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Essays on economics of information: search, networks and price discrimination

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Résumé

Cette thèse se compose de trois chapitres indépendants abordant différentes questions de l'économie de l'information. Le premier chapitre étudie des stratégies optimales des entreprises qui sont présentes sur les marchés en ligne et hors ligne. Nous étudions des stratégies de prix optimales des détaillants en présence de showrooming et leurs décisions sur des canaux de distribution. Le showrooming est une situation où les consommateurs essayent des produits dans des magasins classiques avant de les acheter en ligne à un prix inférieur. Une manière d'empêcher le showrooming est d'utiliser des prix identiques dans le magasin physique et en ligne. Nous montrons que dans l'optique de recherche de prix bas, le choix de prix identiques est, en effet, un choix optimal. Cependant concernant des prix plus élevés, les prix identiques sont suboptimals, et les achats en ligne et en magasins classiques coexistent avec le showrooming. Une entreprise qui fait face à la concurrence en ligne d'un détaillant multicanaux étranger a une incitation au geo-blocking, c.-à-d. qu'elle refuse de servir les clients étrangers, bien que cela amène à une diminution de la demande. Le geo-blocking modère la concurrence en ligne et mène à des prix plus élevés aussi bien en ligne que dans les magasins physiques. L'exigence juridique des prix identiques, aide à éliminer les incitations au geo-blocking et reconstitue ainsi la concurrence en ligne.

Le deuxième chapitre analyse la diffusion de l'information dans les réseaux de la communication où les interactions sociales sont coûteuses. Nous proposons une modèle dynamique avec les agents stratégiques qui décident combien d'effort mettre dans la stratégie publicitaire d'un produit pour une période donnée. Nous montrons que le niveau d'équilibre de l'effort individuel de la communication est convexe avec la proportion des agents avertis, et inférieur le niveau socialement optimal au cause de l'effet substantiel de free-riding. Nous prouvons que pour des coûts de recommandation suffisamment élevées c'est socialement optimal que les agents symétriques exercent le même effort de communication tandis que pour des les coûts de recommandation basses ceci n'est pas vrai. Dans le cadre de notre modèle nous analysons la stratégie de la publicité de l'entreprise lançant un nouveau produit avec des extériorités de réseau positives pour des consommateurs. Les expositions d'analyse montrent que les résultats de la publicité diminuent rapidement en proportion de consommateurs avertis du sa l'effet de free-riding. Ainsi, de façon optimale l'entreprise doit ajuster et réduire le niveau de la publicité par intermittence.

Le troisième chapitre est un papier co-écrit avec Maarten Janssen et Alexei Parakhonyak. Dans cet article nous proposons un nouveau concept d'équilibre "Non-reservation price equilibria" (Non-RPE). "Reservation price equilibria" (RPE) n'évaluent pas exactement la puissance du marché dans les marchés de recherche du consommateur. Sur la plupart des marchés de recherche, les consommateurs ne connaissent pas les éléments importants de l'environnement dans lequel ils font des recherches (comme, pour exemple, le coût pour les entreprises). Nous discutons que RPE souffrent de questions théoriques, telles que la non-existence et la dépendance critique des croyances spécifiques hors-de-équilibre, quand les consommateurs apprennent en faisant des recherches. Nous définissons équilibrée, la situation où les consommateurs choisissent rationnellement des stratégies de recherche qui ne sont pas caractérisées par un prix de réservation. Non-RPE existent toujours et ne dépendent pas des croyances spécifiques hors-de-équilibre. Non-RPE ont pour objectif la recherche active du consommateur et sont compatibles avec les résultats empiriques récents.

Abstract

This thesis consists of three independent chapters addressing different questions of information economics.

The first chapter studies optimal strategies of firms which are present in both offline and online markets. We study optimal pricing strategies of retailers in presence of showrooming and their decisions on distribution channels. Showrooming is a situation where consumers try products at brick-and-mortar stores before purchasing them online at a lower price. One way to prevent showrooming is to use a price matching policy, whereby price is the same in both the physical store and the online channel. We show that for small search costs, a price matching policy is indeed optimal. However for higher search costs price matching is suboptimal, and online and offline purchases coexist with showrooming. A firm which faces online competition from a foreign multichannel retailer has an incentive to geo-block, i.e. refuse to serve foreign customers, even though it leads to a decrease in potential demand. Geo-blocking relaxes online competition and leads to higher prices both online and in brick-and-mortar stores. A legal price parity requirement helps to eliminate incentives to geo-block and thus restores online competition.

The second chapter analyzes information diffusion process in communication networks where social interactions are costly. We provide a dynamic model with strategic agents who decide how much effort to put into the propagation of information about a product in each period. We show that the equilibrium level of the individual communication effort is convex in the proportion of informed agents, and lower than the socially optimal level due to the substantial free-riding effect. We show that for sufficiently high recommendation cost it is socially optimal that symmetric agents exert the same communication effort while for low recommendation cost this is not true. In the context of our model we analyze the advertising strategy of the firm launching a new product with positive network externalities for consumers. The analysis shows that the outcome of advertisement is decreasing fast with the proportion of informed consumers due to the free-riding effect. Thus, optimally the firm has to adjust and reduce the level of advertising in each period.

The third chapter is a co-authored paper with Maarten Janssen and Alexei Parakhonyak. In this paper we propose a new equilibrium concept of Non-reservation price equilibria (Non-RPE). Reservation price equilibria (RPE) do not accurately assess market power in consumer search markets. In most search markets, consumers do not know important elements of the environment in which they search (such as, for example, firms' cost). We argue that when consumers learn when searching, RPE suffer from theoretical issues, such as non-existence and critical dependence on specific out-of-equilibrium beliefs. We characterize equilibria where consumers rationally choose search strategies that are not characterized by a reservation price. Non-RPE always exist and do not depend on specific out-of-equilibrium beliefs. Non-RPE have active consumer search and are consistent with recent empirical findings.

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Introduction

This thesis addresses different questions of information economics. It consists of three independent chapters, where we analyze consumers' and firms' strategies in markets with incomplete information. These chapters study how a search for information, its acquisition, and dissemination affect market structures and observed consumers' behavioral patterns.

The first chapter studies optimal strategies of firms which are present in both offline and online markets. The last decade's growth of the e-commerce sector and rapid development of information and communications technology create new possibilities for firms-consumers interactions. Firms have various opportunities to reach consumers. The more and more firms choose to combine advantages of online retailing with benefits of traditional brick-and-mortar stores. However, the pricing strategies of big multichannel retailers vary a lot depending on the sector and location of firms. We study optimal pricing strategies of retailers in presence of showrooming and their decisions on distribution channels. Showrooming is quite recent phenomena which corresponds to a situation where consumers try products at brick-and-mortar stores before purchasing them online at a lower price. One way to prevent showrooming is to use a price matching policy, whereby price is the same in both the physical store and the online channel. We show that for small search costs, a price matching policy is indeed optimal. However for higher search costs price matching is suboptimal, and online and offline purchases coexist with showrooming.

The important problem related to competition of multichannel retailers is "geo-blocking". It is the situation when firms refuse to serve foreign customers, even though it leads to a decrease in potential demand. This issue is broadly discussed in the last European Commission Report (may 2017) devoted to E-commerce sector. We show that a firm which faces online competition from a foreign multichannel retailer has an incentive to geo-block, because geo-blocking relaxes online competition and leads to higher prices both online and in brick-and-mortar stores. At the same time it facilitates entry of new firms to the online market, which may result in higher online competition and lower prices in brick-and-mortar stores in comparison to the situation when there is no entry. We propose a simple test to distinguish between situations when geo-blocking has an anticompetitive effect and may have a pro-competitive effect.

The second paper is devoted to "word-of-mouth" marketing campaigns where agents share some information about products with their peers. Nowadays this type of campaigns is broadly used, especially for promoting different communication services which provide positive network effects for their users. Examples of these products are social networks as Facebook or Linkedin, messengers as WhatsApp, services for file sharing as Dropbox and etc. The paper analyzes information diffusion process in communication networks where on one side consumers have positive network externalities, and on the other side social interactions are costly. We provide a dynamic model with strategic agents who decide how much effort to put into the propagation of information about a new product in each period. We show that the equilibrium level of the individual communication effort is convex in the proportion of informed agents, and lower than the socially optimal level due to the substantial free-riding effect. For sufficiently high recommendation cost it is socially optimal that symmetric agents exert the same communication effort while for low recommendation cost asymmetric recommendation efforts can be socially desirable. We provide simulation results and show the audience growth of new users has s-shape which can be explained by free-riding effect. In the context of our model we analyze the advertising strategy of the firm launching a new product with positive network externalities for consumers. The analysis shows that the outcome of advertisement is decreasing fast with the proportion of informed consumers due to the free-riding effect. Optimally the firm has to adjust and reduce the level of advertising in each period. The efficiency of direct advertising is increasing in recommendation cost.

The third chapter is a co-authored paper with Maarten Janssen and Alexei Parakhonyak. It is related to the consumer search literature. In this paper we propose a new equilibrium concept of Non-reservation price equilibria (Non-RPE). The standard approach exploited by search literature is so called Reservation price equilibria (RPE) concept. It assumes that consumers follow a reservation price rule when their search for prices: 1) if the observed price is below some threshold they buy the product, 2) if the price is above then they continue to search. RPE do not accurately assess market power in consumer search markets. In most search markets, consumers do not know important elements of the environment in which they search (such as, for example, firms' cost). Thus consumers may learn some underlying market characteristics when they observe a price. We argue that when consumers learn while searching, RPE suffer from theoretical issues, such as non-existence and critical dependence on specific out-of-equilibrium beliefs. We characterize equilibria where consumers rationally choose search strategies that are not characterized by a reservation price. We show that Non-RPE always exist and do not depend on specific out-of-equilibrium beliefs. Moreover Non-RPE have active consumer search and are consistent with recent empirical findings.

Chapter 1

Showrooming in a market of tangible good with heterogeneous agents

1.1 Introduction

Nowadays we observe a stable growth of the E-commerce sector - the web's share of total retail increased by about 4 % in the last five years and continues to grow. The future of traditional brick-and-mortar stores has been widely discussed in the press. Many experts see e-commerce as a substantial threat to traditional brick-and-mortar stores (BMS), while others remain skeptical about BMSs being replaced by online retailers. In 2012 the chief executive of Gilt Groupe, an online retailer selling women's clothing and accessories, expressed his concerns about the future of traditional retail in an interview with the Economist. According to his opinion, there was "no evidence that there were big opportunities for traditional retailers in online retail" as "bricks-and-mortar shops were gravely threatened by Amazon and other online-only retailers".[1] However, four years later in 2016 Gilt Groupe announced its acquisition by Hudson's Bay Company, owner of luxury department store chains. The following year later Amazon opened its first brick-and-mortar store. Another article published in The Economist [2] provides more examples of online retailers which decided to open a BMS or a showroom, as they realized that "customers wanted shops too". The European commission reports that about 59% of retailers selling online make a choice in favor of multichannel retail distribution(see European Commission Report (2017)).

An important problem related to the digital economy is geo-blocking, whereby retailers can refuse to sell to consumers from a "foreign country". It can be implemented by preventing customers from accessing the website and refusing payment or delivery.[3] According to the European Commission Report, 2017 about 36% of online retailers do not sell cross-border for at least one of the relevant product categories. The median proportion is about 47% across the 28 EU Member States. Moreover, a retailer from a large online market is less likely to geo-block than retailers from small markets which mostly focus on domestic sales. Sometimes the choice of geo-blocking strategy is dictated by vertical restraints imposed by a manufacturer. However, this is not the case for most retailers. It is not obvious why firms may prefer to voluntarily concede a part of the online market to their competitors. We build a competition model in order to analyze multichannel retailers' incentives to geo-block.

This paper makes two main contributions to the existing literature. First, it theoretically explains how the magnitude of consumer search costs may explain observed variations in price differences across retail channels, and when price parity is enforced by a retailer. It discusses the role of showrooming as a way to discriminate between different types of consumers, and in the resolution of the hold-up problem which may potentially occur. Second, it provides an explanation of multichannel retailers' incentives to restrict cross-border sales, shows that geo-blocking is likely to happen in equilibrium, and examines policy responses.

[1] "Making it click", *The Economist*, 25th of February 2012.

[2] "Shops to showrooms", *The Economist*, 10 of March 2016.

[3] The retailer can also introduce geographical restraints by re-routing customers to a foreign web page based on their location. This strategy is known as geo-filtering and it is outside the scope of the paper.

In more detail,we develop a model where firms decide on selling through two distribution channels - an online store and a BMS. BM stores provide customer service, which is costly for the firm, but allow consumers to try the product and learn product characteristics. Typically consumers can find out some information about a product on a website, get an impression about a design, colors and a price, but they are able to figure out whether the product fits only after trying it. However, consumers bear a positive search cost to visit the BMS[4]. Online stores allow consumers to buy goods directly without visiting the physical store. However, online shopping is also associated with additional shopping costs, which are subjective and heterogeneous across customers. The literature reports many factors which determine this heterogeneity (see Swinyard and Smith, 2003, Keen et al., 2004). Most of them are related to psychological reasons such as unwillingness to wait for the product to be delivered, uncertainty of delivery dates, reluctance to make online transactions and so forth. Forman, Ghose, and Goldfarb, 2009 provide empirical evidence that online shopping creates some additional disutility for customers compared to shopping in BM stores, which cannot be explained by monetary costs.

In the first part of the paper, we analyze a monopoly case with observable retail prices and study how the presence of search and online shopping costs affects the firm's pricing strategies. Cavallo, 2017 finds that in most cases (about 72 % on average) online and offline prices are identical. The choice of price matching policy is typically explained by the retailers' attempt to increase profit by preventing competition between its own distribution channels (see Kireyev, Kumar, and Ofek, 2017)). However, the percentage of price matching strategies varies a lot across different geographical locations, sectors, and retailers,[5] which can be explained by firms' incentives to price discriminate. We find that for low search frictions, the monopolist prefers to set an online price equal or higher than a price in a brick-and-mortar store. As search frictions are low, there is widespread search - all consumers prefer to try the product in BMS. Thus, there is no efficient way to price discriminate between different types, and the online channel is redundant. All sales are redirected to a BMS store. For moderately high search frictions showrooming co-exists with online and offline purchases. We observe lower online prices than those in a store. For high search frictions all consumers prefer either to purchase online without search, or buy in a BMS. Showrooming is not a part of the equilibrium. We may observe online prices which are higher than store prices. For example, the online tariff for the Louvre museum access exceeds the tariff for tickets sold inside the museum. Both prices are precisely posted on the webpage. Even the museum ticket is not a typical tangible good, which consumers would potentially like to try in the store, the example is still a good illustration of our result. An online ticket assumes that consumers make a commitment for a visit at a certain day and a time slot. The purchase should be done at least one day in advance. In this case the probability of a "bad fit" reflects the uncertainty about whether consumers will visit the museum on that day, and at the time they buy a ticket for. Search costs are associated with the waiting time in the queue which can be high.

We provide a model extension for the monopoly case, where we consider an unobservable BMS price. In this situation consumers are potentially exposed to a hold-up problem: the marginal consumer should be indifferent between visiting the store and abstaining from purchase, but as soon as he is in the store, because the search cost is sunk, the firm has incentives to raise the price. Thus, the market collapses (see Diamond, 1971). This problem, however, can be mitigated by the introduction of an online retail channel. By opening an online shop the monopolist not only simply sells to some consumers online, but also creates artificial competition between the two distribution channels, which drives down prices in the BMS, simultaneously increasing the firm's profits. The firm can increase its profits by committing to the same price online and offline, and increase it even further, when search frictions are high, by publicly and

[4] In this paper we interpret search costs mostly as costs related to the time of visiting the store. Hence the magnitude of the search cost is mostly determined by factors such as locations, quality of public transport connection, waiting time to be served in a store and so forth

[5] Cavallo, 2017 shows that in some countries, i.e. Argentine and Australia, consumers typically face higher online prices.

credibly disclosing the offline price. The price matching policy becomes an important instrument for price advertising when the firm has a lack of commitment power. When search frictions are low, the firm earns the same profit as in the situation where both prices are publicly observed.

The second part of the paper is devoted to competition between retailers. We focus on firms' decisions which retail channels to use and their incentives to introduce restrictions on cross-border sales. We consider competition between two retailers. Consumers are divided geographically: they can only buy from a BMS of their own retailer, as BM stores naturally face some geographical market restrictions. Online stores make it possible for consumers to shop outside of their location. Firms, however, can commit to geo-blocking. We assume that decisions on retail channels and geo-blocking are long-run and precede the price-setting phase of the game. Geo-blocking unambiguously lowers potential demand for a firm imposing this strategy. However, at the later stage competition is much weaker: the competitor understands that its online shoppers are not threatened and raises its prices. This allows the firm which geo-blocks to raise its own BMS price and improve profits compared to the case without geo-blocking. When decisions about geo-blocking are taken simultaneously by both firms, we observe a symmetric equilibrium with two monopolists operating only in their local markets. In this case competition authorities may want to introduce legal restrictions on geo-blocking to restore competition, and increase consumer surplus. If firms' decisions are sequential, we observe an asymmetric market structure in equilibrium, where the follower either geo-blocks or maintains only a BMS, while the leader sells online cross-border. The competition policy authority can use a non-discrimination price policy, which obliges firms to charge the same price in both distribution channels, as an instrument to restore online competition when the market search cost is sufficiently low. It weakens online competition in the absence of geo-blocking, and thus eliminates incentives to ban cross-border sales.

Wang and Wright, 2017 is one of the first papers which study the effect of showrooming and price coherence in the presence of small firms and an online platform. In this context consumers search for lower prices and price observability is the way to eliminate potential showrooming. In our paper, we assume that consumers search for unobservable product characteristics. Consequently, showrooming may exist even with full price transparency. This is quite typical in markets of tangible goods. During the last few years there has been a substantial growth in literature discussing showrooming in the context of tangible goods and competition between online retailers and BMSs.

Mehra, Subodah, and Jagmohan, 2017's framework is the most similar to those considered in this paper. They provide a model of competition between BMS and an online shop, and discuss possible strategies of a BMS store to prevent showrooming, as it has an unambiguously negative effect on BMS profit. The positive effect of showrooming on BMS profit is discussed in the paper by Kuksov and Liao, 2016, where there is a strategic manufacturer. As the manufacturer is interested in a BMS providing customer service, it can propose to it a better wholesale tariff. The manufacturer's strategy to open an online channel in order to motivate a BMS to improve the quality of the customer service is explored in Yan and Pei, 2009. In our paper we don't consider vertical contracts and restraints, but we exploit the same idea of the multichannel retailer being interested in maintaining a BMS in order to provide additional customer service, which results in better matches for customers.

Kireyev, Kumar, and Ofek, 2017 focus on the analysis of multichannel retailers' strategies and consider a price matching policy as the main tool to prevent showrooming. However, they do not take into account the possibility to learn product characteristics and to increase expected utility through the search process. We show that due to the fact that willingness to learn product characteristics may disclose information about consumers' types, it leads to the possibility of price discrimination for a firm. Therefore, a multichannel retailer does not necessarily want to prevent showrooming in a market. Moreover, both the firm and consumers can benefit from higher search costs in presence of the showrooming. This result is related to Taylor, 2017, where the author shows that higher search costs may allow to screen consumers better and to target those who are more interested in buying the product. The intuition behind our result is similar. The difference is that in our consumers benefit not from the higher resulting level of customer service but from better prices proposed by the firm.

The rest of the paper is organized as follows. Section 2 presents the general model description; Section 3 analyzes consumers' behavior. Section 4 provides the solution for the monopoly case and discusses the role of price observability. Section 5 is devoted to the analysis of competition between multichannel retailers. Section 6 analyzes the model extension with unobservable BMS prices. Section 7 concludes with a discussion of the main findings.

1.2 Model

In this section we discuss the general setup of the model. Assume that there are homogenous goods sold in the market. The product fits a consumer with probability π, and realizations of successful matches are independent across customers. A consumer has a product valuation normalized to one in the case of successful match and to zero otherwise. Firms present in the market can have two distribution channels - a brick-and-mortar store and an online shop. Consumers are unaware whether the product fits before they try it. BMS provide customer service, which allows consumers to try the product and thus to learn whether they will get a successful match if they buy it. Visiting the BMS is costly for consumers, and they have to pay search cost s when they come to the BMS. At the same time, the firm has positive cost per visit η of providing customer service, because a store with a higher number of visits needs a higher number of consultants, more capacity and a higher number of provided samples.

Let p_w^i be the online and p_s^i be the store price set by retailer i. Prices are observable by consumers. The multichannel retailer can inform consumers about both prices by posting them directly online, by proposing special tariffs and discounts on online or offline sales or by committing to price parity in different distribution channels. Production cost is normalized to zero. Consumers observe prices and decide whether to search or not and then make a decision about a purchase. Once a consumer buys the product, he leaves the market.

Consumers who decide to buy online bear additional cost of online shopping μ_i, which is distributed according to $F(\mu_i)$. This distribution function has continuous support. $F(0) = 0$ and $F(\pi) = 1$, which means that all consumers want to buy online at zero price, and for any online price below π, there are consumers who have a positive expected utility of buying online. We make the standard assumption that $F(\cdot)$ is log-concave (see Bagnoli and Bergstorm, 2005), which means that $\frac{F'(\mu_i)}{F(\mu_i)}$ is decreasing in μ_i. To simplify the exposition we assume that consumers cannot return the product but we discuss how to relax this assumption in footnote 6 later in the text.

We consider two different market structures:

- Monopoly:

  There is a unique firm in the market. It makes a decision on distribution channel and then sets prices. Consumers make a decision on their buying/searching strategies after observing prices.

- Duopoly with geographical restrictions:

  There are two markets A and B. Consumers are split equally across them. There is one firm in each market. First, both retailers simultaneously decide whether to open an online shop or not. Second, observing each others' decision on a retail channel they simultaneously decide whether to commit to geo-blocking and, thus, refuse to sell abroad and serve only the local market. Third, retailers set prices simultaneously knowing each others' decisions on geo-blocking.

1.3 Consumers' strategies

Let's p_s be the consumer's local store price and p_w be the lowest price available online. Consumers may follow one of three searching and buying strategies. First, they may buy in the BMS. In this case they pay search cost to visit the store and buy the product only if it fits. The expected payoff is $\pi(1-p_s)-s$. We can see that for any price $p_s > \frac{\pi-s}{\pi}$ consumers have a negative payoff from buying in the store and thus do not purchase offline. Second, they can buy online without trying the product beforehand in the BMS. Then they pay the online price plus the additional cost of online shopping, but can learn whether the product fits only after purchasing. As there is no product return[6] the expected payoff is $\pi - p_w - \mu_i$. If the online price is higher than π, consumers never buy online without first searching in a BMS, because they get a negative expected payoff. Third, consumers can showroom, which means that they visit a BMS, try the product and then buy online in case of a successful match. The expected payoff equals $\pi(1-p_w-\mu_i)-s$. Notice that consumers who do showrooming may want to buy online even if the price is above π, but it must be below $1-p_w-\frac{s}{\pi}$.

A choice of the strategy depends on prices, search cost, and online shopping cost. The following Lemma establishes the result.

Lemma 1. *For any prices p_s and p_w consumers' best response strategies are following:*

(i) If $s < (1-\pi)p_w$ then consumers search and buy online if $\mu_i < \min\left\{p_s - p_w, \frac{\pi-s}{\pi} - p_w\right\}$, and buy in the BMS if $\mu_i > p_s - p_w$ and $p_s \leq \frac{\pi-s}{\pi}$.

(ii) If $(1-\pi)p_w \leq s \leq (1-\pi)p_s$ then consumers with cost $\mu_i < \min\left\{\frac{s}{1-\pi} - p_w, \pi - p_w\right\}$ buy directly online, consumers with cost $\frac{s}{1-\pi} - p_w < \mu_i < \min\left\{p_s - p_w, \frac{\pi-s}{\pi} - p_w\right\}$ showroom, consumers with $\mu_i > p_s - p_w$ buy in the store if $p_s \leq \frac{\pi-s}{\pi}$.

(iii) If $s > (1-\pi)\max\{p_s, p_w\}$ then consumers with $\mu_i < \{\pi p_s - p_w + s, \pi - p_w\}$ buy directly online, and consumers with $\mu_i > \pi p_s - p_w + s$ buy in the store if $p_s \leq \frac{\pi-s}{\pi}$.

The proof of Lemma 1 and all other omitted proofs are provided in the appendix.

The choice of prices determine types of consumers' behavior which we observe in the equilibrium. First consider the case where $p_s > p_w$. If search cost is small enough such that all consumers prefer to search in the BMS, then they buy in the BMS. If search cost s is high enough compared to the store price then consumers make a choice between buying directly online and buying in the BMS depending on their online shopping cost. For a moderate search cost there are three types of consumers: consumers with low μ_i buy directly online, consumers with high μ_i showroom and consumers with very high μ_i buy in the store. Second consider the case where $p_w > p_s$, so the online price is higher than the BMS price. This means that consumers never showroom. Thus either everybody buys in the store, when the online price exceeds the threshold $\frac{s}{1-\pi}$, or some consumers buy directly online and another consumers buy in the BMS otherwise. The threshold $\frac{s}{1-\pi}$ is the ratio of the search cost to the uncertainty of a good match. So, intuitively, it is clear that consumers prefer to visit the BMS when the search cost and the probability that the product fits are low.

Consumers' strategies are illustrated in Figure 1.1. The thresholds which separate the regions are not exogenous but defined by the choice of prices. If the online price is equal or higher than the store price, then there are only two regions, where consumers either buy online or in the store. These regions are separated by the decision line defined as $s = p_w + \mu_i - \pi p_s$.

[6] Consider now the case with product returns (see for example Petrikaite, 2017). Suppose that the consumer, who buys online, can return the product, and get reimbursed by αp, where $\alpha < 1$. Then we can rewrite the expected utility as $\pi(1-p) - \mu_i - (1-\pi)(1-\alpha)p = \pi 1 - \mu_i - p(1-\alpha(1-\pi))$. This we can scale by $1-\alpha(1-\pi)$, and introduce $\tilde{\pi} = \frac{\pi}{1-\alpha(1-\pi)}$. So we are still in the framework of our model, where matching probability is $\tilde{\pi}$ and online shopping cost is distributed between 0 and $\tilde{\pi}$. In other words, ex-ante higher probability to be reimbursed is equivalent to higher probability of matching for online sales.

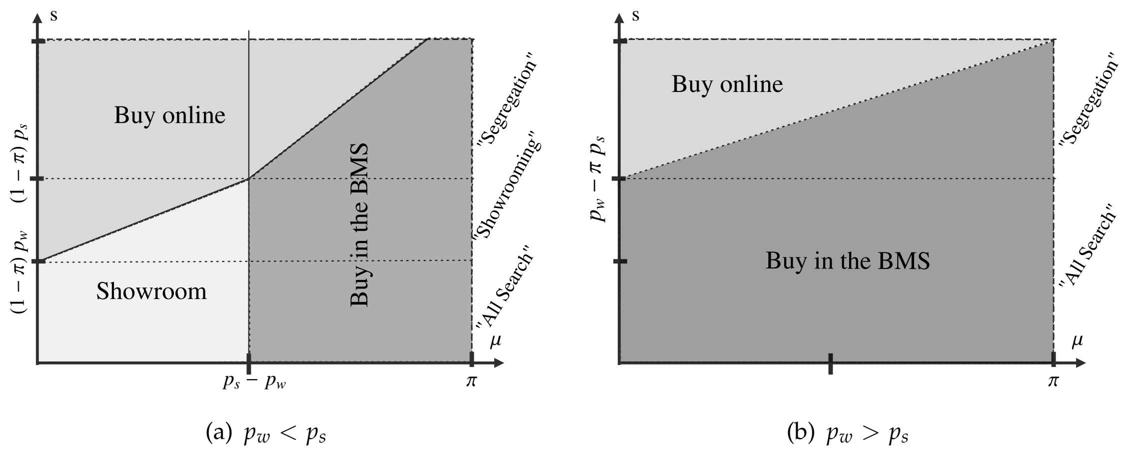


FIGURE 1.1: Consumers' strategies

1.4 Monopoly

We start our analysis with the monopoly case. The results of this section will give us some insights regarding the optimal pricing strategy of the monopolist in the absence of competition and the importance of the possibility to commit to prices in different distribution channels. Later we will use some results for the analysis of geo-blocking in the presence of online competition.

We consider the optimal strategy of the firm which opens both online and BMS and sells to consumers through different channels. Monopoly sets prices p_w in the web store and p_s in BMS. Cost of selling is normalized to zero. The monopolists's profit equals

$$\Pi(p_w, p_s) = D_s p_s + D_w p_w - \eta T V_s, \tag{1.1}$$

where D_s is the demand from consumers who buy in the BMS, D_w is the demand from consumers who buy in the web store, and TV_s is a total number of consumers who visit the store. If per-visit cost η is high then for the firm it is not profitable to maintain the BMS. This gives us a necessary condition for in-store sales.

Remark 1. *The monopoly maintains the brick-and-mortar store only if $s + \eta < \pi$.*

Proof. Consumers visit the store and buy there only if $p_s \leq \frac{\pi - s}{\pi}$. The firm makes a positive profit on in-store sales only if $\pi p_s - \eta > 0$. These two conditions together imply that $\eta \leq \pi - s$ is a necessary condition for positive profit from selling in the BMS. A consumer with online shopping cost μ_i showrooms only if $\pi(1 - p_w - \mu_i) - s > 0$. Thus if $p_w > \frac{\pi - s}{\pi}$, no consumers showroom. The firm makes a positive profit on consumers who showroom only if $\pi p_w - \eta > 0$. Thus, $\pi - s - \eta > 0$ is also a necessary condition for the firm to make a positive profit on selling to consumers who showroom. □

We will focus on the case when the BMS store is viable, and, therefore, assume that $\pi - s - \eta > 0$. Under this assumption we can show that the firm weakly prefers to charge the BMS price equals $\frac{\pi - s}{\pi}$ to any price above this threshold.

Remark 2. *The choice of the BMS price $\frac{\pi - s}{\pi}$ weakly dominates all higher prices.*

The logic behind this result is straightforward. Since consumers have heterogeneous online shopping costs and the distribution function is log-concave, for any given online price the measure of consumers who buy online and have zero expected payoff is zero. Thus the firm has exactly the same online demand for any BMS price equal or above $\frac{\pi - s}{\pi}$. At the same time consumers never buy in the BMS if $p_s > \frac{\pi - s}{\pi}$ and might buy there if $p_s = \frac{\pi - s}{\pi}$. So the firm always prefers to charge $p_s \leq \frac{\pi - s}{\pi}$. We will use this results later in the profit maximization problem of the firm.

In order to analyze the optimal choice of prices we have to split our analysis in three cases which correspond to the best response of consumers to prices (Lemma 1). We divide consumer behavior in three classes (all search, showrooming, segregation), compute optimal prices for each class and check when consumers' searching strategies are indeed consistent with these optimal prices.

These classes are defined as follows:

Case 1 "All search", where all consumers visit the BMS before purchasing online or offline(Lemma 2);

Case 2 "Segregation", where consumers either search and buy in the BMS or buy directly online, and they never showroom(Lemma 3);

Case 3 "Showrooming", where some showroom, while others either buy directly online or in the BMS (Lemma 4).

First, let us consider the case of "all search", where all consumers visit the BMS before purchasing either online or in the store (**Case 1**). From Lemma 1 we know that this happens when $p_w > \frac{s}{1-\pi}$. Consumers with cost $\mu_i < p_s - p_w$ buy online, while consumers with $\mu_i \geq p_s - p_w$ buy in the store if they have non-negative expected utility of buying and the product fits.

$$D_s = \pi(1 - F(p_s - p_w)) \text{ and } D_w = \pi F(p_s - p_w)$$

As long as $p_s \leq \frac{\pi - s}{\pi}$ all consumers are searching in the store and $TV_s = 1$. If the online price is higher than the BMS price, then $D_w = 0$. Therefore, for any price p_s the choice of online price strictly above p_s gives exactly the same profit as the choice of the online price equal to the BMS price. If the firm charges $p_s \leq \frac{\pi - s}{\pi}$ and $p_w \geq \pi$, then all consumers with positive online shopping cost do not buy online, while they still want to visit the BMS. Thus, the firm can also induce "all search" by charging $p_w \geq \pi$. The firm maximizes the expected profit with respect to prices p_s and p_w.

$$\max_{p_w, p_s} \pi(1 - F(p_s - p_w))p_s + \pi F(p_s - p_w)p_w - \eta, \text{s.t. } p_s \leq \frac{\pi - s}{\pi}, \max\left\{\frac{s}{1-\pi}, \pi\right\} < p_w \leq 1$$

We can show that optimally the firm sets the online price equal or above the BMS price.

Lemma 2. *If the firm wishes to induce "all search", it optimally charges $p_s = \frac{\pi - s}{\pi}$ and $p_w \geq \max\left\{\frac{\pi - s}{\pi}, \pi\right\}$.*

As the firm sells only in the BMS, its profit equals $\pi - s - \eta$ which is positive under our assumption on the parameters of the model. In this candidate equilibrium all consumers search in the store, but there is no showrooming. The online shop is therefore redundant.

Second, consider the case which we refer to as "segregation" (**Case 2**). Consumers decide where they buy the product immediately after observing the price. There is no showrooming in the market. From Lemma 1 we know that this requires $p_w < \frac{s}{1-\pi}$ (otherwise everybody visits the BMS) and $p_s < \frac{s}{1-\pi}$ (otherwise some consumers showroom).

Demand functions are

$$D_s = \pi(1 - F(\pi p_s - p_w + s)) \text{ and } D_w = F(\pi p_s - p_w + s)$$

As there is no showrooming $TV_s = 1 - F(\pi p_s - p_w + s)$. The firm maximization problem in this case is

$$\max_{p_w, p_s} (\pi p_s - \eta)(1 - F(\pi p_s - p_w + s)) + F(\pi p_s - p_w + s) p_w,$$
$$\text{s.t. } p_s \leq \min\left\{\frac{\pi - s}{\pi}, \frac{s}{1-\pi}\right\}, p_w \leq \min\left\{\pi, \frac{s}{1-\pi}\right\}$$

Lemma 3. *If the firm wants to induce "segregation", it optimally charges $p_s = \min\left\{\frac{\pi-s}{\pi}, \frac{s}{1-\pi}\right\}$ and p_w, which satisfies*

$$\frac{F[\pi p_s + s - p_w^*]}{F'[\pi p_s + s - p_w^*]} = (\eta + p_w^* - \pi p_s), \tag{1.2}$$

with $p_w^* \in [0, \pi]$.

This implies that for any search cost there exists a candidate equilibrium where consumers either buy directly online without searching in the BMS or search and buy in the store, so they are segregated in two groups in their searching/buying strategies. We have a boundary solution for the store price and interior solution for the online price.

Now let's consider a candidate equilibrium where consumers showroom with positive probability and buy directly online with positive probability (**Case 3**). Here we don't put any restrictions on consumers' behavior, except that the firm charges prices such that $(1-\pi)p_w \leq s \leq (1-\pi)p_s$ in order to induce this type of consumer behavior as follows from Lemma 1. Thus, the candidate equilibrium exists only if the online price is below the store price. Demand functions of the firm are

$$D_s = \pi\left(1 - F\left[p_s - p_w\right]\right) \text{ and } D_w = \pi\left(F\left[p_s - p_w\right] - F\left[\frac{s}{1-\pi} - p_w\right]\right) + F\left[\frac{s}{1-\pi} - p_w\right]$$

and the cost of providing customer service is

$$TV_s = 1 - F\left[\frac{s}{1-\pi} - p_w\right].$$

The profit maximization problem is

$$\max_{p_s, p_w} \pi\left(1 - F\left[p_s - p_w\right]\right) p_s + \pi\left(F\left[p_s - p_w\right] - F\left[\frac{s}{1-\pi} - p_w\right]\right) p_w + F\left[\frac{s}{1-\pi} - p_w\right] p_w - \eta TV_s, \tag{1.3}$$

$$\text{s.t. } p_w \leq \frac{s}{1-\pi}, \frac{s}{1-\pi} \leq p_s \leq \frac{\pi - s}{\pi}$$

We should consider this candidate equilibrium only for search costs below $\pi(1-\pi)$, because otherwise $\frac{\pi-s}{\pi} < \frac{s}{1-\pi}$ and the constraint in equation (1.3) cannot hold. The profit maximization problem always has a solution as the support of prices is bounded and $F[\cdot]$ is log-concave. We don't solve for the explicit solution. It depends on the shape of the online shopping cost distribution function and we can have either an interior solution (implicitly determined by first order conditions) or we can get a boundary solution. The next lemma proves that showrooming strictly dominates "all search" when the search cost is just below $\pi(1-\pi)$.

Lemma 4. *There exists $\varepsilon > 0$, such that for search cost $s \in [\pi(1-\pi) - \varepsilon, \pi(1-\pi))$, there is showrooming in equilibrium.*

The firm may prefer that consumers showroom even if the cost of providing customer service is zero. The reason is that under "showrooming" the firm gets a higher online demand due to additional sales to consumers who buy directly online. These consumers do not pay the search cost and are so unaware about the product fit, and thus may generate higher demand. Therefore, the firm can get a higher profit.

Now we do a pairwise comparison of the firm's profit in all candidate equilibria to derive the equilibrium of the game.

First, consider the interval where $s > \pi(1-\pi)$. We can show that the firm always prefers to induce "segregation". The pair of prices $p_s = \frac{\pi - s}{\pi}$ and $p_w = \pi$ in the "segregation" case gives the same profit as in the "all search" case (where nobody buys online). As we have an interior solution for the online price, which is lower than π, we know that on the interval of search costs $s > \pi(1-\pi)$ the firm gets higher profit in the candidate equilibrium with "segregation" than in the candidate equilibrium with "all search". At the same time, the firm can not induce showrooming for this interval of search costs.

Second, notice that for the search cost $s < \pi(1-\pi)$ the firm also can induce "segregation" by setting both prices below or equal to $\frac{s}{1-\pi}$. We showed in the proof of Lemma 4 that then the monopoly can always reach higher profit if consumers showroom. Thus inducing "segregation" is never the optimal strategy for search costs below $\pi(1-\pi)$.

Third, consider $s < \pi(1-\pi)$. We compare two cases - when the firm prefers to choose $p_w \geq p_s = \frac{\pi - s}{\pi}$ and when it prefers to charge p_w below $\frac{s}{1-\pi}$, so there is a showrooming in the market. If the cost of search s equals zero, the firm charges $p_w \leq \frac{s}{1-\pi} = 0$ only if it gets higher profit when consumers showroom. Hence the following condition must be satisfied:

$$\underbrace{\pi - \eta}_{\Pi, \text{"all search"}} < \underbrace{\pi p_s^*(1 - F[p_s^*]) - \eta}_{\Pi, \text{ showrooming}},$$

where

$$p_s^* = argmax_{p_s}(\pi p_s)(1 - F[p_s]) - \eta.$$

However, $\pi p_s^*(1 - F[p_s^*]) - \eta < \pi p_s^* - \eta \leq \pi - \eta$. Therefore when the search cost is close to zero, there is "all search" in the equilibrium.

Fourth, we can show that there exists some threshold $\tilde{s} > 0$, such that for $s < \tilde{s}$ the firm sets matching prices, and for $s > \tilde{s}$ it prefers to charge different prices online and in the store. We know that for s close to zero there is "all search" in the equilibrium, for s close to $\pi(1-\pi)$ there is showrooming. Hence in order to prove the result it is sufficient to show the following property:

Lemma 5. *If for some search cost $s' < \pi(1-\pi)$ there is showrooming in the equilibrium, then for any search cost $s \in [s', \pi(1-\pi)]$ there is showrooming in the equilibrium.*

We have showed that there exists a threshold $\tilde{s}$, such that it separates two regions where all consumers buy in the store and where there is showrooming in the equilibrium.

The following proposition summarizes these results.

Proposition 1. *There exists $\tilde{s} \in (0, \pi(1-\pi))$ such that in equilibrium the monopolist sets prices p_s and p_w such that:*

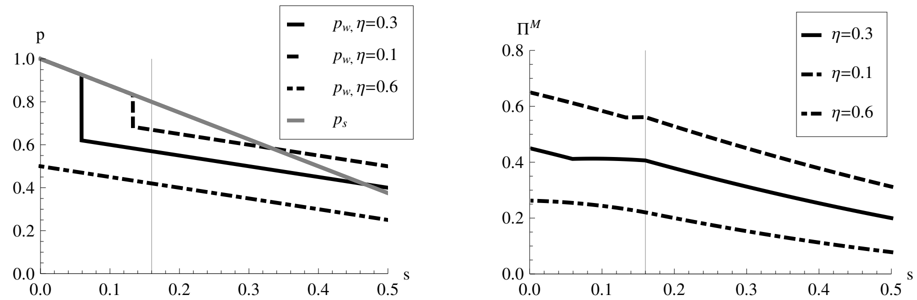
(i) if $s < \tilde{s}$ then $p_w^* \geq p_s^* = \frac{\pi - s}{\pi}$ and consumers buy in the store if product fits;

(ii) if $\tilde{s} \leq s < \pi(1-\pi)$ then $p_w^* \leq \frac{s}{1-\pi} < p_s^* \leq \frac{\pi - s}{\pi}$ and consumers either buy directly online, showroom or buy in the store;

(iii) if $s \geq \pi(1-\pi)$ then $p_s = \frac{\pi - s}{\pi}$ and p_w is defined by equation (1.2), consumers buy either directly online without searching or in the store.

Figure 1.2 illustrates equilibrium prices and profits as a function of the search cost, when the online shopping cost is uniformly distributed on $[0, \pi]$.

We see that the firm chooses $p_w \geq p_s$ for a low search cost. The size of this region of the low search cost is decreasing in the customer service marginal cost. As any choice of online price



(a) Equilibrium prices (b) Equilibrium profit

FIGURE 1.2: Monopoly

above the store price delivers the same profit to the firm, it would be natural to assume that the firm either closes online shop and posts the information about the store price on the webpage or just sets matching prices $p_s = p_w$. As the firm expects that everybody searches in the store, it cannot effectively price discriminate between different types. The fact that a consumer visits the store does not give any signal about his type. The pricing strategy of the firm is to choose equal prices in the store and online. Sales are directed to the store.

This result is robust to search cost heterogeneity, when the upperbound of the search cost distribution is sufficiently low.[7] Thus multichannel retailers with many stores and easy access to them will tend to set the same prices online and in the store, if they maintain both distribution channels. In this case online shops mostly play the role of the information source, where consumers can get informed about prices.

The assumption on common product valuation is not crucial for this result either. Visiting the store reveals to the firm some information about the willingness to buy the product, but does not provide any additional information on preferences of consumers towards online or in-store shopping. Thus heterogeneity in the product valuation will affect the magnitude of the equilibrium prices, but will not affect the optimal choice to set equal prices both online and in the store. We can notice that higher cost of providing customer service leads to a decrease in online prices, as the firm prefers that a bigger proportion of consumers buy directly online.

For high search costs the firm discriminates between two types. The price in the online shop can be both lower and higher than the price in the BMS. This happens due to the fact that the firm has to compensate customers' high search cost if it wants to enhance in-store sales on one side. On the other side higher search costs allow to discriminate better between different types, and thus, the firm can charge a higher online price compared to the store price.

It is less likely to observe $p_w > p_s$ when the customer service cost is high, as the firm has more incentive to decrease the online price. A higher cost of customer service creates incentives for the firm to sell more directly online. Thus the firm has to propose a better online price to prevent showrooming, as benefits of search are increasing in the online price. One can think that if the search cost is sufficiently high it could be a good decision for the firm to shut down the BMS and sell online, as benefits of selling to conservative consumers get outweighed by losses of preventing showrooming. However this is not the case. If there is only an online store, then the optimal choice of online price is such that the expected utility of buying online is positive. Thus, by proposing in the BMS price $p_s = \frac{\pi - s}{\pi}$ the firm can sell to a part of consumers who have negative utility of buying online. This will not affect the demand of the online store, and thus the

[7]We can think about the search cost as a common component s plus an individual heterogeneous component ε_i: $s_i = s + \varepsilon_i$. The common component s characterizes how easily most of consumers can reach the store. As an example, a chain with multiple stores will have a lower common search cost component than a single store. Consumers pay a lower search cost to reach a shop which is located in a city center than a store located out of the city. Longer working hours also facilitate store visits for consumers. In this case, if magnitude and variation of ε is small compared to the common component of search cost, we still find that the firm prefers to set equal prices in both retail channels.

profit of the firm will increase. Later we will show, that this is not the case when the BMS price is not observable.

For moderate search frictions we observe showrooming in equilibrium. On one side, the firm discriminates between different types, on the other side it remains too costly to prevent showrooming in the market. We see that in the equilibrium with showrooming the profit of the firm may increase in the search cost, as it is getting less costly for the firm to discriminate between different types.

1.4.1 Consumer Surplus and Social Welfare

In this section we consider how equilibrium consumer surplus depends on search frictions. When the firm charges price $p_s = \frac{\pi - s}{\pi}$ in the BMS it extracts full consumer surplus from those who buy there. Therefore if the firm does not open an online store and sells only in the BMS, consumer surplus equals zero. This is equivalent to the situation where the search cost is low and the firm sets prices such that everybody visits the store, so consumer surplus is $\pi(1 - \frac{\pi - s}{\pi}) - s = 0$. When the search cost is sufficiently high some consumers buy directly online, and consumer surplus is equal to

$$CS = \int_0^{\pi - p_w} (\pi - p_w - \mu_i)\, dF(\mu_i)$$

For the moderate search cost $\tilde{s} < s < \pi(1 - \pi)$ the equilibrium store price may be below $\frac{\pi - s}{\pi}$, which means that consumers, who buy in the store, also get positive expected utility. In this case consumer surplus is

$$CS = \int_0^{\frac{s}{1-\pi} - p_w} (\pi - p_w - \mu_i)\, dF(\mu_i) + \int_{\frac{s}{1-\pi} - p_w}^{p_s - p_w} (\pi(1 - p_w - \mu_i) - s)\, dF(\mu_i) + \\ + \int_{p_s - p_w}^{\pi} (\pi(1 - p_s) - s)\, dF(\mu_i)$$

As long as p_w is decreasing in search cost, consumers, who buy online, benefit from a lower online price. At the same time the share of online purchases is increasing. So, the total consumer surplus is increasing if there is no showrooming.

