{\nu T}\exp\left(4\overline{N}\ln(1+\overline{N})\right)}\frac{1}{\ln(n_e)^{3p/r}}&\leq\frac{n_e\exp\left(-\Sigma(n_e)\right)}{\widetilde{D}\exp\left(\nu T\right)}\frac{1}{\ln(n_e)^{4p/r}}\\&\leq\frac{n_e\exp\left(-\Sigma(n_e)\right)}{\widetilde{D}\exp\left(\nu\widetilde{D}\exp\left(\Sigma(n_e)\right)\right)}\frac{1}{\ln(n_e)^{4p/r}}\\&\leq\frac{n_e\exp\left(-\Sigma(n_e)\right)}{\widetilde{D}\exp\left(\nu\widetilde{D}\exp\left(\overline{D}\ln(n_e)^{1/r}\right)\right)}\end{aligned}$$

$$\leq \frac{\exp\left(-\Sigma(n_e)\right)}{\widetilde{D}\ln(n_e)^{4p/r}} \leq \frac{1}{\widetilde{D}}.$$

Using (B.68), $\rho \geq r$, $\gamma > \kappa$, and the aforementioned choice of $\underline{a}$, we have, for all $n_e \geq 1$,

$$\frac{n_e \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right)}{\ln(n_e)^{4p/r}\exp\left(\Sigma(n_e)\right)} \leq \frac{C}{M_{10,\mathcal{E},\eta}}\left(2e\left(1 \bigvee \frac{1}{(2\kappa)^{3p/r}}\right)\left(4l^2 + M_{9,\mathcal{E},\eta,1} + M^2\right) + \frac{4l^2}{\widetilde{D}}\right),$$

hence the result. Similar computations yield the result when $q = \infty$.

Proof of (T3.3). Denote by $\theta_1 := \pi/(4Rx_0)$. We start from equation (B.31), where, using that $w = W_{\mathcal{A}}$, we have, for all $N \in \mathbb{N}_0$,

$$\begin{aligned}\text{(B.70)}\qquad & \mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \\ & \leq 2(1 + C(\underline{a}, \epsilon))\left(\int_{\epsilon \leq |t| \leq T} \Delta_q(t, N(t), n, Z_{n_0})dt + 4l^2 \sup_{t\in\mathbb{R}} e^{-2\kappa N(t)} + \frac{4l^2}{(1 \vee T)^{2s}}\right).\end{aligned}$$

Let $q = 1$. By (B.33), for all $|t| \neq 0$,

$$\begin{aligned}\frac{1}{n_e}\exp\left(\frac{\pi(N(t)+p)}{2\,|t|\,Rx_0} + 2\kappa N(t)\right) &\leq \frac{\exp\left(\pi p/(2Rx_0\epsilon)\right)}{n_e}\exp\left(2\left(\frac{\pi}{4\,|t|\,Rx_0} + \kappa\right)\overline{N}(t)\right) \\ &\leq \exp\left(2\pi p\right)\ \left(\text{by definition of } \overline{N}(t) \text{ and } \epsilon = \pi/4Rx_0\right),\end{aligned}$$

and using that $N(t) \leq \ln(n_e)/(2\kappa)$ and (B.30), we obtain, for $|t| > \epsilon$,

$$\text{(B.71)}\qquad \frac{\Delta_1(t, N(t), n, Z_{n_0})e^{2\kappa N(t)}}{e^{2\pi p}((\ln(n_e)/(2\kappa)) \vee |t|)^{2p}} \leq \frac{4eRx_0 c_{\boldsymbol{X}} p^{p-1}}{\pi^{p+1}(p-1)!} + \frac{2M_{1,\mathcal{E},\eta}}{\pi^{p+1}}\ L^2(t) + \frac{2^{p+1}c_{\boldsymbol{X}}M_{1,\mathcal{E},\eta}}{\pi^{p+1}n}.$$

Thus, using (B.71) and $\int_{\epsilon\leq|t|\leq T} L^2(t)dt \leq (2\pi)^{p+1}l^2$, we obtain

$$\text{(B.72)}\qquad \int_{\epsilon \leq |t| \leq T}\frac{\Delta_1(t, N(t), n, Z_{n_0})e^{2\kappa N(t)}}{(\ln(n_e)/(2\kappa) \vee |t|)^{2(p+1)}}dt \leq M_{11,\mathcal{E},\eta},$$

$$M_{11,\mathcal{E},\eta} := e^{2\pi p}\left(\frac{16\kappa^2 eRx_0 c_{\boldsymbol{X}} p^{p-1}}{\pi^{p+1}(p-1)!} + 2^{p+2}l^2(2\kappa)^2 M_{1,\mathcal{E},\eta} + \frac{2^{p+1}(2\kappa)^2 p^p c_{\boldsymbol{X}} M_{1,\mathcal{E},\eta}}{\pi^{p+1}}\right).$$

Then, we have

$$\begin{aligned}&\mathcal{R}^W_{n_0}\left(\widehat{f}^{q,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \\ &\leq 2(1 + C(\underline{a}, \epsilon))\left(\sup_{t\in[\epsilon,T]}\left(e^{-2\kappa N(t)}\left(\frac{\ln(n_e)}{2\kappa}\bigvee t\right)^{2(p+1)}\right)\left(4e^{2\kappa}l^2 + M_{11,\mathcal{E},\eta}\right) + \frac{4l^2}{(1 \vee T)^{2s}}\right).\end{aligned}$$

The next step consists in showing that, denoting by $g : t \mapsto e^{-2\kappa\overline{N}(t)}(\ln(n_e)/(2\kappa) \vee |t|)^{2(p+1)}$, we have $\|g\|_{L^\infty([\epsilon,T])} = e^{-2\kappa\overline{N}(\epsilon)}\left(\ln(n_e)/(2\kappa)\right)^{2(p+1)}$. Indeed, using that, for all $|t| \geq \epsilon$, $\overline{N}'(t) = \theta_1 k_q \ln(n_e)(\kappa\,|t| + k_q\theta_1)^{-2}/2$, where $\theta_1 = \pi/(4Rx_0)$, we have, for all $t \in \mathbb{R}$,

$$g'(t) = \frac{1}{|t|}\left(2(p+1)\mathbb{1}\left\{|t| \geq \frac{\ln(n_e)}{2\kappa}\right\} - \frac{\kappa\theta_1\ln(n_e)\,|t|}{(\kappa\,|t| + \theta_1)^2}\right)g(t).$$

Using that $t \in [0,\infty) \mapsto at/(bt+c)^2$ for all $a,b,c>0$ is decreasing when $t>c/b$ and increasing on $(0,c/b)$ when $t \geq \ln(n_e)/(2\kappa)$, we have (1) if $\ln(n_e)/(2\kappa) \geq \theta_1/\kappa$, then, because $(p+1) > 2\kappa\theta_1$, $g'(t) \geq \frac{2}{|t|}\left((p+1) - \frac{4\kappa\theta_1}{(1+2\theta_1)^2}\right) g(t) > 0$, else (2) if $\ln(n_e)/(2\kappa) < \theta_1/\kappa$, then, because $p+1 > \kappa\theta_1/4$, $g'(t) \geq \frac{1}{|t|}\left(2(p+1) - \frac{\kappa\theta_1}{2}\right) g(t) > 0$ hence g' is positive when $t \geq \ln(n_e)/(2\kappa)$. Thus, g is positive, decreasing on $[\epsilon, \ln(n_e)/(2\kappa)]$ and increasing on $[\ln(n_e)/(2\kappa), \infty)$. This yields

$$\|g\|_{L^\infty([\epsilon,T])} = \left(e^{-2\kappa\overline{N}(\epsilon)}\left(\frac{\ln(n_e)}{2\kappa}\right)^{2(p+1)}\right) \bigvee \left(e^{-2\kappa\overline{N}(T)}T^{2(p+1)}\right).$$

