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Fear of Ruin

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Abstract

This paper offers interpretations and applications of the "fear of ruin" coefficient (Aumann and Kurz, 1977, *Econometrica*). This coefficient is useful for analyzing the behavior of expected utility maximizers when they face binary lotteries with the same worse outcome. Comparative statics results of "more fear of ruin" are derived. The partial ordering induced by the fear of ruin coefficient is shown to be weaker than that induced by the Arrow-Pratt coefficient.

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JEL Classification: D81

This paper offers various interpretations of a risk-aversion coefficient, $[u(w)-u(0)]/u'(w)$ in standard notation, that was first introduced by Aumann and Kurz (1977). This coefficient, coined the "fear of ruin" (*FR*) coefficient, captures an individual's "attitude toward risking his fortune."[1] There has not been to date any systematic analysis of this coefficient. This paper fills this gap by identifying situations in which the *FR* coefficient controls the behavior of expected utility maximizers. These situations involve choices among binary lotteries with a fixed worse outcome.

1. Fear of ruin in the small and in the large

How much would an individual be willing to pay to be fully insured against the possibility of ruin? Suppose that this individual maximizes his expected utility, with an increasing von Neumann Morgenstern utility function u and current wealth w (assume $w > 0$). He may lose his entire wealth w with probability p. The insurance premium $z(p)$ is defined by[2]

$$u(w - z(p)) = (1 - p)u(w) + pu(0). \tag{1}$$

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[1] Aumann and Kurz note that this interpretation is an outcome of a conversation they had with Kenneth Arrow.

[2] The model could allow any arbitrary wealth $\underline{w}$ in the case of ruin. Here, we simply assume, without loss of generality, that $\underline{w} = 0$. We also assume that $u(0)$ is finite.

Assume that u is differentiable. Differentiating (1) with respect to p gives $z'(p) = \frac{u(w)-u(0)}{u'(w-z(p))}$. Suppose that the probability p is small enough so that $z(p)$ may be reasonably approximated by $pz'(0)$ since $z(0) = 0$. A first-order Taylor approximation of the insurance premium $z(p)$ is then given by $z(p) \approx p\frac{u(w)-u(0)}{u'(w)}$.

This insurance premium "in the small" thus depends separately on the characteristics of the risk and of the individual. In accord with intuition, the premium is proportional to the probability of ruin p. Moreover, it depends on the characteristics of the individual's utility function only through the ratio $\frac{u(w)-u(0)}{u'(w)}$. Observe that this ratio is invariant to affine transformations of utility.

As in Aumann and Kurz (1977), we scale the utility function by assuming $u(0) = 0$. This is done for expositional convenience and without loss of generality since the utility function is defined up to an affine transformation. In this case, the insurance premium is

$$z(p) \approx p\frac{u(w)}{u'(w)} \tag{2}$$

The coefficient u/u' corresponds to the "fear of ruin" coefficient, as it was first introduced by Aumann and Kurz (1977). From now, and throughout the paper, we will refer to u/u' as the coefficient of fear of ruin, or *FR*.[3]

Observe that our approximation for the insurance premium does not directly depend on the Arrow-Pratt coefficient, $-u''/u'$. This is because we approximate this premium for a small change in the probability p, not for a small change in the variation of terminal wealth, as in Pratt (1964) and Arrow (1971). Consequently, we can derive a first-order approximation of the insurance premium by simply examining the rate of increase of the insurance premium with respect to p.[4]

Furthermore, observe that $u(w)/u'(w)$ is always strictly positive under $u(\cdot)$ increasing, since $u(0) = 0$. Moreover, under risk-aversion, it is easy to see that this coefficient always increases with wealth w. Intuitively, there are two reasons why the insurance premium increases with wealth. First, when the agent is wealthier, there is more to lose. As a result, the agent is willing to pay more in the face of the risk of losing his entire wealth. This is the effect related to the numerator, $u(w)$, which increases in w. Second, under risk-averse preferences, the marginal value of money is smaller when the agent is wealthier, so he is willing to sacrifice a larger amount of money in face of the same risk. This effect is related to the term $1/u'(w)$, which also increases in w under risk-aversion.

