

2009/23



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CORE DISCUSSION PAPER
2009/23

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April 2009

Abstract

This paper examines the degree of elitism of public education under two different social objectives. It illustrates a potential conflict between welfare and social mobility. In the absence of private supplementary education, utilitarian welfare increases with the degree of elitism of the public education system. On the other hand, elitism decreases the steady state proportion of heterogenous dynasties (those comprised of a skilled parent and an unskilled child, or vice versa) which is our measure of social mobility. Consequently, social mobility is maximized under the least elitist public education system. We then open up the possibility for skilled parents to invest in private supplementary education for their child. We show that when private education is available, the degree of elitism that maximizes social mobility increases, while the welfare-maximizing degree of elitism decreases, provided that the inequality in productivity between the two types of agents is large enough. We provide a numerical example where the ranking between the welfare- and mobility-maximizing degree of elitism is reversed when private education is allowed — i.e., where the public education system that maximizes social mobility is more elitist than the one that maximizes welfare. Finally, we show that utilitarian welfare is always (weakly) higher when private supplementary education is available. However, to maximize social mobility it may be preferable to ban private supplements.

Keywords: elitism, egalitarianism, private education

JEL Classification: H37

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Part of this paper has been written while the second author was visiting Yale University. He thanks Yale's Economics Department for its hospitality.

This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

1 Introduction

The literature on education often advocates “elitist” policies. The standard approach is to consider a population of individuals who differ in their ability to benefit from education. This heterogeneity typically implies a rather regressive distribution of public education: resources are concentrated on the most able individuals in order to get a “cake” as big as possible to share among individuals through income taxation; see e.g., Brett and Weymark (2008), Bruno (1976), Hare and Ulph (1979) and Ulph (1977).¹ This recommendation relies on the assumption that education is not the only channel of policy intervention; there is also in a second stage an income tax that can alter social welfare.² If the exercise is restricted to the first stage, the solution is different and tends to be less regressive. This is shown for instance by Arrow (1971) who studies the optimal distribution of a given amount of public expenditure among individuals differing in their learning ability without accounting for the possibility of subsequent income redistribution. Note that this elitist distribution effect is mitigated when we introduce decreasing returns of educational spending.³

The literature that recommends elitism in education typically concentrates on a single generation. Consequently, the issue of social mobility does not arise. In reality, however, social mobility is often considered as an important issue for the assessment of education policies (Grossman and Kim (2003), Mejia and St-Pierre (2008), Speciale (2007), Iannelli and Paterson (2005) and the references therein). It is often valued for its own sake and independently of efficiency or (intragenerational) equity concerns. To understand the underlying problem suppose (just for the sake of the argument) that learning ability is transmitted by parents. We could end up with an educational policy that indeed maximizes social welfare at each period of time but at the expense of social mobility. Would such an outcome be acceptable? This is the issue dealt with in this paper. But first let us consider some basic facts.

¹In a recent paper Cremer et al (2008) put forward another reason to push for regressive education. It is not linked to heterogeneity in innate ability to benefit from education but to pervasive non-convexities that arise in the optimal income tax problem when individual productivities depend on education.

²See also De Fraja (2002) and Cremer and Pestieau (2006).

³Bovenberg and Jacobs (2005) and Maldonado (2007).

When looking at the educational system of relatively similar countries in terms of GDP, one is surprised by their wide heterogeneity. To characterize it, one can use the amount of expenditure devoted to education, the degree of elitism and the relative involvement of private market. We here focus on the last two characteristics. The design of an educational system is an important issue but also controversial and complex. Important because economic and human development are known to closely depend on the level and the structure of human capital. Controversial because nobody wants to admit that his educational system is elitist even though this is often the case. Complex because it is not easy to measure the degree of elitism of an educational system. Measuring it by the way resources are allocated among students of different origins and learning capacities is not useful. What matters is the effective outcome, for example, the level of knowledge achieved by students of a given age.

Hanushek and Woessmann (2007) take the share of students in each country that reach a certain threshold of basic literacy and the share of students that surpass a threshold of top performance. The first share can be used as a proxy for egalitarianism and the second as a proxy for elitism. Taking, in the sample of Hanushek and Woessmann, countries that have about the same GDP and the same relative level of educational spending, one observes that they differ quite a lot. For instance, the ratio of ninth decile to first decile in the prose literacy test performance varies from 1.4 for Denmark to 1.9 for the USA (p.18). Similar figures can be obtained from the OECD Programme for International Student Assessment (PISA).⁴ As to the heterogeneity in public and private education expenditures, for an OECD average of 4.7% and 1.4% of GDP in 2004, we have 6.0% and 0.1% in Finland and 5.1% and 2.3% in the US.⁵ The heterogeneity is relatively larger for private than for public education, the ranges being 0.1–2.6% and 3.5–6.5% respectively.

In this paper, we assume that society can control both the degree of elitism of public education, and the availability of private supplements to it. More specifically, we state that public authorities can decide through the design of school districts, selection of

⁴See <<http://www.pisa.oecd.org>>

⁵see OECD (2007)

students and teachers, differential investment in schools and students, etc... whether the school system is going to be elitist (i.e., favor the uppertail of the distribution of learning capacities) or to be egalitarian (i.e., aim at equalizing opportunities of successful education). As for private supplementary education, its availability can be affected by either making it tax deductible or by modifying the way public education programs are run. For instance, during the 2007 presidential elections in France, the socialist candidate has proposed to make it more difficult for public sector teachers to moonlight in the private sector and offer parents educational support after the regular public school hours. For simplicity, we assume that the availability of supplementary private education is a binary decision: it can either be allowed or forbidden. The degree of elitism of public education, on the other hand, is a continuous choice variable.

We consider two possible social objectives: utilitarian welfare and social mobility. These two objectives are often referred to in the assessment of education systems. For instance, Jesson (2007) studies both students performance and socioeconomic background in his recent assessment of England's grammar schools. He obtains that, although "one notable feature of the grammar schools is the high performance of their pupils in exams, [... they] do not offer a ladder of opportunity to any but a very small number of disadvantaged pupils."

Our objective is to develop a simple model that illustrates how stark the conflict between welfare maximization and social mobility can be in the determination of the optimal degree of elitism of public education, and how this degree of elitism is affected by the availability of private supplementary education. Within a two-skill setting we assume that skilled parents are more likely to have skilled children than unskilled parents. When the educational policy becomes more elitist, the probability that a skilled parent has a child that is skilled as well increases and the probability that an unskilled parent has a skilled child decreases.

We first study the optimal degree of elitism when private educational supplements are not available. Utilitarian welfare increases with the steady state proportion of skilled agents which, in turn, increases with the degree of elitism of the public education system. On the other hand, elitism decreases the steady state proportion of heterogenous

dynasties (those comprised of a skilled parent and an unskilled child, or vice versa) which is our measure of social mobility. Consequently, social mobility is maximized under the least elitist public education system, in stark contrast with the most elitist system implying maximum welfare.