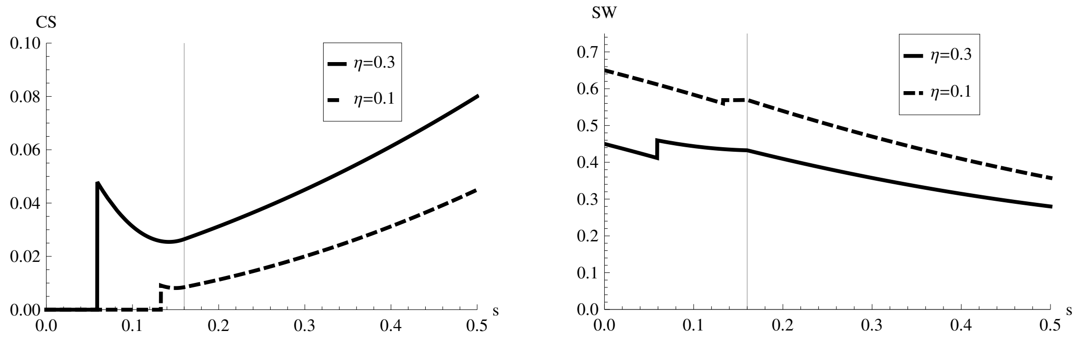
For a moderate search cost the online price offered by the firm is quite high. Consumers have incentives to search in the store and pay an additional search cost, which on one side has a negative effect on consumer surplus. On the other side, the equilibrium prices offered by the firm are decreasing in the search cost, which has a positive effect on consumer surplus. The total effect is ambiguous and CS'_s can be both positive and negative when $\tilde{s} < s < \pi(1 - \pi)$. When the search cost is above $\tilde{s}$ maintaining an online store positively affects both the firm's profit and consumer surplus, and thus it is socially desirable.

These results are illustrated on Figure 1.3.

On the interval where there is a full segregation with an interior solution for price p_w the consumer surplus is increasing in the search cost, but decreasing on the interval where there is boundary solution in the equilibrium. At the same time higher cost of customer service affects consumer surplus in the positive way. This happens due to the incentives of the firm to increase share of direct online sales and, thus, to propose better online price.

1.5 Cross-border competition and geo-blocking

In this section we focus on competition between two retailers which are geographically separated. They can have local BMSs, where only consumers from the local market can buy. At the same time they compete online à la Bertrand since goods are homogenous. We assume that the cost of providing customer service is equal to zero. We analyze decisions of firms on retail



(a) Consumer surplus

(b) Social Welfare

FIGURE 1.3: Monopoly

channels and their incentives to geo-block. Geo-blocking assumes that consumers from one geographical zone have no access to online stores operating in another geographical zone. First, we provide some insights explaining why firms may want to voluntarily refuse to sell in foreign markets. Second, we explain why banning geo-blocking may be not a good decision for the competition authorities, and how a non-discrimination price policy can help to eliminate incentives of firms to refuse to sell abroad.

As decisions on retail channels and geo-blocking are long-run, it is naturally to assume that they precede competition stage. Hence we assume that the game consists of three stages: i) firms decide simultaneously[8] on retail channels; ii) firms decide simultaneously on geo-blocking strategy; iii) firms decide simultaneously on price, competition takes place, and firms' profits are realized. As in the monopoly case we will focus on situations where $s < \pi$, so firms maintain brick-and-mortar stores.

We solve for subgame perfect equilibrium, and thus use backward induction approach. First, we should consider all possible market structures and derive equilibrium profits in each case. Overall we have six possibilities - three cases with a symmetric market structure and three cases with an asymmetric market structure:

- Firms A and B do not open online stores;
- Firms A and B open online stores and geo-block;
- Firms A and B open online stores and do not geo-block;
- Firm A opens online store and does not geo-block, firm B has only the BMS store;
- Firm A opens online store and geo-block, firm B has only the BMS;
- Firms A and B open online stores, firm A does not geo-block and firm B does.

We have already solved for the first two symmetric market structures. The simplest case is when both retailers decide to maintain only BMSs. Both firms are monopolists at the local markets. They charge prices equal $\frac{\pi - s}{\pi}$ in the local stores, as it is the highest price which consumers are ready to pay. The equilibrium profit of each firm is $\Pi^{BMS}(s) = \pi p_s = \pi - s$. If retailers decide to open online shops and both geo-block, then they get monopoly profits derived in the previous section, which are not lower then $\pi - s$ for any search frictions s as we have already shown before. We will denote these profits as $\Pi^{M}(s)$.

The asymmetric market structure, when firm A opens an online store and geo-blocks and firm B has only the BMS, never takes place in the equilibrium. Obviously in this case there is no

[8]Later in this section we will discuss how it changes if firms take decisions on retail channel and geo-blocking sequentially.

competition between firms, and thus in equilibrium firm A gets the monopoly profit $\Pi^M(s)$ and firm B gets profit $\Pi^{BMS}(s)$. Firm A can get at least not lower profit if it sells also in market B online. Thus it does not have any incentives to geo-block.

The remaining symmetric market structure to consider has competition between two multichannel retailers which open online shops and do not geo-block. Since goods are homogeneous consumers who buy online buy from the firm which offers the lowest price. Firms compete online à la Betrand, and therefore each firm has incentives to slightly undercut the online price of the competitor as then it serves all online sales and thus increases its profit. [9] Thus in the equilibrium it should be that $p_w^A = p_w^B = 0$, i.e. firms charge online prices at marginal cost. Store prices should be strictly below $\frac{\pi - s}{\pi}$, as otherwise everybody prefers to buy online.

Consumers in market j prefer buying directly online to showrooming if $\mu_i < \frac{s}{1-\pi}$. Therefore, if $s < \pi(1-\pi)$ and $p_s^j \geq \frac{s}{1-\pi}$, then there is a positive proportion of consumers who showroom. If $s > \pi(1-\pi)$ or $p_s^j < \frac{s}{1-\pi}$, then nobody showrooms. So we can observe either showrooming or "segregation" in equilibrium. The following Lemma establishes the equilibrium outcome.

Lemma 6. *If two multichannel retailers compete online à la Bertrand, then there exists threshold* $0 < \tilde{s} < \pi(1-\pi)$*, such that the equilibrium prices are*

$$p_w^A = p_w^B = 0,$$

$$p_s^A = p_s^B = \begin{cases} p^* & \text{if } s \geq \tilde{s}, \\ \max\left\{p', \frac{s}{1-\pi}\right\}, & \text{if } s < \tilde{s}, \end{cases}$$

where p^* *satisfies*

$$1 - F[\pi p^* + s] - \pi p^* F'[\pi p^* + s] = 0,$$

and p' *satisfies*

$$1 - F[p'] - p' F'[p'] = 0.$$

We can see that equilibrium online prices are at marginal production costs, which are normalized to zero. At the same time store prices are below the optimal price of a single BMS which does not face an online competitor. Thus, we can conclude that in this case each firm gets the profit which is lower than the profit of a single BMS which does not compete with an online store. Let's denote equilibrium profits in this subgame as $\Pi^C(s)$. So, we showed that $\Pi^C(s) < \Pi^{BMS}(s)$.

Competition between a BMS and a multichannel retailer: Now we consider the case of asymmetric competition where the multichannel retailer A, which can sell online in two markets and at the local store, competes with the BMS B, which sells only in the local store in market B. We do not fully derive firms' equilibrium strategies, but we will show that (i) the equilibrium exists and (ii) the equilibrium profit of firm A is not lower than $\Pi^M(s)$ for any search cost s, and the equilibrium profit of firm B is strictly higher than $\Pi^C(s)$ and strictly lower than $\Pi^{BMS}(s)$.

First, we prove equilibirum existence. As firms can set any prices from a continuous support, it is enough to show that firms' profits are continuous[10] in all prices.[11] Suppose firm A sets price p_s^A in BMS A and p_w^A online, and firm B sets price p_s^B in BMS B. The profit of firm B is

$$\Pi^B(p_s^B) = \begin{cases} \pi(1 - F[p_s^B - p_w^A])p_s^B, & \text{if } p_w^A \geq \frac{s}{1-\pi} \\ \pi(1 - F[\pi p_s^B - p_w^A + s])p_s^B, & \text{if } p_w^A \leq \frac{s}{1-\pi}, p_s^B \leq \frac{s}{1-\pi} \\ \pi(1 - F[p_s^B - p_w^A])p_s^B, & \text{if } p_w^A \leq \frac{s}{1-\pi}, p_s^B \geq \frac{s}{1-\pi} \end{cases}$$

[9] This is guaranteed by the continuity of the monopoly profit in both prices.
[10] See the fixed point theorem of Fan-Glicksberg
[11] However we cannot guarantee the existence of a pure strategy Nash equilibrium as the profit function of firm A is not necessarily single-peaked for all possible values of the search cost.

The profit function Π^B is continuous in p_s^B because $\pi\frac{s}{1-\pi}+s=\frac{s}{1-\pi}$. It is obviously continuous in p_w^A, when $p_s^B \geq \frac{s}{1-\pi}$. If $p_s^B < \frac{s}{1-\pi}$ then we have $F[\pi p_s^B + s - p_w^A] = F[\pi p_s^B - \frac{\pi s}{1-\pi}] = F[0] = F[p_s^B - p_w^A] = F[p_s^B - \frac{s}{1-\pi}]$ at $p_w^A = \frac{s}{1-\pi}$. So we can conclude that profit of firm B is continuous in both prices p_s^B and p_w^A.

Now we look at the profit function of firm A. As firm A is the monopolist at market A, we can write its profit as $\Pi^A(p_s^A, p_w^A, p_s^B) = \Pi^{Monop}(p_s^A, p_w^A) + \Pi^O(p_s^B, p_w^A)$, where Π^{Monop} is the monopoly profit at the local market, and $\Pi^O(p_s^B, p_w^A)$ is the profit which comes from online sales in market B. Π^{Monop} is continuous in p_s^A and p_w^A, so we need to show that Π^O is continuous in p_w^A and p_s^B.

$$\Pi^O(p_w^A, p_s^B) = \begin{cases} \pi F[p_s^A - p_w^A]p_w^A, & \text{if } p_w^A \geq \frac{s}{1-\pi} \\ F[\pi p_s^B - p_w^A + s]p_w^A, & \text{if } p_w^A \leq \frac{s}{1-\pi}, p_s^B \leq \frac{s}{1-\pi} \\ \left(\pi F[p_s^B - p_w^A] + (1-\pi)F[\frac{s}{1-\pi} - p_w^A]\right) p_w^A, & \text{if } p_w^A \leq \frac{s}{1-\pi}, p_s^B \geq \frac{s}{1-\pi} \end{cases}$$

The profit function $\Pi^O(p_s^B, p_w^A)$ is obviously continuous in price p_w^A. When $p_s^B = \frac{s}{1-\pi}$ we have $F[\pi p_s^B + s - p_w^A] = F[\frac{s}{1-\pi} - p_w^A] = \left(\pi F[p_s^B - p_w^A] + (1-\pi)F[\frac{s}{1-\pi} - p_w^A]\right) p_w^A$, so the profit is also continuous in p_s^B. We can conclude that function $\Pi^A(p_s^A, p_w^A, p_s^B)$ is continuous in prices p_s^A, p_s^B, and p_w^A. Therefore, there exists a Nash equilibrium.

Second, we show that the equilibrium profit of the firm B (let's denote it as $\Pi^{BC}(s)$[12]) is strictly below Π^{BMS} and strictly above $\Pi^C(s)$. We need to show that the best response of firm A to the competitor's price $p_s^B = \frac{\pi-s}{\pi}$ is such that the positive measure of consumers buy online in market B. Suppose that search cost is above $\pi(1-\pi)$. Then $\pi\frac{\pi-s}{\pi}+s=\pi$, and thus by charging any price $p_w < \pi$ firm A sells online in market B. The proof of Lemma 3 guarantees us that firm A never sets $p_w \geq \pi$ when $s > \pi(1-\pi)$. Now suppose that $s \leq \pi(1-\pi)$. Firm B sells in the BMS at price $\frac{\pi-s}{\pi}$ to all consumers in market B only if $p_w^A > \frac{\pi-s}{\pi}$. So we need to show that the best response of firm A is to set price $p_w < \frac{\pi-s}{\pi}$. If search cost is in the range $(\tilde{s}, \pi(1-\pi)]$, where $\tilde{s}$ is defined in Proposition 1, then firm A sets price $p_w^A < \frac{\pi-s}{\pi}$. If search cost belongs to the range $(0, \tilde{s}]$, then firm A can charge prices $p_w^A = \frac{\pi-s}{\pi} - \varepsilon, p_s^A = \frac{\pi-s}{\pi}$, where $\varepsilon > 0$, and get the profit

$$(\pi - s)(1 - F[\varepsilon]) + 2F[\varepsilon]\,(\pi - s - \pi\varepsilon) = \pi - s + F[\varepsilon]\,(\pi - s - 2\pi\varepsilon) > \pi - s,$$

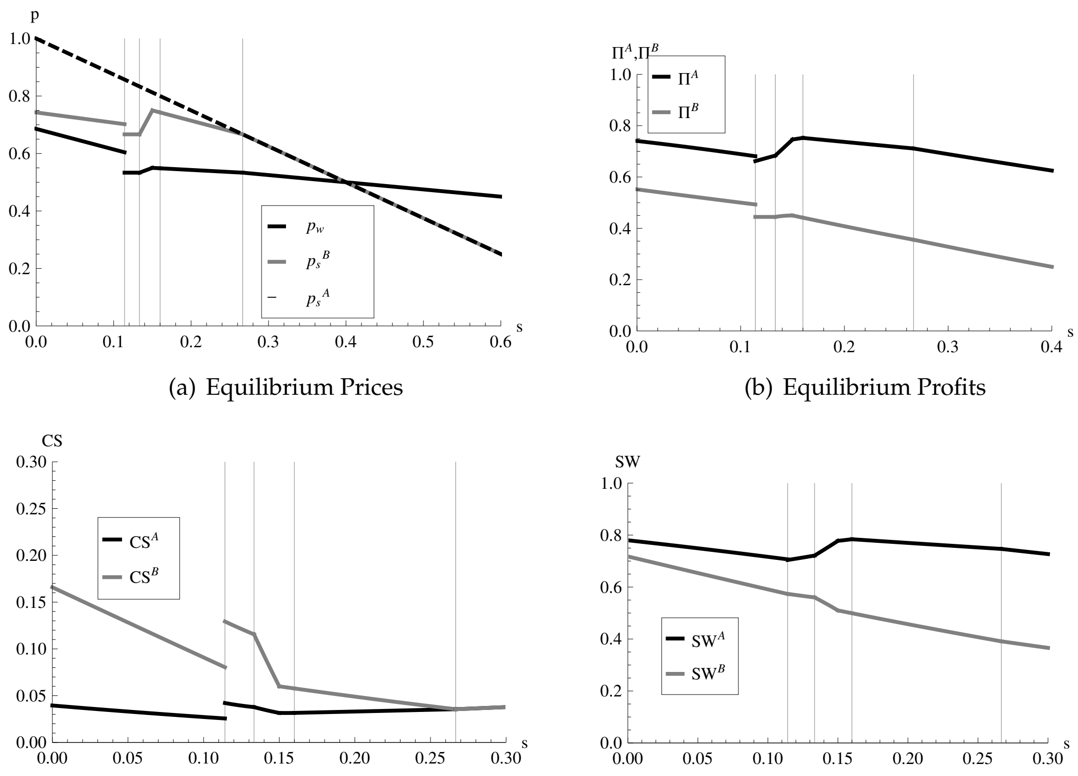
if ε is sufficiently small. Therefore, we can conclude that in the equilibrium firm B gets strictly lower profit than $\Pi^{BMS}(s)$ for any search cost. At the same time firm A can always reach at least the monopoly profit, and therefore it charges a strictly positive online price $p_w^A > 0$ (as otherwise firm A gets profit $\Pi^C < \Pi^M$). Thus, the profit of the firm B has to be higher than Π^C as it faces less online competition. So we showed that $\Pi^C < \Pi^{BC} < \Pi^{BMS}$.

Finally, we can notice that the equilibrium profit of firm A (let's denote it as $\Pi^{MC}(s)$) is not less than $\Pi^M(s)$. Obviously, firm A can always charge optimal monopoly prices as derived in Proposition 1 and get at least the monopoly profit. Thus the equilibrium profit $\Pi^{MC}(s) \geq \Pi^M(s)$, otherwise firm A has a profitable deviation.

On Figure 1.4 we illustrate the equilibrium solution for the linear cumulative distribution function $F[\cdot]$. We can see, that there is a range of search cost where firm A is mixing between two online prices below and above $\frac{s}{1-\pi}$.

Competition between two multichannel retailers, where one retailer geo-blocks: The last possible case is an asymmetric market structure, when both firms open online shops and one firm (let's say B) commits to geo-blocking. First we show that there is no pure strategy Nash equilibrium. We can notice that if price $p_w^B > p_w^A$ then firm B does not sell online to local customers. If it decreases the online price down to p_w^A that will not affect demand for BMS B, plus half of online customers will buy from B. Thus this deviation is strictly profitable. Therefore,

[12]We use notations BC and MC to refer to competition between a single BMS and a multichannel retailer.



(a) Equilibrium Prices
(b) Equilibrium Profits
(c) Consumer Surplus
(d) Social Welfare

FIGURE 1.4: Competition: $F(\mu_i) = \frac{\mu_i}{\pi}$

firm B always prefers to charge $p_w^B \leq p_w^A$. If $p_w^B = p_w^A$ then both firms have incentives to undercut the price as they double the demand from the customers in market B who buy online. Price p_w^A should be strictly positive, otherwise there is a profitable deviation for A to prices which guarantee at least the monopoly profit Π^M. Thus, the existence of pure strategy equilibrium requires that $p_w^B < p_w^A$, which means that firm A does not sell online on market B. If firm A sells online only in the local market it gets profit which is not higher than $\Pi^M(s)$. However, in the equilibrium firm A should get the profit which is not lower than $\Pi^M(s)$ as well. So it should be that firm A gets exactly $\Pi^M(s)$.

As markets are symmetric, another necessary condition, which should be satisfied, is profits' equality $\Pi^A = \Pi^B$, otherwise firm B always has a profitable deviation. If $\Pi^A > \Pi^B$, then firm B can charge price $p_s^B = p_s^A$ and p_w^B slightly below p_w^A, and thus it increases its profit. At the same time the profit of firm B is not higher than $\Pi^M(s)$. Thus, the existence of the pure strategy equilibrium requires that $\Pi^A = \Pi^B = \Pi^M$ and $p_w^B < p_w^A$. That means that monopolistic problem should have multiple equilibria, which contradicts to Proposition 1. So, there is no equilibrium in pure strategies where one multichannel retailer commits to geo-blocking and the other does not. In the equilibrium firms play mixed strategies. The existence of mixed strategy equilibrium is guaranteed by standard results of auction theory applied to the duopoly framework with Bertrand competition (see Dasgupta and Maskin, 1986).

Firms should mix on some interval of prices $p_w^B, p_w^A \in [\underline{p}_w, \bar{p}_w]$, and stores prices are the optimal prices $p_s^B(p_w^B, E(p_w^A))$, $p_s^A(p_w^A)$. We know, that firm A can guarantee at least the monopoly profit Π^M. Firm B never plays $p_w^B = \bar{p}_w$ with strictly positive probability in the equilibrium. If it does, then firm A also should charge the online price at the upper bound with strictly positive probability, but then firm B should charge online price at the upperbound with probability 0. So we have a contradiction. This means that $\Pi^A(p_w^A = \bar{p}_w, p_s^A(\bar{p}_w)) = \Pi^M$. The mixed strategy equilibrium requires that for all prices in the support firms get the same expected profit. So, the lowerbound of the equilibrium price distribution comes from the profit's equality condition.

Profit of firm A charging $p_w^A = \underline{p}_w$ and $p_s^A(\underline{p}_w)$ is equal to the monopoly profit Π^M. Thus, if firm B geo-blocks than its expected profit is increasing as online prices charged by the competitor with positive probabilities are strictly above 0. Let's denote the equilibrium profit of firm B in this subgame as Π^G. We showed that $\Pi^C(s) < \Pi^G(s) < \Pi^M(s)$.

We see that if both firms decide to open online stores at the first round, then each firm has an incentive to commit to geo-blocking at the second stage of the game in order to prevent the tough competition at the stage of competition. The matrix of payoffs of two multichannel retailers is illustrated on Figure 1.5.

Retailer B

Retailer A		Geo-blocking	No
	Geo-blocking	($\underline{\Pi^M}$, $\underline{\Pi^M}$)	(Π^G, Π^M)
	No	(Π^M, Π^G)	(Π^C, Π^C)

FIGURE 1.5: Geo-blocking: Profits of Multichannel Retailers

As geo-blocking is a weakly dominant strategy for both firms, we get three possible weak subgame Nash equilibria. One is symmetric, where both firms geo-block and become monopolists in the local markets. This is a unique trembling hand perfect equilibrium. Both firms get higher profits than in the case of running only brick-and-mortar store. Other two are asymmetric equilibria, where one firm geo-blocks and another bans cross-border sales. They are weak Nash equilibria.

Now we analyze decision of firms at the first stage, when they decide whether to open an online shop. If firm A opens an online store, then

- if firm B opens an online store, both firms geo-block and get profit Π^M at the last stage;
- if firm B does not open an online store, then firm A does not geo-block at the second stage and gets the profit $\Pi^{MC}(s) \geq \Pi^M(s)$.

If firm A decides not to open an online shop at the first stage then

- if firm B opens an online store, it does not geo-block at the second stage, and firm A gets profit $\Pi^{BC}(s)$ at the last stage;
- if firm B does not open an online store, then firm A gets the profit $\Pi^{BMS}(s)$ at the stage of competition.

Thus, the matrix of firms' payoffs is

So, we can see that if decisions are taken simultaneously, then both firms should open online stores at the first stage and commit to geo-blocking at the second stage. So there are two monopolists at the local markets not selling abroad in the equilibrium.

Thus, we can formulate the following result:

Proposition 2. *Both retailers open online stores and commit to geo-blocking in the equilibrium.*

Retailer B

Retailer A	Multichannel	BMS
Multichannel	$(\underline{\Pi^{M}}, \underline{\Pi^{M}})$	$(\Pi^{\text{MC}}, \Pi^{\text{BC}})$
BMS	$(\Pi^{\text{BC}}, \Pi^{\text{MC}})$	$(\Pi^{\text{BMS}}, \Pi^{\text{BMS}})$

FIGURE 1.6: Decisions on Distribution Channels: Profits of Retailers

In this situation competition authorities may be interested in restoring the market competition, and thus they take a decision to forbid geo-blocking strategy and require that firms do not discriminate consumers based on their geographical location. Let's consider how the market equilibrium changes if firms cannot geo-block at the second stage. The matrix of payoffs is illustrated on Figure 1.7. Since firms get profit $\Pi^{C}s$ if both open online stores, at least one firm does not open an online shop in the equilibrium. We should observe asymmetric market structure with only one multichannel retailer. Consumers surplus increases as firms charge lower prices in the equilibrium, compared with the case when there are two multichannel retailers which geo-block. However, this policy leads to the exclusion of one multichannel retailer from the online market.

Retailer B

Retailer A	Multichannel	BMS
Multichannel	(Π^{C}, Π^{C})	$(\underline{\Pi^{\text{MC}}}, \underline{\Pi^{\text{BC}}})$
BMS	$(\underline{\Pi^{\text{BC}}}, \underline{\Pi^{\text{MC}}})$	$(\Pi^{\text{BMS}}, \Pi^{\text{BMS}})$

FIGURE 1.7: Legal Restrictions on Geo-blocking: Profits of Retailers

1.5.1 Sequential decisions on retail channels and geo-blocking

While legal restrictions on geo-blocking lead to unambiguously higher consumer surplus when firms take decisions simultaneously, they may potentially lead to higher expected market prices when decisions are taken sequentially. Suppose that there is a market leader, which decides first on the retail channel and geo-blocking, and the market follower who takes decision on its retailer channel and geo-blocking after observing decisions of the competitor. So the timing of the game is following: i) firm A decides whether to open an online shop and whether to geo-block ii) firm B takes its decisions on a retail channel and geo-blocking iii) competition occurs, and profits realize.

Let's focus on optimal strategies of firm B. Suppose that firm A does not open an online shop, then firm B should open an online store and sell in both markets online, as we have already shown above. If firm A opens an online store and geo-blocks, then the optimal decision of firm B is to open an online store, and it is indifferent between geo-blocking and not geo-blocking. If firm A opens an online store and does not geo-block, then firm B may optimally either open an online store and geo-block or maintain only its BMS. The optimal decision comes from the profit comparison. If $\Pi^{BC}(s) > \Pi^{G}(s)$ then firm B does not open an online store. If $\Pi^{G}(s) \geq \Pi^{BC}(s)$ then firm B opens an online stores, but geo-blocks in equilibrium.

Now we focus on optimal decision of firm A at the first stage of the game. If firm A does not open an online shop it gets profit Π^{BC}. If firm A opens an online shop and geo-blocks then it gets either Π^{M} or Π^{G} depending on geo-blocking decision of firm B. As firm B is indifferent between two options, we can see that if there is some small positive probability ε that it decides not to geo-block, then expected profit of firm A is strictly below Π^{M}. If firm A opens an online store and does not geo-block, then its expected profit is greater or equal Π^{M}. Therefore we can conclude that the optimal decisions of firm A at the first stage is to open an online store and sell in both markets online.

We observe an asymmetric market structure in the equilibrium. The market leader sells online in both markets, and the market follower either does not sell online at all, or sells only in the local market. Suppose that competition policy authorities want to prevent geo-blocking in the second case and introduce legal restrictions on it. Then in equilibrium firm B does not maintain the online shop.

The total effect of geo-blocking restrictions on consumer surplus can differ depending on function $F[\mu_i]$ and market parameters. On one side, we can observe that firm A may charge higher expected online price when firm B maintains an online shop.

Lemma 7. *If $F[\mu_i]$ is convex then firm B opens an online shop and geo-blocks only if firm A charges higher expected price in the case of online competition.*

This result comes from the fact that the multichannel retailer, which faces more tough online competition, would prefer to concentrate more on its local market and thus to charge higher prices. The same idea is well established in Rosenthal, 1980, where the author shows that the increasing number of competing firms leads to higher expected price in the equilibrium. Hence the decision to ban geo-blocking will not lead to a decrease of consumer surplus in market A if function $F[\mu_i]$ is convex.

On the other side consumers in market B buy at the lowest available online price. In the presence of competition firms charge with positive probabilities lower prices than in the case when firm A sells online. Thus, the competition can lead to the lower expected minimum online price in market B. Since the store price B is increasing in the expected minimum online price, in this case we should observe that the expected prices in both channels decrease, and therefore consumer surplus in market B is higher in the presence of online competition.

1.5.2 Non-discrimination price policy

Non-discrimination price policy can be used as an instrument to decrease incentives of multichannel retailers to geo-block. The requirements to charge equal prices online and in a store should weaken a competition between firms when they do not geo-block.

We start our analysis with the case when two multichannel retailers do not geo-block. First of all, we can show that we can not have an asymmetric equilibrium, where $p^A < p^B$. Suppose it exists and prices are $\tilde{p}^A < \tilde{p}^B$.

Firms should have equal profits in the equilibrium because of the same reason which we discussed before in the symmetric competition section. In addition to the profit equality condition there should be that ${\Pi^{A}}'_{p^A}|_{\tilde{p}^A} = 0$ if $\tilde{p}^A \neq \frac{s}{1-\pi}$, so the firm A does not have incentives to deviate. Analogously, ${\Pi^{B}}'_{p^B}|_{\tilde{p}^B} = 0$ if $\tilde{p}^B \neq \frac{\pi-s}{\pi}$. At the same time we can notice, that the profit of the firm B is non-decreasing in price p^A, as higher online price of the competitor cannot negatively affect firm's profit, or $\frac{\partial \Pi^B}{\partial p^A} \geq 0$, and this inequality is strict for online price below store prices.

First, suppose that $\tilde{p}^B < \frac{s}{1-\pi}$. The firm A can deviate and charge $p^A = p^B - \varepsilon > \tilde{p}^A$, where $\varepsilon > 0$. Then $\Pi^A(p^A = \tilde{p}^B - \varepsilon, p^B = \tilde{p}^B) > \Pi^B(p^A = \tilde{p}^B - \varepsilon, p^B = \tilde{p}^B) \geq \Pi^B(p^B = \tilde{p}^B, p^A = \tilde{p}^A)$ for sufficiently small ε. Thus, there is a profitable deviation for firm A. Second, suppose that $\tilde{p}^B > \frac{s}{1-\pi}$ then $\Pi^A(p^A = \tilde{p}^B, p^B = \tilde{p}^B) = \Pi^B(p^A = \tilde{p}^B, p^B = \tilde{p}^B) > \Pi^B(p^B = \tilde{p}^B, p^A = \tilde{p}^A)$. Therefore, we can see that there is also a profitable deviation for the firm A in this case. We conclude that there is no asymmetric equilibrium.

Now let us consider a symmetric equilibrium. If consumers buy with positive probability online (search cost is high), or $p^j \leq \frac{s}{1-\pi}$, then both firms have incentives to slightly undercut the price, as consumers buy online from the firm which charges the lowest price. At the same time firms do not charge prices equal to zero, as they can always get the positive profit charging positive prices even if price of the competitor is equal to zero due to the monopoly power of local BM stores. So we can conclude that in this case there should be a mixed strategy equilibrium where the lower bound of the price support is positive.

When search cost is low (close to zero), consumers prefer to search in the store, profit is equal to πp^j. Firms do not have incentives to slightly undercut price if $1 - \pi p^j F'[0] \geq 0$. At the same time they do not have incentives to charge a bit higher prices if $1 - p^j F'[0] \leq 0$, so the equilibrium price is $p^j = p^j = \min\{\frac{1}{F'[0]}, \frac{\pi-s}{\pi}\}$, subject to $\frac{1}{F'[0]} > \frac{s}{1-\pi}$. Otherwise, there is no pure strategy equilibrium and firm should mix on the interval of prices. So we can conclude that in equilibrium, when nobody geo-blocks, firms charge strictly positive prices. When search cost is sufficiently low, there exists pure strategy equilibrium $p^j = p^j = \min\{\frac{1}{F'[0]}, \frac{\pi-s}{\pi}\}$ if $\frac{1}{F'[0]} > \frac{s}{1-\pi}$, otherwise there is a mixed strategy equilibrium.

Now, let's consider what happens when one firm makes a decision to geo-block. Suppose that firm B decides to ban cross-border sales. First of all, if s is close to zero, then consumers search before purchasing in both markets. Thus, firm B never charges price below p^A. So, it must be that $p^B \geq p^A$.

Let's check first order conditions for the interior solution:

$${\Pi^{B}}'_{p^B} = \pi(1 - F[p^B - p^A] - p^B F'[p^B - p^A]) = 0, \text{ s.t. } p^B < \frac{\pi - s}{\pi}$$

$${\Pi^{A}}'_{p^A} = \pi(1 + F[(p^B - p^A)] - p^A F'[p^B - p^A]) = 0, \text{ s.t. } p^A < p^B$$

We can notice that if the first condition for firm B is satisfied as an equality, or derivative ${\Pi^{B}}'_{p^B}$ is positive, then the ${\Pi^{A}}'_{p^A}$ is positive for any price p^A which is less than p^B. Thus, we should look at the candidate equilibrium where $p^A = p^B$. In this case firm A does not want to undercut the price of the competitor or to charge higher price if

$${\Pi^{A}}'_{p^A}|_{p^A = p^B} = \pi - \pi p^A F'[0] = 0 \Rightarrow p^A = \frac{1}{F'[0]}$$

and firm B does not want to charge higher price if

$${\Pi^{B}}'_{p^B}|_{p^A = p^B} = \pi - \pi p^B F'[0] = 0 \Rightarrow p^B = \frac{1}{F'[0]}$$

Obviously if $\frac{1}{F'[0]} > \frac{\pi-s}{\pi}$, then both firms charge prices $\frac{\pi-s}{\pi}$ and this is an equilibrium. Thus, equilibrium prices are $p^{A*} = p^{B*} = \min\{\frac{1}{F'[0]}, \frac{\pi-s}{\pi}\}$. Therefore, for small search cost close to 0 the equilibrium prices with one-side geo-blocking are not different from those without geo-blocking. Consumers buy in stores, profits are πp^{j*}. This equilibrium does not exist if $\frac{s}{1-\pi} > \min\{\frac{1}{F'[0]}, \frac{\pi-s}{\pi}\}$, or if one of the firms wants to deviate to online price such that online shoppers do not search. There is profitable deviation for any $s > s'$, where s' comes from the equation

$$\underbrace{\pi p^{j*}}_{\text{equil. profit}} = \underbrace{\frac{s\left(\pi + \pi F\left[\left(p^{j*} - \frac{s}{1-\pi}\right)\right]\right)}{1-\pi}}_{\text{deviation profit}}$$

As a result, given search cost being sufficiently low (below s'), firms do not have strict incentives to geo-block. Equilibrium prices are $\min\left\{\frac{1}{F'[0]}, \frac{\pi-s}{\pi}\right\}$. If there proportion of consumers with low μ is higher, or $F'[0]$ is high, then equilibrium prices are below $\frac{\pi-s}{\pi}$, which is lower than in the equilibrium without non-discrimination price policy, when both firms prefer to geo-block.

If search cost is high and consumers do not search before buying online, there is no equilibrium in pure strategies with one-side geo-blocking, where prices are such that $p^A = p^B$. So we look for candidate equilibria with asymmetric pricing. If $p^A > p^B$, then each firm sells online to consumers only form the local market. Using the profit equality argument and the solution of the monopolistic problem, we can show that it can not happen in the equilibrium as one of two firms prefers to deviate. So we should look at the equilibrium candidate such that $p^B > p^A$. First order conditions are

$$\Pi^{B'}_{p^B} = \pi(1 - F[\pi p^B - p^A + s] - \pi p^B F'[\pi p^B - p^A + s]) = 0, s.t. p^B < \frac{\pi - s}{\pi},$$

$$\Pi^{A'} = \pi + F[\pi p^B - p^A + s] + (1-\pi)F[s - (1-\pi)p^A] - p^A F'[\pi p^B - p^A + s] -$$
$$(1-\pi)^2 p^A F'[s - (1-\pi)p^A] = 0.$$

The additional condition is that firm B does not want to deviate and slightly undercut price p^A

$$\pi p^B(1 - F[\pi p^B - p^A + s]) > \pi p^A + (p^A - \pi p^A)F[-p^A - \pi p^A + s].$$

Obviously, if search cost is too high, then firm A has incentives to charge price close to or above $\frac{\pi-s}{\pi}$, thus firms will play mixed strategies. In Lemma 8 we provide the formal proof of this result.

Lemma 8. *If both firms commit to price matching policy and one firm commits to geo-blocking at the preliminary stage of the competition, then there exists a threshold $\tilde{s}$ such that firms play mixed strategies in equilibrium if $s > \tilde{s}$.*

We have to compare profit of the firm A in the case when it geo-blocks and when it does not. If firms play mixed strategies on some support of prices $[\underline{p}, \bar{p}]$, then if firm A charges the price at the upper bound, in the worst case (there is no atom in the price distribution played by firm B), consumers buy online from the firm B with probability 1. Then firm A has to get the profit which is equal to the monopolistic profit, otherwise it can deviate to the optimal monopolistic price. Thus, in the mixed strategy equilibrium firm A gets at least the monopolistic profit. Depending, on the exact shape of online cost distribution function $F(\mu)$ it can get high profit if firm B charges price at the upper bound with positive probability.

For the middle range of search cost there may exist an asymmetric equilibrium in pure strategies, where firm A serves all online sales. Here we can notice that the firm A can always guarantee to itself at least a monopoly profit. The question is whether it gets more when the other firm geo-blocks.

Thus, if one firm geo-blocks another firm in general weakly prefers not to geo-block, and depending on particular parameters and shape of distribution function $F(\mu)$ it may also strictly prefer not to geo-block. Therefore, we can conclude that non-discrimination price policy which imposes price parity in different retailing channels of the same firm, will eliminate incentives of firms to geo-block.

We consider an example of uniform distribution of online purchasing cost in order to illustrate how one-side geo-blocking affects profit of firm B for high search cost.

1.5.3 Example: Uniform Distribution

Let us consider the example with uniform distribution of the online purchasing cost $F(\mu) = \frac{\mu}{\pi}$.

When search cost is low the equilibrium is defined by pricing strategies $p^{j*} = \min\left\{\frac{1}{F'[0]}, \frac{\pi-s}{\pi}\right\}$. As in the case of the linear distribution function $F'[0] = \frac{1}{\pi}$, the solution is $p^A = p^B = \frac{\pi-s}{\pi}$.

The existence of the pure strategy equilibrium, where consumers who buy online do not search, requires that the following conditions are satisfied:

$$\Pi^{B\prime}_{p^B} = \begin{cases} \pi(1-2p^B) + p^A - s = 0, & \text{if } p^B < \frac{\pi-s}{\pi}, \\ \pi(1-2p^B) + p^A - s \geq 0, & \text{if } p^B = \frac{\pi-s}{\pi} \end{cases}$$

$$\Pi^{A\prime}_{p^A} = p^B + \pi(1-2p^A) + 4p^A - s + (-4p^A + 2s)\pi = 0,$$

so firm play best response given this searching/buying strategy of consumers. The last condition requires $p^A < \frac{s}{1-\pi}$ to be satisfied.

The firm A should not be able to get higher profit by charging p^A above $\frac{s}{1-\pi}$, so

$$\frac{(\pi(\pi + p^B - s) + 2s)^2}{4\pi(2 + (-2+\pi)\pi)} \geq \pi - s.$$

This condition defines the threshold of search cost $\underline{s}$, such that for higher cost $s > \underline{s}$ the lowest online price is below $\frac{s}{1-\pi}$ in the equilibrium, and for lower search cost $s < \underline{s}$ it is above, so there is showrooming.

The last condition is that firm B does not want to deviate and slightly undercut the price of firm A, which is

$$\frac{p^A(-p^A + \pi(\pi + 2p^A - \pi p^A - s) + s)}{\pi} \geq \pi p^B \left(1 - \frac{\pi p^B - p^A + s}{\pi}\right)$$

The last equations define the threshold $\bar{s}$, such that for any search cost $s > \bar{s}$ firm should play mixed strategies, as a pure strategy equilibrium does not exist. Clearly, for search cost $\underline{s} < \bar{s}$ there exists an asymmetric pure strategy equilibrium. Equilibrium profits and prices are illustrated on the Figure 1.8.

1.6 Extension: Unobservable store prices

In this section we will discuss the monopoly case with an unobservable BMS price. Suppose that the firm can not credibly commit to different an online and a BMS prices. While the online price is typically easy to observe at no cost on the webpage of the online shop, consumers often remain unaware about the BMS price. We assume that consumers expect to find price $\hat{p}_s$ if they visit the BMS. If consumers are rational then in the equilibrium they should form correct expectations such that $p_s = \hat{p}_s$, where p_s is the actual BMS price.

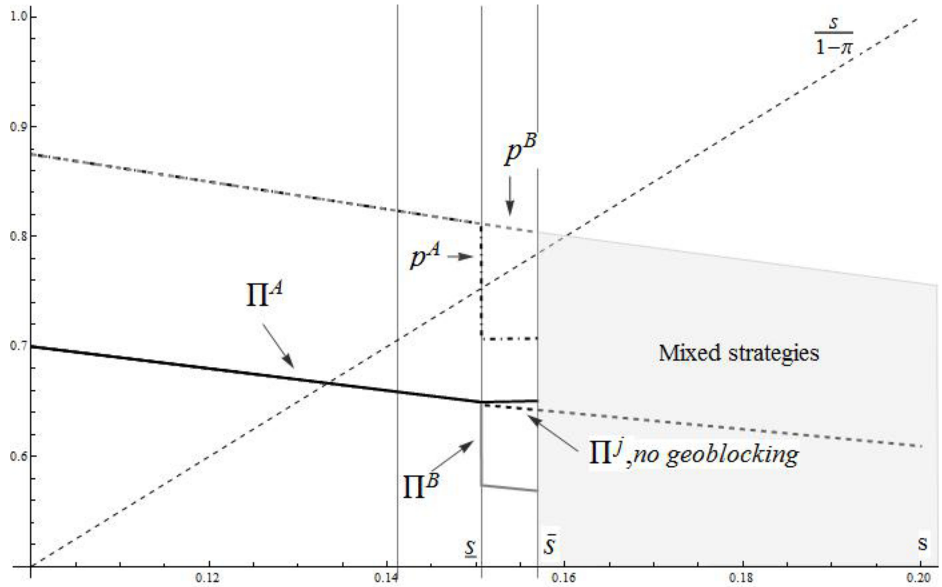


FIGURE 1.8: Equilibrium with one-side geo-blocking and non-discrimination price policy.

When search cost is positive a single BMS suffers from consumers facing a hold-up problem. One way to avoid it is to advertise to consumers the BMS price (see Janssen and Non, 2008 for price advertising incentives). We will show that another way to resolve the hold-up problem is to open an online store and hence create a downward-slopping demand due to artificial competition between different distribution channels. The multichannel retailer can use the online price as an instrument, which affects beliefs of consumers about the store price.

First, we can show that there is no equilibrium with "segregation", where the firm sells in the BMS. As we know from Lemma 1, this candidate equilibrium requires that $p_w \leq \frac{s}{1-\pi}$. Now, suppose that consumers anticipate some price $\hat{p}_s$ in the BMS and some consumers visit the BMS. Consumers visit the store only if they have an intention to buy there in the case of a good match. It means that $\pi(1-\hat{p}_s) - s > \max\{0, \pi - p_w - \mu_i\}$ for the consumer i who prefers to buy in the store. So, consumers with $\mu_i > \pi\hat{p}_s + s - p_w$ go to the store only if they expect price $\hat{p}_s < \frac{\pi - s}{\pi}$. Now we can show that the firm always want to charge $p_s > \hat{p}_s$. Suppose that consumers come to the BMS and find the price $p_s = \hat{p}_s + \varepsilon < 1$. If they buy in the store (when the product fits), then they get utility $1 - \hat{p}_s + \varepsilon > 0$. So they still want to buy. If they buy online, then they get utility $1 - p_w - \mu_i$. So they decide to buy online if $\mu_i < \hat{p}_s + \varepsilon - p_w$. As we are in the case where $\hat{p}_s < \frac{s}{1-\pi}$, we can notice that there exists $\varepsilon > 0$, such that the set $\{\mu : \mu_i \in (\pi\hat{p}_s + s - p_w, \hat{p}_s + \varepsilon - p_w)\}$ is empty. So, the firm always has incentives to slightly increase the store price, because more conservative consumers will still buy in the store after observing $\hat{p}_s + \varepsilon$. The rational consumer should anticipate that, and thus, he does not go to the BMS. The firm sells only online.

Second, we consider the case of "all search" and "showrooming". Notice that in the case where there is no segregation in the market and some consumers may showroom, the final decision about purchasing in the BMS is made by consumers who observe both prices. They prefer to purchase in the BMS if $\mu_i > p_s - p_w$. So the decision whether to buy in the BMS or online is affected by the actual BMS price, while the anticipated price affects the decision to visit the BMS at first place. As the equilibrium condition requires that the anticipated BMS store price equals the actual BMS price, the following equality must be satisfied:

$$\Pi'_{p_s}|_{p_s^*} = 0.$$

There can not be a boundary solution for the BMS price. If $p_s = \hat{p}_s \leq \frac{\pi - s}{\pi}$ and $\Pi'_{p_s}|_{\hat{p}_s} > 0$, then the firm has a profitable deviation to some price above $\hat{p}_s$. This means, that the firm keeps always online price lower than the BMS price in order to affect beliefs of consumers that the BMS price is not too high.

The following proposition formulates a result for the game equilibrium.

Proposition 3. *When the BMS price is unobservable, there exist thresholds s_1, s_2, and s_3 such that in the equilibrium the monopolists sets prices such that*

(i) if $s \le \min\{s_2, s_3\}$, then $p_s = \frac{\pi - s}{\pi}$, $p_w = \frac{\pi - s}{\pi} - \Delta_p$, where $(1 - F[\Delta_p]) - \pi\Delta_p F'[\Delta_p] = 0$, there is "all search" in the equilibrium;

(ii) if $s_2 < s < s_1$, then $p_w^* = \min\left\{\frac{s}{1-\pi}, \frac{\pi - s}{\pi} - \Delta_p\right\}$, $p_s^* = p_w^* + \Delta_p$, where $\pi - \pi F[\Delta_p] - (\pi\Delta_p - \eta)F'[\Delta_p] = 0$, there is showrooming in the equilibrium;

(iii) if $s > \max\{s_1, s_3\}$, then $p_w = p_w^*$, which satisfies $p_w^* = \frac{F[\pi - p_w^*]}{F'[\pi - p_w^*]}$, there are no BMS sales, firm sells only online in the equilibrium.

We covered all possible cases. If $0 < s_3 \le s_2$ that should automatically imply that $s_1 \le s_3$, and, therefore, we observe in the equilibrium either "all search" or only online sales. If $s_2 < s_3$, then it automatically implies that $0 < s_2 \le s_1 \le s_3$, and hence we observe all three possible types of consumers behavior. For high customer service cost we can have that either s_3 or s_2 equal to zero, and the firm sets prices such that a part of consumers always buy online. We can see that the firm can also choose to sell only online for all non-negative search cost, even if η satisfies our assumption in Remark 1. Suppose that search cost equals to zero. Then in order to induce "all search" the firm has to charge p_w strictly below $\frac{\pi - s}{\pi}$, where the price difference $\frac{\pi - s}{\pi} - p_w$ is bounded and finite as it was derived before. Thus, for any η higher than p_w, the firm gets a negative profit on online sales. We can always find ε, such that for $\eta = \pi - \epsilon$ the profit of the firm is below a profit in the case of online sales only. At the same time inducing showrooming requires that $p_w = 0$, so again there are η's sufficiently close to π, such that the profit in the case of showrooming is below the profit in the case of online sales only.

We can conclude that by introducing the online channel the firm can create a downward-slopping demand on BMS purchase and thus to avoid a hold-up problem when search cost is not too high. The possibility of showrooming plays an essential role for this result. However, it can be even more profitable for the firm to shut down the BMS and purely concentrate on online sales when costs of customer service are higher. For search cost above $\pi(1 - \pi)$ this is the only possibility.