Also, simply observe that, under risk neutrality, the *FR* coefficient reduces to w. Moreover, if u is concave,

$$\frac{u(w)}{w} \geq u'(w).$$

This inequality states that the slope of the tangent to the utility function at w is always smaller than the slope of the chord drawn from 0 to w. Multiplying both sides of this

[3] Hence, the reader should remember that the appropriate *FR* coefficient in the general case is $[u(w)-u(0)]/u'(w)$.

[4] The equivalent first-order effect for a variation in terminal wealth is zero in Pratt, so that he examines the second order effect. See Gollier (2001, p. 21–24) for a detailed analysis of Pratt (1964)'s "in the small" approximation.

inequality by $w/u'(w)$ shows that the *FR* coefficient is always larger under risk-aversion than under risk-neutrality. This is consistent with the intuitive requirement that one's fear of ruin is lower when one is risk-neutral.[5]

Previous remarks interpret *FR* as a measure of local risk aversion or local propensity to insure against a *small* chance of ruin. We next examine comparative properties of the *FR* coefficient for any probability of ruin p. To do so, we first introduce two Definitions.

Definition 1. We define $z(u, w, p)$, the insurance premium of agent u facing the risk of losing wealth w with probability p, by

$$u(w - z(u, w, p)) = (1 - p)u(w).$$

Definition 2. We define $c(u, w, p)$, the compensating premium of agent u facing the risk of losing wealth w with probability p, by

$$u(w) = (1 - p)u(w + c(u, w, p)).$$

The quantity $z(u, w, p)$ is the insurance premium that agent u with current wealth w is willing to pay to avoid the possibility that a ruin occurs with probability p. The quantity $c(u, w, p)$ is the compensating premium that agent u is willing to accept to face a possibility of ruin, namely to end up with wealth 0 with probability p or with wealth $w + c(u, w, p)$ otherwise.

Following the approach developed by Pratt (1964), we now compare the *FR* of two individuals u and v for all w and p. Under the normalization at 0 adopted above, we introduce the following natural definition of "more fear of ruin".

Definition 3. Agent v is said to have more fear of ruin (*FR*) than agent u if and only if for all w,

$$\frac{v(w)}{v'(w)} \geq \frac{u(w)}{u'(w)}.$$

Using the three Definitions above, we can now state the first Proposition of this paper.[6]

Proposition 1. *Consider two agents with strictly increasing and differentiable utility functions u and v such that $u(0) = v(0) = 0$. For all $p \in [0, 1]$ and all strictly positive wealth w, the following four conditions are equivalent*:

(i) *Agent v has more I than agent u, namely $\frac{v(w)}{v'(w)} \geq \frac{u(w)}{u'(w)}$*;
(ii) *Agent v has a higher insurance premium than agent u, namely $z(v, w, p) \geq z(u, w, p)$*;
(iii) *Agent v has a higher compensating premium than $\bar{u}$, namely $c(v, w, p) \geq c(u, w, p)$*;

[5] Moreover observe that the fear of ruin is even lower when preferences are risk-seeking.
[6] Detailed proofs are available upon request. See also Foncel and Treich (2003).

(iv) *There exists an increasing and differentiable function* $T(\cdot) = v \circ u^{-1}(\cdot)$ *such that* $T(0) = 0$ *and for all* x, $\frac{T(x)}{x}$ *is decreasing in* x.