We then open up the possibility for skilled parents to invest in private supplementary education for their child. We assume that private education efficiency is larger when the public education system is more egalitarian (i.e., less elitist). We study the impact of allowing for private education on the trade-off between social mobility and welfare. We show that the degree of elitism that maximizes social mobility increases, while the welfare-maximizing degree of elitism decreases provided that the inequality in productivity between the two types of agents is large enough. We provide a numerical example where the ranking between the welfare- and mobility-maximizing degree of elitism is reversed when private education is allowed – i.e., where the public education system that maximizes social mobility is more elitist than the one that maximizes welfare. Finally, we show that utilitarian welfare is always (weakly) higher when private supplementary education is available. On the other hand, to maximize social mobility it may be preferable to ban private supplements.

The rest of the paper is organized as follows. Section 2 presents the model and Section 3 gives the optimal choice of elitism in the absence of private education. Private education is introduced in Section 4 and a numerical example is given in Section 5.

2 The Model

Individuals care for their own consumption (c) and the educational attainment (a , as in altruism) of their (unique) child. All individuals have the same utility function

$$U(c, a) = u(c) + v(a)$$

with $u' > 0, u'' \leq 0, v' > 0, v'' < 0$.

There are two types of individuals: high productivity/wage (w_H) and low productivity (w_L): $w_L < w_H$. The difference between w_H and w_L is called the income gap. Educational attainment a measures the child's probability to achieve a high produc-

tivity. This probability depends on the parent's productivity level and on a parameter $\alpha \in [0, 1]$ which characterizes the degree of elitism of education policy. Observe that education policy is not represented by the level of expenditures which is implicitly assumed to be given. Instead, education policies are differentiated by their degree of elitism.

Formally, a child's probability of achieving a high productivity is given by $\phi_i(\alpha)$, $0 \leq \phi_i(\alpha) \leq 1$, where the index $i \in \{L, H\}$ refers to the parent's ability level. When $\alpha = 0$ the education system is egalitarian in the sense that $\phi_H(0) = \phi_L(0) = \bar{p}$. We assume (i) $\phi_H(\alpha) \geq \phi_L(\alpha)$ and (ii) $\phi'_H > 0$ and $\phi'_L < 0$. Assumption (i) states that a child with a high productivity parent never has a lower probability of achieving w_H than a child with a low productivity parent. In other words, high productivity parents are more likely to have high productivity children than low productivity parents. This illustrates the importance of family background and of social, family-related skills which increase the productivity of formal education (see Introduction). The second assumption implies that the educational attainment function increases with α for the children of high productivity parents and decreases for those of low ability parents.

To complete the characterization of the attainment function define

$$p_H = \phi_H(1) \quad \text{and} \quad p_L = \phi_L(1) \quad (1)$$

with $p_H > \bar{p} \geq p_L$. Figure 1 depicts the relation $\phi_i(\alpha)$ starting at \bar{p} for $\alpha = 0$ and ending at $p_H = \phi_H(1) > \bar{p} \geq p_L = \phi_L(1)$.⁶

Our timeline spans several generations, so that the proportion of high type individuals in one generation is a function of the proportion in the previous generation and of $\phi_L(\alpha)$ and $\phi_H(\alpha)$. Let p denote this proportion of high skilled individuals; its steady state level satisfies

$$p = (1 - p)\phi_L(\alpha) + p\phi_H(\alpha),$$

so that the steady state level $p^*(\alpha)$ is given by

$$p^*(\alpha) = \frac{\phi_L(\alpha)}{1 + \phi_L(\alpha) - \phi_H(\alpha)}. \quad (2)$$

⁶To draw this Figure, we use the functional forms presented in section 5.

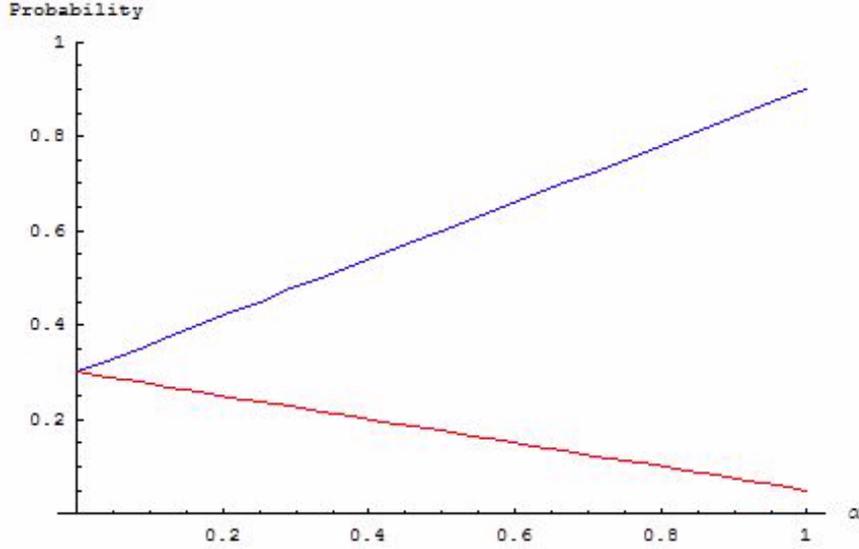


Figure 1: Φ_L (in red) and Φ_H (in blue) in the absence of private education

We have that $p^*(0) = \bar{p}$ and, using (1)

$$p^*(1) = \frac{p_L}{1 + p_L - p_H}.$$

Differentiating (2) shows that p^* is not necessarily a monotonic function of α . To sharpen the contrast between welfare- and social mobility maximization we assume throughout the paper that

$$p^{*\prime}(\alpha) > 0 \quad \forall \alpha \in [0, 1] \quad (3)$$

In words, a more elitist public education system increases the steady state proportion of skilled individuals.

3 The optimal level of α

We use a transition matrix where rows denote the ability level of the parent while columns denote the child's ability. Each cell contains the corresponding proportion of the child population.

The cells LL and HH represent the homogenous dynasties (no social mobility). The remaining cells represent heterogenous dynasties (who experience mobility). The

Table 1: Absolute frequencies

	w_L	w_H
w_L	$(1 - p^*(\alpha))(1 - \phi_L(\alpha))$	$(1 - p^*(\alpha))\phi_L(\alpha)$
w_H	$p^*(\alpha)(1 - \phi_H(\alpha))$	$p^*(\alpha)\phi_H(\alpha)$

proportion of dynasties with upward mobility is given in cell LH while dynasties counted in cell HL experience downward mobility. Observe that, the definition of $p^*(\alpha)$ as steady state level implies $(1 - p^*(\alpha))\phi_L(\alpha) = p^*(\alpha)(1 - \phi_H(\alpha))$. Given that $p^{*\prime}(\alpha) > 0$, $\phi_L'(\alpha) < 0$ and $\phi_H'(\alpha) > 0$, an increase in α increases the proportion of homogenous skilled dynasties and decreases the proportion of heterogeneous dynasties. The impact of α on the proportion of homogenous unskilled dynasties can go either way.

