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Abstract in English

This paper examines two prominent approaches to design efficient mechanisms for debt renegotiation with dispersed bondholders: debt exchange offers that promise enhanced liquidation rights to a restricted number of tendering bondholders (favored under U.S. law), and collective action clauses that allow to alter core bond terms after a majority vote (favored under U.K. law). We use a dynamic contingent claims model with a debt overhang problem, where both hold-out and hold-in problems are present. We show that the former leads to a more efficient mitigation of the debt overhang problem than the latter. Dispersed debt is desirable, as exchange offers also achieve a larger and more efficient debt reduction relative to debt held by a single creditor.

JEL Numbers and Keywords

JEL Nos.: G12, G32, G33.
Keywords: Out-of-court Restructuring, Exchange Offer, Collective Action Clause, Exit Consent, Hold-out problem, Hold-in Problem, Trust Indenture Act.

Résumé en Français

Ce papier examine deux mécanismes mis en avant pour une renégociation éfficace de la dette, lorsque les détenteurs d'obligations sont dispersés: offres d'échange de dette, promettant des droits de liquidation accrus à un nombre limité de créanciers (favorisé par la loi américaine), et clauses d'action collective, permettant de modifier les termes du contrat initial d'obligation, suite à un vote à la majorité (favorisé par la loi britannique). Nous utilisons un model dynamique d'options réelles de l'entreprise surendettée, où des problèmes de hold-in et de hold-out sont présents. Nous montrons que le premier mécanisme conduit à une atténuation plus efficace du problème de surendettement que le deuxieme. La dispersion de dette est souhaitable, car les offres d'échanges permettent également d'obtenir une réduction de dette plus significative et efficace, que lorsque la dette est détenue par un seul créancier.

1 Introduction

In view of rising leverage levels worldwide by corporate borrowers (but also by governments, households and intermediaries), the question how debt can be restructured efficiently when borrowers come under strain and face the threat of default is of paramount importance. Borrowers will frequently resort to out-of-court renegotiation (workouts) in order to avoid formal bankruptcy procedures that are costly. In this paper, we focus on the out-of-court restructuring of corporate bonds, or more generally of widely dispersed debt. The renegotiation of diffusely held debt has received little attention in the dynamic debt pricing literature, but it is of particular importance because of hold-out problems among creditors, i.e. incentives to free-ride on the restructuring effort.[1]

We focus on arguably the two most important mechanisms that facilitate out-of-court restructurings among dispersed creditors, debt exchange offers with seniority transfer and collective action clauses. Both are natural devices to curtail free-rider options: debt exchange offers do so by transferring liquidation rights in the event of default from bondholders that are holding out to bondholders that are consenting, and collective action clauses impose the application of changes in the bond terms to all bondholders, thus effectively thwarting individual holdouts. The question we address in this paper is simple: which of these two procedures leads to more efficient workouts, and hence should be favored by policymakers and equityholders from an ex ante perspective?

In the U.S., exchange offers for corporate bonds are a common workaround against the provision of the Trust Indenture Act that prohibits any alteration of principal, interest or maturity without unanimous consent of bondholders. They are typically structured as a consent solicitation (a.k.a. exit consent), tying the offer to a majority vote on waiving seniority covenants and offering to consenting bondholders to exchange their old bonds for more senior new ones with a reduced coupon.[2] In this way, exchange offers transfer liquidation rights from holdouts to consenting bondholders.

Gertner and Scharfstein (1991) show, however, that hold-in problems may emerge, as the firm can potentially make coercive offers that make bondholders worse off than they would be if no renegotiation was possible. In view of the difficulties imposed by the unanimity clause of the Trust Indenture Act, authors including Roe (1987) and Gertner and Scharfstein (1991) have argued for a change of the rules in favor of allowing to alter core items of a bond by way

[1] In fact, the coordination or free-riding problems among multiple creditors are the primary reason why state-administered bankruptcy procedures are needed (e.g. Bulow and Shoven, 1978).

[2] A simple majority vote of bondholders is required in order to remove protective clauses from the debenture. Thus, exit consents limit an individual bondholder's freedom to retain the original bond, by diluting her claim when rejecting the exchange.

of a majority or supermajority vote among bondholders. Collective action clauses stipulating such majority rules are widely used e.g. for bonds governed by British law. Some countries including Germany and Chile have recently changed their law to allow for such collective action clauses in corporate bond workouts,[3] and the proposal has gained prominence in the discussion how to facilitate sovereign bond workouts. Proposals creating bond governance mechanisms, such as supertrustees (see Amihud, Garbade, and Kahan, 1999), that allow to impose changes of core bond terms on all bondholders are economically equivalent in our analysis.

Our paper analyzes debt renegotiation in a dynamic continuous-time model of the firm with an endogenous bankruptcy decision, in the tradition of Leland (1994). Management takes decisions in the best interest of shareholders and the driving uncertainty is over the revenues of the firm. Leverage pushes management to default too early relative to the first best, driven by debt overhang problems first identified by Myers (1977). Management can decide on the timing of renegotiation, but cannot make infinitely many renegotiation attempts.[4] Thus, when deciding when to launch a restructuring offer, management faces the following trade-off: an early offer means that the debt burden can be alleviated earlier, but a late offer allows to obtain a more drastic reduction. For simplicity we consider that management can make only one renegotiation offer and derive simple closed-form solutions for the timing and structure of renegotiation offers. Inefficiency of the various renegotiation options can conveniently be measured by how close their resulting default point comes to the first best.

We analyze the two out-of-court renegotiation mechanisms in turn. We show that debt exchange offers with seniority transfer are always more efficient at reducing the debt overhang problem than debt restructuring with collective action clauses. For a given initial coupon level, equityholders are ex ante (prior to renegotiation) and ex post better off with the former than with the latter. We argue that these results do not depend on hold-in effects.

We also show that the renegotiation outcome when debt is held by a single creditor is exactly the same as the renegotiation outcome when bonds with collective action clauses are held by dispersed creditors. Thus, we contribute to the literature on the choice between private and public (bond) debt by showing that debt dispersion can be beneficial, as exchange offers with seniority transfer achieve a larger and more efficient debt reduction than debt reorganization with a single creditor.

[3]See Allen (2011) and Berdejo (2016).

[4]Models with infinitely many rounds of renegotiation include Mella-Barral (1999) and Hege and Mella-Barral (2005). Models with a single or a finite number of renegotiation rounds include Christensen et al. (2014) and Moraux and Silaghi (2014).

Throughout, we assume efficient and frictionless renegotiation with dispersed bondholders, provided that they have incentives to make debt concessions. This naturally puts limitations on our analysis. In practice, bond exchange offers frequently fail when multiple creditors or complex debt structures are involved (e.g., Asquith, Gertner, and Scharfstein, 1994). Therefore, collective action clauses are often advocated to facilitate debt negotiation in the presence of difficulties of negotiating with multiple parties. While our analysis fully incorporates holdout (free-riding) incentives of individual dispersed bondholders, it does not address other reasons why renegotiation with many debtholders may fail, e.g. because of imperfect information or investors' behavioral limitations such as bounded rationality or inattention. For example, dispersed bondholders naturally are arm's length investors and hence subject to asymmetric information about the borrower's condition, obstacles that concentrated debtholders can more easily overcome (see Rajan, 1992). Mandated fiduciaries such as supertrustees may enjoy similar advantages or access to better information. We do not model the potential of collective action clauses to facilitate debt renegotiation for other reasons than rational holdout problems. Notably, collective action clauses are also part of formal bankruptcy procedures, for example of Chapter 11 in the U.S., where they help to bypass the deadlock in debt restructurings that require creditor consent.

This paper is related to various strands of the literature. It contributes to the literature on endogenous default decisions in dynamic valuation models, building on the seminal papers by Black and Cox (1976) and Leland (1994),[5] and in particular to work on out-of-court debt restructurings. Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997) Mella-Barral (1999), and Fan and Sundaresan (2000) include debt renegotiation prior to liquidation assuming that equityholders can take advantage of strategic debt service, and Sundaresan and Wang (2007) extends this framework to include investment decisions. Fan and Sundaresan (2000), Sarkar (2013) and Silaghi (2018) consider partial or full debt-for-equity swaps, and Nishihara and Shibata (2016) allow for asset sales. All these papers limit renegotiation to bargaining between equityholders and a single creditor. Our paper addresses renegotiation with dispersed bondholders and is therefore more closely related to Hege and Mella-Barral (2005). In contrast to the present paper, in Hege and Mella-Barral (2005) concentrated debt can be efficiently renegotiated through continuous restructuring, but suffers from a lower debt capacity because of strategic default threats.

Our paper is also related to two-period models addressing out-of-court restructuring with multiple creditors. Jackson (1986), Baird (1986), and Bulow and Shoven (1978) point to difficulties of out-of-court debt restructuring with multiple creditors due to hold-out problems

[5] Besides the debt overhang problem on which we focus in the vein of Leland (1994), the literature has also addressed asset subtitution or risk shifting problems, see Leland (1998) and Ericsson (2000). It has also included releverage options that we do not address, e.g. Goldstein, Ju and Leland (2001).

as an argument why reorganization laws are efficient.[6] Roe (1987) analyzes "exit consents", where in accepting an exchange, a bondholder is required to attach a vote in favour of removing existing protective restrictions on the issuance of new debt. Gertner and Scharfstein (1991) study efficiency implications of restrictions on workouts imposed by the U.S. Trust Indenture Act and Bankruptcy Code, and analyze the hold-in effect that is important in our study.

Our paper contributes to the literature on the choice between public debt (bonds) and private or concentrated (bank) debt, based on differences in their renegotiability. In Bolton and Scharfstein (1996), multiple creditors serve as a commitment device against strategic default. Hege (2003) and Becker and Josephson (2016) demonstrate that the choice also depends on the the efficiency of the reorganization law that works as a backstop. Detragiache and Garella (1996) shows that bondholders' incentives to hold out depend on the probability of being "pivotal" for success or failure of the exchange offer. In dynamic debt valuation models, Hackbarth, Hennessy and Leland (2007) analyze the optimal mix between bank and non-renegotiable public debt.[7]

While we focus on seniority transfer as a means of extorting concessions from a fraction of dispersed debtholders at the detriment of debtholders that are holding out (or not being offered renegotiation), other devices to dilute the value of continuing debt holders exist, including asset stripping from selling or transferring assets, and renegotation with some bondholders for shorter maturity claims that effectively give them precedence over riskier long-maturity claims, as in Brunnermeier and Oehmke (2013). Other models showing that shorter debt maturity offers better protection and limits the bargaining power of equityholders include Leland and Toft (1996) and Dangl and Zechner (2018).

The paper is organized as follows. Section 2 introduces the model and describes the first best and the solution in the absence of renegotiation. Section 3 addresses debt exchange offers with seniority transfer, whereas Section 4 considers collective action clauses. Section 5 compares the outcomes and provides an ordering of their efficiency based on the associated underinvestment costs. Section 6 concludes.

[6]The relationship between out-of-court renegotiation and Chapter 11-style reorganization is addressed in White (1994) and von Thadden, Berglöf, and Roland (2010). Annabi, Breton, and François (2012) analyze bargaining with different classes of debt during bankruptcy.

[7]There is also a vast literature on the choice between public and private debt based on asymmetric information, such as Diamond (1991), Rajan (1992), and Dewatripont and Maskin (1995).

2 Model of the Levered Firm

2.1 Setup

We consider a cash flow model of the firm in continuous time in the tradition for instance of Mella-Barral (1999) or Goldstein, Ju, and Leland (2001). Consider a firm with an instantaneous productive capacity, a flow of costs, $w \in \mathbb{R}_{>0}$, and an uncertain flow of revenue, $x_t \in \mathbb{R}_{>0}$. The revenue is the single source of uncertainty and it follows a geometric Brownian motion:

$$dx_t \quad = \quad x_t \, \mu \, dt \quad + \quad x_t \, \sigma \, dz_t \tag{1}$$

where $\mu \in \mathbb{R}_{>0}$ and $\sigma \in \mathbb{R}_{>0}$ are exogenous parameters.