1.6.1 Price matching policy

From the previous analysis, we can conclude that the multichannel retailer has incentives to commit to the BMS price, when consumers are rational. The simplest way to make a credible commitment to the BMS price is to declare the price matching policy, which requires price parity, $p_s = p_w$. This should immediately remove the possibility of the showrooming and would imply that there is either "all search" in the market, when search cost is low, or full segregation when search cost is high.

When search cost is equal or above $\pi(1 - \pi)$ the monopolist has two strategies. It can either charge price $p_w \le \frac{\pi - s}{\pi}$ in order to sell both online and offline, or to set price $p_w > \frac{\pi - s}{\pi}$, so there will be only online sales. Intuitively, we have to expect, that for very high search cost, it is not profitable to enhance in-store sales, as the monopolist has to compensate high search cost to consumers. So, we will consider these two case, compare profits and then define the equilibrium.

First, suppose that the monopolists decides to sell in both retail channels, which requires that $p_w \le \frac{\pi - s}{\pi}$. He maximizes the profit as follows:

$$\max_{p_w, p_s = p_w} p_w F[\pi p_s - p_w + s] + \pi p_s (1 - F[\pi p_s - p_w + s]) - \eta(1 - F[\pi p_s - p_w + s]), \text{ s.t. } p_w \le \frac{\pi - s}{\pi}.$$

The solution is defined as

$$p_w = \min\{\frac{\pi - s}{\pi}, p'_w\}, \tag{1.4}$$

where p'_w satisfies

$$\pi + (1-\pi)F[s-(1-\pi)p_w] - (1-\pi)(p_w(1-\pi) + \eta F'[s-(1-\pi)p_w] = 0 \tag{1.5}$$

If the firm decides to sell only online than it charges price above $\frac{\pi-s}{\pi}$. The profit maximization problem is

$$\max_{p_w} p_w F[\pi - p_w], \text{ s.t. } p_w < \pi,$$

which has the solution $\tilde{p}_w$ satisfying

$$F[\pi - \tilde{p}_w] - \tilde{p}_w F'[\pi - \tilde{p}_w] = 0$$

Log-concavity of function $F(\cdot)$ guarantees that $\tilde{p}_w \leq \pi$.

Now we need to define the threshold $\tilde{s}$, such that for search cost below the firm charges $p_w \leq \frac{\pi-s}{\pi}$ and for search cost above it charges $p_w = \tilde{p}_w$. This comes from the profit comparison $\Pi_{p_w=p_s=\min\{\frac{\pi-s}{\pi},p'_w\},s=\tilde{s}} = \Pi_{p_s=p_w=\tilde{p}_w,s=\tilde{s}}$. Suppose that s is close to π, than the profit of the firm selling both online and offline goes to zero, while the profit of the firm selling only online is positive. Now suppose that s is close to $\pi(1-\pi)$. Then we can show that $\tilde{p} < \frac{\pi-s}{\pi} = \pi$, and thus optimally the firm sells in both online and BM stores. The profit of the firm selling through both channels is decreasing in s, while the profit of the firm selling only online remains the same for any search cost. Thus, we there exists unique $\tilde{s} > \pi(1-\pi)$, such that for any search cost above this threshold the firm prefers to sell only online, and for search cost below it sells also in the BMS. Hence, when search cost is sufficiently high, price matching policy leads to an exclusion of a part of consumers from the market. The monopolist cannot increase its profit by committing to price parity when search cost is above $\tilde{s}$. However, for search frictions $s \in (\pi(1-\pi), \tilde{s})$ the price matching policy is a profitable strategy for the firm.

If search cost is below $\pi(1-\pi)$ then the multichannel retailer can charge price $p_w > \frac{s}{1-\pi}$, so everybody buys in the store. Optimal choice of prices is $p_w = p_s = \frac{\pi-s}{\pi}$ and profit is equal to $\pi - s - \eta$. We can show that there exists a threshold s', such that for any search cost below $0 \leq s' \leq \pi(1-\pi)$ the firm charges $p_s = p_w = \frac{\pi-s}{\pi}$ and for search cost above the firm charges $p_w = \min\left\{\frac{s}{1-\pi}, p'_w\right\}$[13]. It is easy to prove. First of all, when search cost goes to zero the profit goes to $\pi - \eta$ if all consumers buy in the store (prices above $\frac{s}{1-\pi}$), and to zero, otherwise. At the same time, when search cost approaches to $\pi(1-\pi)$, for any positive η the firm can reach higher profit by charging $p_w = p_s = \frac{s}{1-\pi}$ than by charging $p_s = p_w = \frac{\pi-s}{\pi}$. This happens as $\frac{s}{1-\pi}$ approaches to $\frac{\pi-s}{\pi}$ and the firm gets strictly positive and bounded below savings on customer service if it keeps prices such that a part of consumers buy directly online. The difference in profits $\Pi^{Seg} - \Pi^{AS}$[14] is increasing in s. Thus, there exists a unique threshold s', such that for search cost above the firm sets prices to segregate consumers, and for search cost below everybody goes to the BM store in the equilibrium. The following proposition summarizes results of this section.

Proposition 4. *There exist s' and $\tilde{s}$, such that in the equilibrium the monopolists, who commit to price matching policy, sets prices p_w and p_s such that*

(i) if $s < s'$ then $p_w = p_s = \frac{\pi-s}{\pi}$, consumers buy in the BMS;

[13] We assume that condition form 2 is satisfied when s goes to zero, so $\eta < \pi$. Otherwise, it is obviously never profitable for the monopolist to maintain the BMS

[14] Here Π^{Seg} is the highest profit in the candidate equilibrium with segregation and Π^{AS} is the highest profit in the candidate equilibrium with "all search". $\frac{\partial(\Pi^{Seg}-\Pi^{AS})}{\partial s} = 1 + (\tilde{p}_w(1-\pi) + \eta)F'[(-1+\pi)\tilde{p}_w + s] > 0$, where $\tilde{p}_w = \min\left\{p'_w, \frac{s}{1-\pi}\right\}$

(ii) if $s' \leq s < \tilde{s}$ then $p_w = p_s = \min\left\{\frac{\pi - s}{\pi}, \frac{s}{1-\pi}, p'_w\right\}$, where p'_w is defined as in equation (1.5), consumers either buy directly online or in the BMS;

(iii) if $s \geq \tilde{s}$ then $p_w = \tilde{p}_w$ and there are no sales in the BMS, consumers buy online.

So we can see that for sufficiently low search cost s and η the price matching policy allows to reach the equilibrium profit which is the same as in the case of observable prices. It allows as well to ensure in-store sales, when the firm can not induce showrooming, and its BMS faces a hold-up problem.

Corollary 1. *If the store price is unobservable, then there exists a range of search cost, such that the monopolist strictly prefers to commit to the price matching policy.*

Notice that this range of search cost is not necessarily convex. The firm may prefer to commit to the price matching policy for search cost close to zero and slightly above $\pi(1-\pi)$, but at the same time for search cost slightly below $\pi(1-\pi)$ it may reach the exactly same equilibrium outcome independently of whether the store price is observable. Figure 1.9 illustrates profits' comparison for the uniform distribution of online shopping cost. On Figure 1.10 we illustrate

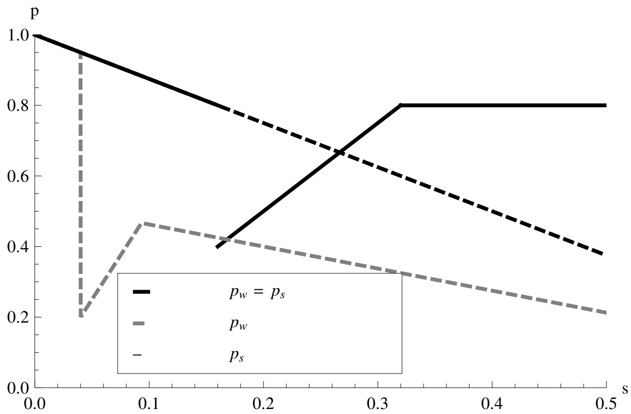


FIGURE 1.9: Price matching policy: Equilibrium Prices.

the comparison of firms profits in there cases assuming uniform distribution of online shopping cost. Obviously, the firm can achieve the highest profit when prices are observable. However, if it cannot credibly commit to different online and the BMS prices, then it increases the profit by applying the price parity commitment, which is efficient for sufficiently low search frictions. Otherwise, the optimal choice of prices is such that all sales are redirected to online shop under price matching policy. In order to induce in-store sales, the firm has to compensate the high

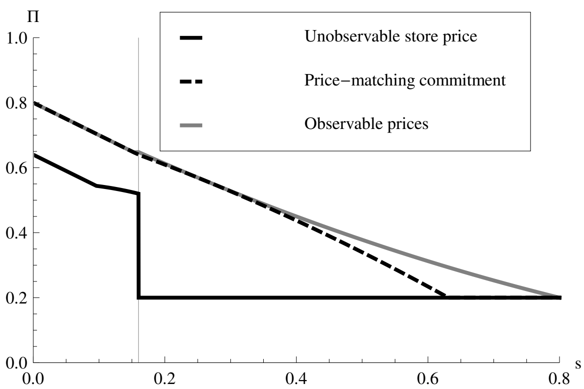


FIGURE 1.10: Monopoly profit and the store price observability.

search cost to consumers, and, therefore, it loses the profit from online sales. In this situation, the optimal decision is to sacrifice in-store sales and to charge higher online price.

1.7 Discussion

This paper analyzed pricing strategies of multichannel retailers in a market of tangible goods, where consumers are heterogeneous in their attitude towards online shopping and have some search cost of visiting a BMS. We show that the choice of price strategies depends on the search frictions present in the market. The monopoly retailer prefers to choose equal prices when search frictions are low, as it can not effectively discriminate between different types, but wants to prevent a competition between its own distribution channels. In this situation, the online shop mostly plays the role of informative source, where consumers can find the information about products and prices. The situation changes for higher search cost. When store visits become substantially costly for consumers, the firm gets an opportunity to screen for consumers' types and, therefore, to price discriminate. By offering a lower online price it creates incentives to showroom for some consumers in the market. For very high search cost, they are split in two groups consisting of more conservative consumers who buy in the store, and consumers who prefer to buy directly online without pre-purchasing search in the store. In this situation we may find higher prices in an online shop than in a BMS.

We show that by opening an online store and becoming a multichannel retailer, the BMS can avoid a hold-up problem, which appears when there are positive search frictions in the market, and the firm can not credibly commit to the store price. The online price is usually easily observable, and thus can be a tool to send a signal about the store price, and to ensure consumers that they will not encounter excessively high prices after coming to a BMS. However, this strategy only works for low or moderate search frictions. In this case the firm can increase its profit by committing to a price matching policy and guarantee consumers that they can buy the product in the store at the price posted online. Thus we can see that price matching plays an important role as a price commitment device, while it can be inefficient if the firm has another possibility to commit to the store price.

The other result of this paper is related to the international competition of online retailers. When two retailers have stores at their local geographical markets (where they are monopolists) and compete in prices online, they have incentives to ban cross-border sales. The firm prefers to concede a part of the international online market in order to prevent severe online competition, which results in low in-store prices as well. Thus, the retailer can obtain higher expected profit at the local market. There exist two types of the weak Nash equilibria. The first one is symmetric, where both firms decide to commit to geo-blocking and operate as monopolists at the local market. The second one is asymmetric, where one firm is present online in both markets, while the other one operates only at the local market. If decisions to geo-block are not taken simultaneously, then only the follower commits to geo-blocking while the leader serves all markets. This can explain while newcomers to the online market are more likely to focus on their home country sales.

In the situation when geo-blocking has an anticompetitive effect and can lead to the monopolization of local markets both online and offline, the competition authorities may want to prevent firms from geo-blocking. However, the full exclusion of a possibility to geo-block creates incentives to shut down online stores for retailers. This leads to the asymmetric market structure where only one firm is present online. First, this relaxes online competition between firms. Second, it is harmful in terms of the growth of e-commerce sector. We show that in this situation authorities can use a non-discrimination price policy, which obliges firms to set equal prices in different distribution channels, in order to prevent geo-blocking and to restore the online competition.

1.A Appendix

Asymmetric competition: uniform distribution of online shopping cost

Here we consider linear distribution function, when there is a range of search cost such that the multichannel has incentives to mix. We suppose that $F(\mu) = \frac{\mu}{\pi}$, thus function $F(\mu)$ is concave, and $F(\pi) = 1$, $F(0) = 0$. We fully solve the asymmetric competition problem and derive equilibria for any possible search cost.

1. When search cost if low, we know the optimal solution for the price $p_s^A = \frac{\pi - s}{\pi}$. So we plug in this price into F.O.C.s for firms A and B and get solution

$$p_w^* = \frac{1}{7}\left(4 + \pi - \frac{4s}{\pi}\right)$$

$$p_s^{B*} = \frac{2}{7}\left(1 + 2\pi - \frac{s}{\pi}\right)$$

So we can notice that indeed $p_w < p_s^B$, and condition $s < (1-\pi)p_w$ is satisfied if

$$s < \frac{(1-\pi)\pi(4+\pi)}{4+3\pi} < (1-\pi)\pi \equiv s'.$$

The interior solution for the price in store B is lower than $\frac{\pi - s}{\pi}$ for any s below $\pi(1-\pi)$. We can check as well, that given this best response of firm B there is a profitable deviation for firm A when it prefers to a price below $\frac{s}{1-\pi}$ for the search cost below $\tilde{s} < s'$.

$$\tilde{s} = \frac{2\left((-8+\pi)\pi(4+\pi) + 7\sqrt{\pi^3(4+\pi)^2}\right)}{-64+\pi}$$

2. As the curvature of function $F(\mu)$ is equal to zero, we know that for any search cost there exists a pure strategy equilibrium.

3. For search cost $\tilde{s} < s < \bar{s}$ there exists an equilibrium with showrooming in both markets. The equilibrium prices derived from the first order conditions are

$$p_s^{A*} = \frac{\pi - s}{\pi},$$

so the store price in market A is defined again by the boundary solution.

The online price is

$$p_w^* = \frac{\pi(4+\pi)}{8-\pi}.$$

The price in store B is

$$p_s^{B*} = \frac{6\pi}{8-\pi}$$

We can notice that in this case prices p_w and p_s^B do not depend on search cost. Due to the linearity of the distribution function, the drop of the store price p_s^A and repartition of the share of consumers who buy directly online and showroom in market A compensate each other in a such way, that the interior solution for the online price remains the same.

At the same time when search cost is equal to $\bar{s} = \frac{6(1-\pi)\pi}{8-\pi}$ the interior solution for firm B is equal to $\frac{s}{1-\pi}$. Thus, for higher search cost we have to consider another type of the equilibrium.

4. When search cost is $\bar{s} < s \leq \pi(1-\pi)$, in the equilibrium there is a showrooming only on the market A. The store price in market B is below or equal to $\frac{s}{1-\pi}$, thus consumers choose whether to buy directly online or in the store.

   The equilibrium prices are

$$p_w^* = \begin{cases} \frac{1}{4}\pi\left(2 - \frac{s}{-1+\pi}\right), & s > \frac{6(1-\pi)\pi}{4+3\pi} \\ \frac{s}{1-\pi}, & s \leq \frac{6(1-\pi)\pi}{4+3\pi} \end{cases}$$

$$p_s^{B*} = \begin{cases} \frac{2(3\pi-2s)}{7\pi}, & s > \frac{6(1-\pi)\pi}{4+3\pi} \\ \frac{s}{1-\pi}, & s \leq \frac{6(1-\pi)\pi}{4+3\pi} \end{cases}$$

$$p_s^{A*} = \frac{\pi - s}{\pi}$$

   Obviously, when search cost is approaching to $\pi(1-\pi)$, consumers in market A are also loosing their incentives to search, as a result we get the equilibrium without showrooming, where consumers on both markets are fully segregated in two subsets both in searching and purchasing strategies.

5. When search cost is high $s > (1-\pi)\pi$, then we have the price in store A should be still equal to $\frac{\pi-s}{\pi}$. Online price and price in store B are

$$p_w = \begin{cases} \frac{1}{7}(5\pi - s), & s < \frac{\pi}{3}, \\ \frac{1}{4}(3\pi - s), & s \geq \frac{\pi}{3}. \end{cases}$$

$$p_s^B = \begin{cases} \frac{2(3\pi-2s)}{7\pi}, & \text{if } s < \frac{\pi}{3}, \\ \frac{\pi-s}{\pi}, & s \geq \frac{\pi}{3} \end{cases}$$

   We can see that for sufficiently high search cost again online price will be higher than prices in stores.

Equilibrium prices and profits are illustrated on the Figure 1.4.

Proofs Consumers:

Proof of Lemma 1. For deriving the optimal buying behavior, we have to do the pairwise comparison of strategies.

Step 1 First of all, we can see that consumers prefer showrooming to outright online purchase when

$$\pi(1 - p_w - \mu_i) - s > \pi - p_w - \mu_i \Rightarrow s < (1-\pi)(p_w + \mu_i)$$

If the online price p_w is above $\frac{s}{1-\pi}$, then all consumers prefer to visit the BMS independently of μ_i. Thus, we have to consider two cases: $p_w \leq \frac{s}{1-\pi}$ and $p_w > \frac{s}{1-\pi}$.

Step 2 Suppose $p_w > \frac{s}{1-\pi}$, then from **Step 1** it follows that consumers always visit the BMS before buying if they have positive expected utility. After visiting the store they buy in the BMS if

$$\pi(1 - p_s) - s > \pi(1 - p_w - \mu_i) - s \Rightarrow \mu_i > p_s - p_w,$$

and $p_s < \frac{\pi-s}{\pi}$. Otherwise they buy online if $\mu_i < 1 - p_w - \frac{s}{\pi}$. This explains part (i) of the Lemma.

Step 3 Suppose that $p_w \leq \frac{s}{1-\pi}$, such that consumers may potentially buy directly online without visiting the BMS, buy in the BMS and showroom. Consumers prefer to buy directly online when $\mu_i \leq \pi - p_w$ and **i)** outright online purchase is better than showrooming $\pi(1 - p_w - \mu_i) - s < \pi - p_w - \mu_i \Rightarrow \mu_i < \frac{s}{1-\pi} - p_w$, and **ii)** outright online purchase is better than shopping in the BMS $\pi(1 - p_s) - s < \pi - p_w - \mu_i \Rightarrow \mu_i < \pi p_s - p_w + s$.

Step 4 Suppose that $p_s > \frac{s}{1-\pi}$ and $p_w < \frac{s}{1-\pi}$, then $\frac{s}{1-\pi} - p_w < \pi p_s - p_w + s$ and $\frac{s}{1-\pi} - p_w < p_s - p_w$. Form **Step 3** we get that consumers with online shopping cost $\mu_i < \frac{s}{1-\pi} - p_w, \pi - p_w$ prefer buying directly online to showrooming and buying in the BMS. From **Step 1** we know that consumers with $\mu_i > p_s - p_w$ prefer the BMS purchase to showrooming. Hence, we get that consumers **i)** buy online if $\mu_i < \min\left\{\frac{s}{1-\pi} - p_w, \pi - p_w\right\}$, **ii)** showroom if $\frac{s}{1-\pi} - p_w \leq \mu_i < \min\left\{p_s - p_w, 1 - \frac{s}{\pi} - p_w\right\}$, **iii)** buy in the BMS if $\mu_i \geq p_s - p_w$ and $p_s < \frac{\pi - s}{\pi}$. This explains part (ii) of the Lemma.

Step 5 Suppose $p_s < \frac{s}{1-\pi}$ and $p_w < \frac{s}{1-\pi}$, then consumers consumers prefer showrooming if $\frac{s}{1-\pi} - p_w < \mu_i < p_s - p_w$ (from **Step 1** and**Step 3**). We can see that if $p_s < \frac{s}{1-\pi}$, then the set of $\{\mu_i\}$, such that $\frac{s}{1-\pi} - p_w < \mu_i < p_s - p_w$ is an empty set. Hence, consumers buy directly online if $\mu_i < \min\{\pi p_s - p_w + s, \pi\}$ (from **Step 3**) or consumers buy in the BMS if $p_s \leq \frac{\pi - s}{\pi}$ otherwise. This explains part (iii) of the Lemma.

We considered all possible combinations of online and store price, and therefore the analysis is complete. □

Proofs Monopoly:

Proof of Lemma 2. We can rewrite the profit function in the case of "all search" as

$$\Pi^{AS} = \pi p_s(1 - F[p_s - p_w]) + \pi p_w F[p_s - p_w] - \eta.$$

It is a weighted sum of πp_s and πp_w, where weights sum up to 1. Thus, the optimal solution of the problem, if no constraints bind, is such that $p_s = p_w$. Depending on which constraint binds first we get that

1. if $s \leq \pi(1-\pi)$, then $\frac{\pi - s}{\pi} \geq \frac{s}{1-\pi}$, $\pi > \frac{s}{1-\pi}$ and $\frac{\pi - s}{\pi} \geq \pi$, and the solution is $p_s = \frac{\pi - s}{\pi}$, $p_w \geq \frac{\pi - s}{\pi}$;
2. if $s > \pi(1-\pi)$, then $\frac{\pi - s}{\pi} < \frac{s}{1-\pi}$ and $\frac{s}{1-\pi} > \pi$, the solution is $p_s = \frac{\pi - s}{\pi}$, $p_w > \pi$;

□

Proof of Lemma 3. The derivatives of the profit function with respect to online price in the case of "segregation" is

$$\Pi'_{p_w} = F[\pi p_s - p_w + s] + (\pi p_s - p_w - \eta)F'[\pi p_s - p_w + s] \tag{1.6}$$

The profit function can be rewritten as a weighted sum of $\pi p_s - \eta$ and p_w:

$$\Pi^{Seg} = (\pi p_s - \eta)(1 - F[\pi p_s - p_w + s]) + p_w F[\pi p_s - p_w + s].$$

Thus, if no constraints bind then optimally $p_w = \pi p_s - \eta$. Depending on which constraint binds first we get that

1. If $s \geq \pi(1-\pi)$, then $\frac{\pi - s}{\pi} \leq \frac{s}{1-\pi}$, $\frac{\pi - s}{\pi} \leq \pi$ and $\pi \leq \frac{s}{1-\pi}$, and thus the first binding constraint is $p_s = \frac{\pi - s}{\pi}$. The store price is equal to $\frac{\pi - s}{\pi}$ and online price p_w^* is defined by the first order condition from equation (1.6):

$$F[\pi - p_w^*] + (\pi - p_w^* - s - \eta)F'[\pi - p_w^*] = 0$$

This equation has a unique solution $p_w^* > 0$ due to log-concavity of function $F[\cdot]$. When p_w goes to π LHS of the equation is negative, so the optimal price satisfies $p_w^* < \pi$.

2. If $s < \pi(1-\pi)$, then $\frac{\pi-s}{\pi} > \frac{s}{1-\pi}$ and $\pi > \frac{s}{1-\pi}$. The first binding constraint is $p_s = \frac{s}{1-\pi}$. Thus the optimal BMS price is $\frac{s}{1-\pi}$. The optimal online price comes from the first order condition
$$F[\frac{s}{1-\pi} - p'_w] + (\frac{\pi s}{1-\pi} - p'_w - \eta)F'[\frac{s}{1-\pi} - p'_w] = 0.$$
It has a unique solution $p'_w > 0$ due to the log-concavity of the function $F(\cdot)$. As LHS of the last equation is negative when p_w approaches to $\frac{s}{1-\pi}$, so $p'_w < \frac{s}{1-\pi}$.

□

Proof of Lemma 4. First of all, we can show that the solution of profit maximization problem when the firm induces showrooming gives higher profit than the solution which induces "all search". Suppose that s is close to $\pi(1-\pi)$, then the firm's profit in the case of "all search" goes to

$$\lim_{s\to\pi(1-\pi)} \Pi\left(p_s = p_w = \frac{\pi - s}{\pi}\right) = \pi^2 - \eta,$$

We can show that there always exists ε such that by choosing prices $p_w = \frac{s}{1-\pi} - \varepsilon$ and $p_s = \frac{\pi-s}{\pi}$, the firm gets higher profit than by setting matching prices.

$$\lim_{s\to\pi(1-\pi)} \Pi\left(p_s = \frac{\pi - s}{\pi}, p_w = \frac{s}{1-\pi} - \varepsilon\right) = \tag{1.7}$$
$$= \pi^2 - \eta + \left(\pi - \pi^2 - \epsilon + \eta\right) F[\epsilon],$$

So, we can see that we can always choose $\varepsilon > 0$, such that firm's profit is higher when consumers showroom and online price is lower than the store price.

At the same time from the proof of Lemma 3 it follows that if the firm wants to induce "segregation" for the range of search cost $s < \pi(1-\pi)$, it has to charge some prices $p_s^* = \frac{s}{1-\pi}$ and $p_w^* < \frac{s}{1-\pi}$, which coincides with the boundary solution for the showrooming case.

□

Proof of Lemma 5. We know that in the case of "all search" the profit of the firm is equal to $\pi - s - \eta$. Suppose that for s' the firm prefers that consumers showroom, so it gets the profit higher than $\pi - s' - \eta$, for some prices $p_w(s') < \frac{s}{1-\pi} < p_s(s')$.

Step 1 If the optimal choice of prices $p_w(s')$ and $p_s(s')$ is such that $p_s(s') = \frac{\pi-s'}{\pi} - \delta_1 = \frac{s}{1-\pi} + \delta_2$, where $\delta_1, \delta_2 > 0$, then for any search cost $s' + \varepsilon$, where $\varepsilon \in (0, \min\{\delta_1\pi, \delta_2\})$, the firm can still induce showrooming and get the same profit as for search cost s' by charging $p_s(s'+\varepsilon) = p_s(s') < \frac{\pi-(s'+\varepsilon)}{\pi}$ and $p_w(s'+\varepsilon) = p_w(s') < \frac{s'}{1-\pi} < \frac{s'+\varepsilon}{1-\pi}$.

Step 2 If the optimal choice of prices is such that $p_s(s') = \frac{\pi-s'}{\pi}$, then the profit at search cost s' is equal to

$$\Pi^{Show}(s') = \pi - s' - \eta + (p_w - \pi p_w)F[-p_w + \frac{s'}{1-\pi}] + (\pi(-1+p_w) + s' + \eta)F[1 - p_w - \frac{s'}{\pi}]$$

We assumed that this profit is higher than $\pi + \varepsilon - s'$, then we can immediately check that for $s = s' + \varepsilon$, the firm can get the profit higher than $\pi - s' - \eta - \varepsilon$ by charging

$p_w(s'+\varepsilon) = p_w(s')$ and $p_s(s'+\varepsilon) = \frac{\pi - s'}{\pi}$.

$$\Pi^{Show}(s'+\varepsilon) = \pi - s' - \varepsilon - \eta + (\pi(-1+p_w) + s' + \varepsilon + \eta)F[-\frac{(\pi(-1+p_w)+s'+\varepsilon]}{\pi}]+$$
$$+ (p_w - \pi p_w)F[-p_w + \frac{s+\varepsilon}{1-\pi}] > \pi - s' - \varepsilon - \eta$$

Thus, if for some search cost $s' < \pi(1-\pi)$ there is a showrooming in the equilibrium, then there is exists $\varepsilon > 0$, such that the firm induces showrooming in the equilibrium for search cost $s' + \varepsilon < \pi(1-\pi)$, where $\varepsilon > 0$.

□

Proof of Proposition 3. First, it has been already shown that in "segregation" candidate equilibrium the firm never sells to rational consumers in the BMS.

Second, let's consider the case of "all search". As we have shown the following condition must be satisfied in equilibrium:

$$\Pi'_{p_s}|_{p_s^*} = 0.$$

From Lemma 2 it follows that in the case of "all search" the firm can not satisfy the first order condition for the store price if $p_w \geq p_s$. Hence if the firm wants to sell in the BMS it has to distort the online price to some value p_w^* such that $p_s^*(p_w^*) \leq \frac{\pi - s}{\pi}$. The optimal choice, which maximizes the firm's profit is $p_s^* = \frac{\pi - s}{\pi}$. Otherwise, the firm can always improve its profit by slightly increasing both prices.

Let's denote the profit function in the "all search" candidate equilibrium as Π^{AS}. The following equation gives the solution for $p_w^*(p_s^*)$:

$$p_w^*(p_s^*) = p_s^* - \Delta_p, \text{ where price difference } \Delta_p \text{ satisfies} \tag{1.8}$$
$$\Pi^{AS'}_{p_s} = \pi(1 - F[\Delta_p]) - \pi \Delta_p F'[\Delta_p] = 0$$

This equilibrium exists only if $p_w \geq \frac{s}{1-\pi}$. Notice that when search cost goes to zero, this candidate equilibrium exist, as the firm can always charge $p_w > 1 - \pi - \frac{s}{\pi} > \frac{s}{1-\pi}$ to satisfy the first order condition for the BMS price.

$$\lim_{s \to 0} \Pi^{AS'}_{p_s}|_{p_s = 1 - \frac{s}{\pi}, p_w = 1 - \pi - \frac{s}{\pi}} = -\pi^2 F[\pi]$$

At the same time when s goes to $\pi(1-\pi)$, this candidate equilibrium does not exist as $\frac{\pi - s}{\pi}$ approaches to $\frac{s}{1-\pi}$ and Δ_p is strictly positive and bounded as follows from the solution for online price $p_w^*(p_s^*)$.

Third, look at the candidate equilibrium with showrooming. It does not exist for any search cost above $\pi(1-\pi)$, as condition $p_s > \frac{s}{1-\pi}$ is not satisfied for the highest possible BMS price $\frac{\pi - s}{\pi}$. Thus, we consider it only for the range of search cost $s \leq \pi(1-\pi)$. Analogously to the previous case, it requires that $\Pi'_{p_s}|_{p_s^*} = 0$. The solution for the optimal online price $p_w^*(p_s^*)$ is

$$p_w^*(p_s^*) = p_s^* - \Delta_p, \text{ where price difference } \Delta_p \text{ satisfies}$$
$$\pi - \pi F[\Delta_p] - (\pi \Delta_p - \eta)F'[\Delta_p] = 0$$

Notice that the profit maximization problem may have an interior solution for both prices, if there exists $\tilde{p}_w$ which satisfies

$$(1-\pi)F[\frac{s}{1-\pi} - \tilde{p}_w] + \pi F[\Delta_p] - ((1-\pi)\tilde{p}_w + \eta)F'[\frac{s}{1-\pi} - \tilde{p}_w] + \pi \Delta_p F'[\Delta_p] = 0$$
$$\text{and } \tilde{p}_w < \frac{s}{1-\pi}, \tilde{p}_w + \Delta_p < \frac{\pi - s}{\pi}.$$

Otherwise there is a boundary solution. Taking into account the price constraint from Lemma 1, we get that the optimal solution is

$$p_w^* = \max\left\{\frac{s}{1-\pi}, \min\left\{\tilde{p}_w, \frac{\pi - s}{\pi} - \Delta_p\right\}\right\}, p_s^* = p_w^* + \Delta_p. \tag{1.9}$$

This candidate equilibrium exists for the whole range of search costs $s \in [0, \pi(1-\pi)]$.

Now we do pairwise comparison of profits to derive the equilibrium outcome.

1. If $s > \pi(1-\pi)$, the firm can not sell in the BMS in the equilibrium, so it has to charge online price p_w^*, such that it satisfies

$$p_w^* = \frac{F[\pi - p_w^*]}{F'[\pi - p_w^*]} \tag{1.10}$$

   Log-concavity of function $F[\cdot]$ guarantees the existence of the interior solution. There are only online sales.

2. For the search cost s below $\pi(1-\pi)$ we compare showrooming strategy and only online sales. The choice depends on cost of providing customer service and shape of function $F[\cdot]$. So we can define some threshold s_2 from the profit equality

$$F[\pi - p_w^{On}]p_w^{On} = \pi p_s^{Sh} - \eta + (\pi(-p_s^{Sh} + p_w^{Sh}) + \eta)F[p_s^{Sh} - p_w^{Sh}] + (p_w^{Sh} - \pi p_w)F[-p_w^{Sh} + \frac{\tilde{s}}{1-\pi}] \tag{1.11}$$

$$\Rightarrow \tilde{s}, s_2 = \min\{\tilde{s}, \pi(1-\pi)\},$$

   where p^{On} corresponds to p_w^* defined in equation(1.10), and p_s^{Sh}, p_w^{Sh} correspond to $p_s^*|_{s=\tilde{s}}, p_w^*|_{s=\tilde{s}}$ defined in equation (1.9), $\tilde{s}$ is defined as a search cost which satisfies the profit equality condition. is For search cost above s_2 the firm prefers to sell only online, and for search cost below s_2 it gets higher profit in the case of showrooming than in the case of online sales only.

3. Now we compare profits in the case of showrooming and "all search". Again we get the threshold s_1 from the following profit equality

$$\begin{aligned}&\pi p_s^{Sh} - \eta + (\pi(-p_s^{Sh} + p_w^{Sh}) + \eta)F[p_s^{Sh} - p_w^{Sh}] + (p_w^{Sh} - \pi p_w)F[-p_w^{Sh} + \frac{\tilde{s}}{1-\pi}] = \\ &= -\eta + \pi p_s^{AS}(1 - F[p_s^{AS} - p_w^{AS}]) + \pi p_w^{AS}F[p_s^{AS} - p_w^{AS}] \Rightarrow \tilde{s}, s_1 = \max\{0, \tilde{s}\},\end{aligned} \tag{1.12}$$

   where p_s^{Sh}, p_w^{Sh} correspond to $p_s^*|_{s=\tilde{s}}, p_w^*|_{s=\tilde{s}}$ defined in equation (1.9) and p_s^{AS}, p_w^{AS} correspond to $p_s^*|_{s=\tilde{s}}, p_w^*|_{s=\tilde{s}}$ defined in equation (1.8), $\tilde{s}$ is defined as a search cost which satisfies the profit equality condition. For search cost above s_1 showrooming is better than "'all search" for the firm. For search cost below s_1 the firm prefers to induce "all search".

4. The last we compare profits in the case of "all search" and online sales only. Analogously to previous two case we derive threshold s_3, such that

$$\begin{aligned}&p^{On}F[\pi - p^{On}] = -\eta + \pi p_s^{AS}(1 - F[p_s^{AS} - p_w^{AS}]) + \pi p_w^{AS}F[pAS_s - p_w^{AS}] \\ &\Rightarrow \tilde{s}, s_3 = \max\{0, \tilde{s}\},\end{aligned} \tag{1.13}$$

   where p^{On} corresponds to p_w^* defined in equation (1.10) and p_s^{AS}, p_w^{AS} correspond to $p_s^*|_{s=\tilde{s}}, p_w^*|_{s=\tilde{s}}$ defined in equation (1.8), $\tilde{s}$ is defined as a search cost which satisfies the profit equality condition.

□

Proofs Competition:

Proof of Lemma 6. We have proved that online prices are equal to zero in the equilibrium. Both firms get zero profits by setting zero store prices and by setting store prices equal to $\frac{\pi-s}{\pi}$, as zero measure of consumers buys in the store in this case. As firms are symmetric, they should charge the same store prices in the equilibrium, so we will derive the optimal store price for some firm j, where $j = \{A, B\}$.

If firm j sets price $p_s^j < \frac{s}{1-\pi}$, then some consumers will buy directly online. So profit of the firm equals $\pi p_s^j(1 - F[\pi p_s^j + s])$. The first order condition of the profit maximization problem is

$$\pi(1 - F[\pi p_s^j + s]) - \pi^2 p_s^j F'[\pi p_s^j + s] = 0. \tag{1.14}$$

If $p_s^j = \frac{\pi-s}{\pi}$, then the derivative of profit function is negative. Hence, if the firm wants to induce "segregation" it charges $\min\left\{\frac{s}{1-\pi}, p^*\right\}$, where p^* is the store price which satisfies equation (1.14).

Firm j can also induce showrooming if it charges $p_s > \frac{s}{1-\pi}$. Profit of the firm is equal to $\pi p_s^j(1 - F[p_s^j])$. The first order condition of the profit maximization problem is

$$1 - F[p_s^j] - \pi p_s^j F'[p_s^j] = 0. \tag{1.15}$$

If $p_s^j = \frac{\pi-s}{\pi}$ and $\frac{\pi-s}{\pi} > \frac{s}{1-\pi}$, then the derivative of profit function is negative. So the interior solution of equation (1.15) is always below $\frac{\pi-s}{\pi}$. Therefore, if the firm wants to induce showrooming, then it charges $\max\left\{\frac{s}{1-\pi}, p'\right\}$, where p' is an interior solution of equation (1.15).

For any search cost $s > \pi(1-\pi)$ the firm cannot induce showrooming as $\frac{\pi-s}{\pi} < \frac{s}{1-\pi}$. So in this case there is a segregation and $p_s^j = p^*$.

When search cost equals zero, firm j prefers induce showrooming as $\frac{s}{1-\pi} = 0$, so it charges price $p_s^j = p'$.

Notice that the derivative of the profit function at $p_s^j = \frac{s}{1-\pi}$ in the case of showrooming is not higher than in the "segregation" case, and profit is continuous on the whole interval of possible prices. Therefore, the profit function is continuous and single-picked. So we can conclude that there exists a unique threshold $0 < \tilde{s} < \pi(1-\pi)$, such that for any $s < \tilde{s}$ firms charge in the equilibrium $p_s^A = p_s^B = \max\left\{\frac{s}{1-\pi}, p'\right\}$, and for $s > \tilde{s}$ they charge $p_s^A = p_s^B = p^*$. □

Proof of Lemma 7. Suppose that firm A charges expected online price $\tilde{p}_w^A$ in subgame equilibrium when firm B does not open an online shop. Suppose that if firm B opens an online shop and geo-blocks then firm A draws an online price form support $[\underline{p}, \overline{p}]$ according to cumulative probability distribution function $G(p)$ in subgame equilibrium. We can show that $\int_{\underline{p}}^{\overline{p}} p dG(p) > \tilde{p}_w^A$.

We know that if firm i plays mixed strategy then it gets the same expected profit for any price set with positive probability. Now suppose that firm B opens online shop and geo-blocks. It does not sell online when it charges price p_w^B at the upperbound of the equilibrium price support. Thus it gets the profit only from in-store sales. The profit function of firm B which charges $p_B^w = \overline{p}$ and $p_s^B(\overline{p})$ is

$$\int_{\underline{p}}^{\overline{p}} \pi \left(1 - \tilde{F}\left[p_w^A, p_s^B(\overline{p})\right]\right) dG(p_w^A),$$

where function $\tilde{F}$ is either $F[p_s^B(\overline{p}) - p_w^A]$ or $F[\pi p_s^B(\overline{p}) - p_w^A + s]$ depending on $p_s^B(\overline{(p)})$. We have already shown that function $\tilde{F}[\cdot]$ is continuous in p_w^A. If $F[x]$ is convex then $F''[x] > 0$, and $(1 - F[x])''_x = -F''[x] < 0$. So we can apply Jensen's inequality and thus

$$\int_{\underline{p}}^{\overline{p}} \pi \left(1 - \tilde{F}\left[p_w^A, p_s^B(\overline{p})\right]\right) dG(p_w^A) < \pi \left(1 - \tilde{F}\left[E(p_w^A), p_s^B(\overline{p})\right]\right)$$

If the expected price of the competitor is below $\tilde{p}^A_w$ then it is not profitable for firm B to open the online shop at first place. Thus firm B opens an online shop and geo-blocks only if $\int_{\underline{p}}^{\overline{p}} pdG(p)$.

As market is fully covered in equilibrium we can conclude that consumer surplus in market A is higher when firm B does not open an online store and geo-block. □

Proof of Lemma 8. We can show that for sufficiently high search cost firms play mixed strategies. First of all, we can notice that for any price $p^A \in [0, \frac{\pi-s}{\pi}]$ condition $p^A < \frac{s}{1-\pi}$ is satisfied if $s > \pi(1-\pi)$. The firm which charges higher price does not charge price above $\frac{\pi-s}{\pi}$, because otherwise it has zero profit as it does not sell neither online nor in the store. Suppose, that there is an equilibrium in pure strategies $(\tilde{p}^A, \tilde{p}^B)$, then as we have already shown the following condition should be satisfied in the equilibrium:

$$\Pi^{A'}{}_{p^A}|_{\tilde{p}^A,\tilde{p}^B} = \pi + F[\pi\tilde{p}^B - \tilde{p}^A + s] + (1-\pi)F[s-(1-\pi)\tilde{p}^A] - \tilde{p}^A F'[\pi\tilde{p}^B - \tilde{p}^A + s] -$$
$$(1-\pi)^2\tilde{p}^A F'[s-(1-\pi)\tilde{p}^A] = 0,$$

where $\tilde{p}^A < \tilde{p}^B$. At the same derivative of the profit Π^B w.r.t. price p^B should be equal to zero if $\tilde{p}^B < \frac{\pi-s}{\pi}$, or not negative if $\tilde{p}^B = \frac{\pi-s}{\pi}$.

$$\Pi^{B'}{}_{p^B}|_{\tilde{p}^A,\tilde{p}^B} = \pi(1-F[\pi p^B - p^A + s]) - \pi p^B F'[\pi p^B - p^A + s]) = 0$$

We can show that for sufficiently high s, these two equations can not be satisfied at the same time for $p^A < p^B$. When s goes to π, profit of the firm A goes to zero, at the same it can get strictly positive profit by selling online at the price $\frac{\pi-s}{\pi} < p^A < \pi$ (obviously, this set is non-empty when $s > \pi(1-\pi)$). Thus firm A has no incentives to play any prices below $\frac{\pi-s}{\pi} \geq p^B$. Thus, we excluded all possible pure strategy candidate equilibria, and firms have to mix in the equilibrium.

□

Chapter 2

Free-riding and word-of-mouth communication

2.1 Introduction

During last years the popularity of “word-of-mouth” marketing campaigns has significantly increased. The main idea behind them is that a firm can use targeted advertising and stimulate consumers to disseminate information about the product in the market instead of a traditional approach of mass advertising where the firm focuses only on direct communication with consumers. We can see how the process of information diffusion works in social networks, i.e. propagation of photos, videos, and articles on Facebook. We try to share an interesting information with our friends, check special sites where we can get some feedback about products before shopping, constantly interact with different people in a variety of social networks. Communication becomes cheaper and more efficient because of new technologies and their availability. In this new environment, the idea to use these highly intense social interactions in order to reduce advertising cost and to put a part of them on consumers became highly exploited. However, in reality, we can see that some "word of mouth" campaigns are successful while others fail. The possible explanation is that a decision of recommendation for consumers is not mechanistic, as it was typically considered in the past by economic literature, but strategic and depends on different factors. We can imagine that a consumer usually wants to tell to his friends about a new car or to recommend an interesting book, but almost nobody wants to inform peers about a new brand of milk or pencils. This means that consumers have different incentives to talk about different categories of goods. The fact that consumers do not want to recommend some products even though they are satisfied with them means that recommendations are costly. The decision whether to communicate some information about the product to peers depends on benefits which consumers get from giving recommendations.

In this paper, we propose a model of information diffusion where consumers share information about a new technology. This technology has some positive network externalities. So the more consumers use the same technology the higher benefits they get. We focus on dynamic information diffusion process, where some consumers are initially informed by a firm and others obtain information through a communication process. We assume that all consumers who get information about the new technology adopt it at zero price. The good examples of this kind of a product are different mobile applications and software which you can get for free for a trial period(i.e. "What’s app", "Dropbox"). A decision to recommend the product is strategic, as on one side communication is costly and on the other side, consumers are interested in informing their peers. So each consumer decides how much effort he wants to put into recommending the technology to friends over time. The decision about a recommendation in each period affects a number of informed consumers not only in current but also in future periods. A consumer benefits from the high number of informed peers in two ways: he gets a higher payoff; he needs to put less effort to recommend in the future as other informed agents will start to recommend the product. We consider a dynamic model in discrete time, so consumers cannot observe decisions of others in the current period, only post factum, as a result, that creates incentives to free-ride.

Each consumer instead of making a recommendation can hope that other consumers will do that instead of him. In some sense, we have the model where information is a public good.

We show the existence of the unique symmetric equilibrium in the model where consumers maximize their individual payoffs and compare results with the case where informed consumers coordinate their efforts and with the case where the social planner assigns to consumers optimal recommendation efforts. We show there is a significant free-riding effect which is non-monotone in the proportion of the uninformed population. In the case where uninformed consumers can cooperate they optimally choose to put equal recommendation efforts only if recommendation cost is sufficiently high, otherwise, it is optimal to choose asymmetric recommendation efforts. We consider a strategy of the firm whose marketing approach combines word-of-mouth marketing with a direct advertising and provide a numerical analysis which shows that the free-riding effect can significantly decrease an efficiency of direct advertising, especially if recommendation cost is low. In the case of sufficient information diffusion the free-riding effect makes the outcome of direct advertisements being almost negligent, which makes it reasonable for the firm to stop direct advertising and to rely on word-of-mouth communication. The effect is stronger for lower recommendation cost.

Our paper is related to the literature on information diffusion in social networks and advertising. In contrast to papers which consider models of dynamic pricing in social networks with non-strategic communication decisions (see Campbell (2013),El Ouardighi et al. (2015)), we focus basically on information dissemination itself in the networks with strategic agents keeping the question of optimal pricing aside. The models of dynamic information diffusion do not take into account cost associated with social interactions, and thus decisions to exchange some information become purely mechanistic. Obviously, this can not explain well an S-shape growth of a new product users audience in highly connected societies when the product has positive network externalities.

There is an increasing number of papers which focus on communication networks where consumers make a strategic communication decisions as social interactions are associated with some costs and benefits. The example of the marketing strategy relying on network structure is considered in Mandjes (2003). The author analyses how a firm can discriminate consumers based on their decisions of assigning into networks with a different class of priority. This paper is also focused on the problem of information diffusion when consumers have direct incentives to propagate the information. A decision to recommend is purely strategic and incentives to recommend are created by positive network externalities. There is a part of literature which focuses on equilibria in networks where payoff of each agent depends on strategies of his neighbor (e.g. Bloch and Querou (2013)), and optimal strategies and equilibria depend on the particular characteristic of a network. The problem of free-riding in networks is analyzed by Bramoulle and Kranton (2007) where authors consider a different type of equilibria which depend on particular network structure. The main difference from these papers is that they focus on static equilibria, while we consider how a decision of consumers to recommend and a number of informed consumers evolute during the time.