We consider two possible social objectives: social welfare maximization and social mobility (maximization of a mobility index). These two objectives are respectively denoted by $W(\alpha)$ and $M(\alpha)$. We now study the two objectives $W(\alpha)$ and $M(\alpha)$ in turn.

Social welfare $W(\alpha)$ is utilitarian and expressed as⁷

$$W(\alpha) = (1 - p^*(\alpha))u(w_L) + p^*(\alpha)u(w_H). \quad (4)$$

Differentiating (4) yields

$$W'(\alpha) = p^{*\prime}(\alpha)(u(w_H) - u(w_L)) > 0.$$

Not surprisingly it thus appears that utilitarian welfare is maximized when the steady state proportion of high type individuals is at its maximum level. With $p^{*\prime}(\alpha) > 0$, the optimal value of α is then given by $\alpha^W = 1$.

Let us now turn to the mobility index $M(\alpha)$ which is defined as the proportion of heterogeneous dynasties in the steady state⁸

$$M(\alpha) = (1 - p^*(\alpha))\phi_L(\alpha) + p^*(\alpha)(1 - \phi_H(\alpha)) = 2(1 - p^*(\alpha))\phi_L(\alpha). \quad (5)$$

⁷Observe that we launder the individual preferences in the sense that we do not take into account the impact of α on $v(\cdot)$. See on this Hammond (1987) and Andreoni (2006)

⁸Alternatively, we could have used the more sophisticated approach advocated by Atkinson and Bourguignon (1982) or Gottschalk and Spolaore (2002) which model the concern for mobility as aversion to multi-period inequality. Our more pedestrian approach is much easier to deal with analytically, especially since we want to contrast the results obtained with a purely welfarist planner and with a planner only interested in social mobility.

Differentiating (5) shows that

$$M'(\alpha) < 0,$$

given that $\phi'_L < 0$ and $p^*(\alpha) > 0$. Social mobility as measured by the index M is thus maximized when $\alpha = \alpha^M = 0$.

These results are summarized in the following proposition.

Proposition 1 *In the absence of supplementary private education, and under assumption (3), the maximization of utilitarian social welfare yields the most elitist public education system ($\alpha^W = 1$) while social mobility is maximum when the least elitist public education system is adopted ($\alpha^M = 0$).*

Observe that one could consider a social objective that is a function of both indexes: $F[W(\alpha), M(\alpha)]$. In Section 5 we present a numerical example where F is a weighted sum of $W(\alpha)$ and $M(\alpha)$.

4 Introducing private education

We now open up the possibility for high ability parents to invest in (supplementary) private education in order to increase the probability that their child be highly productive. We assume that low ability parents never invest in private education. Formally ϕ_H is now (redefined as) a function of expenditure on private education, e , and α , while ϕ_L continues to be a function of the sole variable α .

In the remainder of the paper we will also adopt specific functional forms for these expression. Let

$$\phi_H(\alpha, e) = (1 - \alpha)(\bar{p} + e) + \alpha p_H, \tag{6}$$

so that private education is especially efficient when α is low — i.e., when the public school system is very egalitarian. The rationale for this assumption is that an egalitarian public system does not devote more resources to brighter students, who may then benefit a lot from additional private education. On the contrary, an elitist public system already invests more in brighter kids, so that the marginal benefits they could obtain from

additional private education is low. The function ϕ_L is specified by

$$\phi_L(\alpha) = (1 - \alpha)\bar{p} + \alpha p_L. \quad (7)$$

We further assume from now on that $u(c) = \ln(c)$ and that $v(d) = \ln(d)$.

We start by solving for the individual decision of how much to invest in private education.

4.1 The private education choice

High productivity individuals solve the following problem:

$$\max_e u(w_H - e) + v(\phi_H(\alpha, e)).$$

The first-order condition with respect to e is

$$u'(c) = v'(d)(1 - \alpha),$$

from which we obtain that

$$\begin{aligned} e^o(\alpha) &= \frac{1}{2}(w_H - \bar{p} - \frac{\alpha}{1 - \alpha}p_H) \text{ if } \alpha \leq \tilde{\alpha} = \frac{w_H - \bar{p}}{w_H + p_H - \bar{p}} < 1, \\ &= 0 \text{ if } \alpha > \tilde{\alpha}, \end{aligned}$$

where e^o denotes the most preferred value of e of a high ability individual. We assume that $w_H > \bar{p}$ so that $e^o(0) > 0$.⁹ It is clear that $e^{o'}(\alpha) < 0$ when $\alpha < \tilde{\alpha}$. Intuitively, since the efficiency of private education decreases with α , so does the optimal amount of private education bought by skilled parents. Substituting $e^o(\alpha)$ into $\phi_H(\alpha, e)$ yields

$$\begin{aligned} \phi_H^o(\alpha) &= \phi_H(\alpha, e^o(\alpha)) \\ &= \frac{1}{2}(\bar{p}(1 - \alpha) + w_H(1 - \alpha) + p_H\alpha) \text{ if } \alpha \leq \tilde{\alpha}, \\ &= \phi_H(\alpha, 0) = (1 - \alpha)\bar{p} + \alpha p_H \text{ if } \alpha > \tilde{\alpha}. \end{aligned} \quad (8)$$

This function is linear in two parts. Skilled parents buy private education provided that its marginal productivity is large enough — i.e., that the public education system is not

⁹If it were not the case, nobody would ever buy private education and the analysis contained in the previous section would carry through.

too elitist ($\alpha < \tilde{\alpha}$). As long as $\alpha < \tilde{\alpha}$, increasing α has two effects of opposite signs on ϕ_H^o . On the one hand, the public education system becomes more elitist, which increases ϕ_H for a given value of e . On the other hand, skilled parents buy less private education, which decreases ϕ_H , for a given α . With our formulation, the net effect cannot be signed without additional assumptions on \bar{p} , p_H and w_H . As α increases above $\tilde{\alpha}$, the second effect disappears (since skilled agent do not buy private education for large values of α) and ϕ_H^o increases with α , as in the previous section.

The steady state proportion of high ability individuals is now a function of α and e . However, using $e^o(\alpha)$ we return to a function of the single variable α and can redefine p^* as

$$p^*(\alpha) = \frac{\phi_L(\alpha)}{1 + \phi_L(\alpha) - \phi_H^o(\alpha)}. \quad (9)$$

When $\alpha > \tilde{\alpha}$ (so that $e^o(\alpha) = 0$) and for the functional forms defined by (6) and (7) this steady state proportion is then given by

$$p^*(\alpha) = \frac{\bar{p} - \alpha(\bar{p} - p_L)}{1 - \alpha(p_H - p_L)}. \quad (10)$$

We make the same assumption as in the previous section, namely that the steady state proportion of high type individuals is increasing in α *when the spending on private education is zero*. Differentiating expression (10) shows that this is the case when

$$p_L(1 - \bar{p}) - \bar{p}(1 - p_H) > 0, \quad (11)$$

a condition which is satisfied when p_H or p_L are high enough.