Assume risk neutrality and a constant identical borrowing and lending interest rate $\rho \in \mathbb{R}_{>0}$, where $\rho > \mu$.[8] Denote by $T_y \equiv \inf\{\, T \mid x_T = y \,\}$ the first time at which the state variable x_t hits a lower level $y \in \mathbb{R}_{>0}$ where $y < x_t$, and by $f_t(T_y)$ the density of T_y conditional on information at t. The probability-weighted discount factor at date t for \$1 to be received at date T_y is given by the Laplace transform of $f_t(T_y)$:

$$\left(\frac{x_t}{y}\right)^{\lambda} \quad = \quad \int_t^{\infty} e^{-\rho(T_y - t)} f_t(T_y) \; d\,T_y \ , \tag{2}$$

where λ is the negative root of $\sigma^2/2(\lambda^2 - \lambda) + \mu\lambda = \rho$.[9] It is standard to determine the bankruptcy threshold that is optimal for equityholders given the debt service. Define, for any given constant debt service $z \in \mathbb{R}_{>0}$, this bankruptcy threshold function as:

$$\underline{x}(z) \;\equiv\; \frac{-\lambda}{1-\lambda} \, \frac{\rho - \mu}{\rho} \, (w \,+\, z). \tag{3}$$

Management is assumed to act in the interests of shareholders. Thus, we ignore governance problems (principal-agent conflicts) between management and shareholders that are analyzed e.g. in Hart (1993).

We assume that the firm has issued a unit mass of perpetual bond claims promising an instantaneous flow of coupon payments, $\delta \in \mathbb{R}_{>0}$. With a unit mass of debt, δ measures both the aggregate level of debt service and the coupon. The perpetual bonds have an

[8] $\rho > \mu$ otherwise values are not finite. In a non-risk neutral world the results of the paper would hold for risk-adjusted probabilities (i.e. under an equivalent martingale measure, as discussed in Harrison and Kreps (1979)).

[9] See Karlin and Taylor (1975), page 363.

aggregate face value or principal of F, which for simplicity we set to $F = \frac{\delta}{\rho}$.[10] We consider a level of borrowing such that the bonds are subject to default risk, i.e δ high enough.[11] Furthermore, the corporate bond issue is fully dispersed among a continuum of atomistic bondholders. That is, each bondholder rationally disregards any impact of her decision on the post-renegotiation debt level and ensuing bankruptcy threshold. When discussing debt renegotiation below, we will also discuss the consequences of a debt structure where bondholdings are concentrated, in particular if there is a single creditor. We assume that all debt is initially of identical seniority. By focusing on perpetual debt, we also abstract from variations in debt maturity.

If contractual coupon payments are not met, debtholders have the option to force the firm into bankruptcy. Management can also decide to file for bankruptcy without debt default under the same conditions. In the event of bankruptcy, the firm's assets may be employed under new ownership, but the bankruptcy procedure involves a proportional (direct and indirect) bankruptcy cost of $\alpha \in \mathbb{R}_{>0}$ of the pre-bankruptcy unlevered firm value.[12] Except for the bankruptcy cost α, the bankruptcy procedure avoids further inefficiencies with regard to the future use of the assets.[13] In particular, we assume that the bankruptcy procedure leaves the firm with no debt obligations, and respects the absolute priority rule (APR) so that existing equityholders are fully wiped out and existing debtholders receive compensation or claims worth 100% of the post-bankruptcy asset value. This framework is entirely consistent with a debt-for-equity swap that is a staple in many bankruptcy procedures. However, we remain agnostic about the question whether the procedure captures bankruptcy reorganization (such as Chapter 11 of the U.S. Bankruptcy Code or similar procedures that many countries have adopted over the last 30 years), a liquidation procedure (such as Chapter 7 of the U.S. Bankruptcy Code), or a bankruptcy auction. Our basic assumptions are in fact compatible with any of these procedures as long as the outcome is efficient.

We do not consider taxes in this paper and assume that the corporate tax rate $\tau = 0$.

[10] Corporate perpetual bonds are frequently issued with an explicit face value, since they are often issued as callable (redeemable) bonds, i.e. perpetual bonds where the issuer has the option to redeem them at face value. Although $F = \frac{\delta}{\rho}$ is a natural reference point for the face value, all our results go through for any $F \geq \frac{\delta}{\rho}$.

[11] This assumption is formally stated in (9) below, after we introduce the liquidation value. As $F \geq \frac{\delta}{\rho}$, when the debt is risky, management never has an incentive to redeem the bonds at face value. Hence, we do not need to specify whether the perpetual bond is redeemable (callable) or not, as our analysis would be identical in both cases.

[12] This assumption corresponds to the set-up in Leland (1994). Also, most empirical studies estimate bankruptcy costs as a proportion of the pre-bankruptcy firm value, e.g. Bris, Welch, and Zhu (2006).

[13] This is in contrast to many real-world procedures that are often described as creating distortions, easy to manipulate or advantageous to shareholders and hence triggering early default.

This is a simplifying assumption since our model focuses on a comparison of various debt structures and their capacity to resolve debt overhang problems (Myers, 1977), arguably the most important problem of financial distress in corporations. Debt overhang problems occur when indebted firms (or organizations) are forced to forego value-creating investment (or other) opportunities or to default too early relative to the efficient stopping time. Corporate taxes do not alter the relative efficiency properties of various debt contracts in the face of debt overhang. All of our results go through if we introduce a constant corporate tax rate $\tau > 0$ in the spirit of Leland (1994) who adapts the static trade-off model to his seminal dynamic version of the fundamental trade-off between debt overhang problem and tax benefits. The debt coupon δ is exogenously given; we do not consider the optimal capital structure (in fact, our model lacks a trade-off that would allow us to consider capital structure questions).

We finally turn to the model assumptions concerning the different mechanisms for out-of-court debt renegotiation (also called workouts) that are at the heart of this paper. We assume that debt renegotiation always occurs in the form of a take-it-or-leave-it offer by management to debtholders to permanently reduce the coupon, and that management can make only one renegotiation offer. Specifically, if the offer is rejected, the coupon δ remains in place, and management decides in turn to either continue operating the firm or file for bankruptcy (in the best interests of equityholders). After the offer, irrespective of the outcome, no further offer can be proposed later. This assumption corresponds to the set-up in Christensen et al. (2014) and Moraux and Silaghi (2014).[14] This assumption may limit the scope of our contribution with regard to the comparison between multiple creditors and a single creditor, since for the latter, multiple rounds of renegotiation might be more easily feasible. It means that management decision whether to declare bankruptcy or continue is contingent on the state x_t: there will be a threshold (which we will identify later) such that management will continue the firm if x_t is above this threshold, and file for bankruptcy for values below.

This assumption differs from the strategic debt service assumption, considered in Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997), and Hege and Mella-Barral (2005). In those papers, shareholders are assumed to have such bargaining power that after a debt renegotiation offer, creditors will not receive more than they would obtain if the firm had instead filed for bankruptcy, independently of the creditors' decision on the renegotiation offer. Instead, we assume here that if creditors reject a debt renegotiation offer, the original

[14] Mella-Barral (1999) studies infinitely many successive reorganizations as a sequence of *marginal* reductions of the contractual coupon for a *single creditor*. Hege and Mella-Barral (2005) consider infinitely many successive reorganizations for *multiple creditors*. In these models, the equity value is then the expected discounted stream of equityholders income plus a continuum of options to propose a reduction of the coupon level by a marginal amount. These options are only gradually offered as the revenue falls below its past minimum value.

debt contract remains in place and will be serviced until it is optimal for equityholders to file for bankruptcy.

Many renegotiation offers are structured as debt deferments, where the coupon concession is accompanied by an increase in the face value of the reorganized debt. For simplicity, we only consider negotiated debt deferments. We assume that any reduction of coupon δ is associated with an increase in face value F of the reorganized debt by the capitalized amount of the expected value of the coupon reduction.[15]

2.2 Benchmark: First-Best and Valuation without Debt Reorganization

We begin with the analysis of the first best, that is the firm value and the liquidation or bankruptcy decision if the firm has no leverage.

Denote the value of the unlevered firm by $U(x_t)$ and denote the post-bankruptcy value of the firm by $V^B(x_t)$. We show in the Appendix:

$$U(x_t) \quad = \quad \left[\frac{x_t}{\rho-\mu} \; - \; \frac{w}{\rho}\right] \; + \; \left[V^B(\underline{x}^{fb}) - \left(\frac{\underline{x}^{fb}}{\rho-\mu} \; - \; \frac{w}{\rho}\right)\right]\left(\frac{x_t}{\underline{x}^{fb}}\right)^{\lambda} , \tag{4}$$

where $\underline{x}^{fb}$ is the threshold level of revenues at which bankruptcy is triggered.

The first term in expression (4) expresses the value of the perpetual claim on the firm's net cash flow $x_t - w$, and the second term the put option value of the option to abandon the firm at $\underline{x}^{fb}$. Then the post-bankruptcy value of the firm $V^B(x_t)$ is equal to the fraction of $(1-\alpha)$ of the entreprise value of the unlevered firm:

$$V^B(x_t) \quad = \quad (1-\alpha)\left[\frac{x_t}{\rho-\mu} \; - \; \frac{w}{\rho}\right] \; + \; (1-\alpha)\left[- \; \frac{\underline{x}^{fb}}{\rho-\mu} \; + \; \frac{w}{\rho}\right]\left(\frac{x_t}{\underline{x}^{fb}}\right)^{\lambda} . \tag{5}$$

The first-best bankruptcy threshold, $\underline{x}^{fb}$, maximizes the value of the unlevered firm, $U(x_t)$. Hence

$$\underline{x}^{fb} \; = \; \underline{x}(0) \; , \tag{6}$$

where $\underline{x}(.)$ is defined in (3) and $V^B(\underline{x}(0)) = 0$.

We consider next the levered firm, with continuous coupon obligation δ, for the benchmark case in which debt cannot be reorganized before bankruptcy. Denote by $S(x_t)$ and

[15]This assumption is formally stated in (11) below, after we parameterize a renegotiation offer. Given that we consider a perpetual bond, the only role of F in our model is to determine the allocation of the bankruptcy proceeds between various classes of bondholders and equityholders.

$B(x_t)$ the value of the firm's shares and bonds. If management decides to stop servicing the debt, debtholders take action and trigger the bankruptcy procedure. Thus, the bankruptcy decision is effectively taken by management in order to maximize equity value. We show:

Proposition 1 *In the absence of debt reorganization, the values of equity and bonds are*

$$S(x_t) \quad = \quad \left[\frac{x_t}{\rho-\mu} - \frac{w+\delta}{\rho}\right] + \left[-\frac{\underline{x}(\delta)}{\rho-\mu} + \frac{w+\delta}{\rho}\right]\left(\frac{x_t}{\underline{x}(\delta)}\right)^{\lambda}, \tag{7}$$

$$B(x_t) \quad = \quad \frac{\delta}{\rho} - \left[\frac{\delta}{\rho} - V^B(\underline{x}(\delta))\right]\left(\frac{x_t}{\underline{x}(\delta)}\right)^{\lambda}, \tag{8}$$

where $\underline{x}(\delta)$ is defined as in eq. (3) (with δ replacing z).

The linear part in equity value $S(x_t)$ and bond value $B(x_t)$ represents the asymptotic value of the security, i.e. the value if there was no risk of bankruptcy; the non linear part in $S(x_t)$ and $B(x_t)$, therefore, represents the value of the limited liability put option held by equityholders and written by the debtholders, respectively.

The value of revenues at which management triggers bankruptcy, $\underline{x}(\delta)$, maximizes the value of the equity, $S(x_t)$. The optimal choice of bankruptcy threshold implies $\underline{x}(\delta) > \underline{x}(0)$. A levered firm ($\delta > 0$), therefore, closes down the firm "too early" relative to the first-best closure rule.

Our assumption that the level of borrowing is such that the bonds are subject to default risk can now be formally stated: the instantaneous promised coupon

$$\delta \quad > \quad \rho\, V^B(\underline{x}(\delta)) \, . \tag{9}$$

3 Bond Exchange Offers with Seniority Transfer

3.1 Seniority, Hold-out, and Hold-in

Since the bond issue is dispersed among atomistic shareholders, the take-it-or-leave-it renegotiation offer takes the form of a debt exchange offer made by management to individual bondholders; each bondholder decides independently whether to exchange (tender) or to keep her existing debt claim with coupon δ. When management proposes a *pari passu* exchange offer, i.e. a new reduced coupon of identical seniority, an obvious *hold-out* or free-rider problem emerges that is extensively discussed in Roe (1987), Gertner and Scharfstein (1991), and Hege and Mella-Barral (2005). If some bondholders tender, the probability of bankruptcy is reduced as tendering bondholders accept new bonds with reduced coupons, and hence the total debt service is reduced. As a consequence, the value of the debt claims of hold-outs (non-tendering bondholders) is increased and the offer cannot succeed. This hold-out problem will effectively thwart any attempt to renegotiate debt with a pari passu offer.