This paper is also related to the literature on dynamic advertising (i.e. Doganoglu and Klappere (2006)) and marketing strategies which create incentives for consumers to disseminate information about products. In Arbatskaya and Konishi (2012) it is shown that in the model where a monopolist chooses the price, advertising intensity and a referral fee, referrals lead to higher equilibrium profits and for the certain range of parameters lead to Pareto improvements. The model with referrals and consumers embedded in a small world networks is considered in Tackseung and Jeong-Yoo (2008). Authors provide the numerical simulations and show how effectively a firm uses underlying social network depending on referral cost.

The rest of the paper organized as following - in section II we formulate the model, in section III we show the existence of the solution of the optimization problem and describe the optimal algorithm for the solution. In section IV we consider properties of equilibria where consumers coordinate their efforts. In section V we provide numerical results and comparative statics. Section

VI gives some insights regarding optimal advertising strategies. In the last section we discuss results and make concluding remarks.

2.2 Model

We assume that there is one firm in a market which launches a new product. There is a finite number n of consumers. Information about existence of the new product can reach consumers in two ways: the firm can communicate information about the product to consumers, or consumers may recommend the product to their peers. The firm decides on optimal advertising effort in each period. This effort is characterized by probability w that information reaches each uninformed customer. Direct advertising is costly and cost function is $C(w)$. We assume that the cost function is convex, increasing in w and equal to zero when the firm does not use direct advertising, so $C(0) = 0$, $C'(w) > 0$, $C''(w) > 0$. Consumer who get informed about the product can adopt it at zero cost.

Suppose that in the first period there are some initial consumers who are directly informed by a firm about the new technology. The technology is designed such that after adopting it they can observe who are current users [1], so consumers can always observe who are already informed and who are not. At each stage informed consumers decide to how many of their friends they want to recommend the new product. For simplicity we assume that consumers make a decision not about exact number of recommendations but about probability to inform each uninformed consumer, so then this probability multiplied by number of uniformed consumers gives us exactly number of expected recommendations made by a particular consumer in the current period. Costs of recommendation characterize how much effort a consumer is ready to put to be sure that information reaches his neighbor. They are convex and equal to $C(\rho, k_t) = c\tilde{\rho}_t^2 k_t^2$, where k_t is a number of uninformed consumers at time t, $\tilde{\rho}_t$ is a probability to recommend and c is a positive coefficient. So recommendation cost is convex in the expected number of informed friends.

We suppose that at any period each consumer has a private benefit γ from communication with any other consumer. So if at period t there are m_t informed consumers then each of them gets a private benefit $(m_t - 1)\gamma$ in this period. There is a discount factor δ, so consumers are not fully patient and they are interested in a fast information diffusion process. We assume that there is no price for buying the new technology.[2]

The objective function of consumers is to maximize the expected future payoffs. The objective function of the firm is to maximize the number of informed agents (users) in each period.

Timing in the model is the following:

1. At the first stage some initial consumers get information about a technology, they adopt the new technology and make a decision about a probability to recommend it to their peers in the current period.

2. In the next period newly informed consumers start to use the technology. All informed consumers make their decisions about probability to recommend the product.

3. ...

4. When all consumers are informed, there is no more recommendations in the game, consumers continue to get their private benefit by consuming the new product. Information diffusion process stops.

[1]We can think here about different examples of mobile application or computer software. A person who installs WhatsApp applications automatically learns who many of his mobile contacts use the same application. People who use Facebook and other social networks now how many of their friends are registered in the same network.

[2]The typical examples are WhatsApp, Skype and Dropbox. All these products were monetized either by providing supplementary paid services or by introducing some payments after a long free of charge probation period when sufficient number of customers were already active users.

2.3 Non-strategic firm

We start our analysis with the situation when the firm does not actively participate in the market. Some initial consumers get for free[3] information about the product and start to recommend it to their peers. In this section we consider some properties of information diffusion process and establish general results about existence of equilibrium and presence of the free-riding effect.

2.3.1 Information Diffusion Process

Let us consider a consumer who makes a decision to recommend the product in period t. His expected utility function V depends on the expected number of product's users in period $t+1$ - m_{t+1}, and recommendation cost $C(\rho_{t,i})$. Hence he gets in period $t+1$ a payoff equal to $E(m_{t+1}-1)-C(\rho_{t,i})$. In addition the consumer has also to take into account all discounted future payoffs. So we can write the utility function in period t as

$$V_i(m_t) = \gamma E(m_{t+1}-1) - C(\rho_{t,i}) + \delta E(V_i(m_{t+1})) \tag{2.1}$$

We can notice the recommendation decision of the consumer in a current period is not affected by previous history and depends only on the current state. In other words, information diffusion process is a Markov process with the state variable m_t. Therefore we should solve for the steady state in equilibrium $\rho_t^* = \rho_t^*(m_t)$. As all informed nodes are symmetric in any period we will focus only on symmetric Nash equilibria.

Let us to consider the main equations which describe information diffusion process. Given total number of consumers n as known, state variable uniquely determines number of uninformed consumers $k_t = n - m_t$. So we have the following dynamic problem for a node i:

$$\begin{aligned} & V_i(m_t) = \gamma E(m_{t+1}-1) - C(\rho_{t,i}) + \delta E(V_i(m_{t+1})) \\ & k_{t+1} = n - m_{t+1} \\ & m_{t+1} = m_t + M(\rho_{t,i}|\sigma) \\ & E\left(m_{t+1}\right) = m_t + E\left(M(\rho_{t,i}|\sigma)\right) \\ & \max_{\rho_{t,i}} V_i(m_t) \end{aligned} \tag{2.2}$$

where $M(\rho_t)$ is the function which determines number of newly informed nodes given that all informed nodes play strategy σ.

$$E(M(\rho_t)) = (1-(1-\rho_{t,\sigma_{-i}})^{m_t-1}(1-\rho_{t,i}))k_t \tag{2.3}$$

Strategy of each consumer is a mapping from the set of possible number of uninformed consumer [4] to an interval from 0 to 1 of probabilities to recommend $\sigma_n : \mathbb{N} \mapsto [0,1]$. In symmetric Nash equilibrium we should have:

$$\rho_{t,i} = \rho_{t,j}, \forall i,j \tag{2.4}$$

As we have a Markov process, the optimal $\rho_{t,i}^*(m_t)$ for each consumer does not depend on time but only on the number of informed consumers. So we can skip the time index for the further analysis, $\sigma_n^* = \rho^*(1), ..., \rho^*(n)$.

2.3.2 Algorithm

In this subsection we provide an algorithm to solve dynamic programming problem described above. The state variable in our model is m - number of informed nodes. As there is

[3]These consumers can be considered as shoppers who like to search for the information about new products.

[4]It depends on the model parameters including total number of consumers n, which is considered as exogenously given. The same number of informed consumers for different n's will result in different equilibrium strategies.

a finite number of possible states and each state is defined by $m \in \mathbb{N}$, that means that we have a finite dynamic problem in discrete time. For the rest of the paper we will focus on a solution of our model by optimizing consumer's decision at each state using backward induction. We start backward induction from the last step when $m = n$. The continuation value for each node is

$$V = \gamma(n-1) + \delta V \Rightarrow \frac{\gamma(n-1)}{1-\delta}. \tag{2.5}$$

In state m we can compute the probability to recommend $\tilde{\rho}$ and the continuation value as following:

$$V(m) = \sum_{j=0}^{n-m} ((m+j-1)\gamma + \delta V(m+j))Prob_j(m) - ck^2\tilde{\rho}^2 \tag{2.6}$$

$$Prob_j(m) = P^j(1-P)^{k-j}\frac{k!}{j!(k-j)!} \text{ - probability that } j \text{ nodes will be informed}$$

$$P \equiv (1-(1-\rho)^{m-1}(1-\tilde{\rho}))$$

So we have a transition matrix of probabilities to get from state m to state $m + j$ for any possible j which is given in Table 2.1.

m_t \ m_{t+1}	n	n-1	n-2	...	1
n	1	0	0	...	0
n-1	$Prob_1(n-1)$	$Prob_0(n-1)$	0	...	0
n-2	$Prob_2(n-2)$	$Prob_1(n-2)$	$Prob_0(n-2)$	...	0
...	...	...	...	...	...
2	$Prob_{n-2}(2)$	$Prob_{n-3}(2)$	$Prob_{n-4}(2)$	...	0
1	$Prob_{n-1}(1)$	$Prob_{n-2}(1)$	$Prob_{n-3}(1)$	...	$Prob_0(1)$

TABLE 2.1: Transition matrix of probabilities

Now we consider the penultimate step of the algorithm. The system is in state $m = n - 1$. The continuation value of the game for each informed consumer is $V(n-1)$ which we can compute as following:

$$V(n-1) = (1-(1-\rho)^{n-2}(1-\tilde{\rho}))\frac{\gamma(n-1)}{1-\delta} + (1-\rho)^{n-2}(1-\tilde{\rho})(\gamma(n-2) + \delta V(n-1)) - ck^2\tilde{\rho}^2 \tag{2.7}$$

$$P \equiv (1-(1-\rho)^{m-1}(1-\tilde{\rho})) \tag{2.8}$$

$$V(n-1) = P\frac{\gamma(n-1)}{1-\delta} + (1-P)\gamma(n-2) + \delta V(n-1)(1-P) - ck^2\tilde{\rho}^2 \tag{2.9}$$

$$V(n-1)(1-\delta(1-P)) = P\frac{\gamma(n-1)}{1-\delta} + (1-P)\gamma(n-2) - ck^2\tilde{\rho}^2$$

$$V(n-1) = \frac{\frac{P\gamma(n-1)}{1-\delta} + (1-P)\gamma(n-2) - ck^2\tilde{\rho}^2}{1-\delta(1-P)}$$

Now we have to maximize $V(n-1)$ from equation (2.7) with respect to $\tilde{\rho}$, considering future payoff as given for an agent, and replace $\tilde{\rho}$ by ρ in F.O.C. as we focus on symmetric equilibria.

F.O.C.:

$$\max_{\tilde{\rho}} V(n-1) \Rightarrow \tag{2.10}$$

$$(V(n-1))'_{\tilde{\rho}} = \frac{-2c\tilde{\rho}(-1+\delta)\rho^2 + (\gamma + (-2+n)\gamma\delta + V(n-1)(-1+\delta)\delta)\left(-\rho^2+\rho^n\right)}{(-1+\delta)\rho^2} = 0$$

$$\rho \to \tilde{\rho}: \frac{-2c(-1+\delta)\rho^3 + (\gamma + (-2+n)\gamma\delta + V(n-1)(-1+\delta)\delta)\left(-\rho^2+\rho^n\right)}{(-1+\delta)\rho^2} = 0$$

equation (2.11) $\to$ equation (2.7): (2.11)

$$c(2-\rho)\rho + \frac{\gamma}{\delta} - \frac{2c\rho}{(1-\rho^{-2+n})\delta} = 0$$

We have to prove that the solution of the last equation exists and it is unique. We prove the result in Lemma 9.

Lemma 9. *There exists is a unique solution for recommendation probability $\rho(n-1)$ in symmetric Nash equilibrium.*

Proof of Lemma 9 and all other omitted proofs are in the appendix.

So we have the unique optimal $\rho^*(n-1)$ which maximizes an individual payoff in the symmetric equilibrium. Now we have to establish that $V(n-1) < V(n)$.

$$V(n-1) = \frac{\frac{P\gamma(n-1)}{1-\delta} + (1-P)\gamma(n-2) - ck^2\tilde{\rho}^2}{1-\delta(1-P)} < \frac{\frac{P\gamma(n-1)}{1-\delta} + (1-P)\gamma(n-1)}{1-\delta(1-P)} =$$

$$= \frac{\gamma(n-1)}{1-\delta} = V(n)$$

These results are important for the further analysis as they allow to establish some properties of the solution at the penultimate step of the algorithm.

2.3.3 Symmetric Nash Equilibrium

In this section we will prove existence of the symmetric equilibrium in pure strategies for sufficiently high recommendation costs. In order to show our main results we need the monotonicity property of function $V(m)$. So we assume that $V(m)$ is increasing function in m. Here the logic of backward induction is applied. We know that $V(n-1) < V(n)$. So the property is true for $m \geq n-1$. Thus we can prove uniqueness and existence of a solution for $m = n-2$ given sufficiently high recommendation cost. Then we show that $V(n-2) < V(n-1) < V(n)$. The same procedure is applied for all other steps.

Let us rewrite the optimization problem as following:

$k = n - m$

$C_k^j P^j (1-P)^{k-j}$ - probability to inform j nodes

$$V(m) = \sum_{j=0}^{n-m}\left((\gamma(m+j-1) + \delta V(m+j))P^j(1-P)^{k-j}\frac{k!}{j!(k-j)!}\right) - ck^2\tilde{\rho}^2$$

$$V(m) = \frac{\left(\gamma(m-1)(1-P)^k + \sum_{j=1}^{n-m}\left((\gamma(m+j-1)+\delta V(m+j))P^j(1-P)^{k-j}\frac{k!}{j!(k-j)!}\right)\right) - ck^2\tilde{\rho}^2}{1-\delta(1-P)^k}$$

As we have a Markov process $V(m+j)$ does not depend on $\tilde{\rho}$ for any m, j. The probability to recommend $\tilde{\rho}$ affects only recommendation cost and probability P. So we can rewrite the last equation as follows:

$$V(m) = \sum_{j=0}^{n-m} A_j C_k^j P^j (1-P)^{k-j} - ck_t^2 \tilde{\rho}^2,$$

where

$$A_j \equiv (\gamma j + \delta V(m+j)), j > 0$$
$$Prob_j = \frac{k!}{j!(k-j)!} P^j (1-P)^{k-j}, j > 0.$$

Now we can take a derivative with respect to $\tilde{\rho}$ for $k \geq 2$ (optimal solution for the state when $k = 1$ is considered above) and write the first order condition.

$$\sum_{j=0}^{n-m} A_j \{Prob_j\}'_{\tilde{\rho}} = 2ck_t^2 \tilde{\rho}$$

In Lemma 10 we prove that an individual maximization problem has always a unique solution for probability to recommend $\tilde{\rho}$ when all other consumers recommend with the same probability equal to ρ.

Lemma 10. *For sufficiently high recommendation cost the solution of the individual optimization problem exists and it is unique.*

Existence of the unique solution of the individual maximization problem guarantees us that there is the unique best response strategy for each consumer who believes that all his peers play the same pure strategy. Therefore if we can find fixed point ρ^* such that $\tilde{\rho}(\rho^*) = \rho^*$, then we get a symmetric Nash equilibrium. The next theorem establishes the result.

Theorem 1. *(Existence and uniqueness of symmetric Nash equilibrium) The dynamic programming problem described by the system of equations (2.2) always has a symmetric solution, which is a Nash equilibrium in pure strategies, and it is unique for sufficiently high level of recommendation costs.*

Now we can show that the value function is decreasing in k if consumers play optimal symmetric equilibrium strategies. The result is quite intuitive and means that the more consumers are informed the higher expected future payoff is. So it is always better for consumers when more of their peers are informed.

Proposition 5. *Value function $V(k)$ is decreasing in k.*

This property says us that there is no failure of coordination such that with higher number of informed consumers expected payoff of each consumer decreases. In other words free-riding effect never outweighs benefits of higher number of participants.

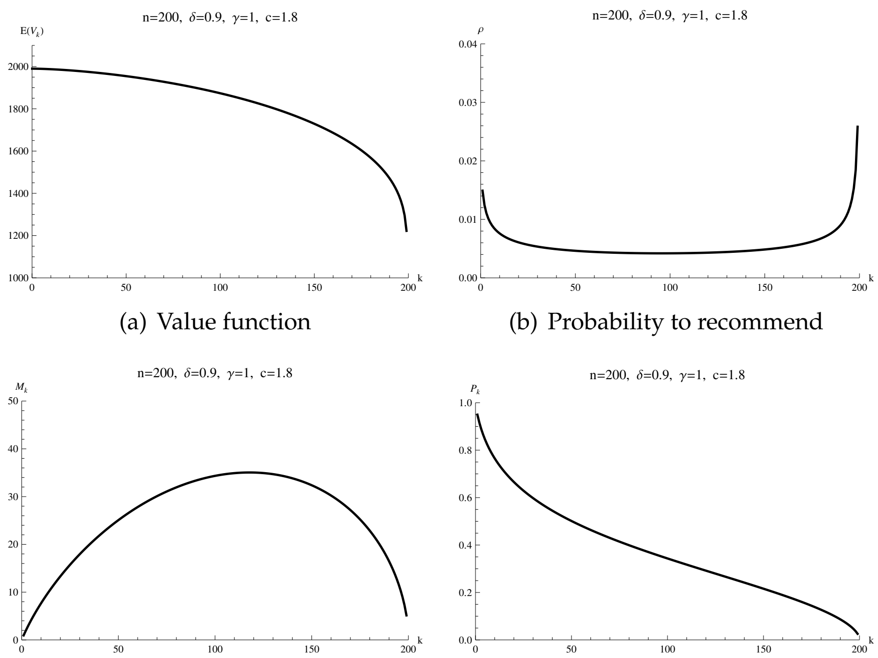
Figures below illustrate the solution of the model and NE, where we can see how expected value, probability to recommend, probability to be informed and expected number of newly informed people depend on state described by the number of uninformed consumers.

Figure 2.1 illustrates how the payoff function depends on number of uninformed consumers. So we can see that it is monotonic and decreasing which means that the more consumers are informed at time t the greater payoff each of them will get, and this property is not distorted by free-riding effect. It also illustrates how expected number of newly informed nodes depends on number of uninformed nodes. We have a non-monotonic function, because the less uninformed nodes we have the less nodes can be potential informed, but the higher probability that information will reach each uninformed node.

The next property follows from the proof of Theorem 1:

Corollary 2. *Probability that all consumers are informed is equal to 1 when $t \to \infty$*

So we can conclude that optimal recommendation effort is bounded below for any n and m.



(a) Value function
(b) Probability to recommend
(c) Informed nodes
(d) Probability to be informed

FIGURE 2.1: NE: Characteristics

2.4 Cooperation

In this section we analyze the optimal recommendation strategy of consumers when they can cooperate and coordinate their actions. We can think about a mediator who decides on probabilities to recommend, and all informed agents follow this decision. We will focus mainly on symmetric strategy to compare with results of Nash equilibrium. First of all we have to discuss the optimal choice of the mediator between symmetric policy, when the same probability to recommend is assigned to each consumer, and asymmetric policy when different probabilities to recommend are assigned to different consumers. Then we will provide some results for a symmetric cooperation strategy, and at the end we will make some comments on the social planner problem. The reason to distinguish between cooperation and the social planner cases is that in the first case we just eliminate free-riding effect on communication. So informed agents cooperate to take the optimal decision. In the second case the social planner also cares about future payoffs of uninformed agents. We will refer to these cases as "the mediator problem" and "the social planner problem".

2.4.1 Symmetric vs. Asymmetric Cooperation Strategy

We start the analysis with the choice of the optimal policy for the mediator and establish when the mediator prefers a symmetric policy and an asymmetric policy. Let us to consider the optimal choice of the cooperation policy at state $m = n - 1$ when there is only one uninformed consumer. We can derive a threshold of recommendation cost, such that the mediator prefers to choose a symmetric strategy for the cost above this threshold and asymmetric below:

$$\bar{c} = -\frac{2(-1+n)\gamma(1-\rho)^n}{\delta(1-\rho)^n - (-1+\rho)\left(-1+(-1+n)\rho^2\right)}$$

On Figure 2.2 the blue line shows the optimal level of recommendation effort depending on recommendation cost. The magenta line shows the level of cost depending on recommendation effort when the value function at state $m = n - 1$ is the same for both cooperation policies -

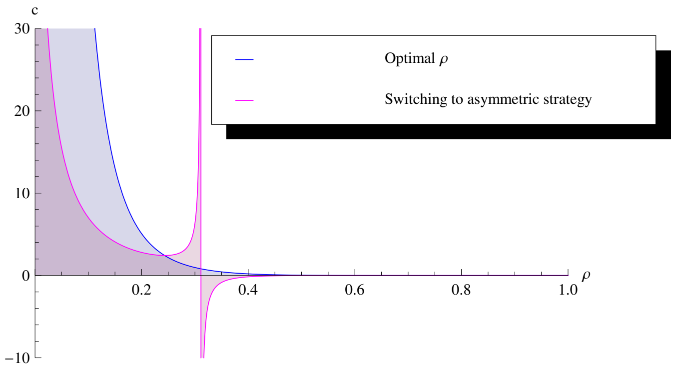


FIGURE 2.2: Partition of optimal symmetric and asymmetric strategies

symmetric and asymmetric. The intersection of blue and magenta lines shows the threshold on recommendation cost such that above this level the mediator chooses symmetric policy where all consumers recommend with an equal probability. We will prove this result formally through several steps.

First of all, we can show that the mediator never chooses to use symmetric policy with $\rho > \frac{1}{2}$. Suppose he chooses the same recommendation effort ρ for all consumers. Then probability that information reaches an uninformed agent is $P = 1 - (1-\rho)^m$. The cost function is proportional to the sum $\sum_{i=1}^{m} \rho^2$ with some positive coefficient. Let's consider whether the mediator can do better by choosing an asymmetric recommendation policy. Let us consider an example where two randomly chosen agents recommend with probabilities ρ_1 and ρ_2. When these probabilities are chosen such that $(1-\rho_1)(1-\rho_2) = (1-\rho)^2$, probability P does not change. So we can solve for the optimal choice of ρ_1 and ρ_2 such that P is fixed.

$$(1-\rho_1)(1-\rho_2) = (1-\rho)^2$$
$$\min_{\rho_1,\rho_2} \rho_1^2 + \rho_2^2 - 2\rho^2$$
$$\text{Set of solutions: } \{\rho_1 = \rho, \rho_2 = \rho\}, \{\rho_1 = \frac{1}{2}\left(1 - \sqrt{1-4(1-\rho)^2}\right),$$
$$\rho_2 = \frac{1}{2}\left(1 + \sqrt{1-4(1-\rho)^2}\right)\}$$

So when $\rho < \frac{1}{2}$ the mediator chooses either symmetric policy $\rho_1 = \rho_2 = \rho$ (if optimal $P < 1$) or asymmetric $\rho_1 = 1, \rho_2 = 0$ (if optimal $P = 1$). When $\rho > \frac{1}{2}$ the optimal strategy is to choose $0 < \rho_1 \neq \rho_2 < 1$. We know that the optimal symmetric cooperation strategy is such that the mediator chooses the highest probability to recommend $\rho(m)$ in state $m = n-1$. So we need to derive a condition on c such that it guarantees that *i)* the optimal choice of ρ under the symmetric strategy is less than one half; *ii)* the mediator cannot do better by playing the asymmetric strategy where one consumer recommends with probability 1[5].

In state $m = n-1$ if all consumers recommend with equal probabilities below 1 then their continuation payoff is equal to

$$\frac{\frac{(n-1)n\gamma\left(1-(1-\rho)^{n-1}\right)}{1-\delta} + (n-2)(n-1)\gamma(1-\rho)^{n-1} - c(n-1)\rho^2}{1-\delta(1-\rho)^{-1+n}}.$$

[5]Obviously, if one consumer recommends with probability 1 then optimally all other consumer should not recommend the product.

If only one consumer recommends with probability 1 then the continuation payoff is equal to

$$-c + \frac{(n-1)n\gamma}{1-\delta}.$$

We solve for the threshold $\bar{c}$ such that the continuation payoff under symmetric policy is higher, and get

$$\bar{c} = \frac{2(n-1)\gamma(1-\rho)^n}{(1-\rho)\left(1-(n-1)\rho^2\right)-\delta(1-\rho)^n},$$

then plug in $\rho = \frac{1}{2}$ and get

$$\bar{c}(\rho = \frac{1}{2}) = -\frac{16(-1+n)\gamma}{2^n(-5+n)+8\delta}.$$

At the same time when

$$c > \frac{8(n-1)\gamma}{2^n - \delta - n\delta}$$

the optimal solution for ρ in the case of symmetric policy is below $\frac{1}{2}$[6]. Comparing two expression for parameter c we can notice that the maximum of two is $\frac{8(n-1)\gamma}{2^n-\delta-n\delta} \equiv \tilde{c}$.

So when $c > \tilde{c}$ the optimal strategy is symmetric such that the optimal probability to recommend in each period for the cooperation strategy is less than $\frac{1}{2}$. Consumers recommend with the same positive probability in the last period. If it is profitable to apply symmetric strategy in the last period then the strategy of the mediator is such that in each period consumers will recommend with an equal probability which is less than 1. So we have a condition on optimal symmetric strategy for the mediator.

$$\lim_{n\to\infty} \frac{8(n-1)\gamma}{2^n - \delta - n\delta} = 0$$

$$\lim_{n\to\infty} \tilde{c} = 0$$

So for high n symmetric strategy is optimal almost for the whole range of recommendation cost parameter.

2.4.2 Symmetric Cooperation Strategy

Further results we derive assuming that $c > \tilde{c}$ and the mediator applies symmetric policy. Mediator's problem is the following:

$$\max_{\rho} V(m) = \sum_{j=0}^{n-m} \left((\gamma(m+j-1) + \delta V(m+j))P^j(1-P)^{k-j}\frac{k!}{j!(k-j)!} \right) - ck^2\rho^2 \tag{2.12}$$
$$P = 1-(1-\rho)^k$$

We can establish monotonicity of continuation payoff in number of informed nodes.

Proposition 6. *Continuation payoff is an increasing function in number of informed consumers or*

$$V(m) \geq V(m'), \text{ if } m > m' \tag{2.13}$$

Proposition 7. *For any n and m_1 two following conditions are satisfied:*

1. *If a consumer recommends with a positive probability in period t, then he recommends with a positive probability in period $t-1$.*

[6]The proof is straightforward. We take a derivative of continuation payoff w.r.t. ρ, plug in $\rho = \frac{1}{2}$ and solve F.O.C. w.r.t. c

2. *While there is at least one uninformed consumer, informed consumers recommend with a positive probability.*

In other words, the last proposition establishes that in any period consumers want to recommend if there is somebody who does not know about the product and the information diffusion process is always active.

Now we can prove the existence of interior equilibrium for the mediator's problem. We have already established that it is true for state $m = n-1$, so we need to show that existence conditions are satisfied for any state $m < n-1$ when $c > \tilde{c}$.

Theorem 2. *(Existence of interior solution in the Mediator's Problem) In each state the mediator chooses the optimal probability to recommend* $0 < \rho < 1$ *assigned to each consumer when recommendation costs are sufficiently high.*

Now we can compare F.O.C. conditions in the mediator's problem and in the individual maximization problem. The LHS of F.O.C. is defined in equation (2.22) in the proof of Theorem 1. So we just need to compare pairwise each term in the sum. We have that

for the NE:

$$\left\{\frac{P^j(1-P)^{k-j}}{1-\delta(1-P)^k}\right\}'_\rho = -\frac{(1-(1-\rho)^m)^{-1+j}\left(j+k\left(-1+(1-\rho)^m\right)-j\delta\left(\left((1-\rho)^m\right)^k\right)\right)\left((1-\rho)^m\right)^{-j+k}}{\left(-1+\delta\left(\left((1-\rho)^m\right)^k\right)\right)^2(-1+\rho)}$$

for the mediator problem:

$$\left\{\frac{P^j(1-P)^{k-j}}{1-\delta(1-P)^k}\right\}'_\rho = -m\frac{(1-(1-\rho)^m)^{-1+j}\left(j+k\left(-1+(1-\rho)^m\right)-j\delta\left(\left((1-\rho)^m\right)^k\right)\right)\left((1-\rho)^m\right)^{-j+k}}{\left(-1+\delta\left(\left((1-\rho)^m\right)^k\right)\right)^2(-1+\rho)}$$

So for $m = n-1$ F.O.C. at the symmetric equilibrium point can be written as $X(\rho) = c$ for the individual optimization problem and $(n-1)X(\rho) = c$ for the mediator problem, where $X(\rho)$ is some function of ρ which monotonically decreases and c is a constant. That means that in Nash equilibrium there is a lack of information comparing to the cooperation problem. Thus we observe the presence of free-riding. We can not say directly that in cooperating equilibria ρ is higher for any state m, because we have different values for continuation function in two games, but given the same continuation payoff in all states $m' > m$ we can observe presence of free-riding and lack of communication in non-cooperative equilibrium for any state m. These results will be illustrated in the next section.

We can compare probability to recommend in the case of the mediator problem and Nash equilibrium [7]. Let's consider F.O.C. in the symmetric equilibrium:

$$\text{F.O.C. (NE): } V'_{NE}(m)|_{\tilde{\rho}=\rho} =$$

$$\sum_{j=0}^{n-m} -A_j^{NE}C_k^j\frac{\left(j+k\left(-1+(1-r)^m\right)\right)\left(1-(1-r)^m\right)^{-1+j}\left((1-r)^m\right)^{-j+k}k!}{(-1+r)j!(-j+k)!} - 2ck^2\rho = 0$$

$$\text{F.O.C. (CS): } V'_{CS}(m) =$$

$$\sum_{j=0}^{n-m} -A_j^{CS}C_k^j\frac{m\left(j+k\left(-1+(1-r)^m\right)\right)\left(1-(1-r)^m\right)^{-1+j}\left((1-r)^m\right)^{-j+k}k!}{(-1+r)j!(-j+k)!} - 2ck^2\rho = 0$$

[7] We will use a notion NE to refer to the Nash Equilibrium in the non-cooperative case and CS for cooperation strategy case.

We know that at each step the optimal policy for the mediator requires the probability to recommend such that the continuation value is at least not smaller than what we get in Nash equilibrium.

So we can see that when $V'_{NE}(m) = 0$ we have that $V'_{CS}(m) > 0$, when $V'_{CS}(m) = 0$ we have that $V'_{NE}(m) < 0$ for the same ρ. Which means that point of maximum is such that $\rho_{NE} < \rho_{CS}$. This implies that due to the free-riding problem consumers always recommend below the socially optimal level.

Figure(2.3) illustrates optimal cooperation strategies for high and low costs:

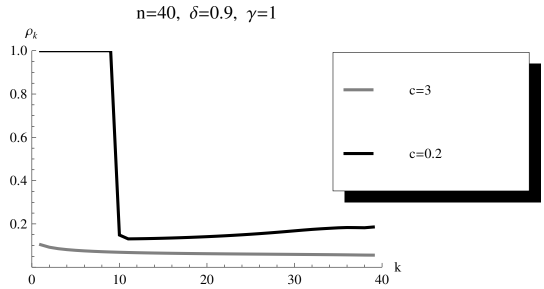


FIGURE 2.3: Cooperation Strategy: Probability to Recommend

2.4.3 Social Planner

Now we briefly describe the case of the social planner. As we have already mentioned above the main difference with a case of the cooperation is that the social planner has to take in account future payoffs of uninformed agents and at each step to maximize the total expected social utility.

So the maximization problem at each step for the social planner is following:

$$\max_{\rho} V(m) = \sum_{j=0}^{n-m} \left((\gamma(m+j-1)(m+j) + \delta V(m+j))P^j(1-P)^{k-j} \frac{k!}{j!(k-j)!} \right) - cmk^2\rho^2 \quad (2.14)$$

$$P = 1 - (1-\rho)^k$$

We can apply the same argument for monotonicity of the value function for the social planner problem as in Proposition 6. The proof of existence of the interior solution for sufficiently high recommendation cost follows the same steps as proof of Theorem 2. We observe the main difference with the cooperation case when the social planner optimally chooses asymmetric strategy. As the social planner cares about future utility of uninformed consumers, the optimal level of recommendation is non-monotonic due to this effect for some range of recommendation cost parameter c. In this case it is optimal for the social planner to choose asymmetric policy with probability to recommend equal to 1 at the first state $m = 1$ in order to maximize social welfare. The result is illustrated on the Figure 2.4.

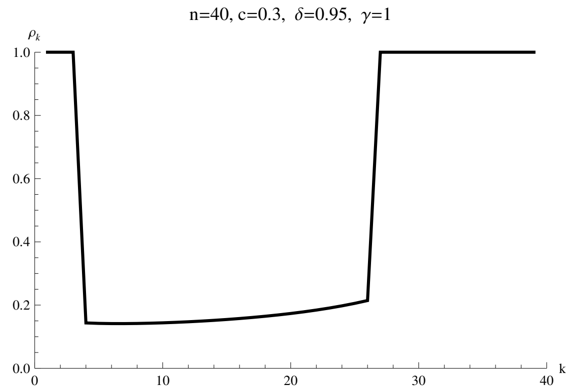


FIGURE 2.4: Social Planner's optimal strategy for low recommendation cost.

2.5 Simulations and Comparative Statics

In this section we provide simulation results and compare consumers' payoff, probabilities to recommend and expected number of newly informed consumers in Nash equilibrium, the mediator and the social planner problems.

Figure 2.5 illustrates probability to recommend with respect to a number of uninformed nodes. We compare cases of cooperation, Nash equilibrium and the social planner. In the cooperation case probability to recommend is a monotone function and it is decreasing in number of uninformed nodes as the more consumers are uniformed the high recommendation cost informed consumers have to pay to reach all uninformed peers. For the case of Nash equilibrium we can see that probability function is not monotone. The difference between two lines illustrates the magnitude of free-riding effect. The more consumers are informed the more they want to free-ride on recommendations of others.

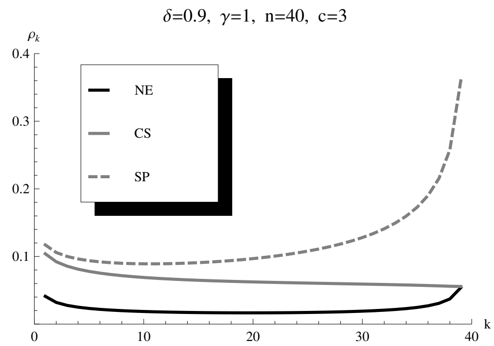
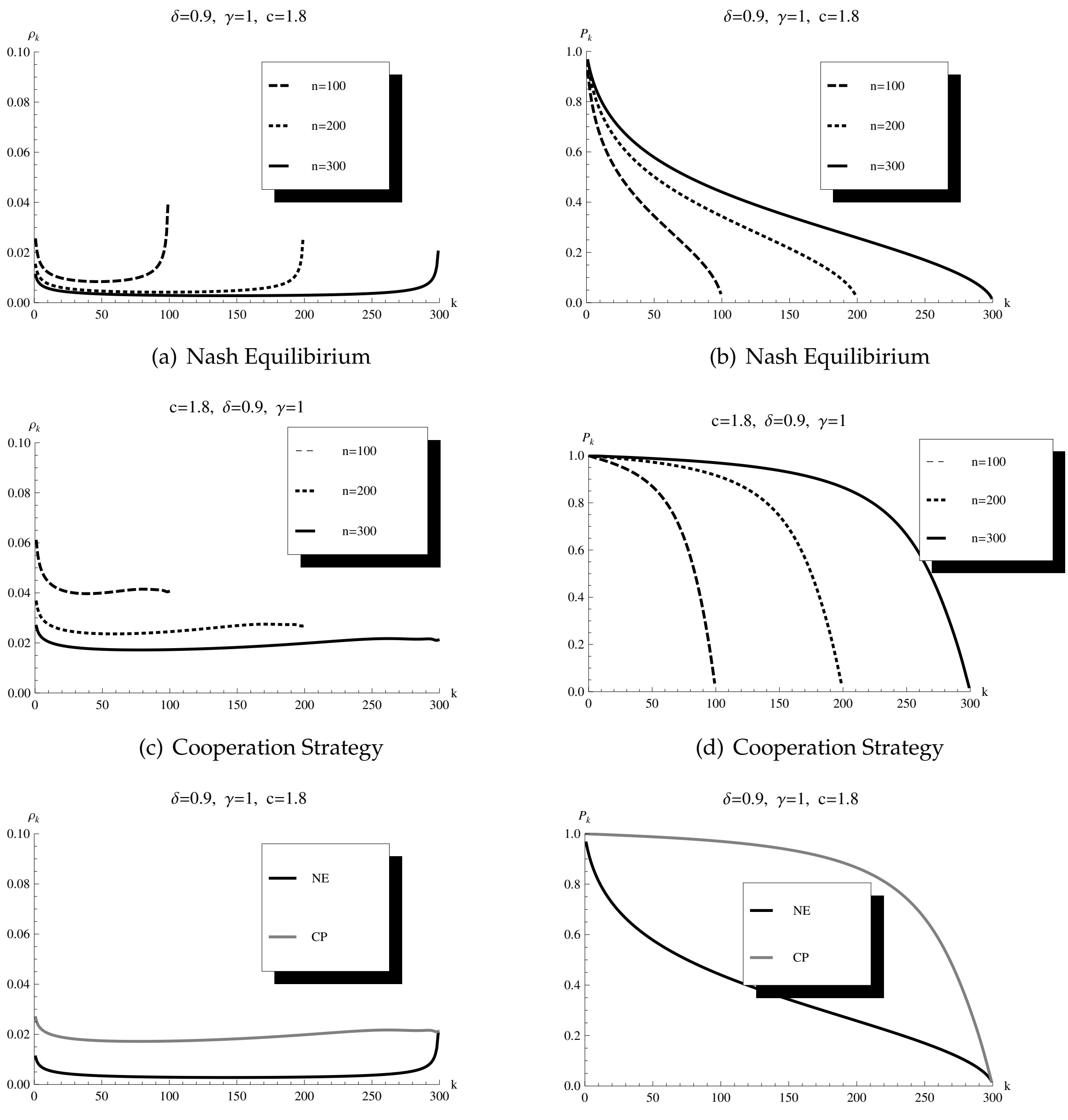


FIGURE 2.5: Probability to recommend

Now we can compare outcomes of the NE and the best symmetric cooperation strategy. We focus on number of uninformed nodes in order to be able to compare how probability to recommend changes with a total number of nodes keeping number of uninformed the same and recommendation cost comparable. Figure 2.6(a) illustrates how probability to recommend changes if we increase the total number of nodes. In this case at each stage each consumer wants to recommend less. The result corresponds to what we have to expect. An increase in number of already informed nodes increases the probability that others will recommend and as a result decreases individual incentives to recommend. Figure 2.6(b) illustrates the probability that each uninformed node will be informed in state $m = n - k$. This probability is increasing in number of informed nodes.

Figures 2.6(c) and 2.6(d) illustrate the same results for the mediator problem and Figures 2.6(e) and 2.6(f) let us to compare results for a social planner problem and individual maximization problem. So we can observe the presence of free-riding in case of individual maximization. The effect is especially strong for middle values of k. Figure (2.7) allows us to compare changes in ratio of the value function in NE and cooperation cases.



(a) Nash Equilibirium

(b) Nash Equilibirium

(c) Cooperation Strategy

(d) Cooperation Strategy

(e) Comparison of NE and CS

(f) Comparison of NE and CS

FIGURE 2.6: Nash equilibrium vs. Cooperation

We can see that probability to recommend ρ is non-monotone in number of uninformed nodes. We get these results because of two opposite effects. On the one hand, we have cost function which is quadratic in k, so the less number of uninformed consumers is the cheaper it is to reach them. On the other hand, the more consumers are informed the less incentives to recommend each particular consumer has because of a free-riding effect. As a result we have a parabolic type function.

Figure 2.8 illustrates how probability to recommend depends on cost coefficient c. An increase in cost function decreases willingness to recommend for each consumer.

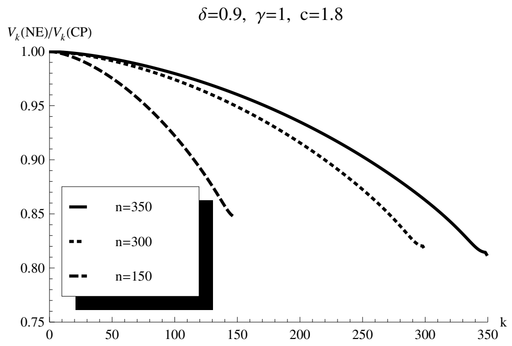


FIGURE 2.7: Ratio of value functions

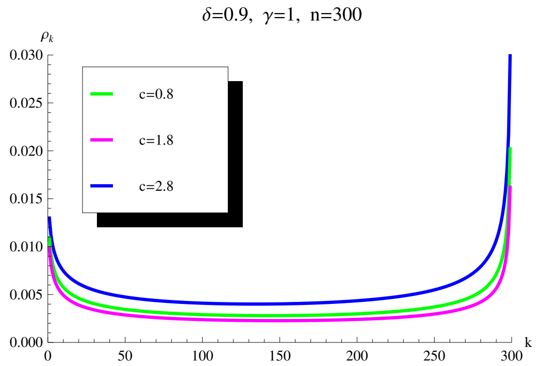


FIGURE 2.8: Probability to recommend depending on cost

We can estimate the expected time $E(T_m)$ to inform all consumers when the system in state m. For that we have to use again backward induction. In the last period expected time is equal to $\frac{1}{P_{1,n-1}}$, where $P_{1,n-1}$ is the probability that one node will be informed in state $n-1$. Then in period m this probability can be computed as following:

$$
\begin{aligned}
&E(T_n) = 0 \\
&E(T_{n-1}) = \frac{1}{P_{1,n-1}} \\
&E(T_m) = 1 + P_{0,m}E(T_m) + P_{1,m}E(T_{m+1}) + ... + P_{n-m-1,m}E(T_{n-1}) \\
&E_(T_m) = \frac{1 + P_{1,m}E(T_{m+1}) + ... + P_{n-m-1,m}E(T_{n-1})}{1 - P_{0,m}}
\end{aligned}
\tag{2.15}
$$

The expected time estimation till the state $m = n$ is illustrated on the picture below. We can also compare how that depends on other parameters and on structure of the graph.

Given the optimal strategy σ^* we can also estimate ex-ante expected number of informed nodes over time. The result is illustrated on Figure 2.10.

On Figure 2.11 we illustrate the audience growth of Facebook users in period 2005-2015 in order to compare our results with what we observe in the reality. Obviously we can not say that Facebook has already reached its maximal potential market size in 2015. There is still a stable audience growth. Moreover, we are able to observe only one particular realization of information diffusion process in this case. However we can see that the overall dynamics coincides with results which we got in theoretical model.

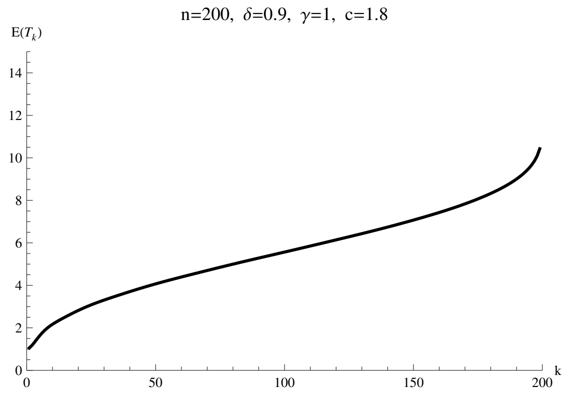


FIGURE 2.9: Expected time of the information diffusion process

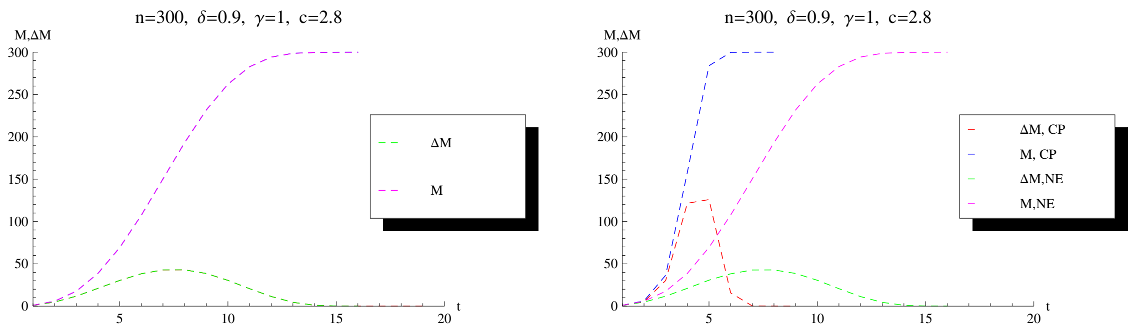


FIGURE 2.10: Expected number of newly informed nodes

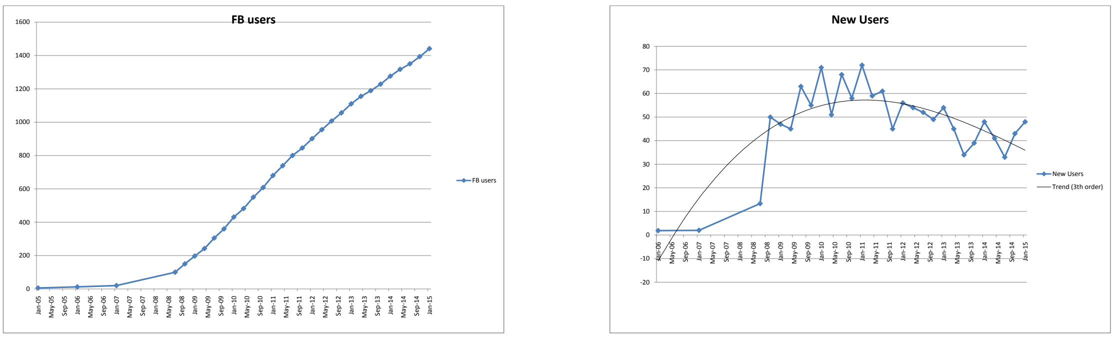


FIGURE 2.11: Facebook users

2.6 Optimal Advertising Strategy

In this section we analyze how free-riding effect can affect efficiency of direct advertising. Suppose that the firm decides to accelerate the information diffusion process by introducing some direct advertising. So it decides on advertising effort w_t in each period t, where w_t is the probability that an advertisement reaches each uninformed consumer. The firm may potentially have different objective functions. It can minimize time of the full information diffusion, maximize number of informed consumers in each period, maximize number of informed consumers in particular periods and so on.

We will consider two different cases:

1. The firm chooses an advertising level w, and maintains the same level in each period *(static strategy)*.

2. The firm chooses an advertising level in each period *(dynamic strategy)*.

The first case corresponds to the situation when information diffusion process is fast, and firm has to chose and to sign long-term contracts with advertisers. So it chooses the level of direct advertising before it launches the campaign. Here we can think about TV advertisement agreements and installing advertisement billboards. The advertising effort is observable for informed consumers, as they can see how often the product is advertised on TV, or how many billboards are installed across a city.