When $\alpha \leq \tilde{\alpha}$, on the other hand, private education spending is positive and the expression for p^* is more complicated and it may well be a decreasing function of α . Figure 2 illustrates this possibility. It depicts $\phi_H^o(\alpha)$, $p^*(\alpha)$ and $\phi_L(\alpha)$ for $p_L = 0.05$, $p_H = 0.9$, $\bar{p} = 0.3$, $w_L = 0.33$ and $w_H = 1$ (the values upon which the simulations of section 5 are based). These values satisfy condition (11) so that p increases when private education spending is zero (when $\alpha > \tilde{\alpha}$). For low values of α on the other hand, when high type individuals buy private education, the steady state proportion of high type individuals is decreasing in α .

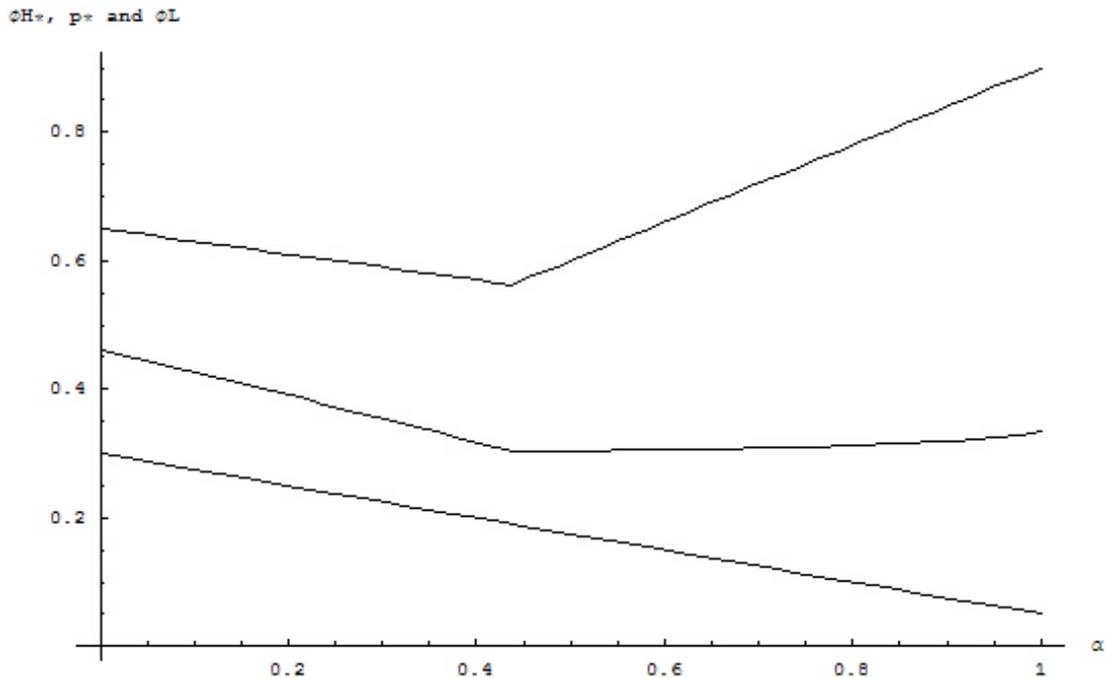


Figure 2: Probabilities of a high productivity child when private education is available.

4.2 The level of elitism

We first consider a utilitarian welfare function given by

$$W^{PE}(\alpha) = (1 - p^*(\alpha))u(w_L) + p^*(\alpha)u(w_H - e^o(\alpha)). \quad (12)$$

This expression differs from (4) in two aspects that reflect the impact of private education spending. First, the utility level of the high productivity individuals now depends on α and, second, $p^*(\alpha)$ is now defined by (9). Consequently, maximizing social welfare is no longer equivalent to maximizing p^* . Observe that as in the previous section, we launder utilities and do not take into account the utility parents obtain from the probability that their kid is of a high type. The (laundered) utility of the high productivity individual increases with α as long as $\alpha < \tilde{\alpha}$, and is constant for higher values of α :

$$\begin{aligned} \frac{\partial u(w_H - e^o(\alpha))}{\partial \alpha} &= \frac{p_H}{(1 - \alpha)((1 - \alpha)(\bar{p} + w_H) + p_H \alpha)} > 0 \text{ if } \alpha < \tilde{\alpha}, \\ &= 0 \text{ if } \alpha > \tilde{\alpha}. \end{aligned}$$

As the public education system becomes more elitist (up to $\tilde{\alpha}$), high type individuals invest less in private education and rely more exclusively on public education. Consequently, their (laundered) utility increases. The utility of low type individuals, on the other hand, continues to be independent of α . Finally, recall that by assumption the steady state proportion of high type individuals increases with α when $\alpha > \tilde{\alpha}$.

Putting these observations together, we obtain

Proposition 2 *With private education and $p^*(\alpha) > 0$ for $\alpha > \tilde{\alpha}$, the level of α (denoted α^W) that maximizes utilitarian welfare, is either equal to one or belongs to $[0, \tilde{\alpha}]$. Consequently, we can exclude $\alpha^W \in]\tilde{\alpha}, 1[$. Moreover, a necessary condition to obtain $\alpha^W \leq \tilde{\alpha}$ is that w_H is large enough.*

Proof: See Appendix

Proposition 2 states that a large wage gap (a large value of w_H) is necessary for a somewhat egalitarian system with effective private education ($\alpha < \tilde{\alpha}$) to yield a higher level of welfare than an elitist system without any private education at equilibrium ($\alpha = 1$). This property can easily be understood. The consumption level of a high ability type is lower when he invests in private education. A low level of α (inducing positive private education spending) can thus only yield a higher level of welfare than $\alpha = 1$ when it implies a larger steady state proportion of high productivity individuals. This proportion increases with w_H because richer parents buy more private education. The Appendix provides a lower bound on w_H that guarantees that total utility is larger with effective private education. This bound depends upon the three determinants of the functions ϕ_L and ϕ_H , namely \bar{p} , p_L and p_H . The next section provides a numerical example where social welfare is larger with an elitist system without effective private education if w_H is low while an egalitarian system with private education is preferred if w_H is large enough.