Using that $T^{2s} = e^{2\kappa\overline{N}(\epsilon)}/(Rx_0)^{2s}$, we have

$$e^{-2\kappa\overline{N}(T)}T^{2(p+1)} = \frac{1}{(Rx_0)^{2(p+1)}} e^{-2\kappa\overline{N}(\epsilon)} e^{-2\kappa N(T) + 2(p+1)\kappa N(\epsilon)/s + 2\kappa\overline{N}(\epsilon)}$$

and using $\overline{N}(t) - 1 \leq N(t) \leq \overline{N}(t)$, $\overline{N}(T) = \ln(n_e)/(2(\kappa + k_q\theta_1/T))$, $\overline{N}(\epsilon) = \ln(n_e)/(2(\kappa+k_q))$, $\epsilon = \theta_1$, and $\kappa/k_q > \pi\left(1 + s/(p+1)\right)/4 - 1$ for the last inequality, we have

$$\begin{aligned}
-2\kappa N(T) + 2(p+1)\kappa N(\epsilon)/s + 2\kappa\overline{N}(\epsilon) &\leq 2\kappa - 2\kappa\overline{N}(T) + 2(p+1)\kappa\overline{N}(\epsilon)/s + 2\kappa\overline{N}(\epsilon) \\
&\leq 2\kappa + \frac{\kappa\ln(n_e)}{\kappa + k_q}\left(\frac{p+1}{s} + 1\right) \\
&\leq 2\kappa + \frac{\kappa\ln(n_e)}{\kappa + k_q}\left(\frac{p+1}{s} + 1 - \frac{\kappa/k_q + 1}{\kappa/k_q + \pi/4}\right) \leq 2\kappa.
\end{aligned}$$

This yields $\|g\|_{L^\infty([\epsilon,T])} = e^{-2\kappa\overline{N}(\epsilon)}\left(\ln(n_e)/(2\kappa)\right)^{2(p+1)}$.
Using $\overline{N}(t) - 1 \leq N(t) \leq \overline{N}(t)$, this yields

$$\frac{n_e^{\kappa/(\kappa+1)}}{\ln(n_e)^{2(p+1)}} \sup_{f_{\alpha,\boldsymbol{\beta}} \in \mathcal{H}^{q,\phi,\omega}_{w,W}(l)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}} \mathcal{R}^W_{n_0}\left(\widehat{f}^{1,N,T,\epsilon}_{\alpha,\boldsymbol{\beta}}, f_{\alpha,\boldsymbol{\beta}}\right) \leq M_{12,\mathcal{E},\eta},$$

where $M_{12,\mathcal{E},\eta} := 2(1 + C(\underline{a},\epsilon))e^{2\kappa}(2\kappa)^{-2(p+1)}\left(4(e^{2\kappa}+1)l^2 + M_{11,\mathcal{E},\eta}\right)$. Similar computations yield the result when $q=\infty$. □

Proof of (T4.1). Step 1. Unlike in the proof of Theorem 2, we do not have to ensure that $f_{1,n}$ and $f_{2,n}$ are densities but only that $f_{1,n}$ and $f_{2,n}$ belong to $\mathcal{H}^{q,\phi,\omega}_{w,W}(l)\cap\mathcal{S}_U$. Using (A.10), we have $f_{2,n} \in L^2\left(w \otimes W^{\otimes p}_{[-R,R]}\right)$. Clearly, $f_{1,n}$ and $f_{2,n}$ belong to $\mathcal{S}_U$. Like for (A.11)-(A.12), we obtain

$$\left(\frac{Rx_0U}{2\pi}\right)^p U\left[2\gamma_n^2\frac{(1\vee U^{2s})}{p+1} \bigvee \frac{2\gamma_n^2 e^{2\kappa N\ln(N)}}{p+1}\right] \leq \pi l^2. \tag{B.73}$$

Step 2. It is the same as for Theorem 2.
Step 3. Let $\xi < \sqrt{2}$. Using (A.19) then Proposition B.1 (ii) we have

$$K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) = \frac{\gamma_n^2 n R^p}{2\sigma^2} \int_{\mathbb{R}} \left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W_{[-1,1]},Rc(t)}\right)^2 \left(\frac{Rc(|t|)}{2\pi}\right)^p \lambda(t)^2 dt.$$

Using Lemma B.4, we have, for all $U/2 \le |t| \le U$ such that $4/(eRc(U)) \ge 1$,

$$\left(\sigma_{\widetilde{\boldsymbol{N}}(q)}^{W_{[-1,1]},Rc(t)}\right)^2 \le \left(\frac{2\pi e^3}{9}\right)^p \exp\left(-2k_q N \ln\left(\frac{4N}{eRc(U)}\right)\right),$$

hence

$$K(\mathbb{P}_{2,n}, \mathbb{P}_{1,n}) \le \left(\frac{R^2 x_0 e^3}{9}\right)^p \frac{U^{p+1}}{(p+1)\sigma^2} \gamma_n^2 n \exp\left(-2k_q N \ln\left(\frac{4N}{eRc(U)}\right)\right).$$

As a result, (iii) is satisfied if

$$\left(\frac{R^2 x_0 e^3}{9}\right)^p \frac{U^{p+1}}{(p+1)\sigma^2} \gamma_n^2 n \exp\left(-2k_q N \ln\left(\frac{4N}{eRc(U)}\right)\right) \le \xi^2. \tag{B.74}$$

Step 4. We take $N = \lceil \overline{N} \rceil$, where $\overline{N} \ln(\overline{N}) = \ln(n)/(2(\kappa + k_q))$, $\gamma_n := C_{\Gamma,2} \exp(-\kappa \overline{N} \ln(\overline{N}))$, $U := 4/(Rx_0 e)$, and

$$C_{\Gamma,2}^2 := \left(\frac{(p+1)\pi \,(2\pi)^p \, l^2}{U(URx_0)^p} \left[\frac{1}{1 \vee U^{2s}} \bigwedge 1\right]\right) \bigwedge \left((\sigma\xi)^2 \left(\frac{9}{Re^3}\right)^p\right)$$

which guarantee that (B.73) is satisfied and $f_{1,n}$ and $f_{2,n}$ belong to $\mathcal{H}_W^{\omega,\phi}(l)$. Moreover, (B.74) is also satisfied because

$$\gamma_n^2 n \exp\left(-2k_q N \ln(N)\right) \le C_{\Gamma,2}^2 n \exp(-2(\kappa + k_q)\overline{N} \ln(\overline{N})) \le C_{\Gamma,2}^2.$$

For this choice of γ_n, we have $h_n^2 = C_{\Gamma,2}^2 n^{-\kappa/(\kappa+k_q)} \left(Rx_0/(2\pi)\right)^p \int_{U/2}^U |t|^p \lambda(t)^2 dt/(4\pi)$.

Proof of (T4.2). Step 1. By (A.10), $f_{2,n} \in L^2\left(w \otimes \cosh\left(\cdot/R\right)^{\otimes p}\right)$ and $f_{1,n}$ and $f_{2,n}$ belong to $\mathcal{H}_{w,W}^{q,\phi,\omega}(l)$ if

$$\left(\frac{URx_0}{2\pi}\right)^p U \left[2\gamma_n^2 \frac{(1 \vee U^{2s})}{p+1} \bigvee \frac{2\gamma_n^2 e^{2\kappa N}}{p+1}\right] \le \pi l^2. \tag{B.75}$$

Step 2. This is the same as for Theorem 2.
Step 3. Let $\xi < \sqrt{2}$. We can check that like for (T2.2), (iii) is satisfied if (A.20) holds.
Step 4. We take $N = \lceil \overline{N} \rceil$, where $\overline{N} = \ln(n)/(2(\kappa+k_q))$, $\gamma_n := C_{\Gamma,3} \exp(-\kappa \overline{N})$, $U := 2/(Rx_0 e)$,

$$C_{\Gamma,3}^2 := \left(\frac{p+1}{2U(URx_0)^p} \left[\frac{\pi^{p+1} l^2}{2\,(1 \vee U^{2s})} \bigwedge \left(\pi \,(2\pi)^p \, l^2\right)\right]\right) \bigwedge \left(\frac{2(\sigma\xi)^2}{U} \left(\frac{\pi(1 - e^{-2})}{R^2 x_0 U e}\right)^p\right)$$

which guarantees (B.75). (A.20) is satisfied because $\gamma_n^2 n \exp\left(-2k_q N\right) \le C_{\Gamma,3}^2 n \exp(-2(\kappa + k_q)\overline{N}) \le C_{\Gamma,3}^2$ and, for such γ_n, $h_n^2 = C_{\Gamma,3}^2 n^{-\kappa/(\kappa+k_q)} \left(Rx_0/(2\pi)\right)^p \int_{U/2}^U |t|^p \lambda(t)^2 dt/(4\pi)$. □