The second case corresponds to the situation when either information diffusion process is slow, or the firm can react fast to the changes in audience growth and adjust the advertising level in each period. The good example is Internet banners. So the firm constantly decides whether it wants to participate in auctions and to place its advertisements and on which webpages.

2.6.1 Static Advertising Strategy

We start our analysis with the case of static advertising. Here we don't specify objective and cost functions of the firm. Obviously the solution for the optimal advertising level for any objective function will give us a point between 0 and 1. In this section we conduct a numerical analysis and consider how the choice of any advertising level affects the increase in the probability to inform an uninformed customer in each period. So we consider the situation when firm commits to particular level of advertising at the beginning of the game[8], and this amount of advertising is known for consumers. The firm provides direct advertising in each period at the same level. Thus probability that the firm reaches each uninformed consumer is w in any period, and it is constant over time. Then we can rewrite an optimization problem for the NE in each period as following:

$$V(m) = \sum_{j=0}^{n-m} ((m+j-1)\gamma + \delta V(m+j)) Prob_j(m) - ck^2\tilde{\rho}^2 \tag{2.16}$$

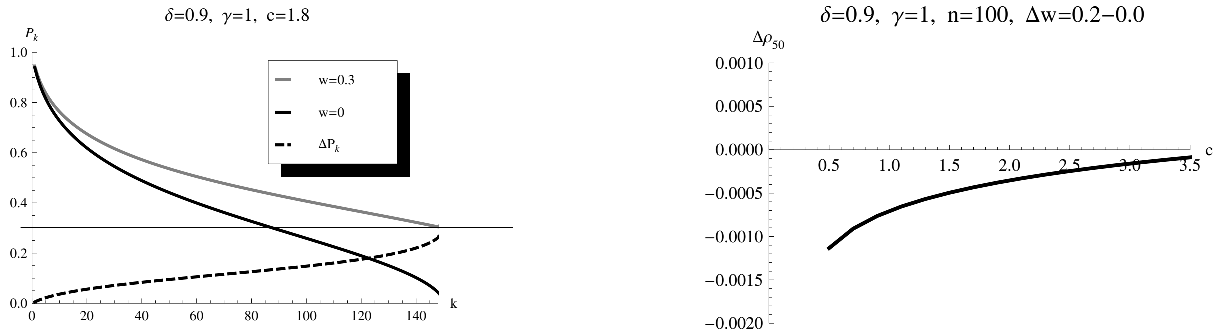
$$Prob_j(m) = P^j(1-P)^{k-j}\frac{k!}{j!(k-j)!} \text{ - probability that } j \text{ nodes will be informed}$$

$$P \equiv (1-(1-w)(1-\rho)^{m-1}(1-\tilde{\rho}))$$

We know that additional advertisements should increase the total probability to be informed for each uninformed consumer in every period. However, at the same time incentives to recommend decreases for each informed consumer. The graph below provides some results regarding an effect of direct advertising on information diffusion process.

Here we see that with an increase in number of informed consumers efficiency of advertising is decreasing as it leads to a decrease in probability to recommend by each informed consumer. So we can conclude that direct advertising is efficient at the beginning of information diffusion process. When there is a high number of informed consumers direct advertising leads to a sufficient increase of free-riding effect. Moreover the effect of direct advertising is smaller in the case of low recommendation cost. It is illustrated in Figure 2.12(b). We calculate the individual recommendation effort of consumers when half of population is informed (50 of out 100 consumers) for different cost c and two advertising levels: $w = 0.2$, $w = 0$. The difference in

[8] So we suppose that the firm makes the choice of the advertising level which is observable by consumers, but we don not specify why the firm chooses this particular level of direct advertising. Consumers consider it as an exogenous parameter when they make their own decisions on recommendation efforts.



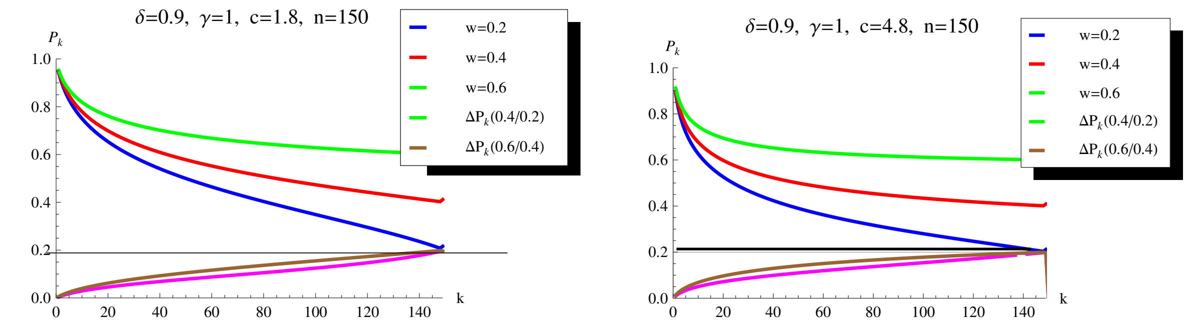
(a) Probability to be informed

(b) Difference between recomendation efforts

FIGURE 2.12: Free-riding effect in the presence of direct advertising

recommendation efforts for two advertising levels is represented by black line. We see that for higher recommendation cost the decrease in the individual recommendation effort is smaller. So the optimal strategy of the should include advertising to increase initial number of informed consumers when recommendation costs are sufficiently high.

Two following pictures illustrate the difference in probability to be informed in each state for different levels of direct advertising. In the considered example there are three levels of direct advertising: $w = 0.2, w = 0.4, w = 0.6$. There was estimated the actual increase in probability to recommend when the level of direct advertising changes. We can see that given the same $\Delta w = 0.2$, for the higher level of advertising increase in probability is higher. It is also possible to see that with an increase of number of informed consumers effectiveness of advertising is decreasing as it leads to a decrease in probability to recommend for each consumer. When we have high number of informed consumers it can lead to sufficient increase of free-riding effect. Moreover the effect of direct advertising is increasing in the value of recommendation cost. The firm can help to consumers to disseminate information when their cost of recommendation are high, and information diffusion is quite costly. In the case of low recommendation cost the free-riding effect almost overwhelms all benefits of additional direct advertising by the firm. So the optimal strategy may include advertising to increase initial number of informed consumers at the beginning of information diffusion process when recommendation costs are sufficiently high.



(a)

(b)

FIGURE 2.13: Probability to be informed

2.6.2 Dynamic Advertising Strategy

In this subsection we consider the situation when firm can react fast and can choose advertising level in each period. That refers to the situation when information diffusion process is slow and firm has short-term contracts with advertisers. We will consider a "myopic" objective function of the firm which maximizes expected number of newly informed consumers in each period. We assume that direct advertising is costly, and cost function is $C_a(w)$, where $C_a(w)$ is continuously differentiable function, $C_a'(w) > 0$, $C_a''(w) > 0$, $C_a(1) = \infty$ and $C_a(0) = 0$. That means that the more customers are informed the more effort firm has to put to increase probability to reach uninformed customers by direct advertising. The firm can never reach all uninformed consumers with probability 1 at finite advertising cost. These properties of function $C_a(w)$ automatically imply that $C_a'(1) = \infty$.

We can write an optimization problem of the firm in each period as

$$\max_{w(k)} kP_k - C_a(w),$$

where k is a number of uninformed consumers, which defines a state, and P_k is probability to inform each uninformed consumer.

$$\max_{w(k)} k(1-(1-\rho)^{n-k}(1-w)) - C_a(w(k))$$

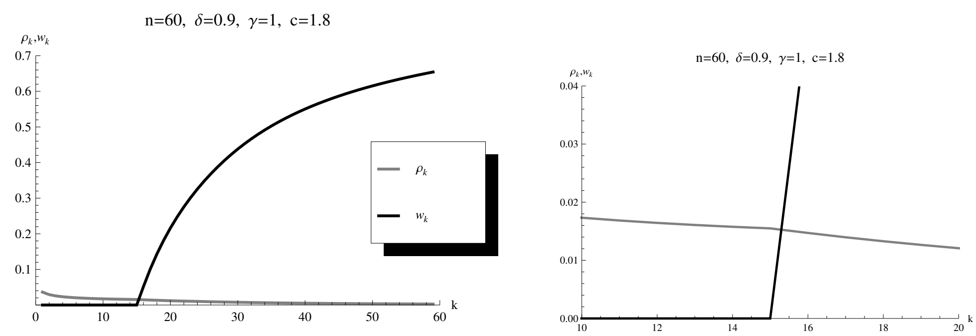
F.O.C.:

$$(1-\rho)^{n-k}k - C_a'(w) = 0$$

$$w = C_a'^{-1}(k(1-\rho)^{n-k})$$

Second order condition is satisfied as $C_a''(w) > 0$. As $C_a'(1) = \infty$ we get that the optimal solution w^* has to belong to the interval $[0,1)$.

The optimal dynamic strategy is illustrated on Figure 2.14 [9].



(a) Advertising Strategy (b) Kink

FIGURE 2.14: Dynamic Advertising Strategy

We can see that effective strategy of the firm assume a decrease in volume of advertising overtime.

On Figure 2.15 we can see the comparison between cases when *i)* there is no any network, consumers do not recommend the product, and firm decides how much to advertise in each period; *ii)* the case when we exclude free-riding due to additional advertising, so consumers do not take into account the amount of advertising provided by firm; *iii)* the case when consumers take into account the amount of direct advertising provided by the firm. We make this comparison to estimate actual reduce in advertising due to free-riding effect. You can see that the firm can sufficiently reduce amount of advertising in presence of "word-of-mouth" communication. However, due to free-riding effect firm has to reduce this level even more as the increase in volume of advertising reduces incentives of agents to recommend.

[9] We used function $C_a(w) = c_w \frac{w}{1-w}$ for the simulations.

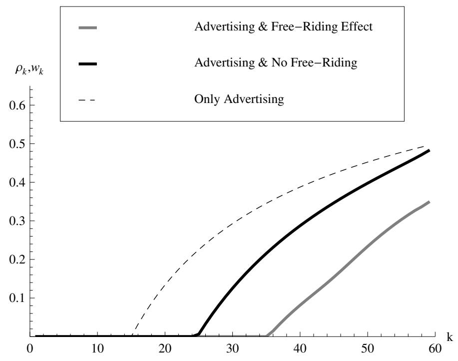


FIGURE 2.15: Dynamic Advertising Strategy

2.7 Concluding Remarks

In this paper, we consider the model of information diffusion process with a strategic decision of each consumer on how much effort to put recommending the new technology to other consumers in the market. We construct a dynamic programming problem and provide an algorithm to get a solution to this problem. The main idea of the paper is to consider a strategic decision on a recommendation by introducing direct benefits for each consumer who interacts with neighbors using the new technology. We focus on information diffusion dynamics over time. The paper shows the presence of a free-riding effect in the market where consumers prefer to recommend less expecting that information can reach their neighbors through other consumers. As a result, this decreases incentives of consumers to promote the new technology. We get a non-monotonic convex function of an individual recommendation effort with respect to the number of uninformed consumers. The reason for observing the non-monotonicity is the presence of two opposite effects: a decrease of total recommendation cost with an increase of a number of informed consumers and an increase of incentives to free-ride. We analyze the optimal strategies of consumers who cooperate in each period by coordinating their recommendation efforts. Symmetric agents should optimally put equal recommendation efforts when recommendation cost is high, while for the low cost it can be optimal that agents exert asymmetric recommendation efforts.

In the paper, we study the efficiency of direct advertising in the presence of "word-of-mouth" communication by providing a numerical analysis. The presence of the free-riding effect in "word-of-mouth" communication sufficiently decreases the efficiency of direct advertising as consumers decrease their individual efforts to recommend observing that the firm actively advertises. In the case of sufficiently small recommendation cost and high proportion of the informed population, the return of direct mass advertising is almost negligent, as it is almost overwhelmed by the free-riding effect. At the same, if we observe high recommendation cost and a tiny proportion of informed consumers the decrease in return of direct advertising is sufficiently small. That result provides some evidence for a choice of the optimal marketing strategy for the firm. If the firm has an opportunity to adjust the amount of advertising from period to period, the optimal advertising level should decrease in the number of informed consumers compared to the advertising level that the firm would choose in the absence of word-of-mouth communication.

2.A Appendix

Proof of Lemma 9. First of all, notice that LHS of equation (2.10) is a continuous function. Now we can check value of that function at the end points.

$$
\begin{aligned}
&FD(\rho) \equiv c(2-\rho)\rho + \frac{\gamma}{\delta} - \frac{2c\rho}{(1-\rho^{-2+n})\delta} \\
&FD(0) = \frac{\gamma}{\delta} > 0 \\
&\lim_{\rho\to 1} FD(\rho) = \lim_{\rho\to 1} c(2-\rho)\rho + \frac{\gamma}{\delta} - \frac{2c\rho}{(1-\rho^{-2+n})\delta} = -\infty < 0 \\
&(FD)'_{\rho} = c\left(2 - 2\rho - \frac{2\rho^2\left(\rho^2 + (-3+n)\rho^n\right)}{\delta\left(\rho^2-\rho^n\right)^2}\right) \\
&\frac{2\rho^2\left(\rho^2 + (-3+n)\rho^n\right)}{\delta\left(\rho^2-\rho^n\right)^2} > 2\frac{\rho^2(\rho^2+(n-3)\rho^n)}{(\rho^2-\rho^n)(\rho^2-\rho^n)} > 2 \\
&\Rightarrow (FD)'_{\rho} < 0
\end{aligned}
\tag{2.17}
$$

So we have that there exists a unique solution of equation (2.7). Now we have to check that it is indeed the maximum.

S.O.C.:

$$(V(n-1))''_{\tilde{\rho}} = -2c < 0$$

□

Proof of Lemma 10. First order condition is:

$$F'_{\tilde{\rho}}(\tilde{\rho},\rho) = 2ck^2\tilde{\rho}$$

$$F'_{\tilde{\rho}}(\tilde{\rho},\rho) = \sum_{j=0}^{k} A_j\{Prob_j\}'_{\tilde{\rho}}$$

$$\{Prob_j\}'_{\tilde{\rho}} =$$

$$= \frac{\left(1+(-1+\tilde{\rho})(1-\rho)^{-1+m}\right)^j\left(k(-1+\tilde{\rho})(1-\rho)^m + (j-k)(-1+\rho)\right)\left((1-\tilde{\rho})(1-\rho)^{-1+m}\right)^{-j+k}k!}{(-1+\tilde{\rho})\left(1+(-1+\tilde{\rho})(1-\rho)^m-\rho\right)j!(-j+k)!}$$

When $\tilde{\rho}\to 0$, RHS of equation (2.18) goes to 0 and

$$\lim_{\tilde{\rho}\to 0}\{Prob_j\}'_{\tilde{\rho}} = -\frac{\left(1-(1-\rho)^{-1+m}\right)^j\left(-k(1-\rho)^m + (j-k)(-1+\rho)\right)\left((1-\rho)^{-1+m}\right)^{-j+k}k!}{\left(1-(1-\rho)^m-\rho\right)j!(-j+k)!}$$

When $j > k(1-1(1-\rho)^{m-1})$ we have that $\{Prob_j\}'_{\tilde{\rho}} > 0$, otherwise it is negative.

$$\sum_{j=0}^{k}\{Prob_j\}'_{\tilde{\rho}}|_{\tilde{\rho}=0} = 0$$

Assuming that $V(k)$ is decreasing function in k we have that A_j is increasing function in j, so we put relatively more weight on positive terms in sum $\sum_{j=1}^{k} A_j\{Prob_j\}'_{\tilde{\rho}}$ which means that this sum is positive and strictly positive when $\tilde{\rho} = 0$.

So we have that LHS>RHS in equation (2.18) when $\tilde{\rho}\to 0$.

Now let us to consider another boundary when $\tilde{\rho} \to 1$, $2ck^2\tilde{\rho}|_{\tilde{\rho}=1} = 2ck^2 > 0$

$$\{Prob_j\}'_{\tilde{\rho}}|_{\tilde{\rho}\to 1} = 0, \forall j < k-1$$

$$\{Prob_j\}'_{\tilde{\rho}}|_{\tilde{\rho}\to 1} = -k(1-\rho)^{m-1}, \forall j = k-1$$

$$\{Prob_j\}'_{\tilde{\rho}}|_{\tilde{\rho}\to 1} = k(1-\rho)^{m-1}, \forall j = k$$

Taking into account that $A_k > A_{k-1}$ we have that the whole sum is positive but not bigger than $k(1-\rho)^{m-1}\frac{\gamma}{1-\delta}$. So if $c > \frac{1}{k}(1-\rho)^{m-1}\frac{\gamma}{2-2\delta}$, then LHS<RHS when $\tilde{\rho} \to 1$, and at least one intersection, such that FOC is satisfied, exists.

Now we can show that for sufficiently high recommendation cost c this intersection is unique and it is point of maximum. So we need to show that:

$$F''_{\tilde{\rho},\tilde{\rho}}(\tilde{\rho},\rho) < 2ck^2$$

It is enough to show that $F''_{\tilde{\rho},\tilde{\rho}}(\tilde{\rho},\rho)$ is bounded above.

Let us to consider the second order derivative:

$$F''_{\tilde{\rho},\tilde{\rho}}(\tilde{\rho},\rho) = \sum_{j=0}^{k} A_j\{Prob_j\}''_{\tilde{\rho},\tilde{\rho}}$$

$$\{Prob_j\}''_{\tilde{\rho},\tilde{\rho}} = \frac{\left(1+(-1+\tilde{\rho})(1-\rho)^{-1+m}\right)^j \left((1-\tilde{\rho})(1-\rho)^{-1+m}\right)^{-j+k} k!}{(-1+\tilde{\rho})^2\left(1+(-1+\tilde{\rho})(1-\rho)^m-\rho\right)^2 j!(-j+k)!}\times \tag{2.18}$$
$$\times\left((-1+k)k(-1+\tilde{\rho})^2(1-\rho)^{2m} - 2(j-k)(-1+k)(-1+\tilde{\rho})(1-\rho)^{1+m} + (j-k)(1+j-k)(-1+\rho)^2\right)$$

$$\lim_{\tilde{\rho}\to 0}\lim_{\rho\to 0}\{Prob_j\}''_{\tilde{\rho},\tilde{\rho}} = 0$$

$$\lim_{\tilde{\rho}\to 0}\lim_{\rho\to 1}\{Prob_j\}''_{\tilde{\rho},\tilde{\rho}} = \begin{cases} 0, & \text{if } j > 2 \\ k(k-1), & \text{if } j = 2 \\ -2k(k-1), & \text{if } j = 1 \\ k(k-1), & \text{if } j = 0 \end{cases}$$

We can see that $F''_{\tilde{\rho},\tilde{\rho}}(\tilde{\rho},\rho)$ is bounded when $\tilde{\rho}$ goes to zero.

$$\lim_{\tilde{\rho}\to 1}\{Prob_j\}''_{\tilde{\rho},\tilde{\rho}} = \begin{cases} 0, & \text{if } j < k-2 \\ k(k-1)(1-\rho)^{2m-2}, & \text{if } j = k-2 \\ -2k(k-1)(1-\rho)^{2m-2}, & \text{if } j = k-1 \\ k(k-1)(1-\rho)^{2m-2}, & \text{if } j = k \end{cases}$$

So when $\tilde{\rho} \to 1$ function $F''_{\tilde{\rho},\tilde{\rho}}(\tilde{\rho},\rho)$ is also bounded above. Obviously for all value of $\tilde{\rho},\rho \in (0,1)$ that is true as well.

So we have that for sufficiently high recommendation cost $F''_{\tilde{\rho},\tilde{\rho}}(\tilde{\rho},\rho) < 2c$, and solution of individual maximization problem is internal and unique.

Notice that when $\rho \to 1$ we have that $\tilde{\rho} \to 0$, because consumer has no incentives to recommend if all uninformed consumers will be informed by his neighbors with probability 1.

When $\rho \to 0$, we have:

$$\{Prob_j\}'_{\tilde{\rho}} = \frac{(j - k\tilde{\rho})(1-\tilde{\rho})^{k-j}\tilde{\rho}^{j-1}k!}{(1-\tilde{\rho})j!(k-j)!}$$

So if $\tilde{\rho} \to 0$ then LHS goes to plus infinity and RHS goes to zero, so obviously $\tilde{\rho} = 0$ can not be a solution when $\rho = 0$, and the optimal solution is strictly positive.

□

Proof of Proposition 5. We will proof this lemma by induction. We know that $V(0) = \frac{(n-1)\gamma}{1-\delta}$[10], $V(1) < V(0)$. We have to show that if for any k the following property is satisfied $V(k) < V(k-1) < V(k-2) < ... < V(1) < V(0)$ then $V(k+1) < V(k)$ for sufficiently high cost c.

$$V(k+1) = \frac{\gamma(m-2)}{1-\delta(1-P^{k+1})} + \frac{C^j_{k+1}P^jP^{k+1-j}}{1-\delta(1-P^{k+1})}(\gamma j + \delta V(k+1-j)) - c(k+1)^2\tilde{\rho}^2$$

From proof of Theorem (1) we know that for sufficiently high recommendation cost c solution is unique.

Let us to prove that for the same ρ the optimal $\tilde{\rho}(k) > \tilde{\rho}(k+1)$. Consider the following ratio:

$$R(k) \equiv \frac{{}^{j+1}_{k}P^{j+1}P^{k-(j+1)}}{1-\delta(1-P^k)}{}^{j}_{k}P^jP^{k-j}1-\delta(1-P^k) = \frac{(k-i)\left(\frac{1}{(1-\tilde{\rho})(1-\rho)^{n-k-1}} - 1\right)}{1+i}$$

$$R'(k)_{\tilde{\rho}} = \frac{(k-i)(1-\rho)^{1+k-n}}{(1+i)(1-\tilde{\rho})^2}$$

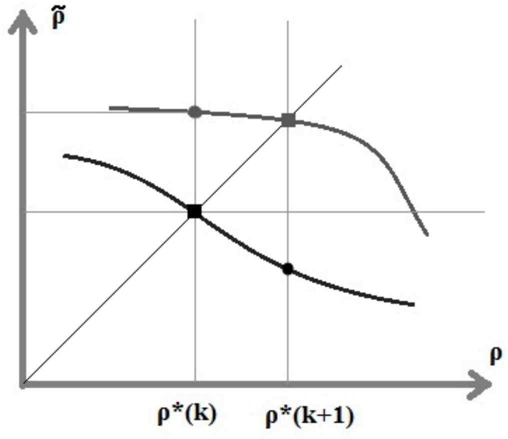
So we can see, that $R(k)$ is increasing function in $\tilde{\rho}$. The more a consumer recommends the more a relative weight of bigger term is in the sum. Now we can check how $R'(k))_{\tilde{\rho}}$ changes with increase of k. We have to consider the ratio for $k+1$ uninformed nodes which corresponds to the same terms as for k uninformed nodes. This ration is equal to:

$$R'(k)_{\tilde{\rho}} = \frac{(k-i)(1-\rho)^{2+k-n}}{(2+i)(1-\tilde{\rho})^2} < \frac{(k-i)(1-\rho)^{1+k-n}}{(1+i)(1-\tilde{\rho})^2}$$

So we conclude that given the same ρ the same increase in $\tilde{\rho}$ implies high relative weight increase for bigger terms in the sum. At the same time total cost function depends on k. That means that for the same ρ with increase in $\tilde{\rho}$ we have that $F'(k)_{\tilde{\rho}} > F'(k+1)_{\tilde{\rho}}$ and $2ck^2\tilde{\rho} < 2c(k+1)\tilde{\rho}$. So we can see that in if in the state $k+1$ consumer has incentives to increase recommendation level at some $\tilde{\rho}$ then in the state k he also has an incentives to increase recommendation level at the same point given the same ρ for the both states. So we can conclude that given the same ρ a consumer chooses higher level of recommendation effort $\tilde{\rho}$ for lower k.

Assume that we have a situation when in a symmetric equilibrium $V(k+1) > V(k)$ which automatically implies that $\rho^*(k+1) > \rho^*(k)$.

[10] Please, pay attention that here we consider $V(\cdot)$ as a function of $k = n - m$, not m.



So we should have an interval where given the same ρ the optimal solution $\tilde{\rho}(k) < \tilde{\rho}(k+1)$, which contradicts to what we got above. So the situation when $V(k+1) > V(k)$ in symmetric equilibrium is impossible $\Rightarrow V(k+1) < V(k)$. So by backward induction we have proved function $V(k)$ is decreasing in k. □

Proof of Theorem 1. Let us to rewrite F.O.C. as following:

$$F'_{\tilde{\rho}}(\tilde{\rho}, \rho) = 2ck^2\tilde{\rho},$$

where

$$F(\tilde{\rho}, \rho) = \sum_{j=0}^{n-m} A_j C_k^j P^j (1-P)^{k-j}$$

$$(F''_{\tilde{\rho},\tilde{\rho}}(\tilde{\rho}, \rho) - ck^2)d\tilde{\rho} + F''_{\tilde{\rho},\rho}(\tilde{\rho}, \rho)d\rho = 0 \tag{2.19}$$

$$\frac{d\tilde{\rho}}{d\rho} = -\frac{F''_{\tilde{\rho},\rho}(\tilde{\rho}, \rho)}{F''_{\tilde{\rho},\tilde{\rho}}(\tilde{\rho}, \rho) - ck^2}$$

$$\frac{d\tilde{\rho}}{d\rho} = \frac{F''_{\tilde{\rho},\rho}(\tilde{\rho}, \rho)}{ck^2 - F''_{\tilde{\rho},\tilde{\rho}}(\tilde{\rho}, \rho)}.$$

From Lemma 10 we know that for sufficiently high recommendation cost a solution of an individual maximization problem exists and it is unique. As we have an argmax, the following property has to be satisfied:

$$ck^2 - F''_{\tilde{\rho},\tilde{\rho}}(\tilde{\rho}, \rho) < 0 \tag{2.20}$$

So we have that both denominator and nominator are continuous bounded functions. Which means that $\frac{d\tilde{\rho}}{d\rho}$ is a continuous and bounded function, which implies that $\tilde{\rho}(\rho)$ is a continuous function. From proof of Lemma (10) we know that when $\rho \to 0$ $\tilde{\rho}$ converges to positive number and when $\rho \to 1$ $\tilde{\rho} \to 0$. So we have at least one interior intersection of function $\tilde{\rho}(\rho)$ with forty-five degree line. Which means that we have a symmetric solution for sufficiently high recommendation cost.

First of all let's consider the following terms of an equation in F.O.C.:

$$\begin{aligned} P &= 1-(1-\rho)^{m-1}(1-\tilde{\rho}) \\ P'_{\rho} &= (m-1)(1-\rho)^{m-2}(1-\tilde{\rho}) \\ P'_{\tilde{\rho}} &= (1-\rho)^{m-1} \end{aligned} \tag{2.21}$$

where

$$F'_{\tilde{\rho}} = \sum_{j=1}^{k} A_j \left(\frac{P^j(1-P)^{k-j}}{1-\delta(1-P)^k}\right)'_{\tilde{\rho}} \tag{2.22}$$

$$\begin{aligned} \left(\frac{P^j(1-P)^{k-j}}{1-\delta(1-P)^k}\right)'_{\tilde{\rho}} &= P'_{\tilde{\rho}}(jP^{j-1}(1-P)^{k-j}-(k-j)P^j(1-P)^{k-j-1}) \\ = -P'_{\tilde{\rho}}\frac{(1-P)^{-1-j+k}P^{-1+j}\left(kP+j\left(-1+(1-P)^k\delta\right)\right)}{\left(-1+(1-P)^k\delta\right)^2} \end{aligned} \tag{2.23}$$

Let us to define

$$F'_{\rho} = \sum_{j=1}^{k} A_j \left(\frac{P^j(1-P)^{k-j}}{1-\delta(1-P)^k}\right)'_{\rho} \tag{2.24}$$

Then

$$\left(\frac{P^j(1-P)^{k-j}}{1-\delta(1-P)^k}\right)'_{\rho} = -P'_{\rho}\frac{(1-P)^{-1-j+k}P^{-1+j}\left(kP+j\left(-1+(1-P)^k\delta\right)\right)}{\left(-1+(1-P)^k\delta\right)^2} \tag{2.25}$$

From (2.21), (2.23) and (2.25) we have that the following property is satisfied:

$$F'_{\rho} = F'_{\tilde{\rho}}(m-1)\frac{1-\tilde{\rho}}{1-\rho} \tag{2.26}$$

Then we can derive following conditions:

$$\begin{aligned} F''_{\rho,\tilde{\rho}} &= F''_{\tilde{\rho},\tilde{\rho}}\frac{1-\tilde{\rho}}{1-\rho}(m-1) - F_{\tilde{\rho}}\frac{m-1}{1-\rho} = \\ &= F''_{\tilde{\rho},\tilde{\rho}}\frac{1-\tilde{\rho}}{1-\rho}(m-1) - \frac{C'(\tilde{\rho},k)(m-1)}{1-\rho} \end{aligned}$$

$$\begin{aligned} \frac{d\tilde{\rho}}{d\rho} &= \frac{F''_{\tilde{\rho},\tilde{\rho}}(m-1)\frac{1-\tilde{\rho}}{1-\rho} - \frac{C'(\tilde{\rho},k)(m-1)}{1-\rho}}{''(\tilde{\rho},k) - F''_{\tilde{\rho},\tilde{\rho}}} = \\ &= \frac{m-1}{1-\rho}\frac{F''_{\tilde{\rho},\tilde{\rho}}(1-\tilde{\rho}) - C'(\tilde{\rho},k)}{C''(\tilde{\rho},k) - F''_{\tilde{\rho},\tilde{\rho}}} \end{aligned}$$

If $\tilde{\rho}$ is an argmax of then we should have that $F''_{\tilde{\rho},\tilde{\rho}} < C''(\tilde{\rho},k)$. So we have that for $\tilde{\rho} > \frac{1}{2}$ $\frac{d\tilde{\rho}}{d\rho} < 0$. Which means that if we have a solution ρ^* then this solution is unique for any costs.

Now assume that if there is a symmetric solution such that $\tilde{\rho} = \rho < \frac{1}{2}$ then for sufficiently high c we have $\frac{d\tilde{\rho}}{d\rho} < 1$ and as we know from above $\rho = 0 \Rightarrow \tilde{\rho} > 0$ which implies that solution is unique. So we have:

$$\frac{m-1}{1-\rho}\frac{F''_{\tilde{\rho},\tilde{\rho}}(1-\tilde{\rho}) - C'(\tilde{\rho},k)}{C''(\tilde{\rho},k) - F''_{\tilde{\rho},\tilde{\rho}}} > 1 \tag{2.27}$$

We have the following condition in asymmetric equilibrium when $\tilde{\rho} = \rho$:

$$F''_{\tilde{\rho},\tilde{\rho}} < \frac{C''\tilde{\rho},k)}{m} + \frac{(m-1)C'(\tilde{\rho},k)}{m(1-\tilde{\rho})} \tag{2.28}$$

If cost function is quadratic and equal to $C(\tilde{\rho},k) = ck^2\tilde{\rho}^2$ then we rewrite this equation as:

$$F''_{\tilde{\rho},\tilde{\rho}} < \frac{ck^2}{m} + \frac{(m-1)ck^2\tilde{\rho}}{m(1-\tilde{\rho})} \tag{2.29}$$

So if function $\frac{F''_{\tilde{\rho},\tilde{\rho}}(1-\tilde{\rho})}{\tilde{\rho}}$ is bounded then $\exists \bar{c}$, s.t. $\forall c > \bar{c}$ and $\forall m,k$ condition (2.27) is satisfied and solution is unique.

Function $F''_{\tilde{\rho},\tilde{\rho}}$ can be represented as:

$$F''_{\tilde{\rho},\tilde{\rho}} = \sum_{j=1}^{k} C_k^j Prob''_{j\ \tilde{\rho},\tilde{\rho}} V(k+j)$$

$$Prob_j = \frac{(1-(1-\rho)^{m-1}(1-\tilde{\rho}))^j((1-\rho)^{m-1}(1-\tilde{\rho}))^{k-j}}{1-\delta((1-\rho)^{m-1}(1-\tilde{\rho}))^k}$$

$$C_k^j\{Prob_j\frac{1-\tilde{\rho}}{\tilde{\rho}}\}''_{\tilde{\rho},\tilde{\rho}} = \frac{k!}{j!(k-j)!}\frac{\left(1+(-1+\tilde{\rho})(1-\rho)^{-1+m}\right)^j\left((1-\tilde{\rho})(1-\rho)^{-1+m}\right)^{-j+k}}{(1-\tilde{\rho})\tilde{\rho}\left(1-\left((1-\tilde{\rho})(1-\rho)^{-1+m}\right)^k\delta\right)^3}\times$$

$$\times\left(\frac{(-1+j)j(-1+\rho)^2\left(-1+\left((1-\tilde{\rho})(1-\rho)^{-1+m}\right)^k\delta\right)^2}{\left(1+(-1+\tilde{\rho})(1-\rho)^m-\rho\right)^2}+\right.$$

$$+\frac{2j(-1+\rho)\left(-1+\left((1-\tilde{\rho})(1-\rho)^{-1+m}\right)^k\delta\right)\left(-1+k+\left((1-\tilde{\rho})(1-\rho)^{-1+m}\right)^k\delta\right)}{-1-(-1+\tilde{\rho})(1-\rho)^m+\rho}+$$

$$\left.+k\left(-1+k+(1+k)\left((1-\tilde{\rho})(1-\rho)^{-1+m}\right)^k\delta\right)\right)$$

We can see that for $\forall j$ every term is bounded for any $\rho \in [0,0.5]$ and $\tilde{\rho} \in (0,0.5]$ and in optimum $\tilde{\rho} > 0$ as $\rho < 1$. So we have that $\frac{F''_{\tilde{\rho},\tilde{\rho}}(1-\tilde{\rho})}{\tilde{\rho}}$ is a bounded, so for sufficiently high communication cost condition (2.27) is satisfied and we have a unique solution.

□

Proof of Theorem 2. We have the following optimization problem:

$$\max_{\rho} V(m) = \frac{\gamma m + \sum_{j=1}^{n-m}(\delta V(m+j)+\gamma j)C_k^j P^j(1-P)^{k-j} - ck^2\rho^2}{1-\delta(1-P)^k}$$

F.O.C.:

$$\sum_{j=0}^{n-m} A_j\left\{\frac{P^j(1-P)^{k-j}}{1-\delta(1-P)^k}\right\}'_\rho - \left\{\frac{ck^2\rho^2}{1-\delta(1-P)^k}\right\}'_\rho = 0$$

$$R \equiv \left\{\frac{ck^2\rho^2}{1-\delta(1-P)^k}\right\}'_\rho = \frac{ck^2\rho\left(2-\delta(1-\rho)^{mk-1}(2+(-2+km)\rho)\right)}{\left(-1+\delta\left((1-\rho)^m\right)^k\right)^2}$$

$$\lim_{\rho\to 0} R = 0$$

$$\lim_{\rho \to 1} R = 2ck^2$$

$$DP_j = \left\{ \frac{P^j (1-P)^{k-j}}{1-\delta(1-P)^k} \right\}'_\rho, j > 0$$

$$DP_0 = \left\{ 11 - \delta(1-P)^k \right\}'_\rho$$

$$DP_0 = -\frac{km\delta(1-\rho)^{mk-1}}{\left(-1+\delta\left((1-\rho)^m\right)^k\right)^2}$$

$$DP_j = \frac{m\left(1-(1-\rho)^m\right)^{-1+j}\left(j+k\left(-1+(1-\rho)^m\right)-j\delta\left((1-\rho)^m\right)^k\right)\left((1-\rho)^m\right)^{-j+k}}{\left(-1+\delta\left((1-\rho)^m\right)^k\right)^2(1-\rho)}$$

$$\lim_{\rho \to 1} DP_j = 0 \Rightarrow V'_\rho(\rho = 1) < 0 \Rightarrow \rho = 1 \text{ is not a global maximum}$$

$$\lim_{\rho \to 0} DP_j = 0, j > 1$$

$$\lim_{\rho \to 0} A_0 DP_0 + A_1 DP_1 = -\frac{km\delta}{(-1+\delta)^2}\gamma m + k\frac{m}{1-\delta}(\gamma j + \delta V(m+1)) >$$

$$> -\frac{km\delta}{(-1+\delta)^2}\gamma m + k\frac{m}{1-\delta}\delta\frac{\gamma(m+1)}{1-\delta} = 0$$

$$\Rightarrow V'_\rho(\rho = 0) > 0 \Rightarrow \rho = 0 \text{ is not a global maximum}$$

So we have that there is an interior solution which satisfies F.O.C. and $\rho = 0$ and $\rho = 1$ are not points of maximum, so we have that a global maximum is an interior solution. □

Proof of Proposition 6. Suppose that $m > m'$, then $V(m) = \gamma(m + E(\Delta m) - 1) - C_t(\rho_t) + E(V(m + \Delta m))$. So whatever is the optimal choice of $\Delta m'$ given m', we can always reach the same state from the state m at lower cost as $m > m' \Rightarrow$ number of people who recommend is higher $\Rightarrow$ it requires lower probability $\rho \Rightarrow$ it is less costly. That means that the optimal choice of continuation strategy from the state m can not give a lower continuation payoff than in the state $m' \Rightarrow V(m) > V(m')$. □

Proof of Proposition 7. The proof of consists of 2 steps:

1. Suppose that the optimal strategy σ^* of a consumer is such that he chooses $\rho_t = 0$ and $\rho_{t+1} = \tilde{\rho} > 0$. Then let's proof that he can do better if deviates to $\{\sigma^d : \rho^*_j = \rho^d_j, \forall j \neq t, t+1, \rho^d_t = \rho k_{t+1}/k, \rho^d_{t+1} = 0\}$.

   Number of informed consumers does not decrease over time, so $m \leq m_{t+1}$. We chose the deviation strategy such that the number of consumers which are informed in both periods t and $t+1$ by directly a particular node is not less for σ^d than for σ^*. Each node starts to get benefits from additionally informed consumers directly in current period. As established in Proposition 6 an increase in number of informed consumers in current period increases continuation payoff. So we have that even for $\delta = 1$, the payoff which a node gets from directly informed consumers in periods t and $t+1$ minus recommendation costs is not less under strategy σ^d and indirect effect of earlier informed consumers does not decrease a continuation payoff, so strategy σ^d weakly dominates strategy σ^* for any δ and strictly dominates for $\delta < 1$.

2. We established that if $\rho_t = 0$ then $\rho_j = 0, \forall j > t$. No suppose that at period t $\rho_t = 0$ and there are $k > 0$ uninformed consumer. Such situation obviously can not appear in

equilibrium as always exists ρ, such that $R(\rho) < \rho\gamma$, because $R'_\rho(0) = \infty$ and $R(\rho)$ is a convex function. That means that if $\rho_t = 0 \Rightarrow$ all nodes are informed.

$\square$

Chapter 3

Non-reservation Price Equilibria and Consumer Search

with Maarten C. W. Janssen and Alexei Parakhonyak

3.1 Introduction

In consumer search markets, firms have market power due to the fact that some consumers do not make price comparisons. Firms take this market power into account when deciding on price. This paper addresses the question of whether, by focusing on consumers following reservation price strategies, the existing consumer search literature accurately evaluates this market power due to search frictions. A reservation price strategy is a cut-off strategy: after observing a price at or below some critical value, consumers decide to buy, otherwise they continue to search.

In markets where there is uncertainty about the underlying factors determining firms' pricing behavior, there are important theoretical reasons to consider other search strategies than reservation price strategies. Rothschild, 1974 drew attention to the fact that when consumers do not know from which distribution of offers they obtain their information, the optimal consumer search rule may well be different from the typical reservation price rule.[1] The main reason is that on the basis of past search observations, consumers learn about the environment in which they search. Depending on the environment, it may well be that, after observing a relatively good outcome, consumers infer that even better outcomes are likely to be observed in the next search round and rationally conclude to continue to search, whereas, after observing a relatively bad outcome, consumers infer that better outcomes are unlikely and thus stop searching.

The consumer search literature has, by and large, neglected this observation. The celebrated models by Stahl, 1989 and Wolinsky, 1986, and much of the literature that takes these models as a starting point, study environments without underlying uncertainty and in such theoretical environments the optimal search rule is indeed a reservation price rule. In consumer search markets where consumers are uninformed about firms' underlying costs (and this probably comprises most markets where consumer search is important), learning is an important part of the search process. There are some papers on learning and consumer search that take consumer uncertainty about firms' costs into consideration (see, Benabou and Gertner, 1993, Dana, 1994, Fishman, 1996 and more recently, Yang and Ye, 2008, Tappata, 2009, Janssen, Pichler, and Weidenholzer, 2011 and Chandra and Tappata, 2011). The observations by Rothschild, 1974 are of immediate concern to these environments, but the relevant economics literature has continued to focus on equilibria where the consumer search rule is characterized by a reservation price.

Some of this literature is inspired by retail gasoline markets where the common wholesale price of crude oil is the most important determinant of the (variation in) costs of retailers, and consumers are uncertain about these costs due to the large fluctuations of this wholesale price on the world market. Although our focus in this paper is on consumer search in retail markets, the issues we address are also relevant for other markets. For example, Benabou and Gertner, 1993 is

[1] Dubra, 2004 studies how optimism and over confidence affect search.

motivated by macroeconomic concerns about inflationary uncertainty and the consequences for firms' mark-ups, while a recent paper by Duffie, Dworczak, and Zhu, 2014 considers over-the-counter (OTC) financial markets and the role of benchmarks in these markets. The current paper is also relevant for the labor search literature where workers search for a better wage. In labor markets, it is natural that the wage distribution depends on the business cycle and that firms are better informed about the business cycle than workers. In that case, workers learn about the wage distribution while searching for another job and their search behavior does not need to follow a reservation wage strategy. In all these markets, there is uncertainty and asymmetric information about a common component that determines the distribution of offers and one needs to understand how the uninformed side (consumers, workers) search and simultaneously learn in such an environment.

Our paper is the first to systematically incorporate Rothschild's observations on non-reservation price strategies into an equilibrium search model with endogenous firm behavior.[2,3] Benabou and Gertner, 1993 also mention the fact that in their model reservation price equilibria (RPE) may not exist. They set up the equations that have to be satisfied in a non-RPE. They perform some numerical analysis for some parameter values, but they neither have an analysis characterizing these non-RPE, nor do they show the conditions under which these equilibria exist.[4]

The literature studying RPE in environments where consumers are uninformed about firms' cost is unsatisfactory for a number of reasons. First, RPE are known to exist only if the search cost is relatively large and/or the uncertainty about costs is relatively small, (cf., Dana, 1994 and Janssen, Pichler, and Weidenholzer, 2011). It is unclear what type of equilibria do exist for small search cost or large uncertainty about common costs. Second, RPE implicitly assume certain out-of-equilibrium beliefs and it is unknown whether these out-of-equilibrium beliefs satisfy game theoretic refinement concepts commonly employed in asymmetric information games. Third, one would expect that when costs are uncertain consumers may engage in active costly search in equilibrium. When consumers observe a high price, they are uncertain about whether this is due to a relatively high (common) production costs or whether this particular firm is charging a high margin. RPE in these homogeneous goods markets have firms charging prices below the consumer reservation price, however, and therefore all consumers buy at the first firm they visit. This lack of consumer search gives firms substantial market power, but it may well be that RPE overestimate the true market power because they underestimate consumers' search intensity.

In response to these points, this paper first sharpens existing results on RPE. We show (i) that independent of out-of-equilibrium beliefs, RPE do not exist when the uncertainty about production cost is relatively large and (ii) that an RPE, even if it exists, is sensitive to the specification of out-of-equilibrium beliefs and do not satisfy, for example, the logic of the D1 equilibrium refinement (hereafter the D1 logic, see Cho and Sobel, 1990). If the uncertainty about cost is relatively large, any equilibrium should have active search. We then continue to characterize non-RPE and show that they exist for all parameter values and that there are parameter values for which multiple non-RPE exist. Thus, non-RPE resolve the non-existence problem that RPE suffer from. Moreover, in any non-RPE, consumers actively search beyond the first firm. In particular, there

[2] Even though our paper focuses on consumer search, similar considerations apply to the labour search literature that uses reservation wage equilibrium as a solution concept (see McCall, 1970 for pioneering work in this direction, and subsequent literature as, for example, surveyed in Rogerson, Shimer, and Wright, 2005).

[3] Lauermann, Merzyn, and Virag, 2018 consider a bargaining and matching environment with uncertainty about the relative scarcity of a commodity. They also do not restrict themselves to studying reservation price strategies, but their set-up is different as they have one side of the market competing in an auction to acquire the good from the other side of the market, instead of markets where firms post prices (as in our setting).

[4] Benabou and Gertner (1993, p.74) state that the "non-reservation price equilibria (if they exist) are too complicated for us to solve" and argue that these equilibria are "somewhat less appealing intuitively than the previous reservation price equilibria" (p. 81) as they think the reservation price property is "required in particular for demand functions to be downward sloping" (p. 74). This is, however, only partially the case. In order to make firms indifferent over the range of prices in a mixed strategy equilibrium, a firm's expected total demand must be downward sloping. In the non-RPE in our model, the demand of an *individual consumer* must have a downward sloping part and *may* have an upward sloping part.

is a region of "high" prices that are set with positive probability such that consumers are indifferent between buying and searching and consumers continue to search with strictly positive probability. When the cost uncertainty is large, market prices may be substantially below the market prices predicted by RPE due to active search by consumers. On the other hand, when cost uncertainty is small, expected market prices are larger in non-RPE. Thus, whether or not RPE overestimate the market power of firms depends on the uncertainty about cost.

In a recent empirical paper, Santos, Hortaçsu, and Wildenbeest, 2012 show that, when buying books online consumers do not follow reservation price strategies. These strategies predict that (i) consumers buy from the last store visited unless all stores have been visited and (ii) the decision whether to continue to search depends on the outcome of the previous search with consumers observing lower prices deciding to buy, and consumers observing larger prices deciding to continue searching. Their evidence contradicts these predictions.[5] In this paper, we show that their empirical findings are consistent with equilibrium behavior under non-reservation price strategies, as follows. When the cost uncertainty is relatively large, non-RPE have a region of intermediate prices where the probability of a sale is lower when the price is low. At lower prices, consumers rationally expect to get lower prices on the next search round and this may induce them to search more. In particular, we show that consumers may accept higher prices in the first search round, while rejecting lower prices. In an extension to oligopolistic markets, we also show that the optimal sequential search behavior of consumers is consistent with consumers going back to previously sampled firms, before they have sampled all firms.[6]

There is also a relationship with the marketing oriented literature on reference price effects (see, e.g., Putler, 1992, Kalyanaram and Winer, 1995 and Mazumdar, Raj, and Sinha, 2005). This literature points to the fact that consumers have particular pricing points around which consumer demand is very sensitive to price changes. This may lead to situations where consumer demand drops significantly if firms price above this reference point, whereas at higher prices, consumers are willing to buy again. Such "reference point" demand behavior can occur in non-RPE when the cost uncertainty is large. After observing intermediate prices above the "reference price", consumers rationally infer that these prices are not chosen by high cost firms. Knowing costs are low, consumers find these prices too high to buy, however, and they continue to search for sure. This inference creates a gap in the equilibrium price distribution of the low cost firms.