A sketch of the proof follows. We prove the equivalence between (i), (ii) and (iv). First, (ii) implies (i) by (2). Second, we show that (i) implies (iv). Observe that since u and v are increasing and differentiable functions, there always exists a unique, increasing and differentiable function $T = v \circ u^{-1}$ such that $v = T \circ u$. Also, $v(0) = T \circ u(0) = T(0) = 0$. Moreover, from $\frac{u}{u'} \leq \frac{v}{v'}$, we have $T'(u) \leq \frac{T(u)}{u}$, which must be true for all u. This latter condition is equivalent to (iv). Third, we show that (iv) implies (ii). By Definition 1

$$v(w - z(u, w, p)) = T(u((w - z(u, w, p))) = T((1 - p)u(w)).$$

Since $T(x)/x$ is decreasing in x, we get $T((1 - p)u(w)) \geq (1 - p)T(u(w))$. We thus have $v(w - z(u, w, p)) \geq (1 - p)v(w) = v(w - z(v, w, p))$. This implies (ii). The proof of the equivalence between (i), (iii) and (iv) is similar.

The result would trivially generalize to a risk premium $\pi(u, w, p)$, namely the insurance premium net of the expected value of the risk $\pi(u, w, p) = z(u, w, p) - pw$.

2. Applications

In this section, we show that the *FR* coefficient is applicable to a wide variety of models. Consistent with the previous section, these applications involve choices among lotteries with just two possible outcomes in which the worse outcome of the lotteries is the same, equal to the "ruin point" (normalized to zero).

2.1. *Value-of-statistical-life*

Let us interpret $u(0)$ in model (1) as the utility when dead. In other words, the ruin point is the death point. The expected utility equals $(1 - p)u(w)$; there is no bequest motive. The value-of-statistical-life (VSL) is usually defined as the rate of substitution between wealth w and mortality risk p (see, e.g., Viscusi, 1993). We have

$$\text{VSL} = \frac{dw}{dp} = \frac{u(w)}{(1 - p)u'(w)} = \frac{FR[u(w)]}{(1 - p)}$$

where $FR[u(\cdot)] \equiv u(\cdot)/u'(\cdot)$. In this simple model, it is clear that there is a one-to-one relation between VSL and *FR*. An individual has, other things being equal, a higher VSL if and only if he has more *FR*.

Let us slightly adapt the model now to allow for insurance opportunities. More precisely, assume that there is an annuity market in which survivors are offered fair tontines shares (Rosen, 1988). In a large group of identical individuals, a proportion p die and their wealth is distributed to $(1 - p)$ survivors. A survivor's consumption thus equals initial wealth w

plus the tontine share $pw/(1-p)$, that is a total of $w/(1-p)$. The state-dependent expected utility thus equals $(1-p)u(w/(1-p))$ and we have

$$\mathrm{VSL} = FR[u(w/(1-p))] - \frac{w}{1-p}$$

There is still a one-to-one relation between VSL and *FR* in this model introduced by Rosen (1988).

Interestingly, *FR* plays an important role in a life-cycle model as well. To see this point, consider the following two-period model

$$V \equiv \max_c u(c) + \beta(1-p)u(R(w-c))$$

where β is a discount factor, R the interest factor, c consumption in period 1 and $(1-p)$ the survival probability from period 1 to period 2. (Observe again that there is no bequest motive.) Then compute the VSL defined by the rate of substitution between wealth w and survival probability p, i.e. $dw/dp = -(\frac{\partial V}{\partial p}/\frac{\partial V}{\partial w})$. Using the Envelope Theorem, it is equal to $1/(1-p)R$ times the *FR* coefficient computed at the optimal period 2 consumption.[7]

2.2. *First-price auctions*

Let us consider the standard first-price auction model. There are N agents, $i = 1, \ldots, N$ each with identical utility function u where $u(0) = 0$. They participate in an auction where they all bid for an indivisible object. Each agent i has a private value x_i for the object. This value is drawn independently from a common distribution $F(\cdot)$ with density $f(\cdot)$ on a support $[\underline{x}, \bar{x}]$. The highest bidder wins the object. His payoff is the value of the object minus the bid, i.e. $x_i - b_i$. Other bidders have payoff 0 (or 0 is the status quo).