We assume, therefore, that bond exchange offers systematically propose an exchange for a new bond with reduced coupon but higher seniority, i.e. include an increase of the tendering bondholders' liquidation right (their claim value in the event of bankruptcy).

Many jurisdictions tolerate exchange offers that grant enhanced liquidation rights relative to the position of bondholders that do not participate in the bond exchange, through techniques including seniority, collateralization, and asset stripping. Section 316(b) of the U.S. Trust Indenture Act of 1939 governs all U.S. corporate bonds, and prohibits any change to a core term of a bond contract, including the amount of the principal, the interest rate, or the maturity unless the proposed contract change receives the unanimous consent of all bondholders. The only feasible form of public debt restructuring, therefore, consists in voluntary exchange offers. In practice, firms tie exchange offers to consent solicitations that require only majority acceptance. Typically, they make a proposal to bondholders, called an exit consent, that simultaneously strips the existing debt of covenants protecting it from dilution through more senior debt issues *and* offers to exchange existing debt for new senior debt claims. Furthermore, the offer to exchange existing debt for new debt is only available to bondholders that have agreed to the exit consent.

Gertner and Scharfstein (1991) show that offering an exchange for new debt which is *senior* to the old one can overcome the hold-out problem. Hege and Mella-Barral (2005) study the same form of exchange offer via exit consent in the context of a dynamic model in the spirit of Leland (1994), and show that if exchange offers can be repeated infinitely often, only *guaranteed liquidation rights* such as collateral can be part of feasible debt renegotiation offers. In our setting, since renegotiation is possible only once, it is sufficient to offer simple, non-guaranteed liquidation rights. This difference is important for firms where a substantial fraction of liquidation rights consists of non-collateralizable assets, such as intangibles.

Moreover, as Gertner and Scharfstein (1991) show, when the exchange offer proposes senior claims, the hold-out problem can even be transformed into a *hold-in* problem in which consenting bondholders receive a lower value than if they collectively rejected the offer.[16]

[16] Gertner and Scharfstein (1991, p. 1205) define a hold-in problem as a situation where "by exchanging for senior debt and leaving holdouts with a junior security, the firm induces public debtholders to tender for a claim that the bank would not accept." They show that hold-in effects can emerge if *(i)* the offer is only valid for a limited pre-announced proportion of tendering bondholders, and *(ii)* the individual loss incurred by non-tendering bondholders is not sufficiently compensated for by the reduced bankruptcy probability since they are being stripped of liquidation rights. To understand point *(ii)*, it is worth noting that non-tendering bondholders are typically rationed (or held out) since the exchange offer is oversubscribed and the proportion

The practical relevance of the hold-in problem, however, is contentious, and, therefore, we abstract from it during our main analysis. However, we discuss in Section 5 the effect when hold-in effects can arise, and show that they will only reinforce our main conclusion on the efficiency ranking of the renegotiation alternatives that we study.

3.2 Exchange Offers with Seniority Transfer and Securities Valuation

Formally, we model a bond exchange offer with seniority transfer as the following take-it-or-leave-it offer to bondholders. The offer is *restricted*, that is valid only for a proportion ϕ of the existing bonds (typically the first bondholders tendering would be accepted but we do not need to specify the rationing rule).[17] The offer consists of exchanging their old debt contracts with coupon δ for new bonds carrying a reduced coupon, θ, but senior to the $(1-\phi)$ existing non-tendered bonds.

A bond exchange offer can be expressed as a triple $R = (\hat{x}, \theta, \phi)$, where $\hat{x}$ is the trigger level for the restricted exchange offer (when the revenue level x_t reaches this level $\hat{x}$ for the first time), and where the debt renegotiation offer proposes to replace a proportion ϕ of the outstanding debt δ for new debt with a reduced coupon of θ. Furthermore, the new debt with coupon θ is senior to the proportion $1-\phi$ of the existing debt that is not renegotiated, and hence acquires a higher level of the liquidation rights $V^B(\underline{x})$ when bankruptcy is declared at revenue level $\underline{x}$. When the renegotiation offer is successful and a proportion ϕ of existing debt is renegotiated, the aggregate coupon of the new debt is $\phi\,\theta$ and that of the non-renegotiated existing debt is $(1-\phi)\,\delta$. We restrict consideration to the efficient equilibrium outcome among all possible equilibrium outcomes for a given exchange offer.[18] Therefore, the results of our analysis do not depend on the question whether we consider unconditional offers (that take effect regardless of the mass of bonds being tendered) or conditional offers (that only take effect if a minimum fraction of bonds (smaller or equal to ϕ) is tendered), and we make no specific assumption in this regard.

For any coupon level of the new bonds θ, and proportion of new bonds ϕ, let b denote the resulting aggregate post-renegotiation coupon level

$$b \quad \equiv \quad \phi\,\theta + (1-\phi)\,\delta\ . \tag{10}$$

of bonds available for exchange is insufficient to cover demand.

[17] Proportional rationing in the event of oversubscription of the offer would yield the same result.

[18] Various equilibrium refinements exist that justify the selection of this equilibrium. Technically, a renegotiation game with multiple creditors is a coordination game with multiple equilibrium outcomes so the best response of each bondholder depends on the decision of other bondholders.

A coupon reduction to b has two counterbalancing effects on debt valuation. On the one hand, for a given threshold when the firm declares bankruptcy, the discounted stream of future debt service payments is reduced. On the other hand, the reduced coupon means that the bankruptcy trigger threshold $\underline{x}(b)$ is lowered, so that the expected time until bankruptcy, and hence the expected duration of coupon payments, is increased. The new bankruptcy threshold $\underline{x}(b)$ is given by (3), where the coupon b replaces the old coupon δ. Accordingly, a successful exchange offer lowers the bankruptcy trigger point $\underline{x}(b)$ and the probability of default.

The permanent coupon reduction by $\delta - \theta$ for a fraction ϕ of the bonds is associated with an increase in face value of the reorganized bonds from ϕF to $\phi F'$. Our assumption that the renegotiation offer takes the form of a negotiated debt deferment implies that

$$\frac{\phi\,(\delta-\theta)}{\rho}\left[1-\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right] \quad = \quad \phi\,(F'-F)\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}. \tag{11}$$

The LHS of (11) is the expected annuity value, at the exchange trigger level $\hat{x}$, of the coupon concession. The RHS of (11) is the expected value, at the exchange trigger level $\hat{x}$, of the increase in the face value. In a negotiated debt deferment, F' is a function of $(\hat{x}, \theta, \phi)$ and is set so that (11) is satisfied.

As far as seniority is concerned, we can limit attention to the case where tendering bondholders get exclusive access to the full bankruptcy value $V^B(\underline{x}(b))$ when bankruptcy is triggered at revenue level $\underline{x}(b)$, whereas non-tendering debtholders receive no part of the bankruptcy value. This assumption is in fact without loss of generality as it represents the maximal reallocation of liquidation rights from non-tendering to tendering bondholders; thus, it is the optimal offer from the perspective of equityholders as it allows for the largest reduction in coupons and hence the largest increase in post-renegotiation equity value.

Let $B_2^{\mathcal{T}}(\hat{x}\,|R)$ denote the value of the new debt at the exchange trigger point $\hat{x}$, and $B_2^{\mathcal{H}}(\hat{x}\,|R)$ the value of the existing debt with coupon β at the same point (superscripts $\mathcal{T}$ and $\mathcal{H}$ denote bondholders that tender and hold out, respectively), and let $B(\hat{x})$ denote the value of the existing debt if renegotiation fails. We obtain the following expression for the securities values (see the Appendix for the derivation):

Lemma 1 *In the presence of a given debt reorganization $R = (\hat{x}, \theta, \phi)$, the values of the equity and the debt, before the debt reorganization (for $\inf_{0\le s\le t}\{x_s\} \ge \hat{x}$), are*

$$S_1(x_t\,|R) \quad = \quad \frac{x_t}{\rho-\mu} - \frac{w+\delta}{\rho} \;+\; \left\{S_2(\hat{x}\,|R) \;-\; \frac{\hat{x}}{\rho-\mu} + \frac{w+\delta}{\rho}\right\}\left(\frac{x_t}{\hat{x}}\right)^{\lambda}, \tag{12}$$

$$B_1(x_t\,|R) \quad = \quad \frac{\delta}{\rho} \;-\; \left[\frac{\delta}{\rho} - B_2^{\mathcal{T}}(\hat{x}\,|R) - B_2^{\mathcal{H}}(\hat{x}\,|R)\right]\left(\frac{x_t}{\hat{x}}\right)^{\lambda}. \tag{13}$$

The values of the equity, the new and the old debt, after the debt reorganization (for $\inf_{0\le s\le t}\{x_s\} \in [\underline{x}(b);\hat{x})$*), are*

$$S_2(x_t\,|R) \;=\; \frac{x_t}{\rho-\mu} - \frac{w+b}{\rho} + \left[-\frac{\underline{x}(b)}{\rho-\mu} + \frac{w+b}{\rho}\right]\left(\frac{x_t}{\underline{x}(b)}\right)^{\lambda}, \tag{14}$$

$$B_2^{\mathcal{T}}(x_t\,|R) \;=\; \frac{\phi\,\theta}{\rho} + \left\{V^B(\,\underline{x}(b)\,) - \frac{\phi\,\theta}{\rho}\right\}\left(\frac{x_t}{\underline{x}(b)}\right)^{\lambda}, \tag{15}$$

$$B_2^{\mathcal{H}}(x_t\,|R) \;=\; \frac{(1-\phi)\,\delta}{\rho}\left[1-\left(\frac{x_t}{\underline{x}(b)}\right)^{\lambda}\right], \tag{16}$$

where $\underline{x}(.)$, $V^B(.)$ *and* b *are defined in equations (3), (5) and (10).*

3.3 Optimal Exchange Offers

Next, we derive the equityholders' optimal exchange offer strategy with seniority transfer. We consider that for an exchange offer $R = (\hat{x}, \theta, \phi)$ to be successful, management faces the following constraints:

1. The exchange offer must satisfy the incentive constraint that tendering bondholders are at least as well off as non-tendering bondholders:

$$B_2^{\mathcal{T}}(\hat{x}\,|R) \;\ge\; \frac{\phi}{1-\phi}\,B_2^{\mathcal{H}}(\hat{x}\,|R)\;; \tag{17}$$

2. The exchange offer must be such that tendering bondholders are not worse off than they would be if the exchange did not take place (no hold-in constraint):

$$B_2^{\mathcal{T}}(\hat{x}\,|R) \;\ge\; \phi\,B(\hat{x})\;; \tag{18}$$

3. Management must make its renegotiation offer before it is optimal for equityholders to trigger bankruptcy if the exchange offer was to be unsuccessful:

$$\hat{x} \;\ge\; \underline{x}(\delta)\;. \tag{19}$$

Constraint (18) excludes that management can engineer an exchange offer that takes advantage of the hold-in problem analyzed in Gertner and Scharfstein (1991) and that would depress the value of debt claims after a successful exchange offer to a level below their value if the offer would not have been made. There are three reasons why we analyze the problem under this constraint:

(i) This condition always needs to hold if there is majority voting, since in a vote that applies to all bondholders, an offer that makes them strictly worse off can never be accepted in equilibrium. So Condition (18) allows us to make an unambiguous comparison of both equity and debt values, and to isolate the effects of seniority transfer from squeezing debtholders below their current value;

(ii) Arguably, bond exchanges that make bondholders strictly worse off are exposed to judicial recourse. In the U.S., there is legal controversy whether policies that strip bondholders aggressively from their liquidation rights constitute a violation of the Trust Indenture Act;[19]

(iii) There are multiple equilibrium outcomes in a debt renegotiation game with atomistic creditors, and they include the outcome of a failed offer in which all creditors reject the offer. There are equilibrium refinements that favor the selection of this outcome when creditors are collectively worse off after a successful offer compared with a failed offer.

We discuss at the end of Section 5 the efficiency properties when relaxing this constraint.