Proof of Lemma A.5. Let $t \in [-T,T] \setminus (-\epsilon,\epsilon)$, $N \in \{0,\dots,N^W_{\max,q}(t)\}$, and $T \in \mathcal{T}_n$. Use $\left(\widehat{F}_1^{q,\widehat{N}(t),T,0} - F_1^{q,\infty,T,0}\right)(t,\cdot) = \sum_{j=1}^3 R_j(t,\cdot)$, where $R_1(t,\cdot) := \left(\widehat{F}_1^{q,\widehat{N}(t),T,0} - \widehat{F}_1^{q,\widehat{N}(t)\vee N,T,0}\right)(t,\cdot)$, $R_2(t,\cdot) := \left(\widehat{F}_1^{q,\widehat{N}(t)\vee N,T,0} - \widehat{F}_1^{q,N,T,0}\right)(t,\cdot)$, and $R_3(t,\cdot) := \left(\widehat{F}_1^{q,N,T,0} - F_1^{q,\infty,T,0}\right)(t,\cdot)$ (for all $|t| \in [\epsilon,T]$, $\mathcal{F}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot) = F_1^{q,\infty,T,0}(t,\cdot)$). Using the Young inequality for products yields

$$\mathcal{R}_{0,q}\left(\widehat{N}(t),t\right) \le (2+c_0)\sum_{j=1}^2 \mathbb{E}\left[\|R_j(t,\cdot)\|^2_{L^2(W^{\otimes p})}\right] + \left(1+\frac{2}{c_0}\right)\mathbb{E}\left[\|R_3(t,\cdot)\|^2_{L^2(W^{\otimes p})}\right].$$

Because $B_1(t,N) = \max_{N'\in\mathbb{N}_0:\ N'\le N^W_{\max,q}(t)} \left(\sum_{N\le|\boldsymbol{m}|_q\le N'\vee N}\left(|\widehat{c}_{\boldsymbol{m}}(t)|/\sigma^{W,tx_0}_{\boldsymbol{m}}\right)^2 - \Sigma(t,N')\right)_+$, $c_1 \ge 1 + C_{c_0}/(2(2+c_0)) = 211/208$ with $c_0 := 1/6$, and by definition of $\widehat{N}$ (see (16)), we have

$$\begin{aligned}
&\mathcal{R}_{0,q}\left(\widehat{N}(t),t\right) + C_{c_0}\mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\right]\\
&\le 2(2+c_0)\left(\mathbb{E}\left[B_1\left(t,\widehat{N}(t)\right)\right] + \mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\right]\right) + \left(1+\frac{2}{c_0}\right)\mathbb{E}\left[\|R_3(t,\cdot)\|^2_{L^2(W^{\otimes p})}\right] + C_{c_0}\mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\right]\\
&\le 2(2+c_0)\left(\mathbb{E}\left[B_1(t,N)\right] + c_1\mathbb{E}\left[\Sigma(t,N)\right]\right) + \left(1+\frac{2}{c_0}\right)\mathbb{E}\left[\|R_3(t,\cdot)\|^2_{L^2(W^{\otimes p})}\right].
\end{aligned}$$

Consider $\mathbb{E}\left[B_1(t,N)\right]$. We obtain

$$B_1(t,N) \le \max_{0\le N'\le N^W_{\max,q}(t)}\left((2+c_0)\sum_{j=1}^2\|K_j(t,\cdot)\|^2_{L^2(W^{\otimes p})} + \left(1+\frac{2}{c_0}\right)\|K_3(t,\cdot)\|^2_{L^2(W^{\otimes p})} - \Sigma(t,N')\right)_+,$$

where $K_1(t,\cdot) := \left(\widehat{F}_1^{q,N\vee N',T,0} - F_1^{q,N\vee N',T,0}\right)(t,\cdot)$, $K_2(t,\cdot) := \left(F_1^{q,N,T,0} - \widehat{F}_1^{q,N,T,0}\right)(t,\cdot)$, and $K_3(t,\cdot) := \left(F_1^{q,N\vee N',T,0} - F_1^{q,N,T,0}\right)(t,\cdot)$. Using $F_1^{q,\infty,T,0}(t,\cdot) = \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\beta}\right](t,\cdot)$, we have

$$\|K_3(t,\cdot)\|^2_{L^2(W^{\otimes p})} = \sum_{N<|\boldsymbol{m}|_q\le(N\vee N')}\left|\frac{c_{\boldsymbol{m}}(t)}{\sigma^{W,tx_0}_{\boldsymbol{m}}}\right|^2 \le \left\|\left(F_1^{q,N,T,0} - \mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\beta}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})},$$

$$\begin{aligned}
B_1(t,N) \le \max_{0\le N'\le N^W_{\max,q}(t)}&\left(2(2+c_0)\left\|\left(F_1^{q,N',T,0} - \widehat{F}_1^{q,N',T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} - \Sigma(t,N')\right)_+\\
&+\left(1+\frac{2}{c_0}\right)\left\|\left(F_1^{q,N,T,0} - \mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\mathcal{R}_{0,q}\left(\widehat{N}(t),t\right) + C_{c_0}\mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\right]\\
&\le 4(2+c_0)^2\mathbb{E}\left[\max_{0\le N'\le N^W_{\max,q}(t)}\left(\left\|\left(\widehat{F}_1^{q,N',T,0} - F_1^{q,N',T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})} - \frac{\Sigma_1(t,N')}{2(2+c_0)}\right)_+\right]
\end{aligned}$$

$$+\,2(2+c_0)c_1\mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\right]+C_{c_0,1}\mathbb{E}\left[\left\|\left(\widehat{F}_1^{q,N,T,0}-\mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}\right].$$

Using Lemma A.3 for the third display, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\max_{0\le N'\le N^W_{\max,q}(t)}\left\|\left(\widehat{F}_1^{q,N',T,0}-F_1^{q,N',T,0}\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}-\frac{\Sigma(t,N')}{2(2+c_0)}\right]\\
\le&\mathbb{E}\left[\max_{0\le N'\le N^W_{\max,q}(t)}\left(\sum_{|\boldsymbol{m}|_q\le N'}\left(\frac{|\widehat{c}_{\boldsymbol{m}}(t)-c_{\boldsymbol{m}}(t)|}{\sigma^{W,tx_0}_{\boldsymbol{m}}}\right)^2-\frac{\Sigma(t,N')}{2(2+c_0)}\right)_+\right]\\
\le&(1+2c_0)\mathbb{E}\left[\max_{0\le N'\le N^W_{\max,q}(t)}\left(\sum_{|\boldsymbol{m}|_q\le N'}\left(\frac{\left|\widetilde{\Delta}_{\boldsymbol{m}}(t)\right|}{\sigma^{W,tx_0}_{\boldsymbol{m}}}\right)^2-\frac{\Sigma(t,N')}{2(2+c_0)}\right)_+\right]\\
&+\left(2+\frac{1}{c_0}\right)\mathbb{E}\left[\max_{0\le N'\le N^W_{\max,q}(t)}\left(S_1(N',t)+S_2(N',t)\right)\right]\\
\le&\frac{(1+2c_0)48N^W_{\max,q}(t)c_{\boldsymbol{X}}\,|t|^p\,\nu_q^W(N^W_{\max,q}(t),tx_0)}{(2\pi)^p n}\Psi_{0,n}(t)+Z_{n_0}\Psi_n(t).
\end{aligned}$$

This yields the result because

$$\begin{aligned}
&\int_{\epsilon\le|t|\le T}\mathcal{R}_{0,q}\left(\widehat{N}(t),t\right)dt+C_{c_0}\int_{\epsilon\le|t|\le T}\mathbb{E}\left[\Sigma\left(t,\widehat{N}(t)\right)\right]dt\\
&\le 4(2+c_0)^2\int_{\epsilon\le|t|\le T}\left(\frac{(1+2c_0)48N^W_{\max,q}(t)c_{\boldsymbol{X}}\,|t|^p\,\nu_q^W(N^W_{\max,q}(t),tx_0)}{(2\pi)^p n}\Psi_{0,n}(t)+Z_{n_0}\Psi_n(t)\right)dt\\
&\quad+C_{c_0,1}\int_{\epsilon\le|t|\le T}\left(\mathbb{E}\left[\left\|\left(\widehat{F}_1^{q,N(t),T,0}-\mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right]\right)(t,\cdot)\right\|^2_{L^2(W^{\otimes p})}\right]+C_{c_0}c_1\mathbb{E}\left[\Sigma(t,N(t))\right]\right)dt.\ \square
\end{aligned}$$