Agent i chooses b_i so as to maximize

$$p_i u(x_i - b_i),$$

with $p_i \equiv \Pr(b_i > B(x_j), \forall j \neq i)$ and where $B(\cdot)$ is the optimal bidding strategy. It is well-known that the first order condition for the Nash equilibrium bidding strategy $B(x)$ is given by the differential equation[8]

$$B'(x) = (N-1)\frac{f(x)}{F(x)}\frac{u(x-B(x))}{u'(x-B(x))}, \quad \text{with} B(x) = \underline{x}.$$

[7] Garber and Phelps (1997) indicate that u/u' is a "central component" in their lifetime medical spending model. Also, in a recent unpublished paper, Bommier (2003) shows that *FR* is a crucial coefficient when one wants to compare lotteries involving lives of different lengths. He calls the *FR* coefficient the "general rate of substitution between the length of life and consumption at the end of life".

[8] See for instance Milgrom (2004, pages 123–125).

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What is the effect of increased risk aversion in the sense of more *FR* on the equilibrium bidding function $B(x)$? Assume that bidders v have more *FR* and let us compare, ceteris paribus, the outcome of a first-price auction populated by bidders v instead of bidders u.[9] Using straightforward notation we find

$$\begin{aligned} B_v'(x) - B_u'(x) &= (N-1)\frac{f(x)}{F(x)}\left[\frac{v(x-B_v(x))}{v'(x-B_v(x))} - \frac{u(x-B_u(x))}{u'(x-B_u(x))}\right] \\ &\geq (N-1)\frac{f(x)}{F(x)}\left[\frac{u(x-B_v(x))}{u'(x-B_v(x))} - \frac{u(x-B_u(x))}{u'(x-B_u(x))}\right], \end{aligned}$$

by $\frac{v(.)}{v'(\cdot)} \geq \frac{u(\cdot)}{u'(\cdot)}$. From that last result, we easily find that, for all x,

$$B_v(x) = B_u(x) \text{implies} B_v'(x) \geq B_u'(x).$$

We thus have obtained a single crossing property. This property means that B_v can only cross B_u from below. Since $B_v(\underline{x}) = B_u(\underline{x}) = \underline{x}$, the function $B_v(x)$ will always be larger than $B_u(x)$ for any x such that $x \geq \underline{x}$. Therefore, more *FR* always raises the bidding price equilibrium. This finding leads to the following Proposition.

Proposition 2. *The equilibrium price of a first-price auction with independent private values increases when bidders have more FR.*

This result extends that of Milgrom and Weber (1982), who showed that introducing risk-aversion raises the bidding price compared to the risk-neutral case.

2.3. *Conflict and bargaining games*

A conflict game may be described as follows (see, e.g., Skaperdas, 1997). Two agents, say 1 and 2, possess one unit of a resource. They may convert this resource and invest it into arms, in quantities y_1 and y_2 respectively. The winner of the conflict gets a prize that depends on the remaining productive resources of both agents, while the loser gets 0. The prize is a function $C \equiv C(1-y_1, 1-y_2)$ which is increasing in both arguments. Let $p \equiv p(y_1, y_2)$ and $1-p$ denote the winning probability of agent 1 and 2 respectively, and u_1 and u_2 their utilities so that they respectively maximize

$$pu_1(C) \quad \text{and} \quad (1-p)u_2(C).$$

It can be shown that, in such a game, an agent with more *FR* always invests more into arms and has a higher probability of winning the conflict when C is symmetrical. See Skaperdas (1997, page 117, equation. 4). Moreover, when two identical agents simultaneously have

[9] The assumption that private values are independent of private characteristics, like risk-aversion, obviously facilitates the comparative statics analysis here.

more *FR*, the total amount invested into arms increases as well. The intuition for this is straightforward. On the one hand, increasing investment into arms decreases payoff C in the case of victory. On the other hand, increasing investment decreases the chance of losing the conflict, and so helps to avoid ruin (notice that the loser's payoff is the ruin point here). The trade-off is thus similar to the one presented in the previous models. It is not surprising that *FR* controls the amount of resources invested into arms in this model.[10]