Constraint (19) takes into account that equityholders are actually making losses before the optimal bankruptcy threshold $\underline{x}(\delta)$ is reached: When the revenue flow x_t deteriorates, the firm is making a loss as soon as $x_t < w+\delta$, and $\underline{x}(\delta) < w+\delta$ from eq. (3). So in order to delay bankruptcy (as x_t approaches $\underline{x}(\delta)$), equityholders are forced to raise additional equity and/or inject cash to enable the firm to continue servicing its debt. Constraint (19) assumes that it becomes impossible for management to raise equity and/or obtain cash injections in order to postpone a renegotiation offer beyond the point where these equityholders would file for bankruptcy if the offer failed.[20]

We use the notation R^E for exchange offers satisfying these constraints. The set of admissible offers proposed by management, $\mathcal{R}^E$, is therefore

$$\mathcal{R}^E \quad = \quad \left\{ R^E = (\hat{x}, \theta, \phi) \quad \middle| \quad \text{Conditions (17), (18) and (19) hold} \right\}. \tag{20}$$

Management faces a trade-off between early and late debt service reductions. On the one hand, with an early exchange offer (high $\hat{x}$), shareholders benefit sooner from a reduction in coupon payment obligation ($\phi\,\theta < \phi\,\delta$). On the other hand, by waiting for revenue to fall before exercising the exchange offer option (low $\hat{x}$), shareholders benefit from being able to obtain a more substantial permanent coupon reduction. There is, therefore, a trade-off between an early but moderate debt service reduction and a deferred but larger one.

[19]See Lubben (2017).

[20]Mella-Barral (1995), Lambrecht (2001), and Moraux and Silaghi (2014) also invoke this constraint.

The constrained optimization problem consists in maximizing the equity value $S_1(x_t \,|R^E)$ subject to the offer $R^E \in \mathcal{R}^E$. The optimal choice of debt exchange offer can be expressed in terms of the following Bellman equation:

$$\rho\, S_1(x_t \,|R^E) \quad = \quad \max_{R^E \in \mathcal{R}^E} \left\{ x_t - w - \delta \; + \; \frac{d}{d\tau} \mathrm{E}_t S_1(x_{t+\tau} \,|R^E) \Big|_{\tau=0} \right\} . \tag{21}$$

We solve problem (21) again using Lagrange's method in the Appendix. We use the superscript E for the solution, and obtain:

Proposition 2 *Equityholders' optimal choice of debt exchange offer with seniority transfer,* $R^E = (\hat{x}^E, \theta^E, \phi^E)$, *is such that:*

$$\frac{\delta}{\rho} \left[1 \; - \; \left(\frac{\underline{x}(\delta)}{\underline{x}(b^E)} \right)^{\lambda} \right] \quad = \quad V^B(\underline{x}(\delta))\,, \tag{22}$$

$$\frac{\phi^E\,(\delta - \theta^E)}{\rho} \left[1 \; - \; \left(\frac{\hat{x}}{\underline{x}(b^E)} \right)^{\lambda} \right] \quad = \quad V^B(\underline{x}(b^E)) \left(\frac{\hat{x}}{\underline{x}(b^E)} \right)^{\lambda} , \tag{23}$$

$$\phi^E \quad = \quad \frac{(1-\alpha)\,\delta}{\alpha\,\theta^E \; + \; (1-\alpha)\,\delta}\,, \tag{24}$$

where $\underline{x}(.)$ *and* $V^B(.)$ *are defined in equations (3) and (5), and* $b^E \;\equiv\; \phi^E\,\theta^E + (1-\phi^E)\,\delta$.

To understand better Proposition 2, we now develop the features of the solution and provide the intuition behind them.

First, we show in the Appendix that shareholders' optimal $(\hat{x}^E, \theta^E, \phi^E)$, is such that both constraints (17) and (18) are binding:

$$B_2^{\mathcal{T}}(\hat{x}^E \,|R^E) \quad = \quad \frac{\phi^E}{1-\phi^E}\, B_2^{\mathcal{H}}(\hat{x}^E \,|R^E)\;, \tag{25}$$

$$B_2^{\mathcal{T}}(\hat{x}^E \,|R^E) \quad = \quad \phi^E\, B(\hat{x}^E)\;. \tag{26}$$

The optimal exchange offer is therefore such that tendering bondholders are (1) only marginally better off than non-tendering bondholders and (2) only marginally better off than they would be if the exchange did not take place.

This implies that the exchange offer is such that, at the time of the exchange offer $\hat{x}^E$, bondholders are only marginally better off than if the debt was non-renegotiable:

$$B_2^{\mathcal{T}}(\hat{x}^E \,|R^E) \; + \; B_2^{\mathcal{H}}(\hat{x}^E \,|R^E) \quad = \quad B(\hat{x}^E)\;. \tag{27}$$

It also implies that, before the exchange offer, the value of debt is equal to its value if it was not renegotiable:

$$B_1(x_t \,|R^E) \quad = \quad B(x_t)\;. \tag{28}$$

Essentially, the exchange offer $(\hat{x}^E, \theta^E, \phi^E)$ allows shareholders to fully capture the economic surplus that the renegotiation offer $(\hat{x}, \theta, \phi) \in \mathcal{R}^E$ generates.

Second, the optimal exchange offer defines a unique fraction of bonds that are offered for exchange, ϕ^E, which satisfies (23). In fact, the ratio of reduced coupons $\phi^E\,\theta^E$ to continuing old coupons $(1 - \phi^E)\,\theta^E$ is uniquely defined by the bankruptcy costs, $(1 - \alpha)/\alpha$.

If there were no bankruptcy costs, i.e. $\alpha = 0$, the fraction ϕ^E would be equal to 1. Clearly, there would be no surplus from renegotiation in this extreme case, and renegotiation offers are close to ineffective.

In the presence of bankruptcy costs, renegotiation is valuable, and here shareholders fully capture the benefits from renegotiation which an offer $(\hat{x}, \theta, \phi) \in \mathcal{R}^E$ can generate. The higher the bankruptcy costs, represented by α, the more the exchange offer is valuable to shareholders, and the more the optimal fraction ϕ^E drops from 1. For an intuition, the more is to be gained from renegotiation, the more useful is it to expand the fraction $(1 - \phi^E)$ of bonds that lose their liquidation rights, and to trade those liquidation rights for coupon concessions from tendering bondholders. Thus, the lower is ϕ^E, the lower will be the reduced coupon θ^E.

Third, the optimal trigger point for the exchange offer may be larger or smaller the bankruptcy trigger level in the absence of renegotiation, $\underline{x}(\delta)$. If it is smaller, then condition (19) binds, and $\hat{x}^E = \underline{x}(\delta)$.

The smaller the bankruptcy costs, the lower the surplus from renegotiation for shareholders, and the less will shareholders be eager to renegotiate early. The less shareholders gain from in an early exchange offer, the lower the optimal trigger point for the offer. Hence, when α is small and ϕ^E is close to 1, the trigger point for the exchange offer $\hat{x}^E$ is small enough for condition (19) to be binding. Then $\hat{x}^E = \underline{x}(\delta)$. The higher α, the lower the implied ϕ^E, and the higher the optimal renegotiation trigger level. Thus, there will be a threshold $\bar{\alpha}$, such that $\hat{x}^E > \underline{x}(\delta)$ for all $\alpha > \bar{\alpha}$.

Fourth, we show in the Appendix that (23) implies that the face value of tendering bondholders exceeds the bankruptcy value of the firm:

$$\phi^E\, F' \quad > \quad V^B(\underline{x}(b^E)). \tag{29}$$

This implies that, in bankruptcy, the tendering bondholders (who are the only senior debtholders after debt renegotiation) hold sufficiently large claims to be allocated the entire bankruptcy value of the firm.

4 Bond Renegotiation with Collective Action Clauses

Roe (1987) and Gertner and Scharfstein (1991) advocate an alternative framework for bond renegotiation: including in the firm's debt covenants a provision that mimics the Chapter 11 voting procedure for bond renegotiation. In the U.S., such clauses are currently prohibited by the Trust Indenture Act, but are possible in other legislations. After a successful vote under such a collective action clause, all bondholders are treated uniformly; those who do not vote in favor of the renegotiation proposal are subject to the voting outcome in the same way as those voting in favor. Thus, holdout problems (and hold-in problems) are not possible since they are predicated on the possibility to differentiate among bondholders, and to transfer liquidation rights between them. In this context, there is no analytical difference whether the bond indenture requires a simple majority vote or a supermajority for a successful debt renegotiation outcome, and we will discuss majority and supermajority provisions jointly, or whether the uniform outcome is established through a procedure other than voting, such as a supertrustee.

In our set-up, modelling a (super-)majority vote in the case of debt renegotiation amounts to imposing the condition $\phi = 1$ in the renegotiation model since discrimination between bondholders is not feasible.

The single debt renegotiation offer consists in management proposing to replace the existing debt contract with coupon δ by a new contract, bearing a reduced coupon obligation θ. Management chooses to make the debt renegotiation offer when the revenue level reaches the threshold $\hat{x}$. Because of the additional condition that $\phi = 1$, the debt renegotiation offer can only be made towards the entire outstanding debt issue. Therefore, the reorganization offer in this case can be characterized by a triple $R = (\hat{x}, \theta, 1)$. Management faces the same sort of trade-off between early contractual debt service reductions and larger but deferred ones as in a debt exchange offer analyzed in Section 3.

Denote $B_2^{\mathcal{T}}(.)$ and $B(.)$ the values of the bondholders' claims when the exchange offer is accepted and rejected, respectively. We consider that for an offer $R = (\hat{x}, \theta, 1)$ to be successful, management faces the following constraints:

1. The offer must be such that the value of the new debt is no less than the value of the existing debt if the offer fails:

$$B_2^{\mathcal{T}}(\hat{x}\,|R) \geq B(\hat{x}) \; ; \tag{30}$$

2. Management must make its offer before it is optimal for equityholders to trigger bankruptcy if the exchange was not to occur:

$$\hat{x} \;\; \geq \;\; \underline{x}(\delta) \; . \tag{31}$$

We use the notation R^C for exchange offers satisfying these constraints. The set of admissible offers proposed by management, $\mathcal{R}^C$, is therefore

$$\mathcal{R}^C \quad = \quad \left\{ \; R^C = (\hat{x}, \theta, 1) \quad \middle| \quad \text{Conditions (30) and (31) hold} \right\} \; . \tag{32}$$

Management's optimization problem consists in maximizing the value of equity subject to the offer $R^C \in \mathcal{R}^C$. The optimal choice of debt renegotiation offer can be expressed in terms of the following Bellman equation:

$$\rho \, S_1(x_t \, | R^C) \quad = \quad \max_{R^C \in \mathcal{R}^C} \; \left\{ x_t - w - \delta \; + \; \frac{d}{d\tau} \mathrm{E}_t S_1(x_{t+\tau} \, | R^C) \Big|_{\tau=0} \right\} \; . \tag{33}$$

We solve problem (33) in the Appendix using Lagrange's method. We obtain:

Proposition 3 *Equityholders' optimal choice of debt renegotiation offer with collective action clauses, $R^C = (\hat{x}^C, \theta^C, 1)$, is such that:*

$$\hat{x}^C \quad = \quad \underline{x}(\delta) \; , \tag{34}$$

$$V^B(\underline{x}(\delta)) \quad = \quad \frac{\theta^C}{\rho} \; - \; \left[\frac{\theta^C}{\rho} \; - \; V^B(\underline{x}(\theta^C)) \right] \left(\frac{\underline{x}(\delta)}{\underline{x}(\theta^C)} \right)^{\lambda} \; , \tag{35}$$

where $\underline{x}(.)$ and $V^B(.)$ are defined in equations (3) and (5).

Here again, to develop the intuition, we develop the main features of the solution in Proposition 3.

First, we show in the Appendix that shareholders' optimal $(\hat{x}^C, \theta^C)$, is such that constraint (30) is binding:

$$B_2^{\mathcal{T}}(\hat{x}^C \, | R^C) \quad = \quad B(\hat{x}^C) \; . \tag{36}$$

The exchange offer is such that, at the time of the exchange $\hat{x}^C$, tendering bondholders are only marginally better off than if the debt was non-renegotiable. This implies that, before the exchange offer, the value of the debt is equal to its value if it was not renegotiable:

$$B_1(x_t \, | R^C) \quad = \quad B(x_t) \; . \tag{37}$$

Here again, the exchange offer $(\hat{x}^C, \theta^C)$ allows shareholders to fully capture the benefits from renegotiation which an offer $(\hat{x}, \theta, 1) \in \mathcal{R}^C$ can generate.

Second, the optimal trigger point for the exchange offer is equal to the bankruptcy trigger level in the absence of renegotiation, $\underline{x}(\delta)$. Proposition 3 states that the optimal debt restructuring with collective action clauses consists of waiting as long as possible and resisting to making early offers. This finding confirms related results in Moraux and Silaghi (2014). It is different from the optimal timing of exchange offers that could occur earlier (see Proposition 2). Under a voting procedure, the best offer $(\hat{x}, \theta, 1) \in \mathcal{R}^C$ is simply the latest possible one.