Lemma B.15. Let $f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)$, $l,M,\epsilon>0$, $q\in\{1,\infty\}$. On the event $E\left(\mathcal{G}_{n_0},\mathcal{E}\right)$, we have that there exists $C_{22,\mathcal{E},\eta}$ such that $\int_{\epsilon\le|t|\le T_{\max}}C_{21,\mathcal{E},\eta}(t)dt\le n^{\zeta_0}(1\vee l^2)C_{22,\mathcal{E},\eta}$, where

$$C_{21,\mathcal{E},\eta}(\star):=M_{1,\mathcal{E},\eta}C_{20}^{W,q}\left(2+\frac{1}{c_0}\right)\left(\frac{c_{\boldsymbol{X}}}{x_0^p}+(2\pi)^p\left\|\mathcal{F}_{1st}\left[f_{\alpha,\boldsymbol{\beta}}\right](\star,\cdot)\right\|^2_{L^2(\mathbb{R}^p)}\right)$$

$$Z_{n_0}\int_{\epsilon\le|t|\le T_{\max}}\Psi_n(t)dt\le\frac{C_{22,\mathcal{E},\eta}\left((p-1)/(e(\zeta-\zeta_0))\right)^{p-1}(1\vee l^2)}{n}\tag{B.76}$$

$$\Pi_1(n,T_{\max},N_{\max,q})\le\frac{C_{24}}{n^{4/3}},\tag{B.77}$$

where $K_{W,p} := \left((p+1+\mathbb{1}\{W=W_{[-R,R]}\})/(e\zeta_0)\right)^{p+1+\mathbb{1}\{W=W_{[-R,R]}\}}$,

$$C_{23} := \frac{\sqrt{2}K_1(\pi k_q)^p(e(1-6p\zeta_0))^{p+3/2}}{H_0^p(4Rx_0)^p(1+x_0^2)^p\,(2p+3)^{p+3/2}}$$

$$C_{24} := \frac{96(1+2c_0)Rx_0c_{\boldsymbol{X}}\zeta_0C_{20}(W,q)x_0^p}{(2\pi)^p\pi k_q\log(2)}\left(1+\frac{294c_{\boldsymbol{X}}^2e^{1/C_{23}}}{(2\pi)^p}\left(\left(\frac{4Rx_0}{\pi}\bigvee 1\right)\frac{H_0(1+x_0^2)}{k_q}\right)^{2p}\right)K_{W,p}.$$

Proof of (B.76). By definition of $N_{\max,q}$, we have, for all $t\neq 0, \left(7eN_{\max,q}^W(t)/(2\,|t|\,Rx_0)\right)^{2N_{\max,q}^W(t)}\leq n$. By Lemma B.13, we have

$$\begin{aligned}Z_{n_0}\Psi_n(t) &\leq M_{1,\mathcal{E},\eta}\left(2+\frac{1}{c_0}\right)\frac{c_{\boldsymbol{X}}}{nx_0^p}\left[\left(\frac{|tx_0|}{2\pi}\right)^p\nu_q^W(N_{\max,q}^W(t),tx_0)\frac{v(n_0,\mathcal{E})}{\delta(n_0)}\right]\\ &\quad+M_{1,\mathcal{E},\eta}\left(2+\frac{1}{c_0}\right)(2\pi)^p\left\|\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)\right\|_{L^2(\mathbb{R}^p)}^2\mathbb{E}\left[\left(\frac{|tx_0|}{2\pi}\right)^p\nu_q^W\left(\widehat{N}(t),tx_0\right)v(n_0,\mathcal{E})\right]\\ &\leq\frac{C_{21,\mathcal{E},\eta}(t)\ln(n)^p}{n^{1+\zeta}}.\end{aligned}$$

Then, using $T_{\max}\leq n^{\zeta_0}$, $\int_{\epsilon\leq|t|\leq T}\left\|\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](t,\cdot)\right\|_{L^2(\mathbb{R}^p)}^2dt\leq 2\pi l^2$, and (A.1) yield the result.

Proof of (B.77). Let $\epsilon\leq|t|\leq T_{\max}$. Using $N_{\max,q}(t)\leq((2\,|t|\,Rx_0/\pi)\vee 1)\ln(n)/k_q$, $T_{\max}\leq n^{\zeta_0}$,

$$\forall t:\ \epsilon\leq|t|\leq T_{\max},\quad K_n(t)\leq T_{\max}^{3p}\left(\left(\frac{4Rx_0}{\pi}\bigvee 1\right)\frac{H_0(1+x_0^2)}{k_q}\right)^p,$$

we obtain

$$\begin{aligned}\frac{K_1\sqrt{p_nn}}{K_n(t)}&\geq\frac{K_1(\pi k_q)^p\sqrt{2\log(n)n}}{H_0^p(4Rx_0)^p(1+x_0^2)^p\ln(n)^pT_{\max}^{3p}}\\ &\geq\frac{\sqrt{2}K_1(\pi k_q)^pn^{(1-6\zeta_0p)/2}}{H_0^p(4Rx_0)^p(1+x_0^2)^p\ln(n)^{p-1/2}}\geq C_{23}\ln(n)^2,\quad\text{(by (A.1))}.\end{aligned}\tag{B.78}$$

By (A.24) and (B.28), we have

$$\begin{aligned}&\Pi_1(n,T_{\max},N_{\max,q})\\ &\leq\frac{(1+2c_0)8c_{\boldsymbol{X}}K_{\max}}{K_1(2\pi)^pn}\int_{\epsilon\leq|t|\leq T_{\max}}N_{\max,q}^W(t)\,|t|^p\,\nu_q^W(N_{\max,q}^W(t),tx_0)dt\sup_{\epsilon\leq|t|\leq T_{\max}}\Psi_{0,n}(t)\\ &\leq\frac{16(1+2c_0)Rx_0c_{\boldsymbol{X}}\zeta_0\log(n)^2}{K_1(2\pi)^p\pi k_q\log(2)n}\int_{\epsilon\leq t\leq T_{\max}}|t|^{p+1}\,\nu_q^W(N_{\max,q}^W(t),tx_0)dt\sup_{\epsilon\leq|t|\leq T_{\max}}\Psi_{0,n}(t)\\ &\leq\frac{16(1+2c_0)Rx_0c_{\boldsymbol{X}}\zeta_0C_{20}^{W,q}}{K_1(2\pi)^p\pi k_q\log(2)}\log(n)^{p+1+\mathbb{1}\{W=W_{[-R,R]}\}}n^{(p+2)\zeta_0}\sup_{\epsilon\leq|t|\leq T_{\max}}\Psi_{0,n}(t)\\ &\leq\frac{96(1+2c_0)Rx_0c_{\boldsymbol{X}}\zeta_0C_{20}^{W,q}}{(2\pi)^p\pi k_q\log(2)}\left(\frac{\log(n)^{p+1+\mathbb{1}\{W=W_{[-R,R]}\}}}{n^{\zeta_0}}\right)\frac{n^2}{n^{2-(p+3)\zeta_0}}\sup_{\epsilon\leq|t|\leq T_{\max}}\Psi_{0,n}(t),\end{aligned}$$

where $2-(p+3)\zeta_0 > 4/3$ and, because $T_{\max}^{6p} \le n$, (B.78), and $\sup_{n>0}\left(e^{-C_{23}\ln(n)^2}n^2\right) = e^{1/C_{23}}$, which yields (B.77). $\square$

We complete the proof of Theorem 5 considering (A.38) in all smoothness cases.
Proof of (T5.2). Let $q=1$, $(n,n_0)\in\mathbb{N}^2$ such that $v(n_0,\mathcal{E})/\delta(n_0)\le n^{-(2+\zeta)}$, $N\in\mathcal{N}_n$, and $T\in\mathcal{T}_n$. Start from (A.38), where, because $w=W_{\mathcal{A}}$, the term $M^2/w(a_0)$ is zero and, using (B.48) this yields

$$\sup_{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,\widehat{N},\widehat{T},\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right)$$
$$\le \frac{C_{25,\mathcal{E},\eta}}{n} + \frac{2\pi C_{c_0,2}^2}{C}\left(\sup_{t\in\mathbb{R}}\frac{4l^2}{e^{2\kappa(N(t)\ln(N(t)+1)}}\int_{\epsilon\le|t|\le T}\widetilde{\Delta}_{2,q}\left(t,N(t),n,\frac{M_{1,\mathcal{E},\eta}v(n_0,\mathcal{E})}{\delta(n_0)}\right)dt + \frac{4l^2}{(1\vee T)^{2s}}\right).$$