The rest of the paper is organized as follows. Section 2 describes the model and the equilibrium concept. Section 3 discusses how RPE depend on assumptions regarding out-of-equilibrium beliefs and why (regardless of these out-of-equilibrium beliefs) they do not exist when cost uncertainty is relatively large. Section 4 describes our analytical results on non-RPE. Section 5 shows, by means of a numerical analysis, the effects of cost uncertainty on profits, expected prices and consumer welfare. Section 6 briefly discusses a generalization of our model to the case of imperfectly correlated production costs and oligopoly markets with N firms. We show that with three or more firms, the optimal search rule may imply that consumers first continue searching another firm, and then go back to a previously sampled firm before all firms are sampled. Section 7 concludes with a discussion, while proofs are given in two Appendices.

3.2 The Model and Equilibrium Concept

The sequential search model we analyze is based on the homogeneous goods models studied in Dana, 1994 and Janssen, Pichler, and Weidenholzer, 2011.[7] The basic model we analyze in

[5] A similar conclusion is drawn in a recent paper on insurance markets (see, Honka and Chintagunta, 2016).

[6] This last observation can also be rationalized by assuming that consumers face an increasing search cost.

[7] Search models for heterogeneous goods typically follow the model developed by Wolinsky (1986). In that model, firms choose pure price strategies as uncertainty resulting from the match distribution of product variety is exogenously imposed. Janssen and Shelegia, 2015b introduce common cost uncertainty in that model and show that firms

the next three Sections incorporates cost uncertainty in a duopoly version of Stahl, 1989.[8] Firms produce a homogeneous good and compete in prices. The production cost of each firm equals c, which can take one of two values, $c \in \{c_L, c_H\}$, $c_L \leq c_H$. Production cost is common for both firms. Denote by α the probability that $c = c_H$, where $0 < \alpha < 1$. Firms observe their production cost, but consumers do not. After observing the realization of cost, firms simultaneously set prices. We denote the (symmetric), perhaps degenerate, price distributions chosen by firms by $F_L(p)$ and $F_H(p)$ when cost is low or high, respectively. The highest price which will be charged by low and high cost firms is denoted by $\overline{p}_L$ and $\overline{p}_H$, respectively, whereas the respective lowest prices will be denoted by $\underline{p}_L$ and $\underline{p}_H$. Each firm's objective is to maximize profits, taking the prices charged by the other firm and consumers' search behavior as given.

The demand side of the market is represented by a unit mass of risk-neutral consumers with identical preferences and unit demand. A fraction $\lambda \in (0,1)$ of consumers, *shoppers*, have a zero search cost. These consumers sample all prices and buy at the lowest price. The remaining fraction of $1 - \lambda$ consumers – *non-shoppers* – search *sequentially* and have a positive search cost $s > 0$ to obtain one additional price quote. They visit each of the two firms at their first search with equal probability. These consumers face a non-trivial problem when searching for low prices, as they have to trade off the search cost with the expected benefit from search. After observing their first price quote, non-shoppers update their beliefs about firms' underlying production costs using Bayes' rule. Consumers can always go back to previously visited firms, incurring no additional cost.[9] We denote the probability that non-shoppers buy after observing price p by $\beta(p)$. With the remaining probability $1 - \beta(p)$ these consumers continue to search. As consumers do not know the underlying production cost, $\beta(p)$ does not depend on the cost realization. Denote by ρ_i the consumers' reservation price if they were to infer that the firms' production cost equals c_i for sure, $i = L, H$.

The timing of the model is as follows. First, Nature chooses c for both firms. After observing c, firms simultaneously decide on their prices. Finally, consumers search and make their purchase decisions.

We consider Perfect Bayesian Equilibria of the game. A (symmetric) equilibrium is a tuple of (*i*) pricing strategies $F_i(p)$, $i = L, H$ such that any p in the support of F_i maximizes firm i's profit, (*ii*) an optimal search strategy $\beta(p)$ minimizing the expected price (including search cost) at which a consumer buys given her beliefs, and (*iii*) beliefs that are consistent with firms' pricing strategies on the equilibrium path.

It is by now a standard argument in the search literature with symmetric information that due to the presence of shoppers and non-shoppers there do not exist equilibria with mass points in the price distributions. This argument continues to hold in our model with asymmetric information as far as pure pricing strategies are concerned: even if all non-shoppers continue to search, an undercutting firm will sell to all shoppers and non-shoppers that first visit that firm. In the present model, this argument does not extend, however, to ruling out pricing distribution with mass points. Given that equilibria have to be in mixed strategies and that we prove that equilibria without mass points always exist, we restrict our attention to mixed pricing strategies without atoms.

As explained in the Introduction, the existing literature focuses on reservation price equilibria, which are defined as follows.

Definition 1. *A Perfect Bayesian Equilibrium is a Reservation Price Equilibrium if there exists a p_0 such that $\beta(p) = 1$ for all $p \leq p_0$ and $\beta(p) = 0$ for all $p > p_0$.*

continue to choose pure strategies and consumers learning the state of the world upon observing a price follow reservation price strategies. Thus, the issues we point at in this paper with the non-existence of RPE are not relevant in that alternative search context.

[8] It is not difficult to extend the analysis to oligopoly situations if we replace sequential search with newspaper search, in which upon paying a search cost (after observing the first search) consumers are immediately informed about all $(N - 1)$ other prices. Extending the analysis with sequential search to oligopoly markets is not straightforward and we discuss the extent to which we can generalize the results in Section 3.6.2.

[9] Janssen and Parakhonyak, 2014 analyze the case where this assumption is replaced by costly recall.

When investigating non-RPE, we focus on equilibria satisfying the logic of the D1 criterion (Cho and Sobel, 1990).[10] The D1 criterion was developed in the context of pure signaling games with one sender. Our model is a two-sender game, where the beliefs of the receivers (the non-shoppers) are only based on the single price they have observed. As firms are of the same type, the out-of-equilibrium belief of non-shoppers is simply a mapping from the observed price to the type distribution of cost, as in the one-sender game.

Consider any perfect Bayesian equilibrium where the equilibrium profit of firm j when it is of type i is given by $\pi_i^{j*}, i = H, L$. Consider any p outside the support of the equilibrium price distribution. This out-of-equilibrium price generates a set of possible optimal actions of the receiver (non-shopper). Let $B_j(p)$ be the set of a firm j's total demand from shoppers and non-shoppers that can be generated by buying probabilities $\beta_j(p)$ of non-shoppers (at firm j at price p) that are best responses to some non-shoppers' belief. A $q_j(p) \in B_j(p) \subset [0,1]$ is an element of this set. For out-of-equilibrium price that are larger than the highest price charged along the equilibrium path, shoppers will not buy and demand at such a price is bounded by $\frac{1-\lambda}{2}$. The D1 refinement compares the sets of demands $\{q_j \in B_j(p) : (p - c_i)q_j \geq \pi_i^{j*}\}$ for which it is gainful for different types i of firm j to deviate to price p. If for $i, i' \in \{H, L\}, i' \neq i$,

$$\{q_j \in B_j(p) : (p - c_i)q_j \geq \pi_i^{j*}\} \subset \{q_j \in B_j(p) : (p - c_{i'})q_j > \pi_{i'}^{j*}\}$$

where "$\subset$" stands for strict inclusion, the D1 logic requires that the out-of-equilibrium beliefs of non-shoppers (upon observing a unilateral deviation by firm j to price p) should assign zero probability to the event that firm j is of type i and thus (as there are only two types and firms have a common type), assign probability one to firm j being of type i'.[11] Intuitively, as type i' has an incentive to deviate to p for a larger set of responses by the non-shoppers than type i, the first type is said to have a stronger incentive to deviate.

In addition, we will focus on equilibria where $\beta(p)$ is continuously differentiable almost everywhere as follows. Let $P = [0, \overline{p}]$, with $\overline{p} = \max\{\overline{p}_L, \overline{p}_H\}$ and define sets $P_{(0,1)} = \{p : p \in P, 0 < \beta(p) < 1\}$, $P_1 = \{p : p \in P, \beta(p) = 1\}$ and $P_0 = \{p : p \in P, \beta(p) = 0\}$.[12] We consider equilibria, where $\beta(p)$ is continuously differentiable in the interior of these sets, and refer to this as $\beta(p) \in C^1$.[13]

Definition 2. *A symmetric Perfect Bayesian Equilibrium is a non-reservation price equilibria (non-RPE) that satisfies the D1 logic and is sufficiently smooth if (i) it does not satisfy Definition 1, (ii) $\beta(p) \in C^1$, (iii) $F_i(\cdot)$ are continuous and (iv) consumers' out-of-equilibrium beliefs are consistent with the D1 logic.*

Thus, from the set of equilibria that do not satisfy Definition 1, we focus on those equilibria that satisfy the D1 logic and that are sufficiently smooth. For easy reference, we refer to such equilibria in the rest of the paper as non-reservation price equilibria (non-RPE). There may exist other equilibria that do not satisfy the properties of a RPE and that do not satisfy the D1 requirement and are not smooth. We do not consider these equilibria in our paper as we show that even with the additional requirements of D1 and smoothness, we can guarantee existence.

[10]Ideally, we want to make sure that the upper bound of the price distribution does not depend on arbitrary out-of-equilibrium beliefs. In the next section we show that this can be achieved when consumers, after observing the highest price charged in equilibrium, believe that they are in a high cost environment. This implies that independent of their beliefs at higher prices consumers prefer to search for a better price after observing such higher price. As we show later the D1 criterion implies that after observing a price equal to the upper bound of the price distribution consumers believe that they are in a high cost environment. Accordingly, if a pricing strategy profile is part of an equilibrium for beliefs that satisfy the D1 criterion, it is also part of an equilibrium for any other out-of-equilibrium beliefs.

[11]A similar treatment is given in Janssen and Roy, 2010 for a more complicated inference problem where consumers observe all prices and there are N firms.

[12]Note that it is not the case that all sets P_i that satisfy the criteria are necessarily convex. In Section 4 we provide examples of equilibria where the set of *all* prices with $\beta = 1$ or with $\beta \in (0,1)$ are non-convex.

[13]Smoothness of the $\beta(p)$ function is needed to assure that the process of Bayesian updating of the underlying cost is continuous on the relevant sets.

Moreover, the equilibria we consider are interesting in their own right and do not depend on arbitrary out-of-equilibrium beliefs or on more technical issues related to non-smoothness.

3.3 Reservation Price Equilibria

In this Section, we summarize some existing results on RPE, (i) prove that they do not exist if the cost uncertainty is large and (ii) prove that they do not satisfy the D1 logic even if they do exist.

Dana, 1994 has characterized RPE in our model, while Janssen, Pichler, and Weidenholzer, 2011 have generalized that analysis to N firms and production cost being distributed according to a continuous distribution function. For $N = 2$, Janssen, Pichler, and Weidenholzer, 2011 showed that the equilibrium price distribution for cost realization c_i is given by

$$F(p|c_i) = 1 - \frac{1-\lambda}{2\lambda}\frac{\bar{p}-p}{p-c_i}, \quad i = L, H \tag{3.1}$$

with support on $[\underline{p}_i, \bar{p}]$ with $\underline{p}_i = \frac{2\lambda}{1+\lambda}c_i + \frac{1-\lambda}{1+\lambda}\bar{p}$, $i = L, H$, and $\bar{p} = \rho$, the consumers' reservation price. The derivation of the mixed strategy distribution follows from the fact that given a firm's own price p, its profit is given by

$$\left[\lambda(1 - F_i(p)) + \frac{1-\lambda}{2}\right](p - c_i), \quad i = L, H,$$

and that in a mixed strategy equilibrium, this profit has to be equal to the profit the firm gets if it sets a price equal to the upper bound ρ of the price distribution, $\frac{1-\lambda}{2}(\rho - c_i)$. It follows that the respective density functions are given by

$$f_i(p) = \frac{1-\lambda}{2\lambda}\frac{\rho - c_i}{(p-c_i)^2}, \quad i = L, H,\text{[14]} \tag{3.2}$$

and that the reservation price ρ is implicitly determined by

$$\rho = s + \frac{\alpha f_H(\rho)}{\alpha f_H(\rho) + (1-\alpha)f_L(\rho)}E(p|c_H) + \frac{(1-\alpha)f_L(\rho)}{\alpha f_H(\rho) + (1-\alpha)f_L(\rho)}E(p|c_L). \tag{3.3}$$

The latter equation basically says that at the reservation price, the consumer must be indifferent between buying now (and paying a price ρ) and continuing to search, which costs s, and paying the expected price, giving the consumer updates her beliefs about whether the underlying cost is high or low. Janssen, Pichler, and Weidenholzer, 2011 show that the expression for the reservation price can be rewritten as $\rho = E(c|\rho) + \frac{s}{1-\gamma}$, where γ is a parameter that only depends on λ and N (which equals two in our case).

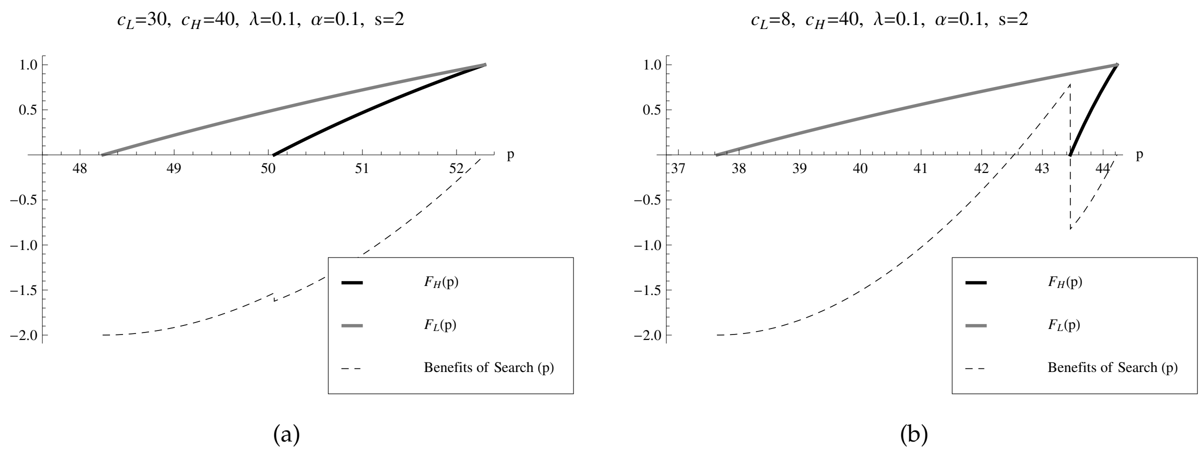
A few results that will be used later follow from this characterization. First, no firm charges prices above the reservation price. Second, the low and high cost densities at the reservation price are positive, i.e., $f_i(\rho) > 0$. This implies that the posterior belief that cost is low after observing the reservation price, $\Pr(c_L|\rho)$, which is given by

$$\Pr(c_L|\rho) = \frac{(1-\alpha)f_L(\rho)}{\alpha f_H(\rho) + (1-\alpha)f_L(\rho)},$$

is strictly larger than 0 (and strictly smaller than 1). Third, for any price p that is in the support of the equilibrium price distributions in both states of the world, the density in the low cost state is smaller than the density in the high cost state. This implies that $F_H(p)$ first-order stochastically

[14] As $p \geq \underline{p}_i > c_i$ it follows that the densities are finite.

FIGURE 3.1: RPE and incentives to search



dominates $F_L(p)$. Thus, the expected price when cost is low is smaller than the expected cost when cost is high, i.e.,

$$E(p|c_L) < E(p|c_H). \tag{3.4}$$

Formally, Dana, 1994 and Janssen, Pichler, and Weidenholzer, 2011 only show that a sufficient condition for the existence of an RPE is that the cost uncertainty $c_H - c_L$ is sufficiently small and that the search cost s is sufficiently large. To show that these are also necessary conditions, we first show that if $c_H - c_L$ is sufficiently large or s is sufficiently small an RPE does not exist irrespective of the out-of-equilibrium beliefs.

Proposition 1. *If $c_H - c_L$ is sufficiently large, or s is sufficiently small an RPE does not exist.*

We illustrate the "$c_H - c_L$ being sufficiently large" part of Proposition 1 in Figure 3.1, where we depict the solutions to equations (3.1)-(3.3) for various cost differences. If an RPE exists, it should be the solution to equations (3.1)-(3.3). As the Figure shows, consumers' beliefs discretely change at $\underline{p}_H$, as for all $p < \underline{p}_H$ consumers infer that they are in a low cost environment. When $c_L = 30$, as on the left pane of the Figure, this does not cause a problem with the equilibrium construction, as the net benefit of search is still negative. However, if $c_L = 8$, the reservation price ρ_L conditional on being in the low cost environment is smaller than $\underline{p}_H$, and for prices just below $\underline{p}_H$ the net benefit of search is positive. Therefore, an RPE does not exist as non-shoppers prefer to continue to search for lower prices after observing a price just below $\underline{p}_H$.[15]

The result of Proposition 1 also holds for heterogeneous search costs. In order to see this, fix some (sufficiently large) c_H and (sufficiently small) s. Then, as it follows from the proof of Proposition 1 there is a sufficiently small c_L such that the reservation price, conditional on knowing that the cost is low, is below c_H. As a high-cost firm will not price below c_H, this implies that consumers would search for prices just below c_H. Now, if search costs are distributed on some interval $[s', s]$ below the search costs considered in Proposition 1, the distribution of prices for low-cost firms cannot be higher than the one for a single search cost s, and therefore consumers would still search actively for prices just below c_H.

We next show that if an RPE does not exist, any equilibrium without mass points should have a region of prices where non-shoppers actively search (search with positive probability).

[15]Note that the non-existence of an RPE for large cost uncertainty (and the "gap in the set of accepted prices") does not depend on the assumption of a binary cost state. If we think of c_H and c_L as the upper and lower bounds of the set of possible cost states with a distribution that is concentrated around these upper and lower bounds, then it continues to be true that the incentive to continue searching may change dramatically (although not discontinuously) when a price just below or just above $\underline{p}_H$ is observed. In fact, Janssen, Pichler, and Weidenholzer, 2011 contains an example where an RPE fails to exist if cost is uniformly distributed.

Proposition 2. *For any search cost $s > 0$, if $c_H - c_L$ is sufficiently large, then consumers search with positive probability in any PBE with continuous $F_i(\cdot)$.*

Basically, from the first part of the proof of Proposition 1 we can use the fact that $\rho_L < \underline{p}_H$ for $c_H - c_L$ being sufficiently large. This implies there should be a region of prices just above $\underline{p}_H$ where there is active search to keep the low cost firm indifferent between charging these prices and ρ_L. The cut-off value of $c_H - c_L$ where an RPE does not exist anymore in Proposition 1 is be exactly the cut-off value where consumers search with positive probability in Proposition 2.

Finally, we show that any RPE, even if it exists for some out-of-equilibrium beliefs, does *not* satisfy the D1 logic. The D1 logic asks which type of firm (high or low cost) has most incentive to deviate to prices above the reservation price. It turns out that high cost firms have stronger incentives to deviate; see the proof of the next Proposition for details. This implies that if a consumer continues to search after observing an out-of-equilibrium price $\rho + \varepsilon$, for small ε, she expects to pay $E(p|c_H) + s$, including the search cost. If the consumer buys immediately, she pays $\rho + \varepsilon$, which using (3.7) can be rewritten as

$$\varepsilon + \Pr(c_H|\rho)E(p|c_H) + \Pr(c_L|\rho)E(p|c_L) + s.$$

This expression is strictly smaller than the expected payment in case of search if, and only if,

$$\Pr(c_L|\rho)\left(E(p|c_H) - E(p|c_L)\right) > \varepsilon.$$

From (3.2) and (3.4) it follows that the LHS is strictly positive. Thus, one can choose ε sufficiently small so that at $\rho + \varepsilon$ it is optimal to buy. Firms would then, however, have an incentive to deviate and set these higher prices defying the notion of equilibrium. The next Proposition formalizes this logic and extends it to all equilibria without active search.

Proposition 3. *All perfect Bayesian equilibria in which non-shoppers buy with probability one in the first search round and in which $F_i(p)$ is continuous, do not satisfy the D1 logic.*

There are two important corollaries, which follow immediately from Proposition 3. First, as in any RPE firms' pricing distributions are atomless (see, e.g. Stahl, 1989), and there is no active search (see Dana, 1994), we immediately have the following corollary.

Corollary 1. *All reservation price equilibria do not satisfy the D1 logic.*

Second, by Definition 2 we have

Corollary 2. *In any non-reservation price equilibrium non-shoppers search with positive probability.*

Note that the D1 logic may be extreme in the sense that it requires that $\Pr(c_L|p) = 0$ for any $p > \rho$. Following the logic of the proof, even weaker restrictions on the out-of-equilibrium beliefs are sufficient to eliminate RPE, however. RPE requires that after observing prices above the reservation price, consumers infer that cost is low with sufficiently high probability. Thus, RPE do not exist either if the out-of-equilibrium beliefs are discountinuous and such that there exists a $k > 0$ such that $\Pr(c_L|p) + k < \Pr(c_L|\rho)$ for any $p > \rho$. RPE can, however, be compatible with out-of-equilibrium beliefs $\Pr(c_L|p)$ that are strictly decreasing for $p > \rho$ and that are continuous at ρ.

3.4 Characterisation and Existence of Non-RPE

In any non-RPE, a firm's profit $\pi(p|c_i)$ when setting price p and cost is $c_i, i = H, L$ can be written as

$$\begin{aligned}\pi(p|c_i) = \Bigg[&\lambda(1 - F_i(p)) + \frac{1-\lambda}{2}\beta(p) + \frac{1-\lambda}{2}(1-\beta(p))(1-F_i(p)) + \\ &\frac{1-\lambda}{2}\int_p^{\overline{p}}(1-\beta(\tilde{p}))f_i(\tilde{p})d\tilde{p}\Bigg](p - c_i).\end{aligned} \tag{3.5}$$

This expression can be understood as follows. First, a firm only attracts shoppers if the other firm charges a higher price, which occurs with probability $1 - F_i(p)$. The number of non-shoppers buying from firm i gives a more complicated expression. There is a fraction $(1-\lambda)/2$ of non-shoppers that randomly first visits firm i, and buys immediately from that firm with probability $\beta(p)$. The remaining non-shoppers that randomly first visit firm i continue searching the other firm and come back to firm i if the other firm has a higher price. Finally, the non-shoppers that first visit the other firm and decide to continue to search buy from firm i if it has a lower price. As firm i does not know which price the other firm charges, this expression involves an expected number of consumers.

To characterize the price distributions of non-RPE, we first show that the upper bounds of the low and high cost price distributions have to be identical. If this were not the case, there would be a region of prices above the upper bound of, say, the low cost distribution that are only chosen by high cost firms, and this would imply that $\beta(p) = 1$. Low cost firms would then have an incentive, however, to deviate to such prices.

Lemma 1. *In PBE with continuous $F_i(p)$, thus, in any non-RPE, $\overline{p}_L = \overline{p}_H \equiv \overline{p}$.*

Without mass points, a firm setting a price equal to the upper bound $\overline{p}$ of the price distribution will not sell to the shoppers and their profits will be equal to $\frac{1-\lambda}{2}\beta(\overline{p})(\overline{p} - c_i)$. As in equilibrium, for any price in the support of the price distribution this expression has to be equal to (3.5), we have that

$$\begin{gathered}\lambda(1 - F_i(p)) + \frac{1-\lambda}{2}\beta(p) + \frac{1-\lambda}{2}(1-\beta(p))(1-F_i(p)) + \\ \frac{1-\lambda}{2}\int_p^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p} = \frac{1-\lambda}{2}\beta(\overline{p})\frac{\overline{p}-c_i}{p-c_i}.\end{gathered} \tag{3.6}$$

At intervals of prices in the support of the price distribution where $\beta(p) = 1$, or, $\beta(p) = 0$, this equation can be solved for $F_i(p)$ in a straightforward manner. As we concentrate on equilibria where $\beta(p)$ is continuously differentiable in the interior of $P_{(0,1)}$, equation (3.6) can be transformed into an exact differential equation that can be solved as shown in the proof of the following Proposition.

Proposition 4. *If $F(p)$ is a price distribution in a non-RPE, then it should be of the following form:*

$$F_i(p) = \begin{cases} \dfrac{2\sqrt{1-(1-\lambda)\beta(\overline{p})} - \int_p^{\overline{p}} \frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(\widetilde{p}-c_i)^2\sqrt{1-(1-\lambda)\beta(\widetilde{p})}}d\widetilde{p}}{2\sqrt{1-(1-\lambda)\beta(p)}} & \text{if } p \in P_{(0,1)} \\ 1 - \frac{1-\lambda}{2\lambda}\left[\beta(\overline{p})\frac{\overline{p}-c_i}{p-c_i} - 1 - \int_p^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right] & \text{if } p \in P_1 \\ 1 - \frac{1-\lambda}{1+\lambda}\left[\beta(\overline{p})\frac{\overline{p}-c_i}{p-c_i} - \int_p^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right] & \text{if } p \in P_0 \end{cases} \tag{3.7}$$

Using the characterization of the price distributions, we can now state that $F_H(p)$ first-order stochastically dominates the low cost distribution $F_L(p)$. Thus, as in RPE, we continue to have the expected price when cost is low, $E(p|c_L)$, being lower than the expected price when cost is high, $E(p|c_H)$.

Corollary 3. *In any non-RPE, for all $p < \overline{p}$, $F_L(p) \geq F_H(p)$ and whenever $0 < F_H(p) < 1$, $F_L(p) > F_H(p)$.*

Using these characterizations of the distribution functions, it is not too difficult to see that if we want that the upper bound of the distributions $\overline{p}$ is not determined by arbitrary out-of-equilibrium beliefs, it must be the case that after observing $\overline{p}$ consumers believe that firms have high cost for sure, and that given this inference, non-shoppers are indifferent between buying now and continuing to search. If this were not the case, and non-shoppers had out-of-equilibrium beliefs such that $\Pr(c = c_H|p) = 1$ for prices $p > \overline{p}$, then they would prefer to buy at these prices, giving firms an incentive to deviate (see the proof of Proposition 3 for details). Thus, the density of the low cost distribution at the upper bound must be equal to zero,

$$f_L(\overline{p}) = 0 \tag{3.8}$$

and the upper bound of the price distributions has to be equal to the reservation price in the case where consumers know cost is high, i.e.,

$$\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp = s. \tag{3.9}$$

As $F_H(p)$ first-order stochastically dominates $F_L(p)$ this implies that if an out-of-equilibrium price larger than $\overline{p}$ is observed, consumers will always want to continue to search independent of their beliefs of the underlying cost.

To fully characterize an equilibrium of the model, we have to inquire into the non-shoppers' equilibrium strategy, $\beta(p)$, with $0 \leq \beta(p) \leq 1$. Optimal search behavior implies that whenever $0 < \beta(p) < 1$ the non-shopper is indifferent between buying now and continuing to search, implying that

$$\frac{(1-\alpha)f_L(p)}{(1-\alpha)f_L(p) + \alpha f_H(p)}\Phi_L(p) + \frac{\alpha f_H(p)}{(1-\alpha)f_L(p) + \alpha f_H(p)}\Phi_H(p) = s, \tag{3.10}$$

where $\Phi_i(p) = \int_0^p F_i(x)dx$. This equation says that after a non-shopper observes price p she will update her beliefs about the underlying cost of the firms and given these updated beliefs concludes that buying now yields the same expected pay-off as continuing to search. Optimal search behavior also implies that the non-shoppers strictly prefer to buy ($\beta(p) = 1$) if the LHS of (3.10) is strictly smaller than s and that the non-shoppers strictly prefer to search ($\beta(p) = 0$) if the LHS of (3.10) is strictly larger than s. Together with (3.7) this behavior characterizes an equilibrium.

As shown in Appendix I, equation (3.10) defines a differential equation which, starting from initial conditions for $\overline{p}$ and $\beta(\overline{p})$, defines the function $\beta(p)$ going downward.[16] This function can continue to satisfy $0 < \beta(p) < 1$ or it may at some price point p reach the boundaries $\beta(p) = 1$ or $\beta(p) = 0$. If for some prices $\beta(p) = 1$, the following Lemma shows that (3.10) implies that in any equilibrium $\beta'(p) = 0$ has to hold at the largest price point p where $\beta(p) = 1$.

Lemma 2. *Let p^* be such that $\beta(p^*) = 1$ and for any sufficiently small $\epsilon > 0$ $\beta(p^* + \epsilon) < 1$. Suppose that p^* is in the interior of the support of $F_i(p), i = L, H$. Then in equilibrium it must be that $\beta'(p^*) = 0$.*

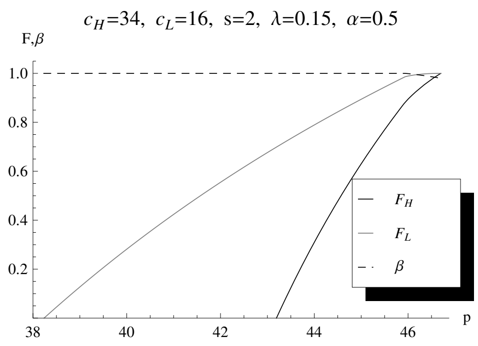
We will now inquire into the existence question. The main question is whether for all parameter values $c_L, c_H, \lambda, \alpha$ and s we can find values of $\overline{p}$ and $\beta(\overline{p})$ such that equation (3.7) defines proper distribution functions that are upward sloping, and that the search strategy of non-shoppers satisfies the optimality condition (3.10).[17] Our main Theorem shows that a non-reservation price equilibrium as defined in Definition 2 exists for all values of the exogenous parameters.

Theorem 1. *For any values of s, λ, c_L, c_H and α a non-reservation price equilibrium as defined in Definition 2 exists. The equilibrium price distributions are characterized by (3.7), while non-shopper's behavior is determined by (3.10) whenever $0 < \beta(p) < 1$.*

[16]Note that (3.10) implies that we should have $\beta'(\overline{p}) = -\beta(\overline{p})/(\overline{p} - c_L)$ as derived in Proposition A.1 in Appendix I.

[17]In Appendix II we show that for given $F_L(p)$ and $F_H(p)$ (3.10) essentially is a differential equation that determines the function $\beta(p)$ up to the boundary condition $\beta(\overline{p})$.

FIGURE 3.2: No-gap equilibrium



The proof is constructive and consists of several Lemmas. It is formally developed in Appendix II. For a range of parameter values the equilibrium is not unique, while for other parameter values it is unique. We now extensively describe how for any set of parameter values we construct an equilibrium.

The four-step procedure works as follows. First, we show that we can always find values $\bar{p}$ and $\beta(\bar{p})$ such that the solution of the system of equations (3.7) and (3.10) satisfies two boundary conditions: $\beta'(p^*) = 0$ and $\int_{\underline{p}_H}^{\bar{p}} F_H(p)dp = s$, where p^* is defined in Lemma 2. Given the values of $\bar{p}$ and $\beta(\bar{p})$, we take $\beta(p) = 1$ for all $p < p^*$ and define ρ_L such that

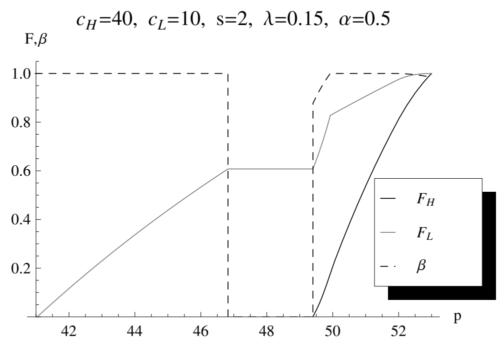
$$\int_{\underline{p}_L}^{\rho_L} F_L(p)dp = s. \tag{3.11}$$

Here we can distinguish between two cases. If $\rho_L \geq \underline{p}_H$ then we claim we have found a non-reservation price equilibrium without a gap in both price distributions. In this so-called no-gap equilibrium, as in any other non-RPE, conditions (3.8) and (3.9) must be satisfied, so after observing $\overline{p}$ non-shoppers are indifferent between buying now and continuing to search. At all prices p with $p^* < p < \bar{p}$ both $f_L(p)$ and $f_H(p)$ are strictly positive and (3.10) guarantees that non-shoppers are indifferent over the whole interval. Finally, Lemma A.3 in Appendix II shows that if we specify $\beta(p) = 1$ for all $p < p^*$, then the equilibrium density functions are such that consumers indeed prefer to buy at all these prices. Lemmas A.1 and A.2 prove that (3.7) always define proper price distributions. An example of a *no-gap equilibrium* is given in Figure 3.2. This Figure illustrates that at high prices $\beta(p) < 1$ and at lower prices $\beta(p) = 1$ and the price distributions do not have a gap. Figure 3.2 also illustrates that the demand of individual consumers is downward sloping.

The second possibility is that the condition $\rho_L \geq \underline{p}_H$ is violated. In that case, it is natural to have out-of-equilibrium beliefs satisfying $\Pr(c_L|p) = 1$ for all $p \in (\rho_L, \underline{p}_H)$ implying that consumers would not like to buy at these prices and $\beta(p) = 0$. This out-of-equilibrium belief not only follows from the D1 logic, but also from the weaker notion of the Intuitive Criterion (Cho and Kreps, 1987).[18] For given values of the other parameters, $\rho_L < \underline{p}_H$ will arise when the cost difference $c_H - c_L$ is sufficiently large, i.e., the same reason why an RPE may fail to exist for any out-of-equilibrium belief (see Proposition 1 in Section 3).

[18]Intuitively, the reason is as follows: by setting a price equal to $\underline{p}_H$ a high cost firm already attracts all shoppers and all non-shoppers that first visited that firm. Of the remaining non-shoppers it will sell to all who continue to search after having visited the first firm. By deviating to a lower price, a firm can never get a higher demand, and lowering the price, can only lower the profits. A low cost firm may have an incentive to deviate to prices $p \in (\rho_L, \underline{p}_H)$ if $\beta(p)$ is sufficiently high. As the high cost type does not have an incentive to deviate and the low cost type may have an incentive (depending on the reaction of the non-shoppers), the Intuitive Criterion implies that $\beta(p) = 0$ for all $p \in (\rho_L, \underline{p}_H)$.

FIGURE 3.3: Monopolistic gap equilibrium



Below, in steps (2)-(4) we indicate different ways an equilibrium with a gap can be constructed. A gap equilibrium configuration presumes that $\beta(p) = 0$ for all $p \in (\rho_L, \underline{p}_H)$ and $F_L(\rho_L) = F_L(\underline{p}')$, where $\rho_L < \underline{p}' \leq \underline{p}_H$. In steps (2)-(3) we claim that $\underline{p}' = \underline{p}_H$ as $\beta(\underline{p}_H) > 0$ requires that for any price $p \in (\rho_L, \underline{p}_H)$ the low cost density function is equal to 0. Moreover, for all $p \in [\underline{p}_L, \rho_L)$ we must have $\beta(p) = 1$ due to the fact that at ρ_L non-shoppers are indifferent between buying and not buying even if they know that the underlying cost is low. Finally, it must be the case that $\beta(\underline{p}_H) < 1$ as otherwise the low quality firms cannot be indifferent between setting $\underline{p}_H$ and ρ_L as the chance to attract shoppers is the same at both prices.

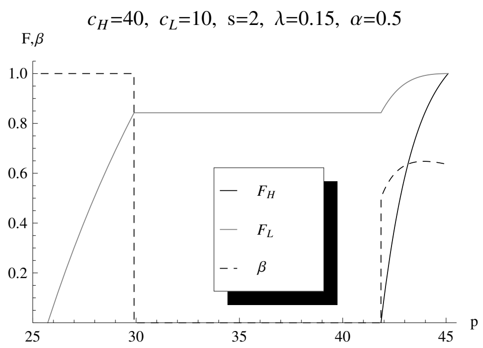
The second step in the equilibrium construction procedure is then to try to construct a non-reservation price equilibrium (in case there is no "no-gap equilibrium") that is close to the "no-gap equilibrium" in that there also exists an interval of prices $p \in [x, p^*]$ with $\underline{p}_H < x < p^*$ where $\beta(p) = 1$. The main difference with the "no-gap equilibrium" is that $\beta(p) < 1$ for prices p with $\underline{p}_H < p < x$. In addition to the two parameter values and two boundary conditions we encountered above in the "no-gap equilibrium", we have one more parameter value that we can choose, namely x to make sure that $\pi_L(\rho_L) = \pi(\underline{p}_H)$, where ρ_L has to satisfy (3.11). In the proof of the main theorem in Appendix II, we show that if there is no "no-gap equilibrium" and $\rho_L < \underline{p}_H$ we can satisfy this third boundary condition as well with a positive size of a gap, provided that we can find an x such that $x \leq p^*$. For lack of a better term, we call such a non-reservation price equilibrium, a *monopolistic gap equilibrium* for the fact that there is a interval of prices $p \in [x, p^*]$ where $\beta(p) = 1$ and firms have some "monopoly power" over non-shoppers as they always buy.

Figure 3.3 illustrates a monopolistic gap equilibrium. At prices close to $\overline{p}$, but also at prices close to $\underline{p}_H$ non-shoppers are indifferent between buying and continuing to search and $\beta(p) < 1$. At prices at and close to $\underline{p}_H$, $\beta(p) > 0$ and $\beta'(p) > 0$ and the low cost distribution function is much steeper in this price region than the high cost distribution function. There is a relatively small gap in the low cost price distribution and $\beta(p) = 1$ for all $p \leq \rho_L$. At the lowest price p such that $\beta(p) = 1$, $\beta(p)$ is not continuously differentiable.[19]

The third step in the procedure starts from the fact that if a no-gap equilibrium does not exist, then a monopolistic gap equilibrium may also fail to exist if we cannot find a value $x \leq p^*$ such that all necessary boundary conditions are satisfied. In this case, we try to construct an equilibrium with $0 < \beta(p) < 1$ for all $p \geq \underline{p}_H$. As there is no price $p \geq \underline{p}_H$ where $\beta(p) = 1$ the condition that $\beta'(p^*) = 0$ is no longer relevant. So we need the two parameter values $\beta(\bar{p})$ and $\bar{p}$ to satisfy the two boundary conditions (3.9) and (3.11) which are relevant for this type of equilibrium, and ρ_L is determined from the indifference condition $\pi_L(\rho_L) = \pi_L(\underline{p}_H)$. Here we can get multiple equilibria which can also coexist with a monopolistic gap equilibrium. We call

[19]Note that Lemma 2 only applies to the largest price p where $\beta(p) = 1$.

FIGURE 3.4: Regular gap equilibrium



this type of equilibrium where $0 < \beta(p) < 1$ for all $p \geq \underline{p}_H$ *a regular gap equilibrium*.[20]

Figure 3.4 illustrates a regular gap equilibrium. In this equilibrium, there are four regions of prices where non-shoppers exhibit different behavior. At high prices (above $\overline{p}$), consumers definitely continue to search. Consumers are indifferent between buying and continuing to search for all prices $p \in [\underline{p}_H, \overline{p}]$ as they update their beliefs about cost being low and the probability of finding lower prices if continuing to search. At prices below $\underline{p}_H$ (and above ρ_L) non-shoppers search for sure. Finally, at prices below ρ_L non-shoppers buy for sure.

The fourth and final step in the procedure starts from the fact that if neither a no-gap equilibrium nor a monopolistic gap equilibrium exists, the only reason why we cannot find parameter values $\beta(\bar{p})$ and $\bar{p}$ to satisfy the two boundary conditions of a regular gap equilibrium is if the constraint $\beta(p) > 0$ for all $p \geq \underline{p}_H$ cannot be satisfied. From Lemmas A.4-A.7 in Appendix II it follows that if this constraint is violated for some prices p, it is certainly violated for all smaller prices. Thus, in our final step we construct an equilibrium where $\beta_H(\underline{p}_H) = 0$. In this case, low cost firms charge prices with positive probability in the interval $p \in [\underline{p}', \underline{p}_H]$ for some $\rho_L < \underline{p}' < \underline{p}_H$ where $\beta(p) = 0$. The third line in equation (3.7) shows the distribution function of low cost prices in this case. In this case, we need that $F_L(\underline{p}') = F_L(\rho_L)$ and $f_L(p) = 0$ for all $p \in (\rho_L, \underline{p}')$ and choose $\beta(\bar{p}), \bar{p}, \underline{p}'$ such that the following three boundary conditions are satisfied: (3.9), (3.11) and $\beta(\underline{p}_H) = 0$. We call such an equilibrium a competitive gap equilibrium (see Figure 3.5) as there is a region of prices that is set by low cost firms where consumers only buy if they know that this is the lowest price in the market. In the last part of the proof of Theorem 1 we show that such a competitive gap equilibrium must exist if none of the other non-reservation price equilibria exist. This finishes the description of the four-step procedure to construct an equilibrium.

The equilibrium construction outlined above is illustrated in Figure 3.6. The Figure shows for given values of s, λ and α how the equilibrium configuration depends on the cost difference $c_H - c_L$. For relatively small values of λ, Figure 3.6(a) shows there are three possible equilibrium configurations, depending on whether the cost difference is small, large or intermediate. If the cost difference is relatively small, there is a unique equilibrium without a gap in the low cost distribution. When c_H is close to c_L, the value of $\beta(\overline{p})$ has to be close to 1 and in the limit, when cost uncertainty disappears, the Stahl, 1989 equilibrium is the only possible equilibrium. If, on the other hand, the cost difference $c_H - c_L$ is relatively large, then there exists a unique regular gap equilibrium. The value of $\beta(\overline{p})$ has to be relatively low to satisfy the equilibrium conditions for such an equilibrium to exist. Finally, if the cost difference $c_H - c_L$ is at intermediate values, a

[20]Obviously, an equilibrium where $\beta(p) = 1$ at just one point $p \geq \underline{p}_H$ is a transition between the monopolistic gap and the regular gap equilibrium.

FIGURE 3.5: Competitive gap equilibrium

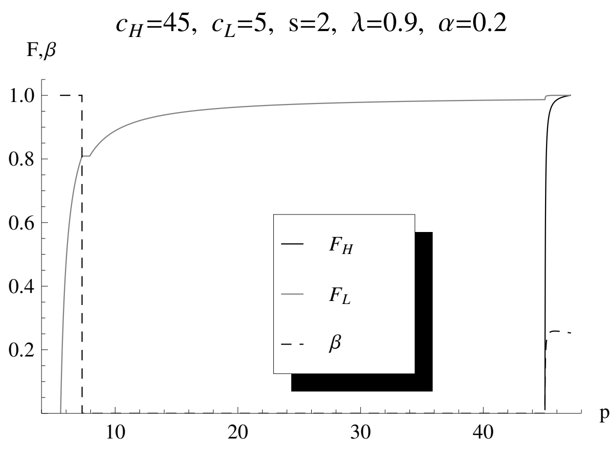
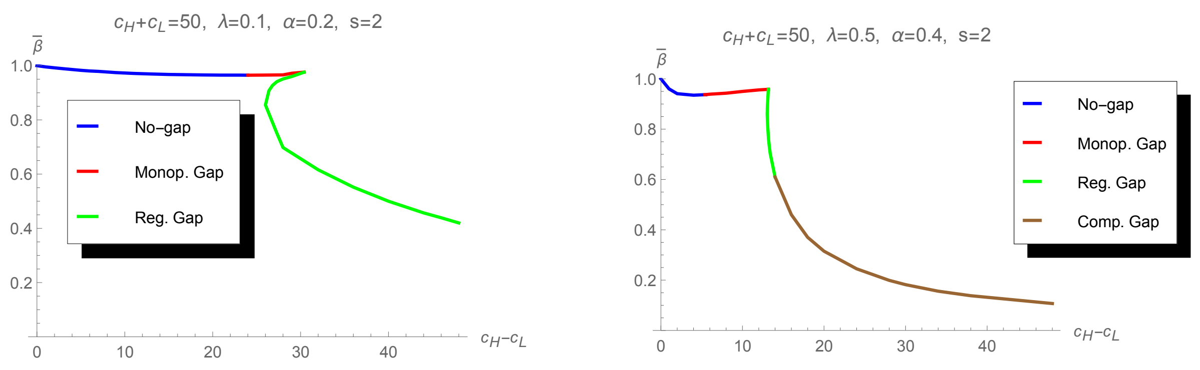


FIGURE 3.6: $\overline{\beta}$ as a function of cost difference.



(a) (b)

monopolistic gap equilibrium co-exists together with two regular gap equilibria.[21] For larger values of λ, Figure 3.6(b) distinguishes four possible equilibrium configurations, while equilibrium is unique for each value of the cost difference. That Figure shows that a regular gap equilibrium may fail to exist if the cost difference $c_H - c_L$ is relatively large and a competitive gap equilibrium emerges.

We end this Section discussing how our analysis may shed some light on the empirical observations we mentioned at the end of the Introduction. Numerically, one can compare for a given cost realization (i) the expected first price observation conditional on the price being accepted and (ii) the expected first price observation conditional on it not being accepted. Santos, Hortaçsu, and Wildenbeest, 2012 observe that in their sample the first conditional expected price is larger than the second, and they rightly claim that this is inconsistent with RPE. For the parameter values used in Figures 3.3-3.5, one can compute and compare both conditional expected prices to conclude that for the high cost realization these non-RPE are consistent with the findings of Santos, Hortaçsu, and Wildenbeest, 2012: for Figure 3.4 the respective numbers are 42.94 and 42.83, for Figure 3.3 the numbers are 50.81 and 49.83, while for Figure 3.5 they are 45.19 and

[21] This multiplicity of equilibria is genuine and we do not know of plausible equilibrium selection arguments that can be used in this context. Fershtman and Fishman, 1992 use a stability argument to argue that one of the equilibria in their search model is unstable. It is difficult to see how a stability argument can be invoked in our context, as the behaviour of consumers is not characterized by a single parameter as in their model, but by the function $\beta(p)$.