For an intuition why management wants to postpone renegotiation, note that any renegotiation offer at $\hat{x} > \underline{x}(\delta)$ cannot achieve a lower ex-ante debt value than the debt value feasible with an offer at $\hat{x} = \underline{x}(\delta)$; it will, however, imply a higher post-renegotiation coupon and thus an earlier bankruptcy threshold, and hence a lower firm and equity value.

Inspecting the problem, it turns out that in the bond exchange offer model analyzed in Section 3, the analysis of the problem with a collective action clause is equivalent to maximizing the exit-consent public debt exchange Lagrangian with respect to $\hat{x}$ and θ, but subject to the additional constraint $\phi = 1$. This yields similar value and threshold expressions to those obtained in Proposition 2.

We point out that the problem, and the incentive constraints, would be exactly the same in the case of concentrated debt held by a single creditor. Thus, the analysis in this section actually encompasses the single-creditor case (such as a bank loan) as well.

5 Comparison of Exchange Offers and Collective Action Clauses

In the previous two sections, we have analyzed the optimal debt renegotiation strategy for a bond exchange offer with seniority transfer (Section 3), and for a bond indenture with a collective action clause (Section 4). Other than the renegotiation set-up, the model is identical, allowing us to compare the properties of the two alternatives, and specifically, the debt obligation reductions and debt overhang costs associated with each type of debt reorganization. As explained earlier, the model set-up and its optimal outcome would be identical if instead of including a collective action clause in the bond indenture, the debt was held by a single creditor.[21] Thus, our comparison also contributes to the longstanding discussion and literature on the comparison of public (bond) and private (bank) debt.

[21] We recognize that the assumption of a single round of renegotiation may be particularly restrictive in the case of a single creditor, where a second attempt to renegotiate can be more easily arranged, than for a dispersed bond issue.

5.1 Debt Obligation Reductions and Efficiency

Bond exchange offers with seniority transfer and bond renegotiation with collective action clauses are both analyzed as constrained maximization problems using the Lagrange method. While the nature of the problem is quite different, a closer inspection of the two solutions reveals that in equilibrium, the attainable solutions can be clearly ranked.

When inspecting the nature of the two problems, it is not obvious the best renegotiation outcome under a collective action clause and under a bond exchange offer can be nested, and that no trade-off emerges. To establish this insight, we pursue a somewhat technical argument that follows the strategy of our proof in the Appendix.

- We first show that the outcome of the Lagrangian for bond exchange offers does not depend on the incentive constraint (17) that tendering bondholders must be at least well off as non-tendering bondholders. In fact, it turns out that the solution to problem (21) if we impose only conditions (18) and (19), is such that condition (17) is always satisfied.
- We then find that the remaining conditions (18) and (19) in problem (21) are identical to conditions (30) and (31) in problem (33), except that problem (21) in addition imposes $\phi = 1$ since discrimination between different bondholders is outlawed.
- Therefore, the outcome with collective action clauses in Section 4, is mathematically equivalent to a *restricted* version of the outcome of the Lagrangian for bond exchange offers in Section 3.
- In conclusion, imposing the extra constraint $\phi = 1$ in the case of collective action clauses can only lead to a lower attainable global level of the objective function compared with the less restricted problem of optimizing exchange offers with seniority transfer.

It follows that, for the same initial coupon level β, the equity value prior to debt renegotiation is lower with collective action clauses than with seniority transfer,

$$S_1(x_t \,|R^C) \quad \leq \quad S_1(x_t \,|R^E)\,. \tag{38}$$

Thus, management prefers being allowed to deploy exchange offers with seniority transfer (as they are under the U.S. Trust Indenture Act) than collective action clauses. Management then prefers issuing dispersed bond contracts than concentrated (single creditor) debt.

Proposition 4 *Bond exchange offers with seniority transfer are more profitable ex ante for equityholders than debt renegotiations with collective action clauses or a single creditor.*

This ordering of ex ante equity values only depends on the magnitude of the global debt service reduction obtained through the exchange. Therefore, the aggregate debt service after debt renegotiation is lower with exchange offers than with collective action clauses:

$$b^E \quad \leq \quad \theta^C \quad < \quad \delta \quad , \tag{39}$$

where $b^E \;\equiv\; \phi^E\,\theta^E + (1-\phi^E)\,\delta$. The ordering of total debt obligations after optimal exchange offers yields the following complete ordering of the implied optimal bankruptcy thresholds:

$$\underline{x}^{fb} \quad \leq \quad \underline{x}(b^E) \quad \leq \quad \underline{x}(\theta^C) \quad \leq \quad \underline{x}(\delta) \; . \tag{40}$$

From this comparison, it follows directly that exchange offers induces lower debt overhang losses and later bankruptcy compared with collective action clauses (or single-creditor debt). This ordering yields the following efficiency result:

Proposition 5 *Bond exchange offers with seniority transfer mitigate debt overhang problems more efficiently than debt renegotiations with collective action clauses or a single creditor.*

5.2 Efficiency of Exchange Offers With Hold-in Effects

We conclude with a short discussion of the efficiency implication if management can take advantage of hold-in problems when making a bond exchange offer. The presence of hold-in problems means that the value of debt claims (tendering and non-tendering) after a successful exchange offer may be lower than prior to the offer since the transfer of liquidation rights from non-tendering to tendering bondholders offers a strong bargaining tool to equityholders in a setting in which dispersed bondholders are unable to coordinate their response to the exchange offer.

Consequently, the constraint (18) ensuring that bondholders are not worse off than in the absence of an exchange offer may be violated, whereas equityholders' exchange offer still needs to satisfy the incentive constraint (17) that ensures that tendering bondholders are better off than hold-outs, and condition (19) that the renegotiation offer cannot be postponed beyond $\underline{x}(\delta)$, the optimal default of non-renegotiable debt. Using the notation I for hold-ins, the set of admissible offers proposed to debtholders, $\mathcal{R}^I$, is then:

$$\mathcal{R}^I \quad = \quad \left\{ R^I = (\hat{x}, \theta, \phi) \quad | \quad \text{Conditions (17) and (19) hold} \right\} . \tag{41}$$

The optimal debt exchange offer solves the Bellman equation:

$$\rho\, S_1(x_t\,|R^I) \quad = \quad \max_{R^I \in \mathcal{R}^I} \; \left\{ x_t - w - \delta \; + \; \frac{d}{d\tau} \mathrm{E}_t S_1(x_{t+\tau}\,|R^I) \bigg|_{\tau=0} \right\} \; . \tag{42}$$

The solution to this problem is different from that of problem (21). In particular, the value of ϕ^I is indeterminate for values of ϕ above a minimum value, whereas ϕ^E is uniquely determined. Since the Lagrangian of problem (42) has fewer constraints than the Lagrangian of problem (21), it follows, similar to the reasoning leading to Proposition 4 and Proposition 5, that taking advantage of hold-in effects allows the optimal exchange offer to achieve a lower post-renegotiation coupon, a lower bankruptcy trigger point and lower debt overhang costs compared with exchange offers that cannot deploy hold-in effects.

5.3 Limits of the Analysis

The scope of our comparison and analysis is clearly limited since we assume that efficient renegotiation with dispersed bondholders is possible. In fact, we only consider one aspect of renegotiation problems with dispersed bondholders, holdout problems in a fully rational environment. In practice, bond exchange offers with multiple bondholders frequently suffer from issues linked to asymmetric information about the firm, about the acceptance probability of other creditors, or from behavioral limitations. Collective action clauses offer advantages in this context, in particular when a majority of bondholders, such as institutional investors, is likely to act rationally and devote enough decision effort to the renegotiation offer, while a minority may reject an offer they should rationally accept, because of limited information or for reasons outside of the scope of a rational decision analysis.

In general, in the presence of uncertainty about the acceptance probability of a renegotiation offer, the higher the probability for an individual bondholder to be pivotal for the success or failure of a conditional renegotiation offer, the more will she be inclined to accept the offer. This means that in reality the ease of renegotiation is more likely to be located on a continuum of debt concentration structures between fully atomistic bondholders and a single creditor. If a majority of debt claims are concentrated among a few large creditors such as institutional investors, it is likely that holdout incentives in the presence of uncertainty are limited, and more so if collective action clauses are in place, improving the efficiency of bond workouts in the presence of collective action clauses. Thus, the overall welfare comparison of bond exchange offers versus collective action clauses is more complex than our limited analysis reveals.

We also add a caveat to our observation that renegotiation with a single creditor (prominently, a bank) is akin to renegotiating with a collective action clause and hence dominated by a bond exchange offer. Besides assuming away any advantage of that a single creditor in accessing or processing information rationally, the comparison also depends on the assumption that the number of renegotiation rounds is limited to one. Obviously, relationship financiers are likely to be more flexible in this regard.

6 Conclusion

This paper analyzes debt renegotiation in a dynamic contingent-claims model of securities that allows for a single round out-of-court renegotiation prior to default, and where default decision is endogenous in the tradition of Leland (1994). The firm faces a debt overhang problem and will default too early relative to the first best when it uses financial leverage. We consider out-of-court restructuring offers with dispersed bondholders where both hold-out and hold-in problems are potentially present. The firm decides on the timing of debt renegotiation prior to bankruptcy, facing the following trade-off: early offers alleviate the debt service sooner, but at the expense of a smaller debt reduction and hence a less efficient reduction of the debt overhang problem.

We analyze two renegotiation mechanisms: debt exchange offers with seniority transfer (as is the case in exit consents), and collective action clauses stipulating that core terms of the bond contract can be altered with majority approval. We derive closed-form expression for debt and equity, and determine the optimal renegotiation offers and their timing, for each mechanism. We find that bond renegotiation with collective actions clauses will always occur (weakly) later compared to exchange offers. Crucially, collective actions clauses lead to a smaller coupon reduction compared with exchange offers since all creditors must be treated equal, hence seniority transfer of liquidation rights cannot be used to enforce creditor concessions. When using collective action clauses, firms default earlier, and debt overhang inefficiencies will be more substantial. Also, for a fixed initial coupon level, equityholders are better off with exchange offers than with majority voting, prior and after renegotiation.

The policy implication of our analysis is that altering Section 316(b) of the Trust Indenture Act of 1939 that prohibits the modification of core terms of the bond without unanimous consent, as well as facilitating the incorporation of collective action clauses may well not be the most urgent policy actions. We recall the limitations of our analysis. We focus on one aspect of the renegotiation problem with multiple debtholders, holdout incentives of dispersed bondholders, but we do so by assuming a frictionless, full-information and fully rational world. In practice, collective action clauses serve to alleviate issues such as imperfect information or investors' behavioral limitations that we do not consider.

Naturally, therefore, there are various avenues to expand on our analysis, in particular accounting for asymmetric information or bounded rationality. In addition, it would be interesting for example to study of mixed debt structures, of the optimal capital structure, the use of collective action clauses in the context of complex debt issues (involving several bond issues or bonds with varying seniority and maturity terms), and the use of other mechanisms to transfer value from one class of bondholders to others, such as asset sales and maturity shortening.