Denote by T^* the closest element in $\mathcal{T}_n$ to the choice of T in (T3.2) and N^* the choice N in (T3.2). This yields $N^*(t)\le N_{\max,q}^W(t)$ hence $N^*\in\mathcal{N}_n$ and that $T^*\le n_e^{1/(2s)}\le T_{\max}=n^{1/(6p)}$ because $s>3p$, hence $T^*\in\mathcal{T}_n$. Thus, we obtain

$$\sup_{f_{\alpha,\boldsymbol{\beta}}\in\mathcal{H}_{w,W}^{q,\phi,\omega}(l,M)\cap\mathcal{D},\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}} \mathcal{R}_{n_0}^W\left(\widehat{f}_{\alpha,\boldsymbol{\beta}}^{q,\widehat{N},\widehat{T},\epsilon}, f_{\alpha,\boldsymbol{\beta}}\right)$$
$$\le \frac{C_{25,\mathcal{E},\eta}}{n} + \frac{2\pi C_{c_0,2}^2}{C}\sup_{t\in\mathbb{R}}\frac{4l^2}{e^{2\kappa(N^*(t)\ln(N^*(t)+1)}}\int_{\epsilon\le|t|\le T^*}\widetilde{\Delta}_{2,q}\left(t,N^*(t),n,\frac{M_{1,\mathcal{E},\eta}v(n_0,\mathcal{E})}{\delta(n_0)}\right)dt$$
$$+\frac{8\pi C_{c_0,2}^2 l^2}{C(1\vee T^*)^{2s}}.$$

This yields the result, using (B.63) with $(1\vee N(t))^p p_n$ replacing $(1\vee N(t))^p$.
Proof of (T5.3). The proof is similar to that of (T5.2), using that with T^* the closest element in $\mathcal{T}_n$ to the choice of T in (T3.3) and N^* the choice N in (T3.3), $T^*\le n_e^{\kappa/(2s(\kappa+k_q))}\le T_{\max}=n^{1/(10p)}$ because $s>5p\kappa/(\kappa+k_q)$, where $\zeta_0\le 1/(10p)$, $N^*\in\mathcal{N}_{\max}$, and replace C_{23} by

$$C_{26} := \frac{\sqrt{2}K_1(\pi k_q)^p(e(1-10p\zeta_0))^{p+3/2}}{H_0^p(4Rx_0)^p\left(1\vee x_0^4\right)^p(2p+3)^{p+3/2}}. \quad \square$$

Appendix B.3. Talagrand inequality for complex functions

Lemma B.16 is the Talagrand inequality (see Lemma 7.1 in [16]) for complex functions.

Lemma B.16. Let $n\in\mathbb{N}$, $\eta>0$, $X_1,\dots,X_n$ be independent random variables and $\nu_n(u) := \sum_{i=1}^n\left(u(X_i)-\mathbb{E}\left[u(X_i)\right]\right)/n$, for all u in a countable class $\mathcal{U}$ of complex measurable functions. If there exist $M_1,H,v>0$ such that

$$\sup_{u\in\mathcal{U}}\|u\|_{L^\infty(\mathbb{R}^p)}\le M_1, \tag{B.79}$$

$$\mathbb{E}\left[\sup_{u\in\mathcal{U}}|\nu_n(u)|\right]\le H, \tag{B.80}$$

$$\sup_{u\in\mathcal{U}}\frac{1}{n}\sum_{i=1}^{n}\max\left(\mathrm{Var}\left(\Re(u(X_i))\right),\mathrm{Var}\left(\Im(u(X_i))\right)\right)\le v, \tag{B.81}$$

then, with $K_2:=1/6$ and $\Lambda(\eta):=(\sqrt{1+\eta}-1)\wedge 1$, we have

$$\mathbb{E}\left[\left(\sup_{u\in\mathcal{U}}|\nu_n(u)|^2-4(1+2\eta)H^2\right)_+\right]\le\frac{8}{K_2}\left(\frac{v}{n}e^{-K_2\eta\frac{nH^2}{v}}+\frac{49M_1^2}{K_2\Lambda(\eta)^2n^2}e^{-\frac{\sqrt{2}K_2\Lambda(\eta)\sqrt{\eta}}{7}\frac{nH}{M_1}}\right).$$

Proof. We use

$$\mathbb{E}\left[\left(\sup_{u\in\mathcal{U}}|\nu_n(u)|^2-4(1+2\eta)H^2\right)_+\right]\le\mathbb{E}\left[\left(\sup_{u\in\mathcal{U}}\Re(\nu_n(u))^2+\sup_{u\in\mathcal{U}}\Im(\nu_n(u))^2-4(1+2\eta)H^2\right)_+\right]$$
$$\le\mathbb{E}\left[\left(\sup_{u\in\mathcal{U}}\Re(\nu_n(u))^2-2(1+2\eta)H^2\right)_+\right]+\mathbb{E}\left[\left(\sup_{u\in\mathcal{U}}\Im(\nu_n(u))^2-2(1+2\eta)H^2\right)_+\right]$$

and apply Lemma 7.1 in [16] to both terms. □

Appendix B.4. Relation to Sobolev ellipsoids.

Define, for $q=1,\infty$ and $\mathcal{F}[f](\cdot,\boldsymbol{k}):=(2R)^{-p/2}\int_{\mathbb{R}}e^{i\cdot a}\int_{[-R,R]^p}e^{i\pi\boldsymbol{k}^\top\boldsymbol{b}/R}\,f(a,\boldsymbol{b})da d\boldsymbol{b}$,

$$H^{q,\sigma,s}(l):=\left\{f:\int_{\mathbb{R}}\sum_{\boldsymbol{k}\in\mathbb{Z}^p}|\mathcal{F}[f](t,\boldsymbol{k})|^2\,|\boldsymbol{k}|_q^{2\sigma}\,(1\vee t^2)^s dt\le 2\pi l^2\right\}.$$

Lemma B.17. For all $\sigma,\delta,l,M,R>0$, $\sigma'=\sigma+1/2+\delta$, $(q',q)\in\{1,\infty\}^2$, $\phi=(1\vee|\cdot|)^s$, where $s>\sigma'+p/2$, and $(\omega_k)_{k\in\mathbb{N}_0}=(k^\sigma)_{k\in\mathbb{N}_0}$, there exists $A>0$ (see (B.87)) such that $H^{q',s,\sigma'}(l)\subseteq\mathcal{H}^{q,\phi,\omega}_{1,W_{[-R,R]}}\left(\sqrt{A}l\right)$.

Proof. In this proof, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the Euclidian scalar product and norm in $L^2([-R,R]^p)$. Take $f\in H^{q',s,\sigma'}(l)$. The Plancherel identity for the second equation yields

$$\int_{\mathbb{R}}\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}|b_{\boldsymbol{m}}(t)|^2\left(1\vee t^2\right)^s dt=\int_{\mathbb{R}}\|\mathcal{F}_1[f](t,\cdot)\|^2\left(1\vee t^2\right)^s dt$$
$$=\int_{\mathbb{R}}\sum_{\boldsymbol{k}\in\mathbb{Z}^p}|\mathcal{F}[f](t,\boldsymbol{k})|^2\left(1\vee t^2\right)^s dt\le 2\pi l^2.$$

Thus, f satisfies the first inequality in the definition of $\mathcal{H}^{q,\phi,\omega}_{1,W_{[-R,R]}}\left(\sqrt{A}l\right)$. Let us now check

$$\int_{\mathbb{R}}\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}|\boldsymbol{m}|_q^{2\sigma}\,|b_{\boldsymbol{m}}(t)|^2\,dt\le 2\pi A l^2. \tag{B.82}$$