45.11, respectively. Thus, we conclude that the observations of Santos, Hortaçsu, and Wildenbeest, 2012 are not necessarily inconsistent with sequential search, although they are inconsistent with reservation price strategies.

Figures 3.3, 3.4 and 3.5 show that non-RPE do not exhibit a simple monotone relationship between price and the probability of buying (or the probability of continuing to search). In a gap equilibrium, the $\beta(p)$ functions have an increasing segment, indicating that at higher prices in that segment the probability of non-shoppers buying at the firm that is visited first is higher (and thus the probability they continue searching is lower). Figure 3.5 shows an extreme case of this where there is a region of prices that are set by low cost firms such that non-shoppers continue to search for sure, while at higher prices the probability of continuing to search is lower. Thus, these Figures indicate that the optimal search behavior may be highly nonmonotonic in price. Santos, Hortaçsu, and Wildenbeest, 2012 empirically find that it is not the case that at higher prices, consumers are more likely to continue to search, while Honka and Chintagunta, 2016 find that it is not the case that consumers with more offers in their consideration sets tend to have higher offers. Again, our analysis shows that this does not rule out that consumers search sequentially.

Finally, equilibria where the low cost price distribution has a non-convex support may be interpreted as a search theoretic foundation for the reference price principle that is discussed in marketing (see the references in the Introduction). In our model, reference prices endogenously arise from the fact that consumers rationally infer that a certain low price will only be set when cost is low, and if the common cost is really low, then the chances of finding low prices are so high that it is rational to continue searching for better deals. Thus, it is better for firms not to set prices just above these reference prices.

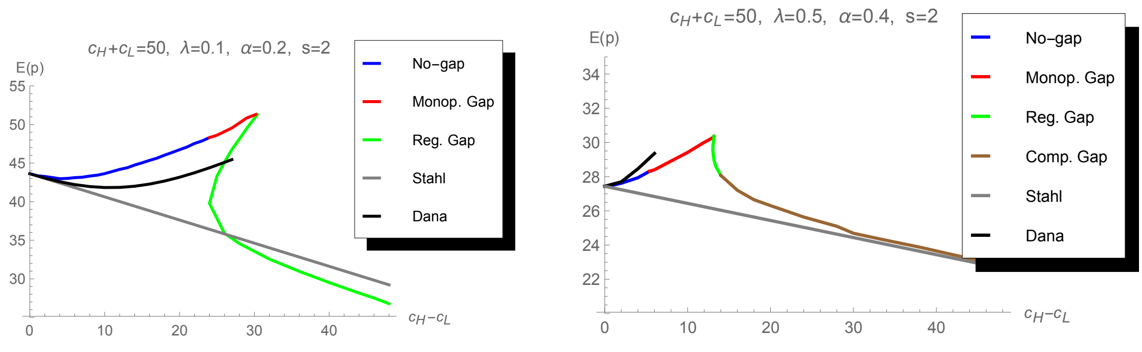
3.5 Comparative Statics and Comparing Models

We are now in a position to compare the equilibrium outcomes of our model with two benchmark models, and to perform some numerical comparative statics analysis. On one hand, we use Stahl, 1989 as a benchmark to show the implications of cost uncertainty. On the other hand, we use Dana, 1994, or equivalently Janssen, Pichler, and Weidenholzer, 2011, as a benchmark for the outcome of RPE with cost uncertainty. As shown in Janssen, Pichler, and Weidenholzer, 2011, the expected price under RPE is larger than the weighted average of the expected price of the high and low cost equilibria as developed by Stahl, 1989 and in that sense, consumers are worse off under cost uncertainty. In this Section we show that this result may well be reversed for non-RPE.

There are several effects that play a role when comparing the outcomes of non-reservation price equilibria with those of RPE. First, for a given upper bound $\overline{p}$, lowering $\beta(\overline{p})$ from an initial value of 1 (which is the value in the case of RPE) implies that there are more consumers making price comparisons. This implies firms tend to lower their prices as a reaction to the increased competition. A second effect is a direct consequence: as for a given upper bound expected prices will be lower, therefore searching for lower prices becomes more beneficial (as the expected prices after a search are lower), the upper bound has to be lower (as it is equal to the high-cost reservation price at which non-shoppers have to be indifferent between searching and buying). The third effect is that in a non-RPE, non-shoppers believe that cost is high after observing the upper bound, while in an RPE as in Dana, 1994 and Janssen, Pichler, and Weidenholzer, 2011, the upper bound equals the weighted average of the reservation prices when cost is certainly low or certainly high. This effect increases the upper bound of the price distributions and thereby also the expected prices.

Figure 3.7 shows the typical effect on ex ante expected price of these three effects. Expected price is a good measure of the surplus of the non-shoppers. When they continue to search, non-shoppers pay the search cost, but they also get to buy at the lowest of two prices. As in equilibrium, when they search twice they are indifferent between buying and searching, and the

FIGURE 3.7: Expected prices as a function of cost difference



(a) (b)

additional expected benefit of the possibility of buying at a lower price is exactly offset by the cost of the additional search. In both panels of Figure 3.7, the average cost is taken to be 25 and the cost difference $c_H - c_L$, measured on the horizontal axis, varies between 0 (implying the cost is known to be 25) and 50 (where $c_L = 0$ and $c_H = 50$). When the cost difference is 0, all models result in the same expected price. In the Stahl, 1989 model where cost is known, the expected price is a fixed number larger than the cost level, where the fixed number depends on λ and s, but not on c. The ex ante expected price reported here for the Stahl model is the weighted costs plus this fixed number. This expected price is thus decreasing in the cost difference $c_H - c_L$, if $\alpha < 0.5$ (as in Figure 3.7). The expected price in Dana, 1994 is known to be higher than the ex ante weighted average of the expected prices in the Stahl, 1989 model. The Figures also show that the RPE analyzed in Dana, 1994 does not exist for larger cost differences. Figures 3.7(a) and 3.7(b) show that for smaller cost uncertainty, expected prices are even larger than the ones reported in Janssen, Pichler, and Weidenholzer, 2011. This is due to the fact that for small cost uncertainty, the third effect outlined above dominates. For small cost uncertainty, RPE tend to underestimate firms' market power (measured by margins). The Figures also show, however, that for larger cost differences the expected price in a non-RPE becomes smaller and that it can even become smaller than the ex ante weighted average of expected prices in the Stahl, 1989 model. For small values of λ, Figure 3.7(a) shows that this difference can be as large as 10%! Cost uncertainty leads here to lower market prices due to the additional search effect resulting in increased competition between firms. In Figure 3.7(b), for large cost differences, the expected price converges to the ex ante expected price in the Stahl, 1989 model. For large cost uncertainty, RPE may thus overestimate the market power due to search frictions.

For better understanding of the mechanism behind the comparison with the Stahl, 1989 model, consider again Figure 3.4 and keep in mind that Janssen, Pichler, and Weidenholzer, 2011 have shown that in the Stahl model with known cost expected price is simply a mark-up of $s/(1-\tau)$ above marginal cost, with $\tau = \int_0^1 \frac{1}{1+\frac{\lambda N}{1-\lambda}\ln\frac{1+\lambda}{1-\lambda}}$ and $N = 2$. For the parameters considered in Figure 3.4, this mark-up approximately equals 12 so that expected prices in the two states would be 22 and 52, respectively. One can clearly see in Figure 3.4 that the expected price (and margin) in the high cost state is significantly lower, while it is significantly higher in the low cost state. Low cost firms can raise prices by pretending to be high cost firms. In non-RPE this leads, however, to active consumer search from which the high cost firms suffer. The high cost margin reduces to 2.88 (from 12), while the low cost margin increases to 20.17. As $\alpha = 0.5$ in Figure 3.4, the average margin is smaller under cost uncertainty in a non-RPE. The potential strength of the additional search effect can be illustrated by comparing the same effects in Figure 3.3 that is drawn for the same parameter values. In the monopolistic gap equilibrium in Figure 3.3, prices are much higher because non-shoppers almost do not search (and when they do at prices just above $\underline{p}_H$ this almost does not affect the high cost distribution, as at these prices it is anyway very likely that searching consumers return to the shop to buy).

In the different panels of Figure 3.8, we perform a numerical comparative static analysis showing how expected price and the probability that non-shoppers search twice, which is given by

$$E(1-\beta(p)) = \alpha \int_{\underline{p}_H}^{\rho_H} (1-\beta(p)) f_H(p) dp + (1-\alpha) \int_{\underline{p}_L}^{\rho_H} (1-\beta(p)) f_L(p) dp,$$

changes with the changes in the different exogenous parameters s, λ and a.

The first two panels (3.8(a) and 3.8(b)) show the dependence on search cost. For small search cost, a large fraction of non-shoppers performs two searches and the expected price is close to the average marginal cost of 25. When the search cost increases from initially low levels, the expected price increases and the fraction of non-shoppers performing two searches decreases (giving firms more market power). At search cost levels close to 2, there are multiple gap equilibria, and it may be that the expected price is decreasing in search cost. When the search cost further increases, a no-gap equilibrium emerges and the probability of non-shoppers searching twice becomes very close to 0. Panel (3.8(b)) also shows that starting from an initially small search cost, non-shoppers will search less when the search cost increases. In this way, non-shoppers partially mitigate the increase in market power typically associated with higher search cost.

The middle two panels (3.8(c) and 3.8(d)) show the dependence on the fraction of shoppers. When λ is small, there are many non-shoppers and a no-gap equilibrium exists. In such an equilibrium, very few non-shoppers perform two searches and the expected price is high. When λ increases, the expected price decreases, but in the area where multiple equilibria exist, the difference in the expected price can be quite large as the fraction of non-shoppers performing two searches differs greatly between the different equilibria. When λ increases further, we enter the area where only competitive gap equilibria exist. In this case, increasing λ leads to a higher probability that low cost firms price in the area where $\beta(p) = 0$ and the average price increases slightly.

The last two panels (3.8(e) and 3.8(f)) show the dependence on the probability that the cost is high. When this probability is high, there is a no-gap equilibrium and consumers search very little, since there is a low probability of obtaining a substantially lower price. In this region the higher the α, the higher the expected price. For lower values of α, there is a monopolistic gap equilibrium with qualitatively similar properties. When α is sufficiently low, there are multiple gap equilibria and the incentives to search can be high, pushing the prices down. The expected price can be both increasing and decreasing in α depending on which of the regular gap equilibria is chosen.

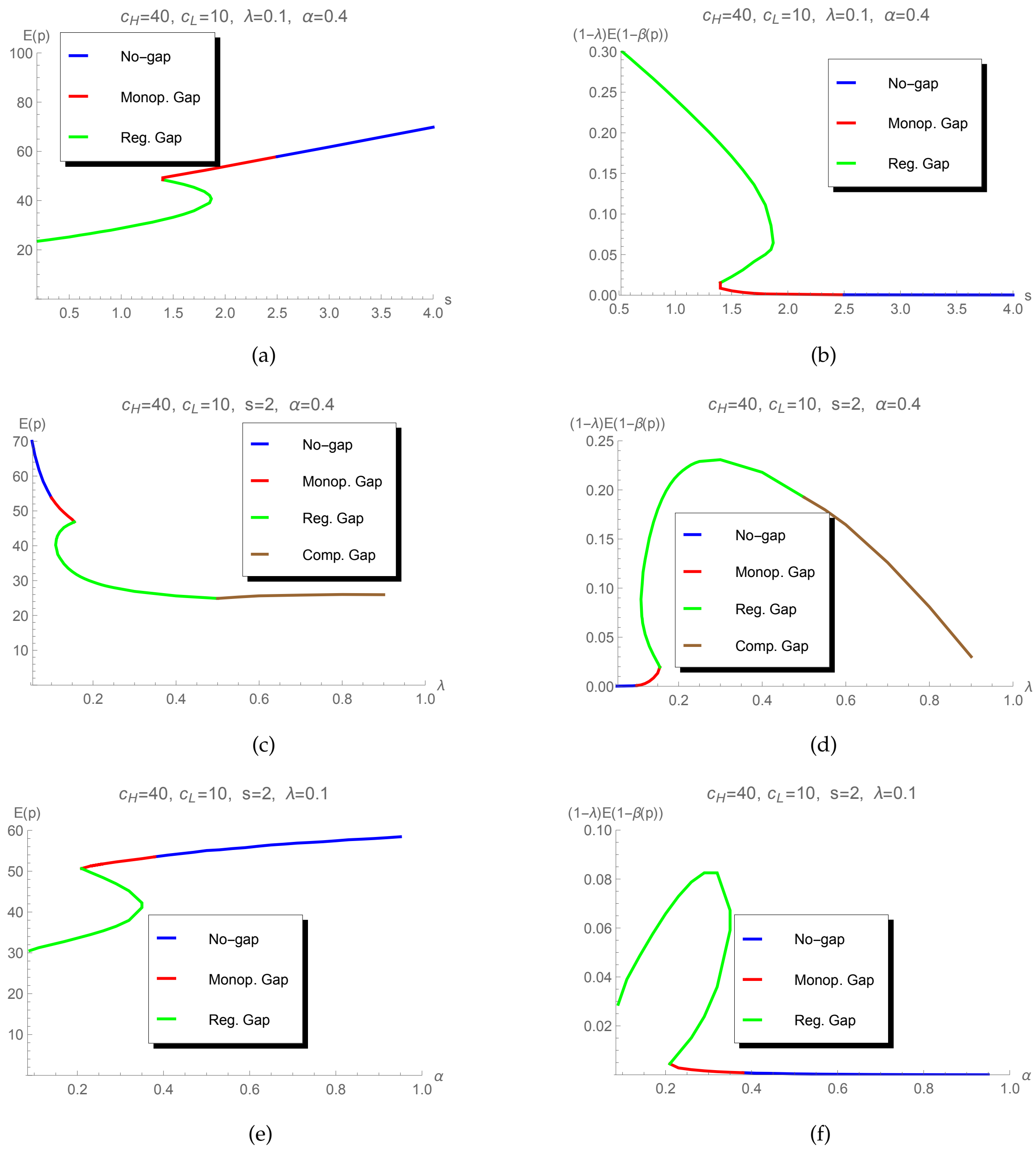
3.6 Extensions

In this Section, we deal with two important extensions of our general analysis. The first relates to introducing more general forms of correlation between firms' costs, the second relates to a first analysis of oligopoly markets with sequential search.

3.6.1 Introducing an Idiosyncratic Cost Component

In this extension we slightly modify the model described in Section 2 by introducing idiosyncratic cost shock in addition to the common cost shock we have analyzed so far. Suppose that each firm has a cost component κ_i, $i = L, H$, which is independent between the firms. Suppose that $\kappa_H \geq \kappa_L$ and the high cost state occurs with probability γ. The idiosyncratic cost shock is private information to the firm, and the common cost shock is, as before, known to both firms, but not to consumers. The total marginal cost of every firm is $c_{ij} = c_i + \kappa_j$ and we refer to the pricing strategy of such firm as $F_{ij}(p)$.

FIGURE 3.8: Comparative statics



In Appendix III, we describe the analysis for the case where there is no uncertainty concerning the common cost component (the "pure idiosyncratic cost shock" case). The main take away from that analysis is that (for the same common cost component) a firm with a low idiosyncratic cost state randomizes prices over a support that is below and does not overlap with the support of the price distribution of a firm with a high idiosyncratic cost state.

Below we describe how we can use these results to combine them with the results we have derived in this paper under common cost uncertainty. We first characterize the distributions $F_{iH}(p)$ in the upper part of the support, i.e., for all $p \geq \underline{p}_{HH}$. From Appendix III, we know that the firms with low idiosyncratic cost will not price in this area. Adapting equation (1), when setting price p firms with high idiosyncratic cost make a profit of

$$\left[\lambda\gamma(1-F_{iH}(p)) + \frac{1-\lambda}{2}\beta(p) + \frac{1-\lambda}{2}(1-\beta(p))\gamma(1-F_{iH}(p)) + \frac{1-\lambda}{2}\gamma\int_p^{\overline{p}}(1-\beta(\widetilde{p}))f_{iH}(\widetilde{p})d\widetilde{p}\right](p-c_{iH})$$

so that using the same technique as in the proof of Proposition 4, the distributions $F_{iH}(p)$ have to satisfy

$$-2\gamma\left[1-(1-\lambda)\beta(p)\right]dF_{iH} + \left[(1-\lambda)\beta'(p)(1-\gamma+\gamma F_{iH}) + (1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_{iH})^2}\right]dp = 0.$$

Solving for $F_{iH}(p)$ gives

$$F_{iH}(p) = \frac{2\gamma\sqrt{1-(1-\lambda)\beta(\overline{p})} - \int_p^{\overline{p}} \frac{(1-\gamma)(1-\lambda)\beta'(\widetilde{p})(\widetilde{p}-c_{iH})^2+(1-\lambda)\beta(\overline{p})(\overline{p}-c_{iH})}{(\widetilde{p}-c_{iH})^2\sqrt{1-(1-\lambda)\beta(\widetilde{p})}}d\widetilde{p}}{2\gamma\sqrt{1-(1-\lambda)\beta(p)}}. \tag{3.12}$$

As in the pure common cost case studied in the main body of the paper, the upper bound $\overline{p}$ is determined purely by high common cost considerations. The same is true here, but now we have to take into account that a firm with a high common and idiosyncratic cost component does not know whether the other firm in the market has a high or a low idiosyncratic cost component. Applying the result on the determination of the reservation price for the idiosyncratic case derived in Appendix III, we have

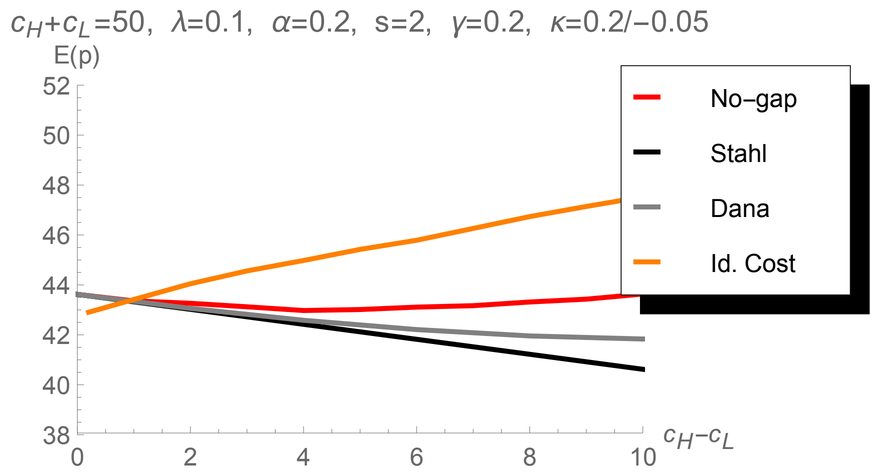
$$\overline{p} = \rho_H = \gamma E_{HH}(p) + (1-\gamma)E_{HL}(p) + s,$$

where now $E_{HH}(p)$ is defined by probability distribution (3.12). To determine ρ_H, we now also need to have $E_{HL}(p)$.

Depending on which type of equilibrium we have, there are different cases to consider. For the no-gap equilibrium, we have that $E_{HL}(p)$ is determined by using (3.28) in Appendix III. This equilibrium holds if at prices $p < \underline{p}_{HH}$ (which are set by both HL and LH firms) consumers prefer to buy rather than continue to search. Depending on the parameters, however, it may well be that consumers prefer to continue to search at such prices (which is in line with the regular or the monopolistic gap equilibrium in the common cost framework). If that is the case, there will be a gap $(x, \underline{p}_H)$ in the $F_{LH}(p)$ distribution where $\beta(p) = 0$ with $F_{HL}(x) = 1$ and $\beta(x) = 1$ such that consumers prefer to buy at prices $p \leq x$.[22]

[22]The determination of x is slightly more complicated than in the common cost case as now after observing x the consumers should still update their beliefs taking into account that both LH and HL firms will choose x. LH firms should be indifferent between charging prices x and $\underline{p}_H$, while the HL firms should (at least weakly) prefer charging x. In these cases, to determine $E_{HL}(p)$ (to be able to determine ρ_H), we should use (3.28), where $\underline{p}_{iH}$ is replaced by x.

FIGURE 3.9: Expected price in a model with idiosyncratic cost component



We conclude that it is entirely possible to extend the analysis in the main body of the paper and deal with situations where firms' cost has a common and an idiosyncratic component. Figure 3.9 shows that if we add idiosyncratic cost uncertainty, the expected market price can be both lower and higher than without this uncertainty. The impact of idiosyncratic cost uncertainty on expected prices is very different than the impact of common cost uncertainty, as there is no consumer learning. The main effect is through the fact that the low and high cost distributions are not overlapping and that the reservation price is based on a weighted average of the expected prices of these two distributions.

3.6.2 Oligopoly Markets

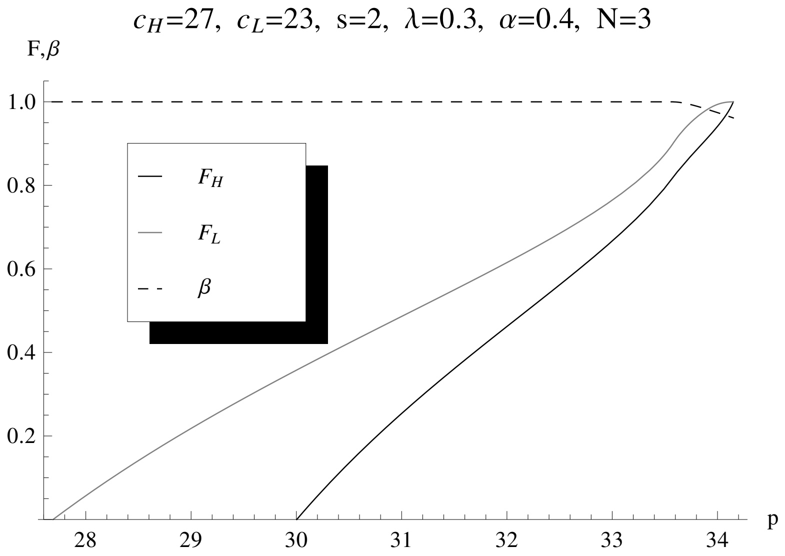
It is not too difficult to reformulate our analysis to an oligopoly model by replacing sequential search with "newspaper search" a la Salop and Stiglitz, 1977 and Dana, 1994. Under newspaper search, a consumer pays a search cost only once to see all remaining prices. Thus, in both the duopoly model with sequential search and the oligopoly model with newspaper search, a consumer effectively has to take the decision whether or not to continue to search only once.

Staying in the sequential search paradigm, it is challenging to give full analytical characterization of non-RPE when there are more than two firms in the market. The difficulty is due to the fact that depending on the prices observed, consumers may perform a different number of searches, creating complications for solving for the price distribution of firms. Nevertheless, the following result on the optimal search behavior of consumers helps to considerably reduce the complexities in analyzing certain types of equilibria under oligopoly. In this result we denote by $p^{(t)}$ the price a non-shopper observes in search round t.

Proposition 5. *Suppose the consumer was indifferent between continuing to search or buying after the first price observation $p^{(1)}$ and $f_H(p) > f_L(p)$ for all $p \in P_{(0,1)}$. Then if the consumer continued, she stops searching after the second price observation $p^{(2)}$ and buys at* $\min\{p^{(1)}, p^{(2)}\}$.

There are two interesting aspects about this Proposition. First, if a non-shopper observes two prices $p^{(1)}$ and $p^{(2)}$, with $p^{(1)} < p^{(2)}$, then the Proposition says the consumer will stop searching and go back to the first firm if the high cost density is larger than the low cost density. Thus, going back to previously sampled firms before all firms are searched may well be consistent with a sequential search. Santos, Hortaçsu, and Wildenbeest, 2012 have observed that consumers do go back to previously sampled firms before having visited all firms. This is inconsistent with reservation price strategies, as they noted, but not necessarily with sequential search.

FIGURE 3.10: Equilibrium price distributions and stopping probability for $N = 3$



Second, if in a non-RPE we have that $f_H(p) > f_L(p)$ in the price region where $\beta(p) < 1$, then we know that non-shoppers will never search beyond the second firm, and the profit function under oligopoly can be written as

$$\begin{aligned}\pi(p|c_i) &= \left[\lambda(1-F_i(p))^{N-1} + \frac{1-\lambda}{N}\beta(p) + \frac{1-\lambda}{N}(1-\beta(p))(1-F_i(p)) + \right. \\ &\left. \frac{1-\lambda}{N-1}\frac{N-1}{N}\int_p^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right](p-c_i)\end{aligned}$$

so that the differential equation which has to be solved to find the distribution functions reduces to

$$-2\left[1+\frac{\lambda N(N-1)}{2(1-\lambda)}(1-F_i)^{N-2}-\beta(p)\right]dF_i + \left[\beta'(p)F_i + \beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}\right]dp = 0. \tag{3.13}$$

This differential equation can be solved numerically, and it can be checked whether $f_H(p) > f_L(p)$ indeed holds for all prices in the price region where $\beta(p) < 1$. In Figure 3.10, we illustrate the distribution functions that solve (3.13) for particular parameter values. It can be checked that the condition on the density functions is satisfied.

In Section 4, we have seen that in the equilibria with a gap in the low cost price distribution (e.g. one depicted in Figure 3.3), $f_H(p) < f_L(p)$ holds for prices close to $\underline{p}_H$. This implies that Proposition 5 does not hold. In that case, it may well be that with N firms in the market, consumers search three or more firms before going back to the lowest price in their sample. This makes the analysis, however, quite tedious, and an analytical treatment of non-RPE is then certainly not feasible.

3.7 Discussion and Conclusion

In this paper we have considered consumer search markets where firms' underlying common cost is unknown to consumers. If consumers do not know the prices different firms charge, it is natural that they also do not know the underlying cost. We have argued that in this environment of cost uncertainty, the standard RPE considered in the consumer search literature suffer from severe limitations. It was already known that RPE do not always exist, but we add that RPE implicitly assume specific out-of-equilibrium beliefs that do not satisfy standard game theoretic refinements. We characterize non-RPE that do not depend on specific assumptions regarding out-of-equilibrium beliefs and show that these equilibria always exist. Non-RPE may provide a significantly different assessment of the market power firms derive from search frictions.

In non-RPE, non-shoppers are indifferent between buying and continuing to search over a range of prices. As prices in this range are set with positive probability, these non-RPE have active search with positive probability in equilibrium. Thus, we extend the Rothschild, 1974 finding by showing in a model with endogenous price setting that in equilibrium firms price in such a way that consumers do not choose reservation price strategies. The fact that consumers rationally search more with cost uncertainty in non-RPE explains why market power may be overestimated in RPE. The additional search has a quantitatively important pro-competitive effect on prices.

Our results on non-RPE also have important consequences for the empirical literature on consumer search models that has recently taken off. Non-RPE may explain the observations of Santos, Hortaçsu, and Wildenbeest, 2012 and Honka and Chintagunta, 2016, as in these equilibria (*i*) consumers may rationally continue to search at lower prices, while they buy at higher prices and (*ii*) consumers may stop searching and buy at a previously visited store, before they have observed all prices in the market (see our oligopoly extension in Section 6). Moreover, the price distributions of non-RPE are quite different from the regular price distributions found in RPE. It would be interesting to see whether these price distributions provide a good fit with empirical data.

As a first inquiry into non-RPE, we have analyzed a stylized model limiting the immediate applicability of this paper to real world markets.[23] In extensions, we have shown that some of the equilibria extend to oligopoly markets, and, importantly, we have dealt with markets where firms' cost consists of an idiosyncratic and a common cost component. Obviously, in an oligopoly framework one may want to consider a continuum of possible cost states. Such an extension of the present paper would be important in enviornments where the firms' cost is determined by an upstream firm (who can choose a continuum of different price levels). In such an environment, Janssen and Shelegia, 2015a have characterized interesting properties of RPE, but they also show such equilibria do not always exist. Non-RPE would solve this non-existence issue and it is natural to inquire into the qualitative properties of such equilibria. Bagwell and Lee, 2014 provide such an analysis for the case where cost has an idiosyncratic component only. An obvious next step is to see whether our analysis on learning about a common cost component can be combined with their analysis.

One important issue that needs to be addressed in the generalizations to oligopoly markets is how consumer inferences after observing two (or more) prices interact with the consumer search decisions. In the oligopoly extension analyzed in this paper, we dealt with the easiest of different possible cases that can arise. In general, however, different possible search behaviors interact in a complicated way with the incentive of firms to choose different prices. This paper made a first step analyzing non-RPE. There are many theoretical and empirical challenges that lie ahead.

3.8 Appendix I: Proofs of Lemmas, Propositions and Corollaries

Proposition 1. *If $c_H - c_L$ is sufficiently large, or s is sufficiently small an RPE does not exist.*

[23]Parakhonyak and Sobolev, 2015 have taken a different line and introduce model uncertainty into the consumer search framework. Consumers do not update beliefs, but choose a stopping rule that minimizes their losses relative to a Bayesian consumer. They show that in such a framework, consumers also choose a non-reservation price strategy and that firms choose prices in a similar way as we have analyzed here.

Proof. Our proof relies on some facts first derived in Janssen, Pichler, and Weidenholzer, 2011, which we first replicate here. The expected price $E(p|c_i)$ is given by

$$E(p|c_i) = \int_{\underline{p}}^{\rho} pf(p|c_i)dp = c_i + \int_0^1 (p - c_i)dF(p|c_i).$$

Introducing

$$z \equiv 1 - F(p|c_i) = \left(\frac{1-\lambda}{2\lambda}\frac{\rho - c_i}{p - c_i}\right),$$

we have that

$$p - c_i = \frac{\rho - c_i}{1 + \frac{2\lambda}{1-\lambda}z}.$$

This allows us to rewrite expression $E(p|c_i)$ as

$$E(p|c_i) = (1-\gamma)c_i + \gamma\rho,$$

where $\gamma \equiv \int_0^1 \frac{1}{1+\frac{2\lambda}{1-\lambda}z}dz \in [0,1]$. As this also yields $E(p|\rho) = (1-\gamma)E(c|\rho) + \gamma\rho$, combining with $\rho = E(p|\rho) + s$, the reservation price is implicitly defined by

$$\rho = E(c|\rho) + \frac{s}{1-\gamma} \tag{3.14}$$

so that

$$E(p|c_i) = c_i + \frac{\gamma}{1-\gamma}s + \gamma\left[E(c|\rho) - c_i\right]. \tag{3.15}$$

Note that a high cost firm charges prices $p \geq c_H$, and therefore in any RPE it must be the case that $\rho \geq c_H$. From the characterization of RPE it is clear that all prices $p \in [\underline{p}_L, \rho]$ are in the support of the equilibrium price distributions. Consider then a consumer who observes a price $p = c_H - \varepsilon$ for some $\varepsilon > 0$. From the expression $\underline{p}_L = \frac{2\lambda}{1+\lambda}c_i + \frac{1-\lambda}{1+\lambda}\bar{p}$ and the fact that as shown in (3.14) $\bar{p} = \rho = E(c|\rho) + \frac{s}{1-\gamma} \leq c_H + \frac{s}{1-\gamma}$ it follows that $\underline{p}_L < c_H$ for $c_H - c_L$ sufficiently large. Thus, observing a price $p = c_H - \varepsilon$ a consumer must believe that this price is set by a low cost firm. If the consumer continues to search the expected cost of buying is certainly smaller than $E(p|c_L) + s$. Then, from (3.15) the expected price conditional on cost being low equals

$$E(p|c_L) = c_L + \frac{\gamma}{1-\gamma}s + \gamma\left[E(c|\bar{p}) - c_L\right].$$

As $E(c|\bar{p}) < c_H$, it is certainly optimal to continue searching after observing such a price (contradicting the reservation price property) if

$$c_L + \frac{\gamma}{1-\gamma}s + \gamma\left(c_H - c_L\right) + s < c_H - \varepsilon.$$

This reduces to

$$\frac{1}{1-\gamma}s + \varepsilon < (1-\gamma)\left(c_H - c_L\right).$$

It is clear that for a given γ and s, this inequality holds for $c_H - c_L$ being sufficiently large, or that for a given γ and $c_H - c_L$ this inequality holds for some $\varepsilon > 0$ for s being sufficiently small. □

Proposition 2. *For any search cost $s > 0$, if $c_H - c_L$ is sufficiently large, then consumers search with positive probability in any PBE with continuous $F_i(\cdot)$.*

Proof. Suppose there is no active search, i.e. $\beta(p) = 1$ for all $\{p : p \in \text{Supp}F_L \cup \text{Supp}F_H\}$. In the proof of Proposition 1, we showed that for any s if $c_H - c_L$ is large enough we have $\rho_L < \underline{p}_H$

for $\rho_L \leq E(p|c_L) + s$.[24] Moreover, as $\beta(p) = 1$ for all $p \leq \rho_L$ the price distribution $F_L(p)$ cannot have any mass points at these prices. As for all $p \in \text{Supp}F_L \cup \text{Supp}F_H$ with $p < \underline{p}_H$ consumers infer that cost is low, prices $p \in (\rho_L, \underline{p}_H)$ cannot be part of an equilibrium without active search. As there cannot be a mass point in the high cost price distribution at $\underline{p}_H$ (as otherwise high cost firms will undercut), there cannot be a masspoint in the low cost price distribution either (as consumers would infer that cost is low with probability 1). This implies that $F_L(\rho_L) = F_L(\underline{p}_H)$ and because $\beta(\rho_L) = \beta(\underline{p}_H) = 1$ we would obtain that

$$\left\{\lambda[1 - F_L(\rho_L)] + \frac{1-\lambda}{2}\right\}(\rho_L - c_L) = \left\{\lambda[1 - F_L(\rho_L)] + \frac{1-\lambda}{2}\right\}(\underline{p}_H - c_L)$$

which is not possible since $\underline{p}_H > \rho_L$. □

Proposition 3. *All perfect Bayesian equilibria in which non-shoppers buy with probability one in the first search round and in which $F_i(p)$ is continuous, do not satisfy the D1 logic.*

Proof. As only non-shoppers buy at the upper bound, in an equilibrium without active search the profits of low and high cost firms are given by $\pi_L = \frac{1-\lambda}{2}(\rho - c_L)$ and $\pi_H = \frac{1-\lambda}{2}(\rho - c_H)$, respectively. If non-shoppers buy with probability $\beta(p)$ after observing an out-of-equilibrium price $p = \rho + \epsilon$, for small $\epsilon > 0$, then the deviating firm sells to $\frac{1-\lambda}{2}\beta(p)$ consumers and makes a profit of $\pi_i = \frac{1-\lambda}{2}\beta(p)(p - c_i), i = L, H$ as shoppers will not buy at these high deviation prices. This deviation profit is larger than the equilibrium profit if

$$\beta(p) > \frac{\rho - c_i}{p - c_i}.$$

Thus, for an out-of-equilibrium price $p = \rho + \epsilon$, for small enough $\epsilon > 0$, the set $\{q_i \in B_i(p) : (p - c_i)q_i \geq \pi_i^{i*}\} = (\frac{1-\lambda}{2}\frac{\rho - c_i}{p - c_i}, \frac{1-\lambda}{2}]$. As the lower bound of this set is decreasing in c_i for all $p > \rho$, $\{q_i \in B_i(p) : (p - c_L)q_i \geq \pi_L^{i*} \text{ and } p > \rho\} \subset \{q_i \in B_i(p) : (p - c_i)q_H \geq \pi_H^{i*} \text{ and } p > \rho\}$. The D1 refinement thus requires that $\Pr(c_L|p) = 0$ for all $p > \rho$.

The remaining part of the proof is given in the main text just above the Proposition. □

Lemma 1. *In PBE with continuous $F_i(p)$, thus, in any non-RPE, $\overline{p}_L = \overline{p}_H \equiv \overline{p}$.*

Proof. If the upper bounds are not equal, it must be the case that $\overline{p}_H > \overline{p}_L$, or vice versa. As the argument in both cases is identical, we only consider the case where $\overline{p}_H > \overline{p}_L$. Due to the fact that the price distributions do not have mass points, it must be the case that in a left neighborhood of $\overline{p}_H$ high cost firms charge prices with strictly positive probability. For any small $\varepsilon > 0$ consider then the interval $(\overline{p}_H - \varepsilon, \overline{p}_H)$. If a low cost firm would not charge prices in this interval, consumers would know that cost is high after observing prices in this interval. Given that consumers are (at least) indifferent between buying and not buying at $\overline{p}_H$ (as, if consumers prefer to continue to search after observing $\overline{p}_H$, no firm would ever charge $\overline{p}_H$), they strictly prefer to buy at prices in the interval $(\overline{p}_H - \varepsilon, \overline{p}_H)$. But then low cost firms would prefer to set prices in this interval as well instead of charging $\overline{p}_L$. Thus, $\overline{p}_L = \overline{p}_H$.

□

[24] Although formally this result was shown only for RPE, it is clear that in any equilibrium without active search (i.e., where consumers buy immediately at the first search) it must hold . Indeed, it follows form the fact that $\underline{p}_L = \frac{2\lambda}{1+\lambda}c_i + \frac{1-\lambda}{1+\lambda}\overline{p}$, which in turn follows from the equal-profit condition in the case when all consumers buy at $\underline{p}_L$ and $\overline{p}$ with probability one (which is true in all equilibria without active search) and the fact that $\underline{p}_H \geq c_H$.

Proposition 4. *If $F(p)$ is a price distribution in a non-RPE, then it should be of the following form:*

$$F_i(p) = \begin{cases} \frac{2\sqrt{1-(1-\lambda)\beta(\overline{p})} - \int_p^{\overline{p}} \frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(\widetilde{p}-c_i)^2\sqrt{1-(1-\lambda)\beta(\widetilde{p})}}d\widetilde{p}}{2\sqrt{1-(1-\lambda)\beta(p)}} & \text{if } p \in P_{(0,1)} \\ 1 - \frac{1-\lambda}{2\lambda}\left[\beta(\overline{p})\frac{\overline{p}-c_i}{p-c_i} - 1 - \int_p^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right] & \text{if } p \in P_1 \\ 1 - \frac{1-\lambda}{1+\lambda}\left[\beta(\overline{p})\frac{\overline{p}-c_i}{p-c_i} - \int_p^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right] & \text{if } p \in P_0 \end{cases}$$

Proof. Assuming the function $\beta(p)$ is differentiable, equation (3.6) can be rewritten as

$$-2\left[1-(1-\lambda)\beta(p)\right]f_i(p) + (1-\lambda)\beta'(p)F_i(p) = -(1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}$$

by taking the derivative of both sides of the equality sign. This equation can be explicitly written as a differential equation:

$$-2\left[1-(1-\lambda)\beta(p)\right]dF_i + \left[(1-\lambda)\beta'(p)F_i + (1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}\right]dp = 0. \tag{3.16}$$

As

$$-2\frac{\partial\left[1-(1-\lambda)\beta(p)\right]}{\partial p} \neq \frac{\partial\left[(1-\lambda)\beta'(p)F_i + (1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}\right]}{\partial F_i}$$

this is an inexact linear differential equation. However, it can be made exact by dividing (3.16) by $\sqrt{1-(1-\lambda)\beta(p)}$:

$$-2\sqrt{1-(1-\lambda)\beta(p)}dF_i + \frac{\left[(1-\lambda)\beta'(p)F_i + (1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}\right]}{\sqrt{1-(1-\lambda)\beta(p)}}dp = 0.$$

The solution to this exact differential function is a function $Z(F_i, p) = C_i$ (where C_i is an integration constant) with $\frac{\partial Z}{\partial p} = \frac{\left[(1-\lambda)\beta'(p)F_i + (1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}\right]}{\sqrt{1-(1-\lambda)\beta(p)}}$ and $\frac{\partial Z}{\partial F_i} = \sqrt{1-(1-\lambda)\beta(p)}$. It follows that the solution $Z(F_i, p)$ is given by

$$-2F_i\sqrt{1-(1-\lambda)\beta(p)} + \int \frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(p-c_i)^2\sqrt{1-(1-\lambda)\beta(p)}}dp + C_i = 0.$$

This equation can be solved explicitly for $F_i(p)$, to yield (3.7), where the integration constant C_i is found by setting $F_i(\overline{p}) = 1$.

If $\beta(p) = 1$ or $\beta(p) = 0$ in an interval of prices $(\widehat{p}, \widetilde{p})$, then the equilibrium price distribution can be simply directly calculated from (3.6). □

Corollary 3. *In any non-RPE, for all $p < \overline{p}$, $F_L(p) \geq F_H(p)$ and whenever $0 < F_H(p) < 1$, $F_L(p) > F_H(p)$.*

Proof. From the previous Proposition, it follows that $F_H(p) < F_L(p)$ if, and only if, $\frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_H)}{(p-c_H)^2\sqrt{1-(1-\lambda)\beta(p)}} > \frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_L)}{(p-c_L)^2\sqrt{1-(1-\lambda)\beta(p)}}$ for all p. This is the case if

$$(p-c_H)^2(\overline{p}-c_L) < (p-c_L)^2(\overline{p}-c_H).$$

This can be rewritten as

$$(c_H - c_L)p^2 - ((c_H - c_L)\overline{p}p + c_L c_H(c_L - c_H) < 0$$

or $p^2 - \overline{p}p - c_L c_H < 0$, which is definitely the case as $p < \overline{p}$. □

Lemma 2. *Let p^* be such that $\beta(p^*) = 1$ and for any sufficiently small $\epsilon > 0$ $\beta(p^* + \epsilon) < 1$. Suppose that p^* is in the interior of the support of $F_i(p), i = L, H$. Then in equilibrium it must be that $\beta'(p^*) = 0$.*

Proof. Suppose, $\beta'(p^*) < 0$. Denote $\Delta f_i = f_i(p^* - \varepsilon) - f_i(p^* + \varepsilon)$. Then, since

$$f_i(p) = \frac{(1-\lambda)\beta'(p)F_i(p) + (1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}}{2\left[1-(1-\lambda)\beta(p)\right]},$$

we have

$$\Delta f_i = \frac{1}{2}\left(\frac{\beta(\overline{p})(\overline{p}-c_i)(1-\lambda)}{(p^*-\varepsilon-c_i)^2\lambda} - \frac{(1-\lambda)\left(\frac{\beta(\overline{p})(\overline{p}-c_i)}{(p^*+\varepsilon-c_i)^2} + F_i(p^*+\varepsilon)\beta'(p^*+\varepsilon)\right)}{1-(1-\lambda)\beta(p^*+\varepsilon)}\right) = \tag{3.17}$$

$$= \frac{1}{2}\left(\frac{\beta(\overline{p})(\overline{p}-c_i)(1-\lambda)}{(p^*-\varepsilon-c_i)^2\lambda} - \frac{(1-\lambda)\left(\frac{\beta(\overline{p})(\overline{p}-c_i)}{(p^*+\varepsilon-c_i)^2}\right)}{1-(1-\lambda)\beta(p^*+\varepsilon)}\right) - \frac{F_i(p^*+\varepsilon)\beta'(p^*+\varepsilon)}{1-(1-\lambda)\beta(p^*+\varepsilon)}$$

As $F_L(p) > F_H(p)$ and $\left\{\frac{\overline{p}-c_i}{(p-c_i)^2}\right\}'_{c_i} > 0$, we get that $f_L, f_H > 0$ for any ε. Now we take a limit with respect to ε.

$$\lim_{\varepsilon\to 0}\Delta f_i = 0 + \lim_{\varepsilon\to 0} -\frac{F_i(p^*+\varepsilon)\beta'(p^*+\varepsilon)}{1-(1-\lambda)\beta(p^*+\varepsilon)} \tag{3.18}$$

The first term in equation (3.17) approaches zero when ε approaches zero, while the second is strictly positive and bounded below. So we can conclude that as $F_L(p) > F_H(p)$, we can always find sufficiently small ε, such that $\Delta f_L > \Delta f_H$.

Denote

$$a_i = (1-\lambda)\beta(\overline{p})(\overline{p}-c_i)$$

Then

$$\frac{a_L}{(p-c_L)^2} < \frac{a_H}{(p-c_H)^2}$$

which implies that $f_L < f_H$ for prices higher than p^*. This gives $\frac{f_L(p^*-\varepsilon)}{(1-\alpha)f_L(p^*-\varepsilon)+\alpha f_H(p^*-\varepsilon)} = \frac{f_L(p^*+\varepsilon)+\Delta f_L}{(1-\alpha)f_L(p^*+\varepsilon)+\alpha f_H(p^*+\varepsilon)+(1-\alpha)\Delta f_L+\alpha\Delta f_H} > \frac{f_L(p^*+\varepsilon)}{(1-\alpha)f_L(p^*+\varepsilon)+\alpha f_H(p^*+\varepsilon)}$. Thus, if consumers are indifferent at $p^* + \varepsilon$, they must strictly prefer to continue searching at $p^* - \varepsilon$, which can not be the case. Therefore, $\beta'(p^*) = 0$ (since it cannot be greater than 0). □

Proposition 5. *Suppose the consumer was indifferent between continuing to search or buying after the first price observation $p^{(1)}$ and $f_H(p) > f_L(p)$ for all $p \in P_{(0,1)}$. Then if the consumer continued, she stops searching after the second price observation $p^{(2)}$ and buys at $\min\{p^{(1)}, p^{(2)}\}$.*

Proof. Consider a consumer who has observed two prices $p^{(1)}$ and $p^{(2)}$. Given that the consumer was indifferent after observing $p^{(1)}$, the optimal stopping rule for the first round gives

$$w_1(p^{(1)})[\Phi_L(p^{(1)}) - s] + [1 - w_1(p^{(1)})][\Phi_H(p^{(1)}) - s] = 0,$$

where

$$w_1(p^{(1)}) = \frac{(1-\alpha)f_L(p^{(1)})}{(1-\alpha)f_L(p^{(1)}) + \alpha f_H(p^{(1)})}.$$

After observing $p^{(2)}$ the decision of the consumer is determined by the sign of

$$w_2(p^{(1)}, p^{(2)})[\Phi_L(p^{(1)}) - s] + [1 - w_1(p^{(1)}, p^{(2)})][\Phi_H(p^{(1)}) - s],$$

where

$$w_2(p^{(1)}, p^{(2)}) = \frac{(1-\alpha)f_L(p^{(1)})f_L(p^{(2)})}{(1-\alpha)f_L(p^{(1)})f_L(p^{(2)}) + \alpha f_H(p^{(1)})f_H(p^{(2)})}.$$

Note, that if $w_2(p^{(1)}, p^{(2)}) < w_1(p^{(1)})$ this sign is always negative and the consumer prefers to stop. This is the case if

$$\frac{(1-\alpha)f_L(p^{(1)})}{(1-\alpha)f_L(p^{(1)}) + \alpha f_H(p^{(1)})} > \frac{(1-\alpha)f_L(p^{(1)})f_L(p^{(2)})}{(1-\alpha)f_L(p^{(1)})f_L(p^{(2)}) + \alpha f_H(p^{(1)})f_H(p^{(2)})},$$

which can be rewritten as

$$\begin{aligned}(1-\alpha)^2 f_L(p^{(1)})f_L(p^{(2)}) + (1-\alpha)\alpha f_L(p^{(1)})f_H(p^{(1)})f_H(p^{(2)}) >\\ (1-\alpha)^2 f_L(p^{(1)})f_L(p^{(2)}) + (1-\alpha)\alpha f_L(p^{(1)})f_H(p^{(1)})f_L(p^{(2)}),\end{aligned}$$

and reduces to

$$f_H(p^{(2)}) > f_L(p^{(2)}).$$

□

Proposition A.1. *In any non-reservation price equilibrium,*

$$\beta'(\overline{p}) \equiv \lim_{p \uparrow \overline{p}} \frac{\beta(\overline{p}) - \beta(p)}{\overline{p} - p} = -\frac{\beta(\overline{p})}{\overline{p} - c_L}$$

and $\beta(\overline{p}) < 1$.