We now turn to the maximization of the social mobility index which continues to be defined by (5) but with $p^*(\alpha)$ redefined by (9). Recall that this index simply measures the proportion of heterogeneous dynasties which at the steady state is given by $2(1 - p^*)\phi_L(\alpha)$ (see Table 1). Social mobility is decreasing on $[\tilde{\alpha}, 1]$ because p^*

Table 2: Main results

Social objective:	Welfare	Social Mobility
No private education	$\alpha^W = 1$	$\alpha^M = 0$
Private education	$\alpha^W = 1$ when w_H is not too large $\alpha^W \in [0, \tilde{\alpha}[$ only when w_H large enough	$\alpha^M \in [0, \tilde{\alpha}]$ and weakly increases with w_H

increases with α over that range, while ϕ_L decreases. When $\alpha < \tilde{\alpha}$, social mobility may increase or decrease with α , because the steady state proportion of unskilled individuals may increase with α , while ϕ_L decreases with α . We then obtain

Proposition 3 *With private education and $p^{*'}(\alpha) > 0$ for $\alpha > \tilde{\alpha}$, the level of α (denoted α^M) that maximizes social mobility belongs to $[0, \tilde{\alpha}]$. Consequently, social mobility is maximized when private education is effective (high type individuals invest in private education). Moreover, the value of α^M is weakly increasing with w_H .*

Proof: See Appendix

Recall that, without private education, both the steady state proportion of low ability types ($1 - p^*$) and the proportion of agents with a low ability parent who achieve a high productivity (ϕ_L) decrease with α . This explains why the steady state proportion of heterogeneous dynasties decreases with α , and why social mobility is maximized with a purely egalitarian public education ($\alpha = 0$). When private education is introduced, an increase in α induces high ability agents to reduce their investment in private education (because its return is lower with a more elitist public system) and this may increase the steady state proportion of low type individuals, resulting in a larger number of heterogeneous dynasties. Furthermore, a larger value of w_H reinforces this impact of α on the steady state proportion of low type individuals, so that α^M weakly increases with w_H .

The following Table summarizes our results.

It is straightforward that utilitarian welfare may only increase when private education becomes available. The welfare level for $\alpha^W = 1$ is the same under both systems (high productivity individuals do not buy private education when $\alpha = 1 > \tilde{\alpha}$). Consequently, when $\alpha^W < 1$ is chosen when private education is available it must yield a

larger welfare level. As for social mobility, the allocation chosen when private education is not available is no longer feasible when private education is introduced. Consequently, we cannot be certain that social mobility increases when the private education becomes available.

These results are summarized in the following proposition.

Proposition 4 *(i) If the income gap is small enough, the availability of private education does not affect α^W nor α^M . (ii) If the income gap is large enough, the availability of private education may decrease α^W while increasing α^M . (iii) The maximum welfare level when private education is available is the same as without private education if the income gap is small enough, and may be larger if the income gap is large enough. Consequently, with a welfarist social objective it is never desirable to forbid private educational supplements. (iv) The maximum level of social mobility may be lower with private education, whatever the income gap. Consequently, when the objective is to maximize social mobility it may be desirable to forbid private educational supplements.*

5 Numerical example

We now resort to numerical simulations, with several objectives in mind. First, we would like to show that the introduction of private education may effectively strictly decrease α^W (provided that the income gap is large enough) and strictly increase α^M . Second, the analytical results do not show whether the ranking of the degrees of elitism achieved under the two social objectives may be reversed when private education is introduced. We do know that $\alpha^M = 0 < \alpha^W = 1$ in the absence of private education. but can we possibly have $\alpha^M > \alpha^W$ once private education is introduced? Finally, we can determine numerically the degree of elitism that maximizes a weighted sum of both objectives,¹⁰

$$F(\gamma, \alpha) = (1 - \gamma)W(\alpha) + \gamma M(\alpha) \tag{13}$$

¹⁰This formulation will prove much easier to handle than the one proposed by Gottschalk and Spolaore (2002), and the results are much easier to explain graphically using the (welfare, social mobility) possibility frontier, as shown below.

Table 3: No private education
 Threshold value of γ below (above) which optimal α is one (zero)

w_H	$\tilde{\gamma}$
1	0.17
1.1	0.18
1.2	0.19
1.25	0.2
1.5	0.22

where γ is the weight put on social mobility.¹¹

We adopt the logarithmic utility functions and the parameters of the linear probability functions (6) and (7) are given by $p_L = 0.05$, $p_H = 0.9$, $\bar{p} = 0.3$. Finally we set $w_L = 0.33$, while w_H varies from 1 to 1.5 (to obtain different levels of the wage gap).

5.1 No private education

Without private education, we know from Section 3 that $\alpha^W = 1$ while $\alpha^M = 0$ whatever the value of w_H . An open question is whether a weighted social objective as given by (13) may yield an interior solution for the degree of elitism. With our specification this is not the case. Instead, we obtain a critical value of γ , denoted by $\tilde{\gamma}$, such that $\gamma < \tilde{\gamma}$ yield $\alpha = 1$, while $\gamma > \tilde{\gamma}$ yields $\alpha = 0$. This is because the possibility frontier in the (welfare, social mobility) space is convex. As w_H increases, the possibility frontier curve shifts to the right since social mobility is not affected by w_H (for any given α) while welfare (as given by (4)) increases with w_H for any given α ; see Figure 3 for a representation of the possibility frontier for $w_H = 1$ (in blue) and $w_H = 1.5$ (in red). Social mobility is at its maximum (and welfare at its minimum) when $\alpha = 0$. As we increase α , we move along the possibility frontier in the south east direction, increasing W and decreasing M . The value of $\tilde{\gamma}$ increases with w_H , as shown in Table 3.

5.2 Private education available

Table 4 gives the optimal level of α with private education as a function of w_H and of γ .

¹¹As mentioned above the linear form for F is not the only conceivable. One can use a strictly convex social indifference curve in the (W, M) plane.

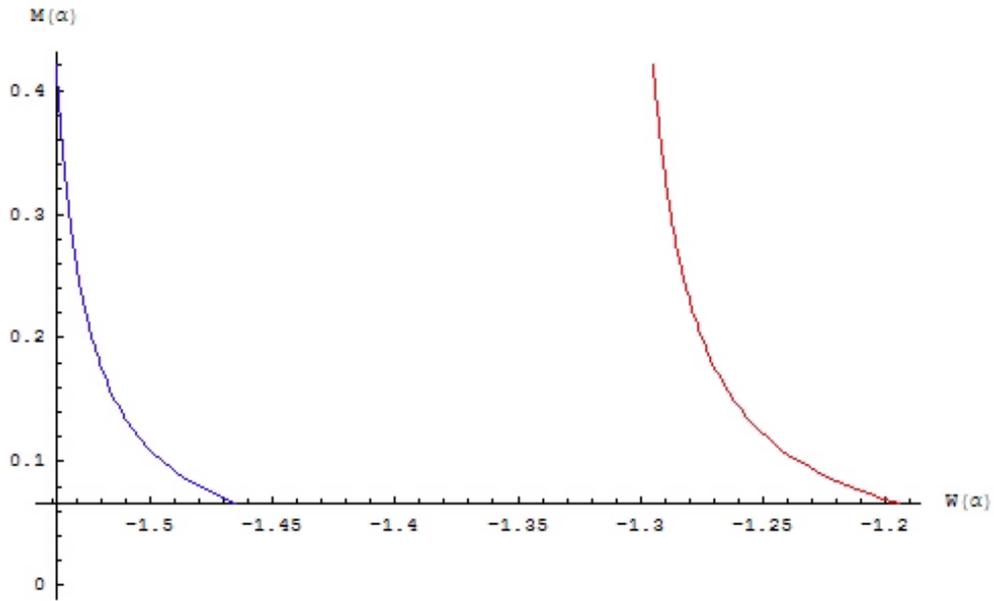


Figure 3: Possibility frontier in the absence of private education, as a function of α , for $w_H = 1$ (blue) and $w_H = 1.5$ (red).