To show (B.82) when $q=\infty$, we show that there exists A' such that

$$\forall N\in\mathbb{N},\ \int_{\mathbb{R}}\sum_{|\boldsymbol{m}|_\infty\geq N}|b_{\boldsymbol{m}}(t)|^2\,dt\leq\frac{2\pi A'l^2}{N^{2\sigma'}},\tag{B.83}$$

which yields (B.82) using $A:=A'\left(k_\infty/k_q\right)^{2\sigma}(1+1/(2\delta))$ because

$$\begin{aligned}\int_{\mathbb{R}}\sum_{\boldsymbol{m}\in\mathbb{N}_0^p}|\boldsymbol{m}|_\infty^{2\sigma}|b_{\boldsymbol{m}}(t)|^2\,dt&=\int_{\mathbb{R}}\sum_{\boldsymbol{m}\in\mathbb{N}_0^p,\boldsymbol{m}\neq\underline{\boldsymbol{0}}}\frac{|\boldsymbol{m}|_\infty^{2\sigma'}}{|\boldsymbol{m}|_\infty^{1+2\delta}}|b_{\boldsymbol{m}}(t)|^2\,dt\\&=\sum_{k=1}^{\infty}\frac{1}{k^{1+2\delta}}\left(\int_{\mathbb{R}}\sum_{|\boldsymbol{m}|_\infty\geq k}|b_{\boldsymbol{m}}(t)|^2k^{2\sigma'}dt\right)\quad\text{(by Fubini-Tonelli's theorem)}\\&\leq 2\pi A'l^2\left(\sum_{k=1}^{\infty}\frac{1}{k^{1+2\delta}}\right)\quad\text{(by (B.83))}\\&\leq 2\pi A'l^2\left(1+\frac{1}{2\delta}\right)\quad\text{(by integral test for convergence)}.\end{aligned}\tag{B.84}$$

We obtain the case $q=1$ using $|\boldsymbol{m}|_1\leq p\,|\boldsymbol{m}|_\infty$. (B.83) holds when $N=1$ so we show it when $N\geq 2$. Denote, for all $N\in\mathbb{N}$ and $c\neq 0$, by P_c^N the projector in $L^2\left(W_{[-R,R]}^{\otimes p}\right)$ onto the vector space spanned by $\left(\psi_{\boldsymbol{m}}^c\left(\cdot/R\right)/(2R)^{p/2}\right)_{|\boldsymbol{m}|_\infty<N}$. Because $\left\|\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)-P_{c(t)}^N\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)\right\|^2=\sum_{|\boldsymbol{m}|_\infty\geq N}|b_{\boldsymbol{m}}(t)|^2$, (B.83) becomes

$$\int_{\mathbb{R}}\left\|\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)-P_{c(t)}^N\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)\right\|^2dt\leq\frac{2\pi A'l^2}{N^{2\sigma'}}.\tag{B.85}$$

We now prove (B.85). Denote by $\mathcal{E}_c^N$ the projector in $L^2\left(W_{[-R,R]}^{\otimes p}\right)$ onto the vector space spanned by $\left(\phi_{\boldsymbol{m}}\left(\cdot/R\right)\right)_{\boldsymbol{m}\in\mathbb{Z}^p}:=\left(e^{i\pi\boldsymbol{m}^\top\cdot/R}/(2R)^{p/2}\right)_{\boldsymbol{m}\in\mathbb{Z}^p}$. Let $\overline{N}:=\lfloor\tau N\rfloor$ for some $\tau>0$ and $t\neq 0$. $\varphi:=\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)$ is such that

$$\begin{aligned}\left\|\varphi-P_{c(t)}^N\varphi\right\|^2&\leq 2\left[\left\|\varphi-\mathcal{E}^{\overline{N}}\varphi-P_{c(t)}^N\left(\varphi-\mathcal{E}^{\overline{N}}\varphi\right)\right\|^2+\left\|\mathcal{E}^{\overline{N}}\varphi-P_{c(t)}^N\mathcal{E}^{\overline{N}}\varphi\right\|^2\right]\\&\leq 2\left[\left\|\varphi-\mathcal{E}^{\overline{N}}\varphi\right\|^2+\left\|\mathcal{E}^{\overline{N}}\varphi-P_{c(t)}^N\mathcal{E}^{\overline{N}}\varphi\right\|^2\right].\end{aligned}$$

$K_R:=(2R)^p\left\|\mathcal{E}^{\overline{N}}\varphi-P_{c(t)}^N\mathcal{E}^{\overline{N}}\varphi\right\|^2$ satisfies, using the Cauchy-Schwarz inequality in the fourth display,

$$K_R=\left\|\sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}}\left\langle\varphi(\cdot),\phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle\phi_{\boldsymbol{k}}\left(\frac{\star}{R}\right)\right.$$

$$
\left. - \sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_\infty\geq N} \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \left\langle \varphi(\cdot), \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle \left\langle \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\star}{R}\right)\right\|^2
$$

$$
= \left\| \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \left\langle \varphi(\cdot), \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle \left(\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_\infty\geq N} \left\langle \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\star}{R}\right)\right)\right\|^2
$$

$$
= \sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_\infty\geq N} \left| \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \left\langle \varphi(\cdot), \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle \left\langle \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle \right|^2
$$

$$
\leq \sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_\infty\geq N} \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \left|\left\langle \varphi(\cdot), \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle\right|^2\right) \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \left|\left\langle \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle\right|^2\right)
$$

$$
\leq \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \left|\left\langle \varphi(\cdot), \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right)\right\rangle\right|^2\right) \left(\sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_\infty\geq N} \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \left|\left\langle \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle\right|^2\right)
$$

$$
\leq \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^p} \left|\mathcal{F}[\varphi](\boldsymbol{k})\right|^2\right) I_{N,\overline{N}}(t),
$$

$$
I_{N,\overline{N}}(t) := \sum_{\boldsymbol{m}\in\mathbb{N}_0^p:\ |\boldsymbol{m}|_\infty\geq N} \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \left|\left\langle \phi_{\boldsymbol{k}}\left(\frac{\cdot}{R}\right), \psi_{\boldsymbol{m}}^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle\right|^2.
$$

Denote, for all $(n,m)\in\mathbb{N}_0^2$, by $\beta_n^m = \left\langle \psi_m^{c(t)}, P_n\right\rangle_{L^2([-1,1])}$ and by J_k the Bessel function of the first kind and order $k>-1$. We have, using (74) in [7] for the third equality,

$$
\begin{aligned}
&\left\langle \phi_k\left(\frac{\cdot}{R}\right), \psi_m^{c(t)}\left(\frac{\cdot}{R}\right)\right\rangle \\
&= \left(\frac{R}{2}\right)^{1/2} \sum_{n=0}^{\infty} \beta_n^m \left\langle e^{i\pi kx}, P_n\right\rangle_{L^2([-1,1])} \\
&= \left(\frac{R}{2}\right)^{1/2} \left(\sum_{n=0}^{\lfloor m/M\rfloor-1} \beta_n^m \left\langle e^{i\pi kx}, P_n\right\rangle_{L^2([-1,1])} + \sum_{n\geq\lfloor m/M\rfloor} \beta_n^m \sqrt{\frac{2}{k}}\sqrt{n+\frac{1}{2}} J_{n+1/2}(k\pi)\right) \\
&\leq (2R)^{1/2}\left(I_{m,k}\vee J_{m,k}\right), \\
&I_{m,k} := \sum_{n=0}^{\lfloor m/M\rfloor-1} \beta_n^m \left\langle e^{i\pi kx}, P_n\right\rangle,\ J_{m,k} := \sum_{n\geq\lfloor m/M\rfloor} \beta_n^m \sqrt{2/k}\sqrt{n+1/2} J_{n+1/2}(k\pi).
\end{aligned}
$$

Using, for all $k \in \mathbb{N}_0$, $\left|\left\langle e^{i\pi kx}, P_n\right\rangle\right| \leq \sqrt{2}$, Proposition 3 in [8] and $2\sqrt{\chi_m^{c(t)}} > c\left(|t|\right)$ for the second inequality, we obtain, for all $m \geq M \vee \left(ec\left(|t|\right)\right)$ and $M := 8/5$,

$$\begin{aligned}
|I_{m,k}| &\leq \sqrt{2} \sum_{n=0}^{\lfloor m/M \rfloor - 1} |\beta_n^m| \\
&\leq \sqrt{\frac{5}{2\pi}} \left|\mu_m^{c(t)}\right| \int_0^{\lfloor m/M \rfloor} \left(\frac{2\sqrt{\chi_m^{c(t)}}}{c\left(|t|\right)}\right)^x dx \leq \frac{\sqrt{5/(2\pi)}\left|\mu_m^{c(t)}\right|}{\ln\left(2\sqrt{\chi_m^{c(t)}}/c\left(|t|\right)\right)} \left(\frac{2\sqrt{\chi_m^{c(t)}}}{c\left(|t|\right)}\right)^{\lfloor m/M \rfloor}.
\end{aligned}$$