Proof. As it follows from equations (3.8) and (3.9), non-shoppers have to be indifferent between buying and continuing to search after observing $\overline{p}$.

From Lemma 1, it follows that in the interval $(\overline{p} - \varepsilon, \overline{p})$ both types of firms charge prices with strictly positive probability. By differentiating (3.5) with respect to p and setting $f_L(\overline{p}) = 0$ and $F_L(\overline{p}) = 1$ we obtain

$$\beta'(\overline{p})(\overline{p} - c_L) + \beta(\overline{p}) = 0,$$

which can only be the case when $\beta(\overline{p}) < 1$. □

3.9 Appendix II: Proof of Theorem 1

Here, we prove the existence of equilibrium (Theorem 1) in several lemmas and a final main result. In general, and as explained in the main text, we need to prove that the two functional equations characterizing the distribution functions and the optimality condition for the search rule of non-shoppers, i.e., equations (3.7) and (3.10), have an economically meaningful solution. Thus, the distribution functions should be well-defined, i.e. the densities are positive, and $0 \leq \beta(p) \leq 1$. If $\beta(p) = 0$ it should be optimal for non-shoppers to continue searching after observing these prices, while at prices where $\beta(p) = 1$ non-shoppers should prefer to buy. In addition, two boundary conditions need to be satisfied and we have two parameters to satisfy them: $\overline{p}$ and $\beta(\overline{p})$. First, we need that $f_L(\overline{p}) = 0$, which implies that $\int_0^{\overline{p}} F_H(x)dx = s$. The second boundary condition is different for different parameter values. For the purpose of formulating this second boundary condition, implicitly define $p_0 \leq \underline{p}_H$ as $\pi_L(p_0) = \pi_L(\underline{p}_H)$,or

$$\left[\lambda(1-F_L(p_0)) + \frac{1-\lambda}{2} + \frac{1-\lambda}{2}\int_{p_0}^{\underline{p}_H} f_i(\widetilde{p})d\widetilde{p} + \frac{1-\lambda}{2}\int_{\underline{p}_H}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right](p_0 - c_L) =$$
$$\left[\lambda(1-F_L(\underline{p}_H)) + \frac{1-\lambda}{2} - \frac{1-\lambda}{2}(1-\beta(\underline{p}_H))F_L(\underline{p}_H) + \frac{1-\lambda}{2}\int_{\underline{p}_H}^{\overline{p}}(1-\beta(\widetilde{p}))f_L(\widetilde{p})d\widetilde{p}\right](\underline{p}_H - c_L).$$

That is, p_0 is the largest price smaller than $\underline{p}_H$ that makes low cost firms indifferent between (*i*) setting this price and having uninformed consumers immediately buy at this price and not buying for sure at any price in the interval $(p_0, \underline{p}_H)$ and (*ii*) choosing $\underline{p}_H$ and having uninformed consumers buying with probability $\beta(\underline{p}_H)$. To see that p_0 is uniquely defined, consider the following two cases. If low cost firms do not charge prices in the interval $(p_0, \underline{p}_H)$ with positive probability, then the demand at p_0 is independent of p_0 and thus the profit expression is increasing in p_0. In that case, if $\beta(\underline{p}_H) = 1$, then $p_0 = \underline{p}_H$, while if $\beta(\underline{p}_H) < 1$, then $p_0 < \underline{p}_H$. If, on the other hand, with positive probability low cost firms do charge prices in the interval $(p_0, \underline{p}_H)$, then the profit at p_0 can be written as

$$\pi_L(p_0) = \left[\frac{1+\lambda}{2}(1-F_L(p_0)) + \frac{1-\lambda}{2}F_L(\underline{p}_H) + \frac{1-\lambda}{2}\int_{\underline{p}_H}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right](p_0 - c_L),$$

which using (3) for $\beta(p_0) = 1$, can be written as

$$\frac{1+\lambda}{2}\left(\frac{1-\lambda}{2\lambda}\left[\beta(\overline{p})\frac{\overline{p}-c_L}{p_0-c_i} - 1 - \int_{p_0}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right]\right)(p_0 - c_L)$$
$$+\left(\frac{1-\lambda}{2}F_L(\underline{p}_H) + \frac{1-\lambda}{2}\int_{\underline{p}_H}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right)(p_0 - c_L),$$

or

$$\frac{1-\lambda^2}{4\lambda}\left[\beta(\overline{p})(\overline{p}-c_L) - \left(1 + \int_{p_0}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right)(p_0 - c_L)\right]$$
$$+\left(\frac{1-\lambda}{2}F_L(\underline{p}_H) + \frac{1-\lambda}{2}\int_{\underline{p}_H}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right)(p_0 - c_L),$$

which is clearly increasing in p_0.

The second boundary condition can then be stated as follows.

(*i*) If $\rho_L \geq \underline{p}_H$, then $\beta'(p) = 0$ when p is such that $\beta(p) = 1$ (Lemma 2);

(*ii*) If $\beta(p) < 1$ for all $p \in [\underline{p}_H, \overline{p}]$, then $p_0 = \rho_L$;

(*iii*) If $\rho_L < \underline{p}_H$ and there is an interval $[x, y]$ of prices p such that $\beta(p) = 1$ for all $p \in [x, y]$, then $\lim_{p \downarrow y} \beta'(p) = 0$ (Lemma 2), and $p_0 = \rho$.

To simplify notation, we rewrite the distribution functions as

$$F_i(p) = \frac{2g(\overline{p}) - \int_p^{\overline{p}} \frac{a_i}{(x-c_i)^2 g(x)}dx}{2g(p)} \qquad i = L, H, \tag{3.19}$$

where $g(p) = \sqrt{1-(1-\lambda)\beta(p)}$ and $a_i = (1-\lambda)\beta(\overline{p})(\overline{p}-c_i)$, and proceed as follows. We first note that (3.10) and (3.7) only need to hold in an interval of prices where $\beta(p) < 1$ and that this is a subset of $(\rho_L, \overline{p}]$. Lemma A.1 shows that this implies that $f_L(p)$ and $f_H(p)$ are either both positive or both negative over the relevant interval. We next show that $f_H(\overline{p}) > 0$. Together with Lemma A.1, this shows that if the indifference equation for consumers has a solution, then the distribution functions in (3.7) are well-defined, increasing functions.

Lemma A.1. *For any $p \in P_{(0,1)}$, $f_L(p) \cdot f_H(p) \geq 0$.*

Proof. As $\Phi_L(\rho_L) = \int_0^{\rho_L} F_i(x)dx = \Phi_H(\rho_H) = \int_0^{\rho_H} F_i(x)dx = s$, and $\Phi_i(p)$ are increasing functions it follows that $\Phi_L(p) > s$ and $\Phi_H(p) < s$ for all $\rho_L < p < \rho_H$. As (3.10) can be rewritten as

$$\frac{(1-\alpha)f_L(p)}{(1-\alpha)f_L(p) + \alpha f_H(p)}(\Phi_L(p) - s) + \frac{\alpha f_H(p)}{(1-\alpha)f_L(p) + \alpha f_H(p)}(\Phi_H(p) - s) = 0$$

it follows that both the weights $\frac{(1-\alpha)f_L(p)}{(1-\alpha)f_L(p)+\alpha f_H(p)}$ and $\frac{\alpha f_H(p)}{(1-\alpha)f_L(p)+\alpha f_H(p)}$ have to be of the same sign, which can only be the case if $f_L(p)$ and $f_H(p)$ have the same sign. □

Lemma A.2. *For all $p \in [\max(\rho_L, \underline{p}_H), \overline{p}] \cap P_{(0,1)}$, $f_i(p) > 0$, $i = L, H$.*

Proof. As the function $\beta(p)$ is differentiable, equation (3.6) can be rewritten as

$$-2\left[1 - (1-\lambda)\beta(p)\right] f_i(p) + (1-\lambda)\beta'(p)F_i(p) = -(1-\lambda)\beta(\overline{p})\frac{\overline{p} - c_i}{(p - c_i)^2},$$

which at $\overline{p}$ reduces to

$$-2\left[1 - (1-\lambda)\beta(\overline{p})\right] f_i(\overline{p}) = -(1-\lambda)\left[\frac{\beta(\overline{p})}{\overline{p} - c_i} + \beta'(\overline{p})\right].$$

As the RHS of this expression equals 0 for $c_i = c_L$, the RHS is clearly negative for $c_i = c_H$ for any choice of $0 < \beta(\overline{p}) < 1$. Thus, $f_H(\overline{p}) > 0$. By continuity, there exists $\varepsilon > 0$ such that for all $p \in [\overline{p} - \varepsilon, \overline{p}]$ $f_H(p) > 0$. Then, by Lemma A.1 $f_L(\cdot)$ is also positive in the interior of this interval. Moreover, Lemma A.1 implies that if $f_L(\cdot)$ and $f_H(\cdot)$ change sign it must happen at the same price, which we denote as $q \in [\max(\rho_L, \underline{p}_H), \overline{p}]$. By differentiating (3.7) and taking the ratio of the derivatives, we obtain

$$\frac{(\overline{p} - c_H)(q - c_L)^2}{(q - c_H)^2(\underline{p} - c_L)} = \frac{F_H(q)}{F_L(q)}.$$

Note, that the LHS of this expression is larger than 1 (since $q < \overline{p}$), while by Corollary 3 the RHS is smaller than 1. Therefore, there is no such q and both densities must be positive. □

Thus, it directly follows from Lemma A.1 and A.2 that both density functions have to be positive for all $p \in [\max(\rho_L, \underline{p}_H), \overline{p}] \cap P_{(0,1)}$. As for all other prices $\beta(p) = 0$ or $\beta(p) = 1$, the density functions are positive for all prices.

We also need that consumers prefer to buy as long as $\beta(p) = 1$. The proof is a simple adaptation of a proof given by Dana, 1994 that in a reservation price equilibrium uninformed consumers strictly prefer to buy at all prices in the support of the price distribution of the high cost firm that are strictly smaller than the reservation price.

Lemma A.3. *If $\beta(p) = 1$ on a certain interval $[x, y]$ and uninformed consumers weakly prefer buying to continuing searching at $p = y$, then these consumers strictly prefer buying to continuing searching at $p \in [x, y)$.*

Proof. If $\beta(p) = 1$, then

$$\frac{f_H(p)}{f_L(p)} = \frac{(p - c_H)^2(\overline{p} - c_L)}{(p - c_L)^2(\overline{p} - c_H)}.$$

This expression is decreasing in p. Thus, after observing a larger price, updating beliefs results in uninformed consumers believing it is more likely that cost is high. The expected pay-off of continuing to search is thus larger at larger prices. At the same time, the pay-off of buying at a higher price decreases. Thus, if a consumer is indifferent between the two options at $p = x$, then she must strictly prefer buying at $p < x$. □

We also need that along the equilibrium path we construct, consumers prefer to continue searching when $\beta(p) = 0$. This is, however, trivial, as $\beta(p) = 0$ only occurs along the equilibrium path when $\rho_L < p < \underline{p}_H$, but in that case consumers infer that it is only low cost firms that charge such prices, and non-shoppers prefer to search on as these prices are above ρ_L.

The next five lemmas establish that our system can be rewritten into five proper differential equations and invoke the Pickard-Lindelof theorem of differential equations to show that the system has a (mathematical) solution that it is locally unique. To make sure that the conditions of this theorem apply, we first need to eastblish some properties of the function $g'(p)$.

Lemma A.4. *The solution to the indifference equation* (3.10) *can be written as*

$$g'(p) = \frac{(1-\alpha)\frac{a_L}{2(p-c_L)^2 g(p)}(\Phi_L(p) - s) + \alpha\frac{a_H}{2(p-c_H)^2 g(p)}(\Phi_H(p) - s)}{(1-\alpha)F_L(p)(\Phi_L(p) - s) + \alpha F_H(p)(\Phi_H(p) - s)}. \tag{3.20}$$

Proof. Taking the derivative of (3.19) gives

$$f_i(p) = \frac{1}{g(p)}\left(\frac{a_i}{2(p-c_i)^2 g(p)} - F_i(p)g'(p)\right).$$

Then the optimal stopping rule can be rewritten as

$$\begin{aligned} 0 \;=\; & \frac{(1-\alpha)\left(\frac{a_L}{2(p-c_L)^2 g(p)} - F_L(p)g'(p)\right)}{(1-\alpha)\left(\frac{a_L}{2(p-c_L)^2 g(p)} - F_L(p)g'(p)\right) + \alpha\left(\frac{a_H}{2(p-c_H)^2 g(p)} - F_H(p)g'(p)\right)}(\Phi_L(p) - s) + \\ & \frac{\alpha\left(\frac{a_H}{2(p-c_H)^2 g(p)} - F_H(p)g'(p)\right)}{(1-\alpha)\left(\frac{a_L}{2(p-c_L)^2 g(p)} - F_L(p)g'(p)\right) + \alpha\left(\frac{a_H}{2(p-c_H)^2 g(p)} - F_H(p)g'(p)\right)}(\Phi_H(p) - s), \end{aligned}$$

which can easily be rewritten as the equation in the statement of the Lemma. □

We proceed with some facts about the function $g'(p)$. Define $g'(p) = \frac{A(p)}{B(p)}$, where

$$A(p) \equiv (1-\alpha)\frac{a_L}{2(p-c_L)^2 g(p)}(\Phi_L(p) - s) + \alpha\frac{a_H}{2(p-c_H)^2 g(p)}(\Phi_H(p) - s) \tag{3.21}$$

and

$$B(p) \equiv (1-\alpha)F_L(p)(\Phi_L(p) - s) + \alpha F_H(p)(\Phi_H(p) - s). \tag{3.22}$$

Lemma A.5. *The equation $A(p) = 0$ has at most one root on $Q = \{p : \Phi_L(p) > s\}$.*

Proof. From the definition of $A(p)$ it follows that $A(p) = 0$ if, and only if,

$$\frac{\alpha(s - \Phi_H(p))}{(1-\alpha)(\Phi_L(p) - s)} = \frac{\frac{a_L}{(p-c_L)^2}}{\frac{a_H}{(p-c_H)^2}} = \frac{(p-c_H)^2(\overline{p} - c_L)}{(p-c_L)^2(\overline{p} - c_H)}.$$

Note that the denominator of the LHS is always positive. For all prices p such that $\Phi_H(p) \leq s$ the numerator is also positive and the LHS is decreasing, while the RHS is increasing in p. Thus, there is at most one p where $A(p) = 0$. Morevoer, for all p such that $\Phi_H(p) > s$ we have $A(p) > 0$, so there is no solution of $A(p) = 0$ for these prices. □

Lemma A.6. *For all $p \in Q = \{p : \Phi_L(p) > s\}$ such that $A(p) > 0$ we have $B(p) > 0$.*

Proof. Note that for all p such that $\Phi_H(p) \geq s$ we have both $A(p) > 0$ and $B(p) > 0$. Now, consider prices such that $\Phi_H(p) < s$. Then, $A(p) > 0$, if and only if,

$$\frac{(1-\alpha)(\Phi_L(p)-s)}{\alpha(s-\Phi_H(p))} > \frac{\frac{a_H}{(p-c_H)^2}}{\frac{a_L}{(p-c_L)^2}} = \frac{(p-c_L)^2(\overline{p}-c_H)}{(p-c_H)^2(\overline{p}-c_L)}. \tag{3.23}$$

Similarly, $B(p) > 0$, if and only if,

$$\frac{(1-\alpha)(\Phi_L(p)-s)}{\alpha(s-\Phi_H(p))} > \frac{F_H(p)}{F_L(p)}. \tag{3.24}$$

The LHS of (3.23) and (3.24) are identical; as the RHS of (3.23) is larger than 1, while by Corollary 3 the RHS of (3.24) is smaller than 1, the statement follows. □

Now note, that if $\rho_L \geq \underline{p}_H$ then due to Lemma 2 it follows from Lemmas A.5 and A.6 that $B(p) > 0$ for all $p = [\rho_L, \overline{p}]$. The next lemma establishes the same result for the case where $\rho_L < \underline{p}_H$.

Lemma A.7. *Suppose $\rho_L < \underline{p}_H$. Then $B(p) > 0$ for all $p \in P_{(0,1)}$.*

Proof. Note that

$$B'(p) = (1-\alpha)f_L(p)[\Phi_L(p)-s] + \alpha f_H(p)[\Phi_H(p)-s] + (1-\alpha)F_L(p)^2 + \alpha F_H(p)^2.$$

As $p \in P_{(0,1)}$ we have that the sum of the first two terms must be equal to zero due to (3.10). Thus, $B'(p) > 0$. Note, that $B(\underline{p}_H) \geq 0$ as $F_H(\underline{p}_H) = 0$. Thus, as $P_{(0,1)} \subseteq SuppF_H$ we get that $B(p) > 0$ on this set. □

In the proof of the Theorem, we use the fact that the system of differential equations (3.7) and (3.10) has a unique solution. To this end, we prove in the next Lemma that this is the case by applying the Pickard-Lindelof theorem.

Lemma A.8. *For any p_1 the system of differential equations given by (3.7) and (3.10) with boundary values $\Phi_i(p_1), F_i(p_1), \beta(p_1), i = L, H$ such that*

$$(1-\alpha)F_L(p_1)(\Phi_L(p_1)-s) + \alpha F_H(p_1)(\Phi_H(p_1-s) > 0$$
$$\beta(p_1) \leq 1$$

has a unique solution in a neighborhood of p_1.

Proof. To apply the Pickard-Lindelof theorem, we need to rewrite our system in the form where the derivatives of certain functions are expressed as functions of these functions themselves. We do this in the following way: we define a function $\Phi_i(p) = \int_{\underline{p}}^{\overline{p}} F_i(p)dp$, so that $\Phi_i'(p) = F_i(p)$, and a function $z_i(p) = \int_0^p \frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(x-c_i)^2 g(x)}dx$, so that $z_i'(p) = -\frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(p-c_i)^2 g(p)}$. Using these transformations, we can rewrite our system as

$$z_i'(p) = -\frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(p-c_i)^2 g(p)}, i = L, H;$$

$$\Phi_i'(p) = \frac{2g(\overline{p}) - z_i(p)}{2g(p)}, i = L, H;$$

and

$$g'(p) = -\frac{(1-\alpha)\frac{a_L}{(p-c_L)^2}(\Phi_L(p)-s) + \alpha\frac{a_H}{(p-c_H)^2}(\Phi_H(p)-s)}{(1-\alpha)\left(z_L(p)-2g(p)\right)(\Phi_L(p)-s) + \alpha\left(z_H(p)-2g(p)\right)(\Phi_H(p)-s)},$$

whenever $g(p) > \sqrt{\lambda}$ $(\beta(p) < 1)$. Note that the expression for $g'(p)$ is equivalent to (21) and that $g'(p) = 0$ if $g(p) = \sqrt{\lambda}$.

To apply the Pickard-Lindelof theorem, we need that the RHS of this system of differential equations is Lipschitz-continuous with respect to $(g, z_i, \Phi_i), i = L, H$.

Denoting $b_i = -\frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(p-c_i)^2}, i = L, H$, the derivatives of the vector-function representing the RHS of the system of five differential equations for $z'_L, z'_H, \Phi'_L, \Phi'_H, g'$ with respect to g, z_i, Φ_i is summarized in the matrix

$$\nabla = \begin{pmatrix} \frac{b_L}{g^2} & 0 & 0 & 0 & 0 \\ \frac{b_H}{g^2} & 0 & 0 & 0 & 0 \\ -\frac{2g(\overline{p})-z_L}{2g^2} & -\frac{1}{2g} & 0 & 0 & 0 \\ -\frac{2g(\overline{p})-z_H}{2g^2} & 0 & -\frac{1}{2g} & 0 & 0 \\ D_1 & D_2 & D_3 & D_4 & D_5 \end{pmatrix},$$

where

$$D_1 = \frac{2((1-\alpha)b_L(\Phi_L - s) + \alpha b_H(\Phi_H - s))((1-\alpha)(\Phi_L - s) + \alpha(\Phi_H - s))}{[(1-\alpha)z_L(\Phi_L - s) + \alpha z_H(\Phi_H - s) - 2g((1-\alpha)(\Phi_L - s) + \alpha(\Phi_H - s))]^2}$$

$$D_2 = -\frac{(1-\alpha)(\Phi_L - s)\left[(1-\alpha)b_L(\Phi_L - s) + \alpha b_H(\Phi_H - s)\right]}{[(1-\alpha)z_L(\Phi_L - s) + \alpha z_H(\Phi_H - s) - 2g((1-\alpha)(\Phi_L - s) + \alpha(\Phi_H - s))]^2}$$

$$D_3 = -\frac{\alpha(\Phi_H - s)\left[(1-\alpha)b_L(\Phi_L - s) + \alpha b_H(\Phi_H - s)\right]}{[(1-\alpha)z_L(\Phi_L - s) + \alpha z_H(\Phi_H - s) - 2g((1-\alpha)(\Phi_L - s) + \alpha(\Phi_H - s))]^2}$$

$$D_4 = \frac{(2b_L g - 2b_H g + b_H z_L - b_L z_H)\alpha(1-\alpha)(\Phi_L - s)}{[(1-\alpha)z_L(\Phi_L - s) + \alpha z_H(\Phi_H - s) - 2g((1-\alpha)(\Phi_L - s) + \alpha(\Phi_H - s))]^2}$$

$$D_5 = -\frac{(2b_L g - 2b_H g + b_H z_L - b_L z_H)\alpha(1-\alpha)(\Phi_H - s)}{[(1-\alpha)z_L(\Phi_L - s) + \alpha z_H(\Phi_H - s) - 2g((1-\alpha)(\Phi_L - s) + \alpha(\Phi_H - s))]^2}.$$

Due to the condition $(1-\alpha)F_L(p_1)(\Phi_L(p_1) - s) + \alpha F_H(p_1)(\Phi_H(p_1) - s) > 0$ all D_i's are bounded and our vector-function is continuously differentiable. It is known that if a function is continuously differentiable on $[\underline{p}_H, \overline{p}]$, then it is Lipschitz-continuous on this interval.[25] The statement of the Lemma then is an application of the Pickard-Lindelof theorem. □

Note that as the five equations in the proof of Lemma A.8 are just a different representation of the system of differential equations (3.7) and (3.10), this system with boundary conditions $F_i(p_1)$ and $\beta(p_1)$ also has a unique solution as long as the condition of the Lemma is satisfied. Later in the proof we show that Lemmas A.5 - A.7 guarantee that the conditions of Lemma A.8 are satisfied in the relevant domains.

Theorem 1. *For any values of s, λ, c_L, c_H and α a NRPE as defined in Definition 2 exists. The equilibrium price distributions are characterized by (3.7), while non-shopper's behavior is determined by (3.10) whenever $0 < \beta(p) < 1$.*

[25]Proof of this statement can be found on this web-page:
http://unapologetic.wordpress.com/2011/05/04/continuously-differentiable-functions-are-locally-lipschitz/

Proof. Fix some $\overline{p} > \max(\rho_L^{NU}, c_H + s)$, where ρ_L^{NU} is the standard Stahl reservation price where there is no ex ante cost uncertainty and cost is known to be low. We first show that for any $\overline{p}$, all equilibrium conditions except $\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp = s$ can be satisfied. The second part of the proof shows that this last condition can always be satisfied by an appropriate choice of $\overline{p}$. Then, by applying Lemmas A.1 and A.2 we guarantee that the distribution functions are well-defined.

We write the solution of the system of differential equations with boundary conditions $\overline{\beta}, F_L(\overline{p}) = F_H(\overline{p}) = 1$ as $\hat{\beta}(p, \overline{\beta})$ and use $\beta(p)$ whenever we refer to the equilibrium stopping probability. The solution $\hat{\beta}(p, \overline{\beta})$ does not necessarily belong to $[0, 1]$, while $\beta(p)$ does. Note that according to Lemmas A.6 and A.7 for any price in the support of the high-cost distribution the condition of Lemma A.8 is satisfied. Then, from Lemma A.8 the solution is unique for any $\overline{\beta}$. As solution paths cannot intersect, $\hat{\beta}(p, \overline{\beta})$ is monotone in the second argument.

Next, we argue that there exists a unique $\overline{\beta}_0$ such that $\max_{p\in[\underline{p}_H,\overline{p}]} \hat{\beta}(p, \overline{\beta}_0) = 1$. First, Lemma A.5 together with $\hat{\beta}'(\overline{p}, \overline{\beta}) < 0$ guarantee that the function $\hat{\beta}(p, \overline{\beta})$ has a unique maximum on $[\underline{p}_H, \overline{p}]$. Therefore, for any $\overline{\beta}$ the solution $\hat{\beta}(p, \overline{\beta})$ either attains its maximum in the interior of $[\underline{p}_H, \overline{p}]$ or at $\underline{p}_H$.[26] In the latter case $\hat{\beta}(p, \overline{\beta})$ is monotonically decreasing on $[\underline{p}_H, \overline{p}]$. Second, independent of whether the maximum is attained in the interior or at the lower bound, for any $b > 0$ there is a value of $\overline{\beta}$ such that $\max_{p\in[\underline{p}_H,\overline{p}]} \hat{\beta}(p, \overline{\beta}) < b$. To see this, note that if b_0 is the largest value of β on this interval, then, as $\underline{p}_H > c_H$, we have $\pi_L(p) > \frac{1-\lambda}{2} b_0(p - c_L) > \frac{1-\lambda}{2} b_0(c_H - c_L)$. At the same time $\pi_L = \frac{1-\lambda}{2}\overline{\beta}(\overline{p} - c_L)$. Thus,

$$b_0 < \frac{\overline{p} - c_L}{c_H - c_L}\overline{\beta} \equiv b$$

and b can be chosen arbitrary small by an appropriate choice of $\overline{\beta}$. Finally, there exists a value $\overline{\beta}$ such that $\max_{p\in[\underline{p}_H,\overline{p}]} \hat{\beta}(p, \overline{\beta}) \geq 1$. This follows immediately from the continuous differentiability of $\hat{\beta}$ and the fact that $\hat{\beta}'(\overline{p}, \overline{\beta}) < 0$. Thus, as $\hat{\beta}(p, \overline{\beta})$ is monotone in the second argument there is a unique $\overline{\beta}_0$ such that $\max_{p\in[\underline{p}_H,\overline{p}]} \hat{\beta}(p, \overline{\beta}_0) = 1$.

Now, we show that it is always possible to choose $\overline{\beta}$ such that a least one of the four types of NRPE exist. We start with considering the case when the maximum of $\hat{\beta}(p, \overline{\beta}_0)$ is attained in the interior of $[\underline{p}_H, \overline{p}]$.

No-gap equilibrium. Consider a solution to the system of differential equations for the $\overline{\beta}_0$ defined above. Recall, from Lemma 2 that $p^*(\overline{\beta}_0)$ is the largest price such that $\beta(p) = 1$. We consider a candidate NRPE, such that for all prices $p > p^*(\overline{\beta}_0)$ the consumer indifference condition is satisfied and $\beta(p) = \hat{\beta}(p, \overline{\beta}_0)$, and for all $p \leq p^*(\overline{\beta}_0)$ we set $\beta(p) = 1$. If $\rho_L(\overline{\beta}_0) \geq \underline{p}_H$, then it is clear from Lemmas A.1-A.3 that all conditions for a NRPE to exist (apart from $\Phi_H = s$) are satisfied.

Monopolistic gap equilibrium. Suppose then that $\rho_L(\overline{\beta}_0) < \underline{p}_H$. Take some price $\hat{p} \in [\underline{p}_H, p^*(\overline{\beta}_0)]$, and construct a solution from that price $\hat{p}$ for all $p \in [\underline{p}_H, \hat{p})$, using $F_i(\hat{p}), \Phi_i(\hat{p})$ and $\hat{\beta}(\hat{p}) = 1$ as boundary conditions. We denote this solution path as $\hat{\beta}_{\hat{p}}(p, 1)$ to make clear that this is the mathematical solution to (10) starting from $\hat{p}$ with $\hat{\beta}(\hat{p}) = 1$. Recall, that p_0 is the highest price smaller than $\underline{p}_H$ (such that $\beta(p_0) = 1$) which makes the low-cost firm indifferent between charging this price and charging $\underline{p}_H$. To construct an NRPE, it is necessary that $p_0 = \rho_L$. From Lemmas A.5 and A.6, it follows that for any $p \in [\underline{p}_H, \hat{p})$ $\frac{\partial \hat{\beta}_{\hat{p}}(p,1)}{\partial p} > 0$. Together with Lemma A.8 this implies that $\hat{\beta}_{\hat{p}}(\underline{p}_H, 1)$ is decreasing in $\hat{p}$ and that p_0 is continuous in $\hat{p}$. Note that $\lim_{\hat{p}\downarrow\underline{p}_H} p_0 = \underline{p}_H$, which implies that $\lim_{\hat{p}\downarrow\underline{p}_H} \int_{\underline{p}_L}^{p_0} F_L(p)dp > s$ (as otherwise a no-gap equilibrium would exist).

[26] With slight abuse of notation we write $\underline{p}_H$ instead of $\underline{p}_H(\overline{\beta})$.

We derive the following two conditions from the equal profits condition (taking into account that $F(\underline{p}_H) = F_L(p_0)$)

$$\underline{p}_L = c_L + \frac{\frac{1-\lambda}{2}\overline{\beta}(\overline{p} - c_L)}{\frac{1+\lambda}{2} + \frac{1-\lambda}{2}\int_{\underline{p}_H}^{\overline{p}}(1-\beta(p))f_L(p)dp} \tag{3.25}$$

and

$$\underline{p}_H - p_0 = \frac{(1-\hat{\beta}_{\hat{p}}(\underline{p}_H, 1))F_L(\underline{p}_H)\frac{1-\lambda}{2}(\underline{p}_H - c_L)}{\lambda(1-F_L(\underline{p}_H)) + \frac{1-\lambda}{2}(1+\int_{\underline{p}_H}^{\overline{p}}(1-\beta(p))f_L(p)dp)} \tag{3.26}$$

It is clear that $\underline{p}_H - p_0 > 0$ for any $\hat{\beta}_{\hat{p}}(\underline{p}_H, 1) < 1$. As $F_L(\underline{p}_H)$, $\int_{\underline{p}_H}^{\overline{p}}(1-\beta(p))f_L(p)dp$ and $\int_{\underline{p}_L}^{p_0} F_L(p)dp$ are all continuous in $\hat{p}$ it must be the case that either

- there either exists a $\hat{p} \in [\underline{p}_H(\overline{\beta}_0), p^*(\overline{\beta}_0)]$ such that $p_0 = \rho_L$ (or $\int_{\underline{p}_L}^{p_0} F_L(p)dp = s$) and $\hat{\beta}_{\hat{p}}(p, 1) \geq 0$ for all $p \in [\underline{p}_H, p^*(\overline{\beta}_0)]$, meaning that there exists an NRPE;
- there exists $\hat{p}_0 \in [\underline{p}_H(\overline{\beta}_0), p^*(\overline{\beta}_0)]$ such that $\hat{\beta}_{\hat{p}_0}(\underline{p}_H, 1) = 0$ while $\int_{\underline{p}_L}^{p_0} F_L(p)dp > s$, and we deal with this case under the competitive gap equilibrium;
- or $\int_{\underline{p}_L}^{p_0} F_L(p)dp > s$ for any $\hat{p} \in [\underline{p}_H, p^*]$ and $\hat{\beta}_{\hat{p}}(p, 1) > 0$ for all $p \in [\underline{p}_H, \hat{p}]$ and $\hat{p} \in [\underline{p}_H, p^*(\overline{\beta}_0)]$, and we deal with this case in the regular gap equilibrium.

Regular gap equilibrium. Suppose then that for any $\hat{p} \in [\underline{p}_H, p^*]$, $\int_{\underline{p}_L}^{p_0} F_L(p)dp > s$, or equivalently, $p_0 > \rho_L$. Note that if $\hat{p} = p^*(\overline{\beta}_0)$ there exists a solution path $\hat{\beta}(p, \overline{\beta}_0)$ with $\hat{\beta}(p, \overline{\beta}_0) \leq 1$ and we can set $\beta(p) = \hat{\beta}(p, \overline{\beta}_0)$. Moreover, $\hat{\beta}(p, \overline{\beta}) < 1$ for any $\overline{\beta} < \overline{\beta}_0$. Now, note that $\lim_{\overline{\beta}\to 0} \pi_L = 0$ and therefore $\lim_{\overline{\beta}\to 0} p_0 = c_L$. As $\hat{\beta}(p, \overline{\beta})$ is continuous in both arguments, $p_0(\overline{\beta}_0) > \rho_L$ and $\rho_L \geq c_L + s$, there either exists $\overline{\beta}_1$ such that $p_0 = \rho_L$ with $\hat{\beta}(p, \overline{\beta}_1) > 0$ for all $p \in [\underline{p}_H, \overline{p}(\overline{\beta}_1)]$, meaning that the equilibrium exists, or there is a $\overline{\beta}_2 < \overline{\beta}_0$ such that $\hat{\beta}(\underline{p}_H, \overline{\beta}_2) = 0$ and $p_0(\overline{\beta}_2) > \rho_L$, and we deal with this case in the competitive gap equilibrium.

Competitive gap equilibrium. Suppose, that either $\hat{p}_0$ in the monopolistic gap equilibrium or $\overline{\beta}_2$ in the competitive gap equilibrium exist, meaning that $\hat{\beta}(\underline{p}_H) = 0$ and $\int_{\underline{p}_L}^{p_0} F_L(p)dp > s$. In both cases we take the solution on $[\underline{p}_H, \overline{p}]$ from the corresponding case and construct a NRPE such that $\beta(p) = 0$ for all $p \in (\rho_L, \underline{p}_H]$ and $p_0 = \rho_L$ and where low cost firms still choose prices in a left region of p_H, denoted by $[\underline{p}', \underline{p}_H]$. Note that as by definition $\beta(\underline{p}') = 0$ and $\beta(p_0) = 1$ it follows from the equal profit condition for the low cost firms that $p_0 < \underline{p}'$. However, by choosing $\underline{p}'$ sufficiently low, $F_L(\underline{p}')$ can be chosen arbitrarily close to zero, which as $\int_{\underline{p}_L}^{\rho_L} F_L(p)dp = s$ implies that $\rho_L > \underline{p}' > p_0$. Therefore, there must exist a $\underline{p}'$ that $p_0 = \rho_L$. This completes the proof for a given $\overline{p}$ for the case when $\beta(p, \overline{\beta}_0)$ reaches its maximum in the interior of $[\underline{p}_H, \overline{p}]$.

Consider then the case where the maximum of $\hat{\beta}(p, \overline{\beta}_0)$ is not attained in this interior, From Lemma A.5, it follows then that the maximum must be reached at $\underline{p}_H$ and $\hat{\beta}(\underline{p}_H, \overline{\beta}_0) = 1$. If in this case $\rho_L \geq \underline{p}_H$, then, as previously, a no-gap equilibrium exists as we can simply set $\beta(p) = 1$ for all $p < \underline{p}_H$. Suppose then that $\rho_L < \underline{p}_H$. If for all $\overline{\beta} < \overline{\beta}_0$ the maximum of $\hat{\beta}(p, \overline{\beta})$ is attained at $\underline{p}_H$, then a regular gap equilibrium exists with $\overline{\beta}_1 < \overline{\beta}_0$ as $\hat{\beta}(p, \overline{\beta}_1) > 0$ for all p due to the monotonicity of $\hat{\beta}(p, \overline{\beta}_1)$ in price. If for some $\overline{\beta} < \overline{\beta}_0$ the maximum of $\hat{\beta}(p, \overline{\beta})$ is attained in the interior of the support of the high cost distribution, then, using our analysis in the previous case we conclude that either a regular gap or a competitive gap equilibrium exists.

We have now proved that for any fixed $\overline{p} > \max(c_H + s, \rho_L^{NU})$ we can satisfy all equilibrium conditions apart from the fact that in an NRPE we should have $\int_{\underline{p}_H}^{\overline{p}} F_H(p) = s$. We now prove that we can always choose $\overline{p}$ such that this indifference condition is also satisfied. To do so, we first realize that as $c_H > c_L$ we have

$$\lim_{\overline{p}\downarrow\rho_L^{NU}} \int_{\underline{p}_H}^{\overline{p}} F_H(p)dp < s.$$

We next show that for sufficiently large $\overline{p}$, the other equilibrium conditions imply that

$$\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp > s.$$

As $\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp$ is continuous in $\overline{p}$, it follows then that there must be a $\overline{p}$ such that $\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp = s$. Thus, the only thing to be proved is that for $\overline{p}$ sufficiently large, $\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp > s$. To this end, it follows from

$$\pi(\overline{p}|c_H) = \frac{1-\lambda}{2}(1-\beta(\overline{p}))(\overline{p}-c_H) < \frac{1-\lambda}{2}(\overline{p}-c_H)$$

and

$$\pi(\underline{p}_H|c_H) > \frac{1+\lambda}{2}(\underline{p}_H - c_H)$$

and the fact that a firm has to be indifferent between charging the upper and lower bound of the price distribution that

$$\overline{p} - \underline{p}_H > \frac{2\lambda}{1+\lambda}(\overline{p}-c_H). \tag{3.27}$$

Thus, the support of the mixed strategy distribution grows without bound when $\overline{p}$ becomes larger. Suppose then that $\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp < s$ even for large $\overline{p}$. This would imply that for all $\epsilon > 0$ there exist a large $\overline{p}$ such that $F_H(\frac{\overline{p}+\underline{p}_H}{2}) < \epsilon$. Let us then consider the profit a firm makes when setting prices $\underline{p}_H$ and $[\overline{p}+\underline{p}_H]/2$:

$$\pi(\underline{p}_H|c_H) = \left[\frac{1+\lambda}{2} + \frac{1-\lambda}{2}\left(\int_{\underline{p}_H}^{\frac{\overline{p}+\underline{p}_H}{2}} (1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p} + \int_{\frac{\overline{p}+\underline{p}_H}{2}}^{\overline{p}} (1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right)\right](\underline{p}_H - c_H),$$

and

$$\pi\left(\frac{\overline{p}+\underline{p}_H}{2}|c_H\right) = \left[\frac{1+\lambda}{2} - \left[\lambda + \frac{1-\lambda}{2}(1-\beta(\frac{\overline{p}+\underline{p}_H}{2}))\right] F_H\left(\frac{\overline{p}+\underline{p}_H}{2}\right) + \frac{1-\lambda}{2}\int_{\frac{\overline{p}+\underline{p}_H}{2}}^{\overline{p}} (1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right]\left(\frac{\overline{p}+\underline{p}_H}{2} - c_H\right).$$

As by choosing $\overline{p}$ we can make $F_H\left(\frac{\overline{p}+\underline{p}_H}{2}\right)$ arbitrarily small and as $1-\beta(\widetilde{p}) < 1$, it is clear that

$$\pi(\underline{p}_H|c_H) < \left[\frac{1+\lambda}{2} + \frac{1-\lambda}{2}F_H\left(\frac{\overline{p}+\underline{p}_H}{2}\right) + \frac{1-\lambda}{2}\int_{\frac{\overline{p}+\underline{p}_H}{2}}^{\overline{p}} (1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right](\underline{p}_H - c_H),$$

so that

$$\pi\left(\frac{\overline{p}+\underline{p}_H}{2}|c_H\right)-\pi(\underline{p}_H|c_H)>$$
$$\left[\frac{1+\lambda}{2}+\frac{1-\lambda}{2}\int_{\overline{p}/2}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}-\frac{1-\lambda}{2}F_H\left(\frac{\overline{p}+\underline{p}_H}{2}\right)\right]\frac{\overline{p}-\underline{p}_H}{2}-$$
$$\lambda F_H\left(\frac{\overline{p}+\underline{p}_H}{2}\right)\left(\frac{\overline{p}+\underline{p}_H}{2}-c_H\right).$$

using (3.27) it follows that

$$\pi\left(\frac{\overline{p}+\underline{p}_H}{2}|c_H\right)-\pi(\underline{p}_H|c_H)>$$
$$\left\{\left[\frac{1+\lambda}{2}+\frac{1-\lambda}{2}\int_{\overline{p}/2}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}-\frac{1-\lambda}{2}F_H(\frac{\overline{p}+\underline{p}_H}{2})\right]\frac{\lambda}{1+\lambda}-\lambda F_H(\frac{\overline{p}+\underline{p}_H}{2})\right\}(\overline{p}-c_H),$$

which is clearly positive for large $\overline{p}$. This implies that for large $\overline{p}$ a high cost firm cannot be indifferent over the whole support of the price distribution if $\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp < s$.

□

3.10 Appendix III: Idiosyncratic Cost

Consider the case where firms are known to have a high common cost. The low cost case is identical. The first, preliminary, but important, result is that the price distributions of the firm with low and high idiosyncratic cost cannot overlap, and that the upper bound of the low cost distribution should be no larger than the lower bound of the high cost distribution, denoted by $\underline{p}_{iH}$. To see this, suppose that both the high and low cost firm have a range of prices p in the interior of their support and that they sell with probability $q(p)$. As the high cost firm has to be indifferent between charging these different prices, it follows that $q(p)(p-c_{iH})$ equals a constant K that is independent of p. But then the profit of the low cost firm equals $q(p)(p-c_{iL}) = K(p-c_{iL})/(p-c_{iH}), i = L, H$, which is decreasing in p. As, because of the shoppers, there cannot be mass points, it follows that the distributions do not overlap.

We then derive the equilibrium price distribution functions for both types of players. The distribution function $F_{iH}(p)$ of the firm with high idiosyncratic cost has to satisfy the following

$$\pi_{iH}(p)=\left[\lambda\gamma(1-F_{iH}(p))+\frac{1-\lambda}{2}\right](p-c_{iH})=\left(\lambda\gamma+\frac{1-\lambda}{2}\right)(\rho_i-c_{iH}),$$

so that

$$F_{iH}(p))=1-\frac{1-\lambda}{2\lambda\gamma}\frac{\rho_i-p}{p-c_{iH}}.$$

These are the standard formulae, except for the factor γ and the fact that the determination of the reservation price ρ_H is different (see below). Note that as the low cost distribution is below the high cost price distribution, the high cost firm only attracts the shoppers if the other firm also has a high idiosyncratic cost component.

The distribution function $F_{iL}(p)$ of the firm with low idiosyncratic cost has to satisfy the following

$$\pi_{iH}(p)=\left[\lambda\left\{\gamma+(1-\gamma)(1-F_{iL}(p))\right\}+\frac{1-\lambda}{2}\right](p-c_{iL})=\frac{1-\lambda}{2}(\underline{p}_{iH}-c_{iL}),$$

so that

$$F_{iL}(p) = 1 - \frac{1-\lambda+2\gamma\lambda}{2\lambda(1-\gamma)}\left(\frac{\underline{p}_{iH} - p}{p - c_{iL}}\right). \tag{3.28}$$

Note here that the low cost firm always attracts the shoppers if the other firm has a high idiosyncratic cost component. Note also that if the idiosyncratic cost shock is the only uncertainty, it has to be the case that the upper bound of the low cost price distribution equals the lower bound of the high cost price distribution.

It remains to determine the reservation price ρ_i. As under idiosyncratic cost uncertainty there is no learning, consumers have to be indifferent between buying and continuing to search at the reservation price, i.e.,

$$\rho_i = \gamma E_{iH}(p) + (1-\gamma)E_{iL}(p) + s,$$

where $E_{iH}(p)$ is the expected price of a firm with idiosyncratic cost $j = L, H$. It may happen that $\rho_i \leq c_{iH}$, in which case the high cost firm's price distribution is degenerate and $E_{iH}(p) = c_{iH}$. Using the proof of Lemma 1 in Janssen, Pichler, and Weidenholzer, 2011, these expected prices can be written as

$$E_{iH}(p) = (1-\alpha_{iH})c_{iH} + \alpha_{iH}\rho_i, \text{ and } E_{iL}(p) = (1-\alpha_{iL})c_{iL} + \alpha_{iL}\underline{p}_{iH},$$

respectively, where $\alpha_{iH} = \int_0^1 \frac{1}{1+\frac{2\lambda\gamma}{1-\lambda}z}dz, \alpha_{iL} = \int_0^1 \frac{1}{1+\frac{2\lambda(1-\gamma)}{1-\lambda+2\lambda\gamma}z}dz$, and $\underline{p}_{iH} = \frac{c_{iH}+\frac{1-\lambda}{2\lambda\gamma}\rho_i}{1+\frac{1-\lambda}{2\lambda\gamma}}$. Thus, the reservation price is defined by

$$\rho_i = \frac{\left[\gamma(1-\alpha_{iH}) + (1-\gamma)\frac{2\lambda\gamma}{2\lambda\gamma+1-\lambda}\alpha_{iL}\right]c_{iH} + (1-\gamma)(1-\alpha_{iL})c_{iL} + s}{1 - \gamma\alpha_{iH} - (1-\gamma)\frac{1-\lambda}{2\lambda\gamma+1-\lambda}\alpha_{iL}}.$$

This fully characterizes the equilibrium under idiosyncratic cost uncertainty only.

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