Table 4: Private education
Optimal value of α for $\gamma =$

w_H	0	0.25	0.5	0.6	0.8	1
1	1	0.39	0.20	0.12	0	0
1.1	1	0.07	0.04	0.03	0.01	0
1.2	0	0	0	0	0.06	0.14
1.25	0	0	0	0	0.08	0.19
1.5	0	0	0	0	0.14	0.39

With private education, one observes interior solutions and a certain sensitivity to the wage gap. When the wage gap is small ($w_H \leq 1.1$), $\alpha^W = 1$ and $\alpha^M = 0$ as in the case without private education. However, we now obtain interior optimal values of α for a range of values of γ . Observe moreover that the optimal value of α is (weakly) decreasing in γ . When the wage gap is large ($w_H \geq 1.2$), we have $\alpha^W = 0$ while $0 < \alpha^M < 1$. With our specification, α^W is a corner solution for all values of w_H , with α^W jumping from 1 to 0 as w_H becomes large enough. On the other hand, α^M is not a corner solution when the wage gap is large enough. We also obtain interior optimal values of α for a range of values of γ , and these optimal values of α now (weakly) increase with the weight put by the planner on social mobility. Finally, with a large wage gap ranking of the optimal degree of elitism between “extreme” social objectives is reversed (compared with the case without private education, or with private education but with a low wage gap) and we obtain $\alpha^W < \alpha^M$.

Figures 4–7 present the possibility frontier in the (welfare, social mobility) plane for different values of w_H ranging from 1 to 2.5. The blue curve in each figure represents the case without private education, as in Figure 3, with α increasing from 0 to 1 as one moves downward. If α is sufficiently large ($\alpha > \tilde{\alpha}$), individuals choose zero private education spending and the blue curve also represents the possibility frontier in presence of private education. The red curve represents the frontier when private education is allowed and bought at equilibrium ($\alpha < \tilde{\alpha}$). The intersection between blue and red curves occurs when $\alpha = \tilde{\alpha}$, and the value of α decreases as one moves along the red curve away from this intersection. The point where $\alpha = 0$ is attained at the extremity of the red curve.

When the wage gap is low ($w_H \leq 1$), the red curve stands to the left of the blue curve. This illustrates the property that private education may be detrimental to both welfare and social mobility when the public education system is egalitarian ($\alpha < \tilde{\alpha}$). On the contrary, the red curve stands to the right of the blue curve when the wage gap is large enough ($w_H \geq 1.1$). Decreasing α from $\tilde{\alpha}$ then allows to increase both welfare and social mobility.

The optimal pair (welfare, social mobility) is obtained by looking for the tangency

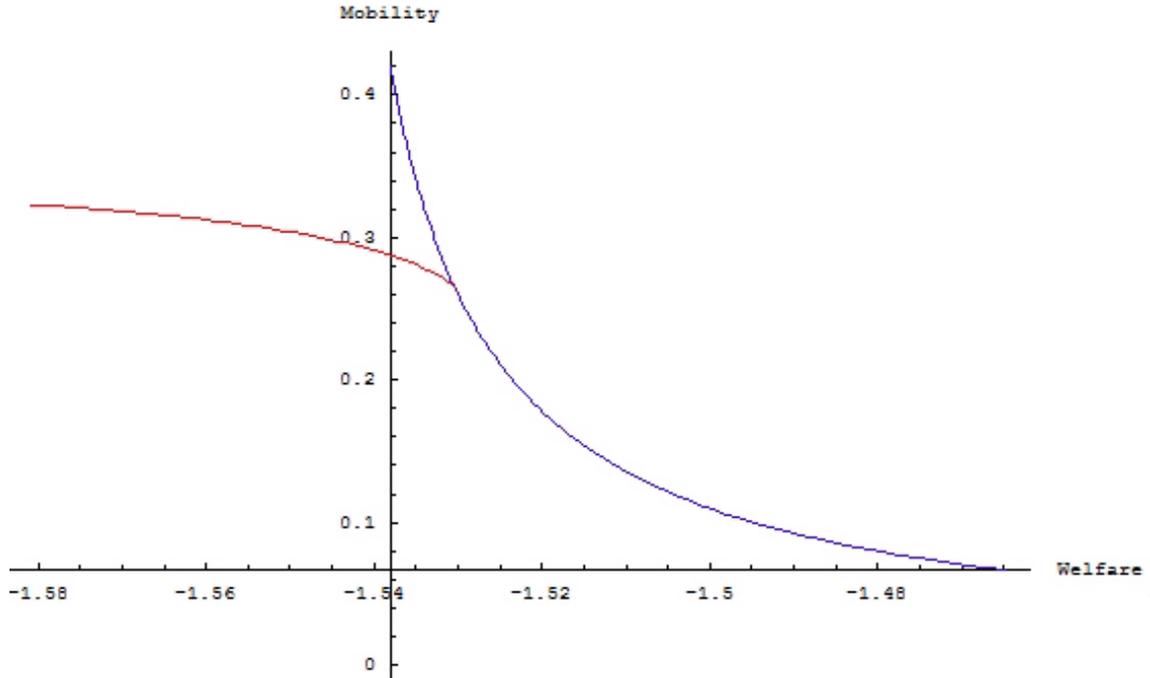


Figure 4: Possibility frontier with private education and $w_H = 1$.

point between possibility frontier and social indifference curves with a slope of $-(1 - \gamma)/\gamma$. The corresponding value of the optimal α is given in Table 4. As can be seen from Figures 4–7, the maximum value of welfare attainable is higher with private education provided that the wage gap is large enough; a welfarist social planner then wishes to allow for private education (and chooses an egalitarian public education system). In the numerical examples we have looked at, maximum social mobility is always lower when private education is allowed than when it is forbidden. Consequently, when the social objective puts a sufficiently large weight on social mobility it may be desirable to make private education unavailable.