Thus, for all $m \geq M \vee \left(e^2 c\left(|t|\right)\right)$, using Lemma B.4 and (3.4) page 34 in [44] which yields $\ln\left(2\sqrt{\chi_m^{c(t)}}/c\left(|t|\right)\right) \geq \ln(2) + 2$ for the first inequality, and decomposing the exponent as $m = m/M + (1 - 1/M)m$ for the third, we obtain

$$\begin{aligned}
|I_{m,k}| &\leq \sqrt{\frac{5e^3}{9}} \frac{1}{\ln(2)+2} \left(\frac{2\sqrt{m(m+1)+c(t)^2}}{c\left(|t|\right)}\right)^{m/M} \left(\frac{ec\left(|t|\right)}{4(m+1/2)}\right)^m \\
&\leq \sqrt{\frac{5e^3}{9}} \frac{1}{\ln(2)+2} \left(\frac{2\sqrt{1+1/e^2}(m+1)}{c\left(|t|\right)}\right)^{m/M} \left(\frac{ec\left(|t|\right)}{4m}\right)^m \\
&\leq \sqrt{\frac{5e^3}{9}} \frac{1}{\ln(2)+2} \left(\frac{\sqrt{e^2+1}(m+1)}{(2m+1)}\right)^{m/M} \left(\frac{1}{4}\right)^{(1-1/M)m} \exp\left(-\left(1-\frac{1}{M}\right) m \ln\left(\frac{m}{ec\left(|t|\right)}\right)\right) \\
&\leq \sqrt{\frac{5e^3}{9}} \frac{\sqrt{e^2+1}(M+1)}{(\ln(2)+2)4^{M-1}(2M+1)} \exp\left(-\left(1-\frac{1}{M}\right) m \ln\left(\frac{m}{ec\left(|t|\right)}\right)\right) \\
&\leq \alpha_M \exp\left(-\left(1-\frac{1}{M}\right) m \ln\left(\frac{m}{ec\left(|t|\right)}\right)\right),
\end{aligned}$$

$\alpha_M := \sqrt{5e^3/9}\sqrt{e^2+1}(M+1)/\left((\ln(2)+2)4^{M-1}(2M+1)\right)$. Using, for all $\alpha > -1/2$ and $x \in \mathbb{R}$, $|J_\alpha(x)| \leq |x|^\alpha / \left(2^\alpha \Gamma(\alpha+1)\right)$ (see 9.1.20 in [1]) and $|\beta_n^m| \leq 1$ for the first display, and $m > M$ and $n! \geq (n/e)^n \sqrt{2\pi n}$ for the third, we obtain, for all $k \in \mathbb{N}_0$,

$$\begin{aligned}
|J_{m,k}| &\leq \sum_{n \geq \lfloor m/M \rfloor} \frac{1}{n!} \left(\frac{k\pi}{2}\right)^n \\
&\leq \frac{1}{\lfloor m/M \rfloor !} \left(\frac{k\pi}{2}\right)^{\lfloor m/M \rfloor} \exp\left(\frac{k\pi}{2}\right) \leq \sqrt{\frac{m\theta_{M/m}}{2\pi M e^2}} \exp\left(-\frac{m}{M} \ln\left(\frac{2m\theta_{M/m}}{k\pi e M}\right)\right) \exp\left(\frac{k\pi}{2}\right),
\end{aligned}$$

where $\theta_{M/m} := 1 - M/m$. We have, taking $\overline{N} = \lfloor \tau N \rfloor$,

$$\forall t: \ N \geq M \vee \left(e^2 c\left(|t|\right)\right), \ I_{N,\overline{N}}(t) \leq (2R)^p \left(I_1(t) \vee J_1\right),$$

where $I_1(t) := \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \sum_{m\ge N} m^p \alpha_M^p \exp\left(-2p\left(1-1/M\right) m \ln\left(m/(ec\left(|t|\right))\right)\right)$ and $J_1 := \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \sum_{m\ge N} m^p \prod_{j=1}^p \left(m\theta_{M/m}/(2\pi M e^2)\exp\left(-2m\ln\left(2m\theta_{M/m}/(k_j\pi e M)\right)/M\right)\exp\left(k_j\pi\right)\right)$. Then, denoting by $\gamma_M := \ln\left(2/(\tau\pi e M)\right)/M$ and using that $\overline{N}-1\le\tau N$ and that $\sup_{t\ge1} m^{2p}e^{-\gamma_M m} = e((2p-1)/(p\gamma_M e))^{2p}$ for the second display, taking $\tau = 2/(\pi e^2 M)$ for the last display, and $\gamma = \gamma_M - \tau\pi > 3/4$, we obtain, denoting by $\alpha_{M,J} := \left(2\theta_{M/m}/\left(\pi M\gamma_M^2 e^4\right)\right)^p/(p\gamma_M)$,

$$
\begin{aligned}
J_1 &\le \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \sum_{m\in\mathbb{N}_0:\ m\ge N} \left(\frac{m^2\theta_{M/m}}{2\pi M e^2}\exp\left(-\frac{2m}{M}\ln\left(\frac{2m}{\overline{N}\pi e M}\right)\right)\exp\left(\overline{N}\pi\right)\right)^p \\
&\le \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \left(\frac{\theta_{M/m}\exp\left(\tau N\pi\right)}{2\pi M e^2}\right)^p \left(\frac{2}{e\gamma_M}\right)^{2p} \int_N^\infty \exp\left(-pm\gamma_M\right) dm \\
&\le \sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \alpha_{M,J}\exp\left(-p\gamma N\right) \le (2\tau N+1)^p \alpha_{M,J}\exp\left(-p\gamma N\right).
\end{aligned}
$$

Denoting by $\kappa_M(t) := -2\left(1-1/M\right)\ln\left(ec\left(|t|\right)/N\right)$ and using $\sup_{m\ge1} m^p e^{-p\kappa_M(t)m} = (\kappa_M(t)e)^{-p}$ for the second display, we obtain, for all $N > M \vee \left(e^2 c\left(|t|\right)\right)$,

$$
\begin{aligned}
I_1(t) &\le (2\overline{N}+1)^p\alpha_M^p \sum_{m\in\mathbb{N}_0:\ m\ge N} m^p \exp\left(-2p\left(1-\frac{1}{M}\right) m \ln\left(\frac{N}{ec\left(|t|\right)}\right)\right) \\
&\le \left(\frac{(2\overline{N}+1)\alpha_M}{\kappa_M(t)e}\right)^p \int_N^\infty \exp\left(-p\kappa_M(t)m\right) dm \le \left(\frac{(2\tau N+1)\alpha_M}{\kappa_M(t)e}\right)^p \frac{\exp\left(-p\kappa_M(t)N\right)}{p\kappa_M(t)}, \\
I_{N,\overline{N}}(t) &\le (2R(2\tau N+1))^p \left(\alpha_{M,J}e^{-p\gamma N} \bigvee \left(\left(\frac{\alpha_M}{\kappa_M(t)e}\right)^p \frac{1}{p\kappa_M(t)} e^{-p\kappa_M(t)N}\right)\right) \\
&\le N^p\left(2R(2\tau+1)\right)^p \left(\alpha_{M,J} \bigvee \left(\left(\frac{\alpha_M}{\kappa_M(t)e}\right)^p \frac{1}{p\kappa_M(t)}\right)\right) e^{-p(\gamma\wedge\kappa_M(t))N},
\end{aligned}
$$

(B.86)

$$
\sup_{t:\ |t|\le N/(e^2x_0)} I_{N,\overline{N}}(t) \le N^p\left(2R(2\tau+1)\right)^p \left(\alpha_{M,J} \bigvee \left(\frac{(\alpha_M/((1-1/M)e))^p}{2p(1-1/M)}\right)\right) e^{-p(\gamma\wedge(2(1-1/M)))N}.
$$