Appendix

Proof of Proposition 1: The value of the firm's equity, $S(x_t)$, and bonds, $B(x_t)$ must satisfy the following equilibrium or no-arbitrage conditions:

$$\rho\, S(x_t) \quad = \quad x_t - w - \delta \; + \; \frac{d}{d\tau} \mathrm{E}_t S(x_{t+\tau}) \bigg|_{\tau=0} \tag{43}$$

$$\rho\, B(x_t) \quad = \quad \delta \; + \; \frac{d}{d\tau} \mathrm{E}_t B(x_{t+\tau}) \bigg|_{\tau=0} \tag{44}$$

Applying Ito's lemma inside the expectations operator in equations (43) and (44) yields:

$$\rho\, S(x_t) \quad = \quad x_t - w - \delta \; + \; \frac{d\, S(x_t)}{d\, x_t} \mu x_t \; + \; \frac{d^2\, S(x_t)}{d\, x_t^2} \frac{\sigma^2}{2} x_t^2 \tag{45}$$

$$\rho\, B(x_t) \quad = \quad \delta \; + \; \frac{d\, B(x_t)}{d\, x_t} \mu x_t \; + \; \frac{d^2\, B(x_t)}{d\, x_t^2} \frac{\sigma^2}{2} x_t^2 \tag{46}$$

$$\rho\, V^B(x_t) \quad = \quad (1-\alpha)(x_t - w) \; + \; \frac{d\, V^B(x_t)}{d\, x_t} \mu x_t \; + \; \frac{d^2\, V^B(x_t)}{d\, x_t^2} \frac{\sigma^2}{2} x_t^2 \quad . \tag{47}$$

The general solutions to equations (45), (46) and (47) are:

$$S(x_t) \quad = \quad k_0^s + k_1^s x_t + k_2^s x_t^{\lambda} + k_3^s x_t^{\lambda'} \tag{48}$$

$$B(x_t) \quad = \quad k_0^b + k_1^b x_t + k_2^b x_t^{\lambda} + k_3^b x_t^{\lambda'} \tag{49}$$

$$X(x_t) \quad = \quad k_0^x + k_1^x x_t + k_2^x x_t^{\lambda} + k_3^x x_t^{\lambda'} \tag{50}$$

where λ and λ' are the negative and positive roots respectively of the characteristic equation: $\sigma^2/2(\lambda^2 - \lambda) + \mu\lambda = r$. Take derivatives and substitute back in equations (45), (46) and (47). Equating coefficients on like terms gives:

$$k_0^s + k_1^s x_t \quad = \quad \frac{x_t}{\rho - \mu} - \frac{w + \delta}{\rho} \tag{51}$$

$$k_0^b + k_1^b x_t \quad = \quad \delta/\rho \tag{52}$$

$$k_0^x + k_1^x x_t \quad = \quad (1-\alpha) \left[\frac{x_t}{\rho - \mu} - \frac{w}{\rho} \right] . \tag{53}$$

The remaining coefficients, as well as $\underline{x}(\delta)$ and $\underline{x}(0)$ are obtained with the following boundary conditions:
(i) Ruling out bubbles, the coefficients of the power terms of λ', the positive root, must be equal to zero, as the power terms of λ, the negative root, vanishes when x_t becomes high.

$$k_3^s \; = \; 0, \qquad k_3^b \; = \; 0, \qquad k_3^x \; = \; 0. \tag{54}$$

(ii) There are no arbitrage opportunities at the bankruptcy trigger levels of revenues $\underline{x}(\delta)$ and $\underline{x}(0)$. This yields so called "value matching conditions" because it matches the value of the assets in the case of continuation to those in the case of bankruptcy:

$$S(\underline{x}(\delta)) = 0, \qquad B(\underline{x}(\delta)) = V^B(\underline{x}(\delta)), \qquad V^B(\underline{x}(0)) = 0. \tag{55}$$

(iii) Equityholders decide to trigger bankruptcy in a non-cooperative fashion, the determination of the optimal bankruptcy thresholds $\underline{x}(\delta)$ and $\underline{x}(0)$ becomes a problem of optimal stopping of a Brownian motion, as discussed in Dixit (1991), Dumas (1991) and Dixit and Pindyck (1993). Such a problem of choice under uncertainty of one optimal binary decision yields so called "high order contact" or"smooth-pasting conditions" as it requires also derivatives of the equity value, in the case of continuation to match those in the case of bankruptcy, when the firm is respectively levered or unlevered

$$\frac{d\,S(\underline{x}(\delta))}{d\,x_t} = 0, \qquad \frac{d\,V^B(\underline{x}(0))}{d\,x_t} = 0\ . \qquad \square \tag{56}$$

Proof of Lemma 1: Drop the notation R. Similarly to the proof of Proposition 1, one obtains the coefficients of the linear part of the general solutions. Denote then as follows the coefficients of the non-linear part:

$$S_1(x_t) = \frac{x_t}{\rho-\mu} - \frac{w+\delta}{\rho} + k^s_{2,1}x_t^{\lambda} + k^s_{3,1}x_t^{\lambda'} \tag{57}$$

$$B_1(x_t) = \delta/\rho + k^b_{2,1}x_t^{\lambda} + k^b_{3,1}x_t^{\lambda'} \tag{58}$$

$$S_2(x_t) = \frac{x_t}{\rho-\mu} - \frac{w+b}{\rho} + k^s_{2,2}x_t^{\lambda} + k^s_{3,2}x_t^{\lambda'} \tag{59}$$

$$B_2^{\mathcal{T}}(x_t) = \phi\theta/\rho + k^n_{2,2}x_t^{\lambda} + k^n_{3,2}x_t^{\lambda'} \tag{60}$$

$$B_2^{\mathcal{H}}(x_t) = (1-\phi)\delta/\rho + k^o_{2,2}x_t^{\lambda} + k^o_{3,2}x_t^{\lambda'}\ . \tag{61}$$

These 10 unknowns and $\underline{x}(b)$ are solved for using the following boundary conditions:

$$k^s_{3,1} = 0, \qquad k^b_{3,1} = 0, \qquad k^s_{3,2} = 0, \qquad k^n_{3,2} = 0, \qquad k^o_{3,2} = 0, \tag{62}$$

$$S_1(\hat{x}) = S_2(\hat{x}), \qquad (vi) \quad B_1(\hat{x}) = B_2^{\mathcal{T}}(\hat{x}) + B_2^{\mathcal{H}}(\hat{x}), \tag{63}$$

$$B_2^{\mathcal{T}}(\underline{x}(b)) = V^B(\underline{x}(b)), \qquad B_2^{\mathcal{H}}(\underline{x}(b)) = 0, \quad \text{New seniority conditions} \tag{64}$$

$$S_2(\underline{x}(b)) = 0, \qquad \frac{d\,S_2(\underline{x}(b))}{d\,x_t} = 0. \tag{65}$$

Preliminary Calculations: The following elements are only called upon in the proofs of Propositions 2 and 3 which follow. Drop the notation R. We calculate the following partial derivatives of $S_1(x_t)$, $B_2^{\mathcal{T}}(\hat{x})$ and $B(\hat{x})$.

– Develop the expression of $S_1(x_t)$.

$$S_1(x_t) = \frac{x_t}{\rho-\mu} - \frac{w+\delta}{\rho} + \left\{S_2(\hat{x}) - \left[\frac{\hat{x}}{\rho-\mu} - \frac{w+\delta}{\rho}\right]\right\}\left(\frac{x_t}{\hat{x}}\right)^{\lambda}\ . \tag{66}$$

$$S_2(\hat{x}) = \frac{\hat{x}}{\rho-\mu} - \frac{w+b}{\rho} - \left[\frac{\underline{x}(b)}{\rho-\mu} - \frac{w+b}{\rho}\right]\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\ . \tag{67}$$

Replacing gives

$$S_1(x_t) = \frac{x_t}{\rho-\mu} - \frac{w+\delta}{\rho} + \left\{\frac{\delta-b}{\rho} - \left[\frac{\underline{x}(b)}{\rho-\mu} - \frac{w+b}{\rho}\right]\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right\}\left(\frac{x_t}{\hat{x}}\right)^{\lambda}\ . \tag{68}$$

Given that $\underline{x}(b) = \frac{-\lambda}{1-\lambda}\,\frac{\rho-\mu}{\rho}\,(w+b)$, we have $\frac{\underline{x}(b)}{\rho-\mu} - \frac{w+b}{\rho} = \frac{-(w+b)}{(1-\lambda)\,\rho}$. Hence,

$$\left[\frac{\underline{x}(b)}{\rho-\mu} - \frac{w+b}{\rho}\right]\left(\frac{x_t}{\underline{x}(b)}\right)^{\lambda} \quad = \quad \frac{-1}{(1-\lambda)\,\rho}\left[\frac{(1-\lambda)\,\rho}{-\lambda\,(\rho-\mu)}\right]^{\lambda}\ (w+b)^{1-\lambda}\ x_t^{\lambda}\ . \tag{69}$$

Therefore, given that $\delta - b = \delta - \{\phi\,\theta + (1-\phi)\,\delta\} = \phi(\delta-\theta)$,

$$S_1(x_t) \quad = \quad \frac{x_t}{\rho-\mu} - \frac{w+\delta}{\rho} + \frac{\phi(\delta-\theta)}{\rho}\left(\frac{x_t}{\hat{x}}\right)^{\lambda} + \frac{(1-\alpha)}{(1-\lambda)\,\rho}\left[\frac{(1-\lambda)\,\rho}{-\lambda\,(\rho-\mu)}\right]^{\lambda}(w+\theta)^{1-\lambda}x_t^{\lambda}\ . \tag{70}$$

We then obtain

$$\frac{\partial S_1(x_t)}{\partial \hat{x}} \quad = \quad \frac{-\lambda\,\phi(\delta-\theta)}{\rho\,\hat{x}}\left(\frac{x_t}{\hat{x}}\right)^{\lambda}\ , \tag{71}$$

$$\frac{\partial S_1(x_t)}{\partial \theta} \quad = \quad \frac{-\phi}{\rho}\left[1-\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right]\left(\frac{x_t}{\hat{x}}\right)^{\lambda}\ , \tag{72}$$

$$\frac{\partial S_1(x_t)}{\partial \phi} \quad = \quad \frac{\delta-\theta}{\rho}\left[1-\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right]\left(\frac{x_t}{\hat{x}}\right)^{\lambda}\ . \tag{73}$$

– Develop the expression of $B_2^{\mathcal{T}}(\hat{x})$.

$$B_2^{\mathcal{T}}(\hat{x}) \quad = \quad \frac{\phi\,\theta}{\rho} + \left\{V^B(\underline{x}(b)) - \frac{\phi\,\theta}{\rho}\right\}\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\ . \tag{74}$$

This gives

$$\frac{\partial B_2^{\mathcal{T}}(\hat{x})}{\partial \hat{x}} \quad = \quad \frac{\lambda}{\hat{x}}\left\{V^B(\underline{x}(b)) - \frac{\phi\,\theta}{\rho}\right\}\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda} \quad = \quad \frac{\lambda}{\hat{x}}\left\{B_2^{\mathcal{T}}(\hat{x}) - \frac{\phi\,\theta}{\rho}\right\}\ . \tag{75}$$

Develop the expression of $V^B(\underline{x}(b))$ to write

$$V^B(\underline{x}(b)) - \frac{\phi\,\theta}{\rho} \quad = \quad (1-\alpha)\left[\frac{\underline{x}(b)}{\rho-\mu} - \frac{w}{\rho}\right] - \frac{\phi\,\theta}{\rho} \ - \ (1-\alpha)\left[\frac{\underline{x}(0)}{\rho-\mu} - \frac{w}{\rho}\right]\left(\frac{\underline{x}(b)}{\underline{x}(0)}\right)^{\lambda}\ . \tag{76}$$

From $\underline{x}(b) = \frac{-\lambda}{1-\lambda}\,\frac{\rho-\mu}{\rho}\,(w+b)$, we have

$$\frac{\underline{x}(b)}{\rho-\mu} - \frac{w}{\rho} \quad = \quad \frac{-\lambda\,(w+b)}{(1-\lambda)\,\rho} \ - \ \frac{w}{\rho}\ . \tag{77}$$

So,

$$(1-\alpha)\left[\frac{\underline{x}(b)}{\rho-\mu} - \frac{w}{\rho}\right] - \frac{\phi\,\theta}{\rho} \quad = \quad \frac{-\lambda\,(1-\alpha)}{(1-\lambda)\,\rho}\,(w+b) \ - \ \frac{(w+\phi\,\theta)}{\rho} \ + \ \frac{\alpha\,w}{\rho}\ , \tag{78}$$

$$\quad = \quad \left[\frac{-\lambda\,(1-\alpha)}{(1-\lambda)\,\rho} \ - \ \frac{1}{\rho}\right](w+b) \ + \ \frac{w\alpha + (1-\phi)\delta}{\rho}\ . \tag{79}$$

It follows that

$$V^B(\underline{x}(b)) - \frac{\phi\,\theta}{\rho} = \left[\frac{-\lambda\,(1-\alpha)}{(1-\lambda)\,\rho} - \frac{1}{\rho}\right](w+b) + \frac{w\alpha + (1-\phi)\delta}{\rho} - (1-\alpha)\left[\frac{\underline{x}(0)}{\rho-\mu} - \frac{w}{\rho}\right]\left(\frac{\underline{x}(b)}{\underline{x}(0)}\right)^{\lambda}. \tag{80}$$

Therefore,

$$B_2^{\mathcal{T}}(\hat{x}) \;=\; \frac{\phi\,\theta}{\rho} + \left[\frac{-\lambda\,(1-\alpha)}{(1-\lambda)\,\rho} \;-\; \frac{1}{\rho}\right](w+b)\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}$$
$$+\;\left[\frac{w\alpha+(1-\phi)\delta}{\rho}\right]\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda} - (1-\alpha)\left[\frac{\underline{x}(0)}{\rho-\mu} - \frac{w}{\rho}\right]\left(\frac{\hat{x}}{\underline{x}(0)}\right)^{\lambda}\;. \tag{81}$$