Another application in strategic games is the Nash bargaining problem, as first noticed by Aumann and Kurz (1977, p. 1149). To see that, consider two agents u_1 and u_2 who bargain over the division of a cake of size w. The well-known Nash solution to this problem calls for maximizing

$$u_1(y_1)u_2(y_2) \quad \text{subject to} \quad y_1 + y_2 = w.$$

It is easy to show that this solution equates the two individuals' *FR* computed at the optimal bargaining points. The intuition is as follows (see also Svejnar, 1986): In the bargaining problem, the ruin point $u(0) = 0$ can be interpreted as the threat utility if the bargaining process fails. As a result, at each stage of the bargaining process, each agent considers a gamble in which he risks losing the entire net gain which he has won so far against an additional gain of a small amount. More fear of ruin thus reduces the willingness to accept this gamble and so is a disadvantage in bargaining. See Roth and Rothblum (1982) for a general analysis.

2.4. *Contingent background risk*

Take model (1) but replace the term $u(w)$ by the term $u_\varepsilon(w) \equiv E_\varepsilon u(w + \tilde{\varepsilon})$ and assume $E\tilde{\varepsilon} = 0$. The individual thus faces a background risk $\tilde{\varepsilon}$ only if ruin does not occur. What is the effect of this contingent background risk? From Proposition 1, it is clear that the insurance premium always decreases if and only if

$$\frac{u_\varepsilon}{u'_\varepsilon} \leq \frac{u}{u'}$$

Observe that, given risk-averse preferences, $E\tilde{\varepsilon} = 0$ implies $u_\varepsilon(\cdot) = Eu(. + \tilde{\varepsilon}) \leq u(\cdot)$ by the Jensen inequality. Similarly, given prudence, $E\tilde{\varepsilon} = 0$ implies $u'_\varepsilon(\cdot) = Eu'(. + \tilde{\varepsilon}) \geq u'(\cdot)$ by the Jensen inequality. Hence, under the conditions of positive risk-aversion and prudence, *FR* decreases with a contingent background risk. Some implications directly follow. For instance, the VSL of risk-averse and prudent individuals decreases in face of a background risk contingent on being alive.[11]

[10]The *FR* coefficient is also at play in contest games (Skaperdas and Gan, 1995), or in rent-augmenting and rent-seeking games (Konrad and Schlesinger, 1997). However, more *FR* is not enough to control the comparative statics of more risk-aversion in those games as the loser's payoff generally depends on the agents' actions, and so the ruin point varies.

[11]This observation relates to Eeckhoudt and Hammitt (2001)'s analysis of the effect of a financial background risk on the VSL.

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Let us now consider an implication of this observation concerning the first-price auction model. This implication arises when the value of the auctioned object is uncertain. Here, we follow (Eso and White, 2004). Take the standard model of Section 3.2. Assume that the private value of the auctioned object is no longer x_i but instead is $x_i + \varepsilon_i$, where ε_i is the realization of a random variable $\tilde{\varepsilon}_i$. Random variables $\tilde{\varepsilon}_i$ are identically distributed as $\tilde{\varepsilon}$, and are independent of private values x_i. Thus, the highest bidder now receives an ex post payoff $x_i + \varepsilon_i - b_i$. Losing bidders still receive payoff of 0. This model implies that the background risk is contingent upon winning the auction. Ex ante, agent i chooses b_i so as to maximize

$$p_i E_\varepsilon u(x_i + \tilde{\varepsilon} - b_i) \equiv p_i u_\varepsilon(x_i - b_i)$$

with p_i defined as above. It is immediately clear that the differential equation characterizing the equilibrium strategy in the noisy auction takes on the following form

$$B'(x) = (N-1)\frac{f(x)}{F(x)}\frac{u_\varepsilon(x - B(x))}{u'_\varepsilon(x - B(x))}, \quad \text{with } B(\underline{x}) = \underline{x}.$$

In other words, analyzing the effect of the noise $\tilde{\varepsilon}$ on the equilibrium bidding price amounts to comparing the equilibrium with utilities $u_\varepsilon(\cdot)$ to the equilibrium with utilities $u(\cdot)$. This leads to the following Proposition.