Denote by $\tilde{\tau} := (2\tau+1)^p\left(\alpha_{M,J}\bigvee\left(((1-1/M)e/\alpha_M)^{-p}/(2p(1-1/M))\right)\right)$ and $\tilde{\gamma} := p\gamma\wedge(2(1-1/M))$. Using, for all $|t| > N/(e^2x_0)$, $I_{N,\overline{N}}(t) \le R^{2p}\sum_{\boldsymbol{k}\in\mathbb{Z}^p:\ |\boldsymbol{k}|_\infty<\overline{N}} \|\phi_{\boldsymbol{k}}(\cdot)\|^2_{L^2([-1,1]^p)} \le R^{2p}(2\overline{N}+1)^p$ and $f\in H^{q',s,\sigma'}(l)$, $s > \sigma' + p/2$ for the first display and (A.1) for the second, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \left\|\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot) - P_{c(t)}^N\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)\right\|^2 dt \\
&\le \frac{2}{(2R)^p}\sup_{t:\ |t|\le N/(e^2x_0)} I_{N,\overline{N}}(t)\int_{-N/(e^2x_0)}^{N/(e^2x_0)} \sum_{\boldsymbol{k}\in\mathbb{Z}^p} \left|\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\boldsymbol{k})\right|^2 dt \\
&\quad + \frac{2(2\tau+1)^pN^pR^p}{2^p}\frac{1}{(1\vee(N/(e^2x_0))^2)^s}\int_{|t|>-N/(e^2x_0)} \sum_{\boldsymbol{k}\in\mathbb{Z}^p} \left|\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\boldsymbol{k})\right|^2 (1\vee t^2)^s dt
\end{aligned}
$$

$$+ 2\int_{\mathbb{R}}\left\|\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)-\mathcal{E}^{\overline{N}}\mathcal{F}_{1\mathrm{st}}\left[f\right](t,\cdot)\right\|^2dt$$
$$\le 4\tilde{\tau}N^pe^{-\tilde{\gamma}pN}\pi l^2+\frac{4(2\tau+1)^pR^p\pi l^2}{2^p}\frac{N^p}{(1\vee(N/(e^2x_0))^{2\sigma'+p})}+\frac{4\pi l^2\tau^{2\sigma'}}{(2R)^pN^{2\sigma'}}.$$

This yields (B.85) hence (B.83) and (B.82) with
(B.87)

$$A:=2\left(\frac{k_\infty}{k_q}\right)^{2\sigma}\left(1+\frac{1}{2\delta}\right)\left(\tilde{\tau}\left(\frac{p+2\sigma'}{\tilde{\gamma}pe}\right)^{p+2\sigma'}+\frac{(2\tau+1)^pR^p}{2^p}(e^2x_0)^{2\sigma'+p}+\frac{\tau^{2\sigma'}}{(2R)^p}\right).\square$$

Appendix B.5. Estimation of the marginal $f_{\boldsymbol{\beta}}$

For all $(\omega_m)_{m\in\mathbb{N}_0}$ increasing, $\omega_0=1$, $l,M>0$, $q\in\{1,\infty\}$, consider

$$\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M):=\left\{f:\|f\|_{L^2(w\otimes W^{\otimes p})}\le M,\ \sum_{k\in\mathbb{N}_0}\omega_k\|\theta_{q,k}\|^2_{L^2(\mathbb{R})}\le 2\pi l^2\right\}.$$

For the sake of brevity, we consider one of multiple sets of assumptions on the estimand and the estimator $\widehat{f}^{q,N,\epsilon}_{\beta}:=\sum_{|\boldsymbol{m}|_q\le N(\epsilon)}\widehat{c}_{\boldsymbol{m}}(\epsilon)\varphi^{W,\epsilon x_0}_{\boldsymbol{m}}/\sigma^{W,\epsilon x_0}_{\boldsymbol{m}}$. It does not involve integration or interpolation and is based on $f_{\boldsymbol{\beta}}=\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](0,\cdot)$. An alternative is to rely on $\int_{\mathbb{R}}\widehat{f}^{q,N,T,\epsilon}_{\alpha,\beta}(a,\cdot)da$.

Proposition B.3. Take $q\in\{1,\infty\}$, $W=W_{[-R,R]}$, $\phi(\cdot)=1\vee|\cdot|^{2s}$, $(\omega_k)_{k\in\mathbb{N}_0}=\left(k^{2\sigma}\right)_{k\in\mathbb{N}_0}$, $w=1\vee|\cdot|^{3+\delta_1}$, $\sigma>2$, $R,\delta_1,s>0$, $0<l,M<\infty$, $\epsilon=(\ln(n_e)/\ln_2(n_e))^{-\sigma}$, $n_e=n\wedge(\delta(n_0)/v(n_0,\mathcal{E}))$, $N=\lfloor\overline{N}\rfloor$, and $\overline{N}=\ln(n_e)/(4k_q(1+\sigma)\ln_2(n_e))$, we have

$$\sup_{f_{\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap D,\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}}\mathbb{E}\left[\left\|\widehat{f}^{q,N,\epsilon}_{\boldsymbol{\beta}}-f_{\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^p)}\right]\left(\frac{\ln(n_e)}{\ln_2(n_e)}\right)^{2\sigma}=O_p(1).$$

Proof. We assume $f_{\boldsymbol{X}|\mathcal{X}}$ is known. The general case can be handled like in the proof of (T1.1). Denote by $f^{\epsilon}_{\boldsymbol{\beta}}:=\mathcal{F}_{1\mathrm{st}}\left[f_{\alpha,\boldsymbol{\beta}}\right](\epsilon,\cdot)$ and define $f^{q,\epsilon,N}_{\boldsymbol{\beta}}$ like $\widehat{f}^{q,\epsilon,N}_{\boldsymbol{\beta}}$ with $\widetilde{c}_{\boldsymbol{m}}(t)$ (see above Lemma A.2) instead of $\widehat{c}_{\boldsymbol{m}}(t)$. Use $\left\|\widehat{f}^{q,N,\epsilon}_{\boldsymbol{\beta}}-f_{\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^p)}\le 3\sum_{j=1}^3\|R_j\|^2_{L^2(\mathbb{R}^p)}$, where $R_1:=\widehat{f}_{\boldsymbol{\beta}}-f^{q,N,\epsilon}_{\boldsymbol{\beta}}$, $R_2:=f^{q,N,\epsilon}_{\boldsymbol{\beta}}-f^{\epsilon}_{\boldsymbol{\beta}}$, and $R_3:=f^{\epsilon}_{\boldsymbol{\beta}}-f_{\boldsymbol{\beta}}$. Let $q=1$. The case $q=\infty$ can be treated similarly. By (A.24) and $(N+p-1)^p\le(pN)^{p-1}$, we have

$$\mathbb{E}\left[\|R_1\|^2_{L^2(\mathbb{R}^p)}\right]\le\frac{B}{n}N^p\exp\left(2N\ln\left(\frac{7e(N+1)}{2Rx_0\epsilon}\bigvee 1\right)\right), \tag{B.88}$$

where $B:=c_{\boldsymbol{X}}2p^{p-1}/\left[(2\pi)^p(p-1)!\right]$. We clearly have $\|R_2\|^2_{L^2(\mathbb{R}^{p+1})}\le 2\pi l^2/\omega_N$,

$$\|R_3\|^2_{L^2(\mathbb{R}^p)}\le\int_{[-R,R]^p}\left|\int_{\mathbb{R}}\left|e^{i\epsilon a}-1\right|f_{\alpha,\boldsymbol{\beta}}(a,\boldsymbol{b})da\right|^2d\boldsymbol{b}$$
$$\le\epsilon^2\int_{[-R,R]^p}\left|\int_{\mathbb{R}}|a|\,f_{\alpha,\boldsymbol{\beta}}(a,\boldsymbol{b})da\right|^2d\boldsymbol{b}\le\epsilon^2M^2\left(1+\frac{1}{\delta_1}\right). \tag{B.89}$$

The result follows from (B.45) and, by (B.88) and (B.89),

$$\sup_{f_{\boldsymbol{\beta}}\in\mathcal{H}^{q,\phi,\omega}_{w,W}(l,M)\cap D,\ f_{\boldsymbol{X}|\mathcal{X}}\in\mathcal{E}} \mathbb{E}\left[\left\|\widehat{f}^{q,N,\epsilon}_{\boldsymbol{\beta}}-f_{\boldsymbol{\beta}}\right\|^2_{L^2(\mathbb{R}^p)}\right] \le \frac{3}{N^{2\sigma}}\left(\frac{BN^p}{n}\left(\frac{7e(N+1)^{1+2\sigma}}{2Rx_0}\bigvee 1\right)^{2N} + l^2 + M^2\left(1+\frac{1}{\delta_1}\right)\right). \qquad \square$$