Given that $\underline{x}(0) \;=\; \frac{-\lambda}{1-\lambda}\,\frac{\rho-\mu}{\rho}\,w$, we have

$$\left[\frac{\underline{x}(0)}{\rho-\mu} - \frac{w}{\rho}\right]\left(\frac{\hat{x}}{\underline{x}(0)}\right)^{\lambda} \;=\; \frac{-1}{(1-\lambda)\,\rho}\left[\frac{(1-\lambda)\,\rho}{-\lambda\,(\rho-\mu)}\right]^{\lambda}(w)^{1-\lambda}\,\hat{x}^{\lambda} \;=\; \frac{-w}{\rho\,(1-\lambda)}\left(\frac{\hat{x}}{\underline{x}(0)}\right)^{\lambda}. \tag{82}$$

Therefore,

$$B_2^{\mathcal{T}}(\hat{x}) \;=\; \frac{\phi\,\theta}{\rho} + \left[\frac{-\lambda\,(1-\alpha)}{(1-\lambda)\,\rho} \;-\; \frac{1}{\rho}\right]\left[\frac{(1-\lambda)\,\rho}{-\lambda\,(\rho-\mu)}\right]^{\lambda}(w+b)^{1-\lambda}\,\hat{x}^{\lambda}$$
$$+\left[\frac{w\alpha+(1-\phi)\delta}{\rho}\right]\left[\frac{(1-\lambda)\,\rho}{-\lambda\,(\rho-\mu)}\right]^{\lambda}(w+b)^{-\lambda}\hat{x}^{\lambda} + \frac{(1-\alpha)\,w}{\rho\,(1-\lambda)}\left(\frac{\hat{x}}{\underline{x}(0)}\right)^{\lambda}. \tag{83}$$

We then obtain

$$\frac{\partial B_2^{\mathcal{T}}(\hat{x})}{\partial\theta} \;=\; \frac{\phi}{\rho}\left[1-\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda} - \lambda\left[\frac{(1-\phi)\delta \,-\, \alpha\, b}{w+b}\right]\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right], \tag{84}$$

$$\frac{\partial B_2^{\mathcal{T}}(\hat{x})}{\partial\phi} \;=\; \frac{\delta}{\rho} - \frac{\delta}{\rho}\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda} - \frac{(\delta-\theta)}{\rho}\left[1-\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda} - \lambda\left[\frac{(1-\phi)\delta \,-\, \alpha\, b}{w+b}\right]\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right] \tag{85}$$

– Develop the expression of $B(\hat{x})$.

$$\phi\, B(\hat{x}) \;=\; \frac{\phi\,\delta}{\rho} \;+\; \phi\left\{V^B(\underline{x}(\delta)) - \frac{\delta}{\rho}\right\}\left(\frac{\hat{x}}{\underline{x}(\delta)}\right)^{\lambda}\;. \tag{86}$$

We then have

$$\frac{\partial\phi\, B(\hat{x})}{\partial\hat{x}} \;=\; \frac{\phi\,\lambda}{\hat{x}}\left\{V^B(\underline{x}(\delta)) - \frac{\delta}{\rho}\right\}\left(\frac{\hat{x}}{\underline{x}(\delta)}\right)^{\lambda} \;=\; \frac{\lambda}{\hat{x}}\left\{\phi\, B(\hat{x}) - \phi\,\frac{\delta}{\rho}\right\}, \tag{87}$$

$$\frac{\partial\phi\, B(\hat{x})}{\partial\theta} \;=\; 0\;, \tag{88}$$

$$\frac{\partial\phi B(\hat{x})}{\partial\phi} \;=\; B(\hat{x})\;. \tag{89}$$

– Let $F(\hat{x}) \equiv B_2^{\mathcal{T}}(\hat{x}) - \phi B(\hat{x})$. We therefore have

$$\frac{\partial F(\hat{x})}{\partial\theta} \;=\; \frac{\phi}{\rho}\left[1\,-\,\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda} - \lambda\left[\frac{(1-\phi)\delta \,-\, \alpha\, b}{w+b}\right]\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right], \tag{90}$$

$$\frac{\partial F(\hat{x})}{\partial\phi} \;=\; -\,\frac{(\delta-\theta)}{\rho}\left[1\,-\,\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda} - \lambda\left[\frac{(1-\phi)\delta \,-\, \alpha\, b}{w+b}\right]\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right]$$
$$+\;\frac{\delta}{\rho} - \frac{\delta}{\rho}\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda} -\; B(\hat{x})\;, \tag{91}$$

$$\frac{\partial F(\hat{x})}{\partial\hat{x}} \;=\; \frac{\lambda}{\hat{x}}\left\{F(\hat{x}) + \frac{\phi\,(\delta-\theta)}{\rho}\right\} \;=\; \frac{\lambda}{\hat{x}}\,\frac{\phi\,(\delta-\theta)}{\rho}\;. \tag{92}$$

– Let $\Lambda \equiv -\frac{\partial S_1(x_t)}{\partial \theta} \;/\; \frac{\partial F(\hat{x})}{\partial \theta}$. Replacing yields

$$\Lambda \quad = \quad \frac{\left[1-\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right]\left(\frac{x_t}{\hat{x}}\right)^{\lambda}}{1\;-\;\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\;-\;\lambda\left[\frac{(1-\phi)\delta\,-\,\alpha\,b}{w+b}\right]\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}}\;. \tag{93}$$

Proof of Proposition 2: In order to be later able to prove Proposition 2, we pursue the following proof strategy: We assume that the solution is such that condition (17) is always satisfied and consider a variant of problem (21), where condition (17) is relaxed; We then verify that the solution to this problem is indeed such that condition (17) is satisfied; The solution to this variant problem is then also the solution to problem (21).

Drop the notation R and superscripts E. Assume that the solution is such that $B_2^{\mathcal{T}}(\hat{x}) \geq \frac{\phi}{1-\phi}\,B_2^{\mathcal{H}}(\hat{x})$. Consider the variant problem

$$\max_{\hat{x}\geq\underline{x}(\delta),\theta\geq 0,\phi\in[0,1]} \quad \mathcal{L}(x_t) \quad = \quad S_1(x_t)\,+\,\Lambda\,F(\hat{x}) \tag{94}$$

$$\text{where} \quad F(\hat{x}) \quad = \quad B_2^{\mathcal{T}}(\hat{x}) - \phi B(\hat{x})\,.$$

The first order conditions are:

$$\begin{aligned}
(i) \qquad \frac{\partial \mathcal{L}(x_t)}{\partial \theta} \quad &= \quad \frac{\partial S_1(x_t)}{\partial \theta}\;+\;\Lambda\,\frac{\partial F(\hat{x})}{\partial \theta}\;=\;0\;,\\
(ii) \qquad \frac{\partial \mathcal{L}(x_t)}{\partial \phi} \quad &= \quad \frac{\partial S_1(x_t)}{\partial \phi}\;+\;\Lambda\,\frac{\partial F(\hat{x})}{\partial \phi}\;=\;0\;,\\
(iii) \qquad \frac{\partial \mathcal{L}(x_t)}{\partial \Lambda} \quad &= \quad F(\hat{x})\;=\;0\;,\\
(iv) \qquad \frac{\partial \mathcal{L}(x_t)}{\partial \hat{x}} \quad &= \quad \frac{\partial S_1(x_t)}{\partial \hat{x}}\;+\;\Lambda\,\frac{\partial F(\hat{x})}{\partial \hat{x}}\;=\;0\;.
\end{aligned} \tag{95}$$

Develop the expression of $\frac{\partial \mathcal{L}(x_t)}{\partial \Lambda}$: From $F(\hat{x}) \;=\; B_2^{\mathcal{T}}(\hat{x}) - \phi\,B(\hat{x})$, we can write the first order condition (iii) $\frac{\partial \mathcal{L}(x_t)}{\partial \Lambda} = 0$ as

$$\phi\,\frac{\theta}{\rho}\;-\;\left[\phi\,\frac{\theta}{\rho} - V^B(\underline{x}(b))\right]\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda} \;=\; \phi\,\frac{\delta}{\rho}\;-\;\phi\,\left[\frac{\delta}{\rho}\;-\;V^B(\underline{x}(\delta))\right]\left(\frac{\hat{x}}{\underline{x}(\delta)}\right)^{\lambda}\;. \tag{96}$$

From (i) we have $\Lambda \;=\; -\frac{\partial S_1(x_t)}{\partial \theta} \;/\; \frac{\partial F(\hat{x})}{\partial \theta}$. Develop the expression of $\partial \mathcal{L}(x_t)/\partial \phi$, replacing Λ using the expression in (93) and those developed in the Preliminary calculations:

$$\begin{aligned}
\frac{\partial \mathcal{L}(x_t)}{\partial \phi} \quad = \quad & \frac{(\delta-\theta)}{\rho}\left[1-\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right]\left(\frac{x_t}{\hat{x}}\right)^{\lambda}\\
& +\Lambda\,\left\{-\frac{(\delta-\theta)}{\rho}\left[1-\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}-\lambda\left[\frac{(1-\phi)\,\delta\;-\;\alpha\,b}{w+b}\right]\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right]\right.\\
& \qquad \left.-\;\frac{\delta}{\rho}\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}-\;\left[V^B(\underline{x}(\delta))\;-\;\frac{\delta}{\rho}\right]\left(\frac{\hat{x}}{\underline{x}(\delta)}\right)^{\lambda}\right\}\;,
\end{aligned} \tag{97}$$

$$= \quad \Lambda\,\left\{-\,\frac{\delta}{\rho}\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}-\;\left[V^B(\underline{x}(\delta))\;-\;\frac{\delta}{\rho}\right]\left(\frac{\hat{x}}{\underline{x}(\delta)}\right)^{\lambda}\right\}\;. \tag{98}$$

We can therefore write the first order condition (ii) $\frac{\partial \mathcal{L}(x_t)}{\partial \phi} = 0$ as

$$\frac{\delta}{\rho}\left[1 \; - \; \left(\frac{\underline{x}(\delta)}{\underline{x}(b)}\right)^{\lambda}\right] \quad = \quad V^B(\underline{x}(\delta))\,. \tag{99}$$

Using (99), we can rewrite the first order condition (iii) $\frac{\partial \mathcal{L}(x_t)}{\partial \Lambda} = 0$ in (96) as

$$V^B(\underline{x}(b))\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda} \quad = \quad \phi\,\frac{(\delta-\theta)}{\rho}\left[1 \; - \; \left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right]\,. \tag{100}$$

Develop similarly expression of $\partial \mathcal{L}(x_t)/\partial \hat{x}$:

$$\frac{\partial \mathcal{L}(x_t)}{\partial \hat{x}} \quad = \quad \frac{-\lambda\,\phi(\delta-\theta)}{\rho\,\hat{x}}\left(\frac{x_t}{\hat{x}}\right)^{\lambda} \; + \; \Lambda\,\frac{\lambda}{\hat{x}}\,\frac{\phi\,(\delta-\theta)}{\rho}\,, \tag{101}$$

$$= \quad \frac{\lambda^2\,\Lambda\,\phi\,(\delta-\theta)}{\rho\,\hat{x}}\,\frac{\left(\frac{\hat{x}}{\underline{x}(\theta)}\right)^{\lambda}}{1-\left(\frac{\hat{x}}{\underline{x}(\theta)}\right)^{\lambda}}\left[\frac{(1-\phi)\delta \; - \; \alpha\,b}{w+b}\right]\,. \tag{102}$$

The first order condition (iv) $\frac{\partial \mathcal{L}(x_t)}{\partial \hat{x}} = 0$, can be written $(1-\phi)\delta \, - \, \alpha\, b = 0$. Using $b = \phi\,\theta + (1-\phi)\delta$, it becomes

$$\frac{\phi\,\theta}{1-\phi} \quad = \quad \frac{1-\alpha}{\alpha}\,\delta\,. \tag{103}$$

The best offer is therefore the triple $(\hat{x}^E, \theta^E, \phi^E)$ solving (99), (100) and (103). If $\hat{x}^E < \underline{x}(\delta)$, the constraint that $\hat{x} \geq \underline{x}(\delta)$ is violated. In that case, the best offer is the triple $(\hat{x}^E, \theta^E, \phi^E)$ solving (99), (100) and $\hat{x}^E = \underline{x}(\delta)$.