Proposition 3. *Consider a first-price auction with independent private values and with risk-averse and prudent bidders. Then uncertainty over the value of the auctioned object decreases the equilibrium price.*

The intuition is two-fold. First, when preferences are risk-averse, utility is reduced if one wins the object $u_\varepsilon(\cdot) \leq u(\cdot)$. Hence, the object is less desirable. Second, given prudence, the marginal utility of income increases $u'_\varepsilon(\cdot) \geq u'(\cdot)$. Individuals thus bid less aggressively in the noisy auction because they value an extra dollar of income more. This Proposition shows that Eso and White (2004)'s result that decreasing absolute risk-averse (DARA) individuals bid smaller amounts in a noisy first-price auction also holds for any risk-averse prudent bidders (i.e., DARA is sufficient for prudence but the converse is not true).

Overall, these applications suggest that *FR* may be useful to sign various comparative statics results in a large class of models used throughout the economics literature.

3. Comparison with the Arrow-Pratt and the asymptotic risk-aversion coefficients

We have demonstrated in the previous section that more risk-aversion in the sense of *FR* increases the bidding price in a model of first-price auctions, and also controls risk-aversion motives in other models. This raises the question of the effect of an increase in risk-aversion à la Arrow-Pratt in those models. The answer to the question is given in the present section, as we precisely examine the link between *FR* and the Arrow-Pratt coefficient.

Following Jones-Lee (1980), it is useful to distinguish three different risk-aversion coefficients

$$\begin{aligned} FR[u(\cdot)] &\equiv \frac{u(\cdot)}{u'(\cdot)} \\ AP[u(\cdot)] &\equiv \frac{-u''(\cdot)}{u'(\cdot)} \\ AS[u(\cdot)] &\equiv \frac{u'(\cdot)}{u^* - u(\cdot)}. \end{aligned} \tag{3}$$

The last coefficient corresponds the asymptotic risk aversion coefficient (AS) introduced by Jones-Lee (1980). The AS coefficient measures the individual's willingness to participate in a "small-stake large-prize gamble." It assumes that u is bounded above, where u^* is the supremum of u.[12]

The complementarity of these three coefficients is apparent when one approximates insurance premia "in the small." Indeed, it is well-known that the Arrow-Pratt coefficient appears when considering risks with small gains and small losses. On the other hand, we have seen that the *FR* coefficient appears when the risk is a small probability of ruin. Finally, the AS risk aversion coefficient appears for a small loss/large gain risk, like gambling for the jackpot. See Jones-Lee (1980) for an interesting presentation and discussion.

In this section, we ask: to what extent is an individual v who is more risk-averse than an individual u in one specific sense also more risk-averse with respect to another sense? In other words, we want to compare the partial orderings induced by these three risk-aversion coefficients. To do so, let the statement "v is more risk averse than u in the sense of I" be condensed into $v \supseteq_I u$ and defined as follows.

Definition 4. Consider the three coefficients $I = \{FR, AP, AS\}$ as they are introduced in (3). Then

(i) $v \supseteq_{FR} u$ holds if and only if $FR[v(w)] \geq FR[u(w)]$ for all w,
(ii) $v \supseteq_{AP} u$ holds if and only if $AP[v(w)] \geq AP[u(w)]$ for all w,
(iii) $v \supseteq_{AS} u$ holds if and only if $AS[v(w)] \geq AS[u(w)]$ for all w.

Conversely, the ordering $v \not\supseteq_I u$ means that there exists w such that u is locally more risk averse than v in the sense of I.