6 Conclusion

We have considered two alternative objectives of education policy: utilitarian welfare and social mobility. These two issues are often referred to in the assessment of education

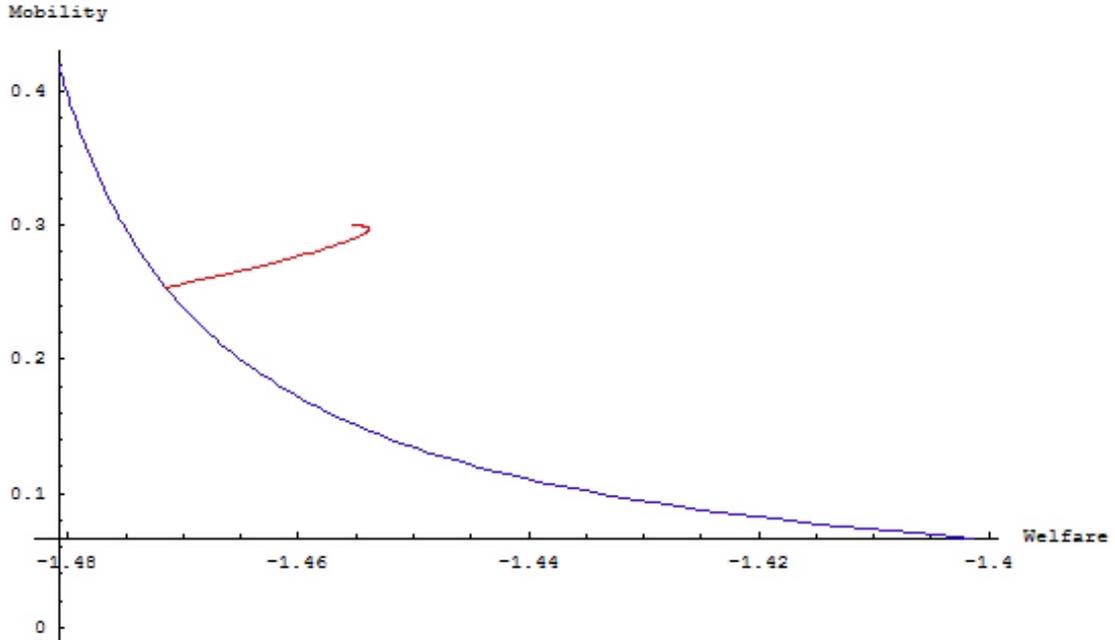


Figure 5: Possibility frontier with private education and $w_H = 1.1$

systems. We have developed a simple model that studies the determination of the degree of elitism of public education. It has shown that there may be a stark conflict between welfare maximization and social mobility.

We have first studied the optimal degree of elitism when private educational supplements are not available. Utilitarian welfare increases with the degree of elitism of the public education system. On the other hand, elitism decreases the steady state proportion of heterogenous dynasties (those comprised of a skilled parent and an unskilled child, or vice versa) which is our measure of social mobility. Consequently, social mobility is maximized under the least elitist public education system.

Next, we have introduced the possibility that skilled parents invest in private supplementary education and studied its impact on the trade-off between social mobility and welfare. We have shown that the degree of elitism that maximizes social mobility increases, while the welfare-maximizing degree of elitism decreases provided that the inequality in productivity between the two types of agents is large enough. We have

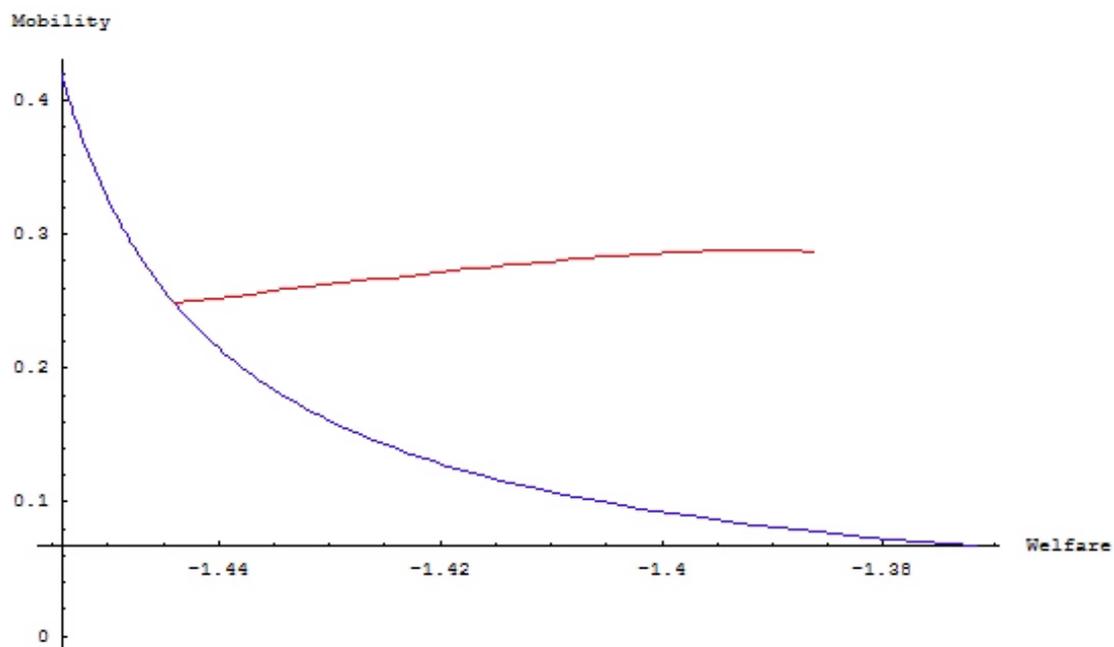


Figure 6: Possibility frontier with private education and $w_H = 1.15$.

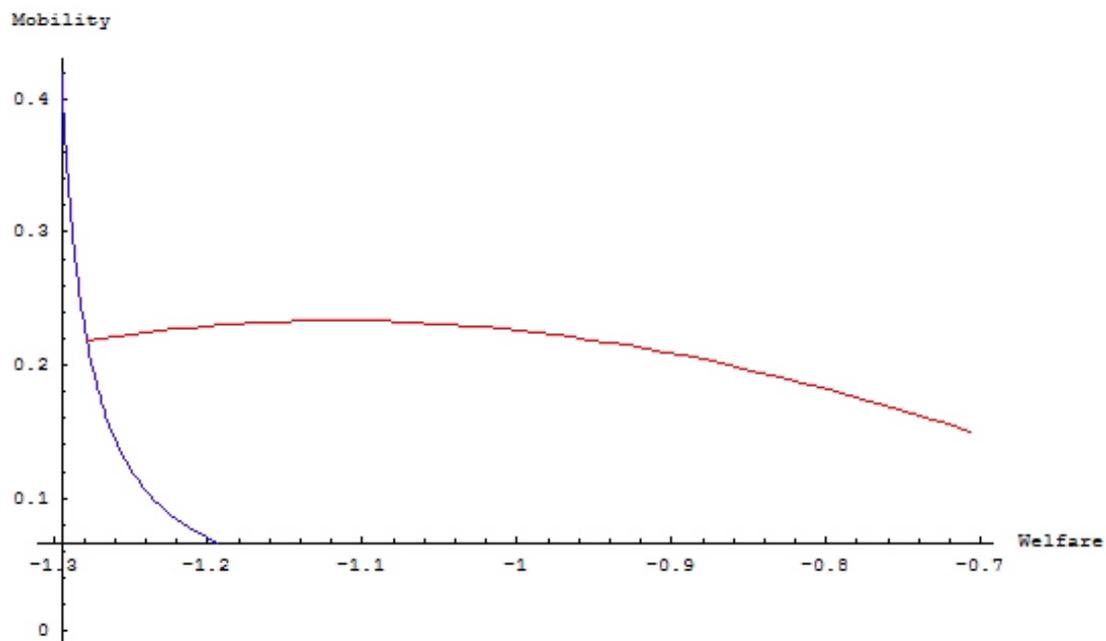


Figure 7: Possibility frontier with private education and $w_H = 1.5$.