We now verify that the solution triple $(\hat{x}^E, \theta^E, \phi^E)$ to the variant problem (94) is such that condition (17) is satisfied. From (15) and (16), $B_2^{\mathcal{T}}(\hat{x}) \geq \frac{\phi}{1-\phi}\, B_2^{\mathcal{H}}(\hat{x})$ can be written

$$\frac{\phi\,\theta}{\rho} \; + \; \left\{V^B(\,\underline{x}(b)\,) \; - \; \frac{\phi\,\theta}{\rho}\right\}\left(\frac{x_t}{\underline{x}(b)}\right)^{\lambda} \quad \geq \quad \frac{\phi\,\delta}{\rho}\left[1 \; - \; \left(\frac{x_t}{\underline{x}(b)}\right)^{\lambda}\right]. \tag{104}$$

Rearranging, this inequality can be written

$$V^B(\underline{x}(b))\left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda} \quad \geq \quad \phi\,\frac{(\delta-\theta)}{\rho}\left[1 \; - \; \left(\frac{\hat{x}}{\underline{x}(b)}\right)^{\lambda}\right]\,. \tag{105}$$

Given that $(\hat{x}^E, \theta^E, \phi^E)$ is such that (100) hold, inequality (105) is indeed always satisfied. The solution $(\hat{x}^E, \theta^E, \phi^E)$ to the variant problem (94) is then also the solution to problem (21). $\square$

Proof of (25) to (28): From (8), we have

$$B(x_t) \quad = \quad \frac{\delta}{\rho} \; - \; \left[\frac{\delta}{\rho} \; - \; V^B(\underline{x}(\delta))\right]\left(\frac{x_t}{\underline{x}(\delta)}\right)^{\lambda}. \tag{106}$$

Replacing the expression of $V^B(\underline{x}(\delta))$ in (22) into (106) gives

$$B(x_t) \;=\; \frac{\delta}{\rho} \;-\; \left[\frac{\delta}{\rho} \;-\; \frac{\delta}{\rho}\left[1 \;-\; \left(\frac{\underline{x}(\delta)}{\underline{x}(b^E)}\right)^{\lambda}\right]\right]\left(\frac{x_t}{\underline{x}(\delta)}\right)^{\lambda}, \tag{107}$$

$$\;=\; \frac{\delta}{\rho}\left[1 \;-\; \left(\frac{x_t}{\underline{x}(b^E)}\right)^{\lambda}\right]. \tag{108}$$

From the expression of $B_2^{\mathcal{H}}(x_t\,|R)$ in (16), it follows that

$$B_2^{\mathcal{H}}(x_t\,|R^E) \;=\; (1-\phi^E)\,B(x_t)\,. \tag{109}$$

From (23), we have

$$\phi^E\,\frac{\delta}{\rho}\left[1 \;-\; \left(\frac{\hat{x}^E}{\underline{x}(b^E)}\right)^{\lambda}\right] \;=\; \phi^E\,\frac{\theta^E}{\rho}\left[1 \;-\; \left(\frac{\hat{x}^E}{\underline{x}(b^E)}\right)^{\lambda}\right] \;+\; V^B(\underline{x}(b^E))\left(\frac{\hat{x}^E}{\underline{x}(b^E)}\right)^{\lambda}\,. \tag{110}$$

Hence, from the expressions of $B_2^{\mathcal{T}}(x_t\,|R)$ and $B_2^{\mathcal{H}}(x_t\,|R)$ in (15) and (16), (110) can be written

$$\frac{\phi^E}{1-\phi^E}\;B_2^{\mathcal{H}}(\hat{x}^E\,|R^E) \;=\; B_2^{\mathcal{T}}(\hat{x}^E\,|R^E)\,. \tag{111}$$

It follows that

$$B_2^{\mathcal{T}}(\hat{x}^E\,|R^E) \;+\; B_2^{\mathcal{H}}(\hat{x}^E\,|R^E) \;=\; \frac{B_2^{\mathcal{H}}(\hat{x}^E\,|R^E)}{1-\phi^E}\,. \tag{112}$$

Replacing $B_2^{\mathcal{H}}(x_t\,|R^E)$ in (109) into (112) gives

$$B_2^{\mathcal{T}}(\hat{x}^E\,|R^E) \;+\; B_2^{\mathcal{H}}(\hat{x}^E\,|R^E) \;=\; B(\hat{x}^E)\,, \tag{113}$$

which is (27). From (109) and (113), we have (25) and (26).

From (13), we have

$$B_1(x_t\,|R^E) \;=\; \frac{\delta}{\rho} \;-\; \left[\frac{\delta}{\rho} \;-B_2^{\mathcal{T}}(\hat{x}^E\,|R^E) - B_2^{\mathcal{H}}(\hat{x}^E\,|R^E)\right]\left(\frac{x_t}{\hat{x}^E}\right)^{\lambda}. \tag{114}$$

Using (113), (114) can be written

$$B_1(x_t\,|R^E) \;=\; \frac{\delta}{\rho} \;-\; \left[\frac{\delta}{\rho} \;- B(\hat{x}^E)\right]\left(\frac{x_t}{\hat{x}^E}\right)^{\lambda}. \tag{115}$$

From (8), we have

$$B(\hat{x}^E) \;=\; \frac{\delta}{\rho} \;-\; \left[\frac{\delta}{\rho} \;-\; V^B(\underline{x}(\delta))\right]\left(\frac{\hat{x}^E}{\underline{x}(\delta)}\right)^{\lambda}. \tag{116}$$

Replacing the expression of $B(\hat{x}^E)$ in (129) into (128) gives

$$B_1(x_t\,|R^E) \;=\; B(x_t)\,, \tag{117}$$

which is (28).

Proof of (29): From (11),

$$\phi\,(F'-F) \;=\; \frac{\phi^E\,(\delta-\theta^E)}{\rho}\left[1\;-\;\left(\frac{\hat{x}}{\underline{x}(b^E)}\right)^{\lambda}\right]\left(\frac{\hat{x}}{\underline{x}(b^E)}\right)^{-\lambda}. \tag{118}$$

(23) can be written

$$\frac{\phi^E\,(\delta-\theta^E)}{\rho}\left[1\;-\;\left(\frac{\hat{x}}{\underline{x}(b^E)}\right)^{\lambda}\right]\left(\frac{\hat{x}}{\underline{x}(b^E)}\right)^{-\lambda} \;=\; V^B(\underline{x}(b^E))\,. \tag{119}$$

Replacing (119) in (118), gives $\phi^E\,(F'-F) = V^B(\underline{x}(b^E))$. Then (29) surely holds. $\square$

Proof of Proposition 3: Drop the notation R and superscripts C. Suppose the solution to problem (33) is such that the condition $\hat{x} \geq \underline{x}(\delta)$ is slack. Then (33) can be written

$$\begin{aligned} \max_{\theta\geq 0}\;\mathcal{L}(x_t) \;&=\; S_1(x_t)\;+\;\Lambda\,F(\hat{x})\;, \\ \text{where}\quad F(\hat{x}) \;&=\; B_2^{\mathcal{T}}(\hat{x})-B(\hat{x})\,. \end{aligned} \tag{120}$$

The first order conditions are

$$\begin{aligned} (i)\qquad \frac{\partial\mathcal{L}(x_t)}{\partial\theta} \;&=\; \frac{\partial S_1(x_t)}{\partial\theta}\;+\;\Lambda\,\frac{\partial F(\hat{x})}{\partial\theta}\;=\;0\;, \\ (ii)\qquad \frac{\partial\mathcal{L}(x_t)}{\partial\Lambda} \;&=\; B_2^{\mathcal{T}}(\hat{x})-B(\hat{x})\;=\;0\,, \\ (iii)\qquad \frac{\partial\mathcal{L}(x_t)}{\partial\hat{x}} \;&=\; \frac{\partial S_1(x_t)}{\partial\hat{x}}\;+\;\Lambda\,\frac{\partial F(\hat{x})}{\partial\hat{x}}\;=\;0\;. \end{aligned} \tag{121}$$

In the following, set $\phi = 1$ (hence $b = \theta$) in the expressions developed in the Preliminary calculations.

Develop the expression of $\frac{\partial\mathcal{L}(x_t)}{\partial\Lambda}$: From $F(\hat{x}) \;=\; B_2^{\mathcal{T}}(\hat{x})-B(\hat{x})$, we can write the first order condition (ii) $\frac{\partial\mathcal{L}(x_t)}{\partial\Lambda} = 0$ as

$$\left[\frac{\theta}{\rho}-V^B(\underline{x}(\theta))\right]\left(\frac{\hat{x}}{\underline{x}(\theta)}\right)^{\lambda} \;=\; \frac{\theta}{\rho}-\;\frac{\delta}{\rho}\;+\;\left[\frac{\delta}{\rho}\;-\;V^B(\underline{x}(\delta))\right]\left(\frac{\hat{x}}{\underline{x}(\delta)}\right)^{\lambda}\;. \tag{122}$$

From (i) we have $\Lambda \;=\; -\,\frac{\partial S_1(x_t)}{\partial\theta}\;/\;\frac{\partial F(\hat{x})}{\partial\theta}$. Develop the expression of $\partial\mathcal{L}(x_t)/\partial\hat{x}$, replacing Λ using the expression in (93):

$$\frac{\partial\mathcal{L}(x_t)}{\partial\hat{x}} \;=\; \frac{\lambda^2\,\Lambda\,(\delta-\theta)}{\rho\,\hat{x}}\,\frac{\left(\frac{\hat{x}}{\underline{x}(\theta)}\right)^{\lambda}}{1-\left(\frac{\hat{x}}{\underline{x}(\theta)}\right)^{\lambda}}\left[\frac{\alpha\,\theta}{w+\theta}\right]\;. \tag{123}$$

The first order condition (iii) $\frac{\partial\mathcal{L}(x_t)}{\partial\hat{x}} = 0$, can only be satisfied if $\theta = 0$. However, the first order condition (ii) $B_2^{\mathcal{T}}(\hat{x})-B(\hat{x}) = 0$, implies $\theta > 0$ (because $B(\hat{x}) \geq V^B(\hat{x}) > 0$). Hence, there cannot be $\hat{x} > \underline{x}(\delta)$ such that (iii) is satisfied. This contradicts the assumption that condition $\hat{x} \geq \underline{x}(\delta)$ is slack. The constraint $\hat{x} \geq \underline{x}(\delta)$ is therefore binding.

In the Lagrangian including the condition $\hat{x} = \underline{x}(\delta)$, the solution to the first order condition (ii) $\frac{\partial \mathcal{L}(x_t)}{\partial \Lambda} = 0$ is unchanged. Hence (122) is still valid. Replacing $\hat{x} = \underline{x}(\delta)$ in (122), the best offer is therefore such that $\hat{x} = \underline{x}(\delta)$ and

$$V^B(\underline{x}(\delta)) \quad = \quad \frac{\theta}{\rho} \;-\; \left[\frac{\theta}{\rho} \;-\; V^B(\underline{x}(\theta))\right] \left(\frac{\underline{x}(\delta)}{\underline{x}(\theta)}\right)^{\lambda} . \qquad \Box \tag{124}$$

Proof of (36) to (37): From the expression of $B_2^{\mathcal{T}}(x_t\,|R)$ in (15), we can write (35) as

$$V^B(\underline{x}(b^C)) \quad = \quad B_2^{\mathcal{T}}(\hat{x}^C\,|R^C) \;. \tag{125}$$

From (8), we therefore have

$$B_2^{\mathcal{T}}(\hat{x}^C\,|R^C) \quad = \quad B(\hat{x}^C) \;, \tag{126}$$

which is (36). From (13), we have

$$B_1(x_t\,|R^C) \quad = \quad \frac{\delta}{\rho} \;-\; \left[\frac{\delta}{\rho} \;-B_2^{\mathcal{T}}(\hat{x}^C\,|R^C)\right] \left(\frac{x_t}{\hat{x}^C}\right)^{\lambda} . \tag{127}$$

Using (126), (127) can be written

$$B_1(x_t\,|R^C) \quad = \quad \frac{\delta}{\rho} \;-\; \left[\frac{\delta}{\rho} \;-B(\hat{x}^C)\right] \left(\frac{x_t}{\hat{x}^C}\right)^{\lambda} . \tag{128}$$

From (8), we have

$$B(\hat{x}^C) \quad = \quad \frac{\delta}{\rho} \;-\; \left[\frac{\delta}{\rho} \;-\; V^B(\underline{x}(\delta))\right] \left(\frac{\hat{x}^C}{\underline{x}(\delta)}\right)^{\lambda} . \tag{129}$$

Replacing the expression of $B(\hat{x}^C)$ in (129) into (128) gives

$$B_1(x_t\,|R^C) \quad = \quad B(x_t) \;, \tag{130}$$

which is (28).

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