The claim that an individual v is more risk averse than an individual u (in the sense of *AS*, *AP* or *FR*) can be fully characterized by setting the corresponding properties on a function T such that $v = T \circ u$. First, from Pratt (1964, Theorem 1) we know that $v \supseteq_{AP} u$ if and only if $v = T \circ u$ with T increasing, twice differentiable and concave. Second, from Proposition 1, we know that $v \supseteq_{FR} u$ if and only if $v = T \circ u$ where T is increasing, differentiable, $T(0) = 0$ and $T(x)/x$ decreasing in x. Finally, it is easy to show that $v \supseteq_{AS} u$ if and

[12] What is actually important is that both the ruin point and u^* are very bad and very good points beyond which it is not possible to go. In particular, the utility u needs not be bounded above if we set an upper limit for wealth.

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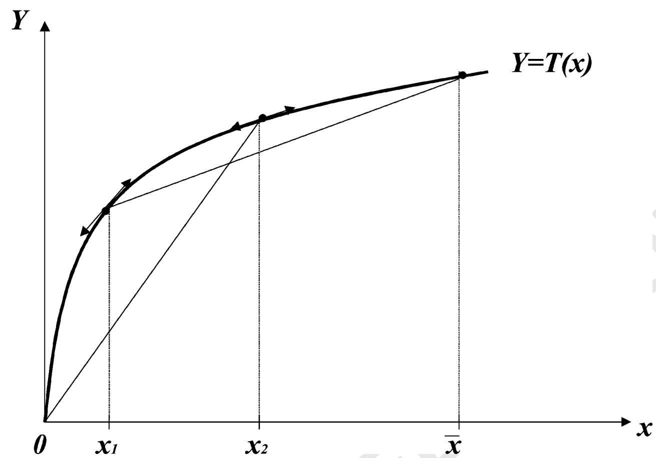

Figure 1. Represents an increasing and concave function T with $T(0) = 0$. This function is such that $T(x)/x > T'(x)$ together with $T'(x) > \frac{T(\bar{x})-T(x)}{\bar{x}-x}$ for all $x \in [0, \bar{x}]$.

only if $v = T \circ u$ where T is increasing, differentiable on $[0, \bar{x}]$ and $T'(x) \geq \frac{T(\bar{x})-T(x)}{\bar{x}-x}$, with $\bar{x} = u^* = \sup_w u(w)$. Hence, the comparative analysis of the different risk-aversion coefficients can be presented by equivalent characterizations on such transformations T without any reference to the underlying utility functions u and v.[13]

It is immediate that if a function T is concave on $[0, \bar{x}]$ then $T(x)/x$ is decreasing in x, which is equivalent to $T(x)/x > T'(x)$, and to $T'(x) \geq \frac{T(\bar{x})-T(x)}{\bar{x}-x}$. Figure 1 illustrates this result. First, observe on the figure that the slope of the chord drawn from the end-point $\bar{x}$ to any point x_1, i.e. $\frac{T(\bar{x})-T(x_1)}{\bar{x}-x_1}$, is always lower than the slope of the tangent at this point $T'(x_1)$. Second, observe that the slope of the chord drawn from the origin to any point x_2, i.e. $T(x_2)/x_2$, is larger than the slope of the tangent at this point $T'(x_2)$. The results for partial orderings are summarized as follows.

Proposition 4. *Let u and v be two strictly increasing, twice differentiable and concave functions that are bounded above with $u^* = \sup_w u(w)$ and $v^* = \sup_w v(w)$. Moreover, assume that $u(0) = v(0) = 0$. Then $v \supseteq_{AP} u$ implies*:
(i) $v \supseteq_{FR} u$
(ii) $v \supseteq_{AS} u$.

This Proposition shows that if an agent is more risk-averse in the classical sense of *AP* then he is also more risk-averse in the sense of *AS* and *FR*. Proposition 4 is of clear mathematical significance. As mentioned above, v is more risk-averse than u in the sense of *AP* if and only if v is obtained by a concave transformation of u. This is a very intuitive mathematical property, as any coefficient of curvature of a function should in principle increase when one

[13] In order to compare these different characterizations T, we need to restrict our attention to any increasing, differentiable T defined over $[0, \bar{x}]$ and such that $T(0) = 0$.

"concavifies" a function. The Proposition shows that this is actually the case for *AS* and *FR* coefficients.