provided a numerical example where the ranking between the welfare- and mobility-maximizing degree of elitism is reversed when private education is allowed – i.e., where the public education system that maximizes social mobility is more elitist than the one that maximizes welfare. Finally, we have shown that utilitarian welfare is always (weakly) higher when private supplementary education is available. On the other hand, to maximize social mobility it may be preferable to ban private supplements.

In the literature maximum social welfare has lead many authors to recommend an elitist and regressive educational policy. In this paper we have shown that this recommendation can be challenged when introducing considerations of social mobility. This is not the only way an elitist policy can be questioned. Here are two examples. First, one can deem that education not only brings more productivity but has a value per se. If education were introduced as an argument in the individual utility function, the case for a regressive educational policy would be weakened. Second, there is also the effect of earnings distribution on growth. There exists a rich literature comparing the growth incidence of two polar systems of education that can be labeled for short egalitarian and elitist. The former one tries to induce equalization of human capital while the latter tends to perpetuate or even exacerbate its initial inequality. Benabou (1996) addresses the question of which system promotes faster growth. He shows that their short run effects are ambiguous, but that in the long run the egalitarian system is clearly desirable. This suggests that when growth is accounted for the conflict between welfare and mobility may be less drastic than in our setting.

APPENDIX

A Proof of Proposition 2

First, the optimal value of α cannot belong to $]\tilde{\alpha}, 1[$ because the utility of both types of individuals is constant over this interval while the proportion of high type (and thus high utility) individuals increases with α over this interval.

Second, the necessity of a large w_H to obtain $\alpha^W < 1$ is established as follows. A necessary condition to have $\alpha^W < 1$ is

$$p^*(\alpha^W) > p^*(1) = \frac{\bar{p}}{1 + \frac{\bar{p} - w_H}{2}},$$

because $\alpha^W < \tilde{\alpha} < 1$. Using equations (2), (7) and (8), we obtain that

$$p^*(\alpha^W) = \frac{2\bar{p}(1 - \alpha^W) + 2p_L\alpha^W}{2 + \bar{p} - w_H(1 - \alpha^W) - \alpha^W(\bar{p} + p_H - 2p_L)},$$

which increases with w_H . Combining these two expressions shows that a necessary condition to obtain $\alpha^W < 1$ is that

$$w_H > 2 - \frac{\bar{p}(2(1 - p_H) + p_L)}{p_L} + \frac{\alpha^W}{1 - \alpha^W} p_H.$$

Observe that this lower bound on the value of w_H increases with α^W . If this condition is satisfied for some $\alpha^W < \tilde{\alpha}$, it thus must also hold for $\alpha^W = 0$, so that a necessary condition to obtain $\alpha^W < 1$ is

$$w_H > 2 - \frac{\bar{p}(2(1 - p_H) + p_L)}{p_L}.$$

B Proof of Proposition 3

The problem is given by

$$\max_{\alpha} M(\alpha) = [1 - p^*(\alpha)]\phi_L(\alpha),$$

where $p^*(\alpha)$ is given by (9) and $\phi_L(\alpha)$ by (7). Consequently, when α^M is an interior solution it is determined by

$$M'(\alpha) = -p^{*\prime}(\alpha)\phi_L + (1 - p^*)\phi_L'(\alpha) = 0.$$

We have that¹²

$$\begin{aligned} \text{sign}\left(\frac{\partial\alpha^M}{\partial w_H}\right) &= \text{sign}\left(\frac{\partial^2 M}{\partial\alpha\partial w_H}\right) \\ &= \text{sign}\left(-\frac{\partial^2 p^*}{\partial\alpha\partial w_H} \phi_L - \frac{\partial p^*}{\partial w_H} \frac{\partial\phi_L}{\partial\alpha}\right) \end{aligned} \quad (14)$$

so that $\partial^2 p^*/\partial\alpha\partial w_H < 0$ is sufficient to yield $\partial\alpha^M/\partial w_H > 0$.

Differentiating (9), the expression for p^* yields

$$\frac{\partial p^*}{\partial\alpha} = \frac{\frac{\partial\phi_L}{\partial\alpha}(1 - \phi_H^*) + \phi_L \frac{\partial\phi_H^*}{\partial\alpha}}{(1 - \phi_H^* + \phi_L)^2} < 0$$

and

$$\begin{aligned} \frac{\partial^2 p^*}{\partial\alpha\partial w_H} &= \frac{\left(\phi_L \frac{\partial^2\phi_H^*}{\partial\alpha\partial w_H} - \frac{\partial\phi_L}{\partial\alpha} \frac{\partial\phi_H^*}{\partial w_H}\right)(1 - \phi_H^* + \phi_L)^2}{(1 - \phi_H^* + \phi_L)^4} \\ &\quad + \frac{2\left(\frac{\partial\phi_L}{\partial\alpha}(1 - \phi_H^*) + \phi_L \frac{\partial\phi_H^*}{\partial\alpha}\right)(1 - \phi_H^* + \phi_L) \frac{\partial\phi_H^*}{\partial w_H}}{(1 - \phi_H^* + \phi_L)^4} \\ &< 0 \text{ if } \phi_L \frac{\partial^2\phi_H^*}{\partial\alpha\partial w_H} - \frac{\partial\phi_L}{\partial\alpha} \frac{\partial\phi_H^*}{\partial w_H} < 0. \end{aligned}$$

Using the definitions of ϕ_L and ϕ_H^o , (7) and (8) we obtain that

$$\phi_L \frac{\partial^2\phi_H^*}{\partial\alpha\partial w_H} - \frac{\partial\phi_L}{\partial\alpha} \frac{\partial\phi_H^*}{\partial w_H} = -\frac{p_L}{2} < 0,$$

which completes the proof because from (14) $\partial^2 p^*/\partial\alpha\partial w_H < 0$ implies $\partial\alpha^M/\partial w_H > 0$.

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¹²We slightly abuse notation by expressing M and p^* as functions of both α and w_H and using the symbol ∂ to denote their derivatives.

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