Proposition 4 shows that $\supseteq_{FR}$ and $\supseteq_{AS}$ are weaker orderings than $\supseteq_{AP}$. Moreover, it is possible to show that these orderings are strictly weaker. To see this, take the function $T_1(x) = (x-2)^3 + 8$ over the interval $[0, \bar{x}]$ where $\bar{x} = 2.8$. This function is such that $T_1(x)/x$ is decreasing in x over this entire interval while $T_1''(x) > 0$ together with $\frac{T_1(\bar{x})-T_1(x)}{\bar{x}-x} > T_1'(x)$ for some x over this interval. In other words, there exists an individual v who has globally more *FR* than u but who is also locally less risk-averse than u in the sense of *AS* and of *AP*. Similarly, let $T_2(x) = \bar{x} - 2 - [T_1(\bar{x}) - x - 8]^{1/3}$ over the interval $[0, T_1(\bar{x})]$ where T_1 is the function just defined above.[14] Function T_2 is such that $\frac{T_2(T_1(\bar{x}))-T_2(x)}{T_1(\bar{x})-x} < T_2'(x)$ over this entire interval while $T_2(x)/x$ is increasing in x and $T_2''(x) > 0$ for some x over this latter interval. In other words, there also exists an individual v who is globally more risk-averse in the sense of *AS* than u but who is locally less risk-averse in the sense of *FR* and of *AP*.

To conclude this section, we remark that $\supseteq_{FR}$, $\supseteq_{AS}$ and $\supseteq_{AP}$ are equivalent for some important classes of utility functions. This is the case if we restrict our attention to power functional forms. Technically, the curvature of power functions is often captured by one single parameter and the *AP*, *FR* and *AS* coefficients may vary monotonically with this parameter.[15] The equivalence result follows.

4. Conclusion

In this paper, we have investigated the basic properties of the "fear of ruin" (*FR*) coefficient introduced by Aumann and Kurz (1977). First, we have derived an approximation of the insurance premium that an individual would be willing to pay in face of a small chance of losing his entire wealth. This premium has been shown to be proportional to the *FR* coefficient. We have then provided equivalent characterizations for comparing the *FR* of two agents. Specifically, we have shown that an agent v has globally more *FR* than an agent u if and only if v's premium to insure against the risk of ruin is always larger than u's premium. We also have given a characterization of more *FR* in terms of the properties required of an increasing transformation T, such that $v = T \circ u$. Furthermore, we have shown that the *FR* coefficient plays a crucial role in strategic games with risk-averse players. For instance, in first price auctions, we have demonstrated that the equilibrium bidding price of an auctioned object is always higher if auctioneers have more FR, and that uncertainty over the value of the auctioned object always leads the equilibrium bidding price to decrease under prudence. In addition, we have shown that the *FR* coefficient may be instrumental in simple mortality risk models. Finally, we have compared the FR's coefficient with other

[14] Observe that T_1 appears in the characterization of T_2. This can be easily understood once we explain how these counter-examples were generated. In short, we used the fact that finding T_1 such that we have $\supseteq_{FR}$ and $\supseteq_{AS}$ is equivalent, up to a change of reference axes, to finding T_2 such that $\supseteq_{FR}$ and $\supseteq_{AS}$. Mathematically, the change of reference axes is such that $T_2(x) = \bar{x} - T_1^{-1}(T_1(\bar{x}) - x)$.

[15] Foncel and Treich (2003) derive a formal proof of this equivalence for a generic class of power utility functions. This generic class U_z is the class of increasing and concave function of w that takes the form $\frac{(z+w)^{1-m}}{1-m} - \frac{z^{1-m}}{1-m}$, and defined for all positive parameter $m \neq 1$ and over the interval $[0, \overline{w}]$. (This result does not hold for all functions with a single parameter of power form.)

coefficients of risk-aversion. In particular, we have shown that if an agent v is more risk-averse than u in the sense of Arrow-Pratt, then v has more FR than u.

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