Efficient and Optimal Capital Accumulation under a Non Renewable Resource Constraint

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Abstract

Usual resource models with capital accumulation focus upon simple one to one process transforming output either into some consumption good or into some capital good. We consider a bisectoral model where the capital good, labor and a non renewable resource are used to produce the consumption good and the capital good. Capital accumulation is an irreversible process and capital is depreciating over time. In this framework we reconsider the usual results of the efficient and optimal growth theory under an exhaustible resource constraint. We show that the usual efficiency condition relates to the investment good production function and not to the consumption good production function as in the canonical model of Dasgupta and Heal. We show then that the standard Hotelling rule relating the growth rate of the consumption good to the growth rate of the marginal productivity of the resource remains valid independently of the multisectoral specification of the model. Last we explore different forms of the Hartwick rule in the context of efficient paths and optimal paths.

JEL classification : O30, O41, Q01, Q32

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1 INTRODUCTION

The basic framework of numerous aggregate models with a man made good and a non renewable resource has been laid down by Dasgupta and Heal (DH in the sequel) in their well known seminal paper (DH, 1974). In this setting the man made good can be either consumed or accumulated to increase the capital stock. Capital accumulation is a reversible process: It is possible to transform back the capital into consumption good at no cost. Furthermore, there is no physical depreciation of capital or equivalently, the production process of the man made good is expressed in net terms. We build here the minimally differentiated production model permitting to disentangle basic relationships which are blurred in a single production sector model. Thus we assume that there exist two production sectors: The consumption good production sector and the capital good production sector which both use labor, capital and some non renewable resource. Furthermore, capital is a specific good which cannot be consumed and is depreciating over time. The production process of each good is expressed in net terms, thus capital depreciation is made explicit when used in the consumption good production sector.

In this framework we reconsider the usual results of the efficient and optimal growth theory with a non renewable resource. We assume that the exhaustible resource is an essential input in both production sectors. We put a particular emphasis upon physical efficiency. Efficiency is a fundamental problem, rooted in the issue of minimizing the cumulated use of the resource to sustain as long as possible a positive consumption rate. The possibility of a constant utility level, or equivalently of a constant consumption rate, has strong connections with different forms of Hartwick's rule[1].

To study the efficiency problem we resort to a standard two stages procedure: First solve the static efficiency problem having to be solved at each point of time before attacking the pure dynamical problem, the solution of which is linking through time the optimized values of the sequence of static problems. The static problem may be given different formulations. Here we

[1]See Dixit, Hammond and Hoel (1980), Dasgupta and Mitra (1983), Cass and Mitra (1991), Mitra (2002), Asheim, Buchholz and Withagen (2003), Cairns and Yang (2000), Cairns and Long (2006), for important contributions to this issue.

maximize the net capital good production for a given aggregate resource use, available capital stock and consumption rate having to be achieved. This is one amongst the many possible ways to describe the production frontier at time t, a frontier which is assumed to exist in most disaggregated models, like in the Dixit *et al.* (1980) paper. We have to assume that the capital good production function be concave with respect to the set of inputs together with the strict quasi-concavity of the consumption good production function to obtain the unicity of the solution of the static problem and thus a well defined static efficiency frontier. Proceeding along these lines is permitting to formulate the dynamical efficiency problem as an elementary control problem in which the only command variable is the instantaneous exhaustion rate of the non renewable resource.

Using this reduced form of the capital accumulation process we solve next the truly dynamic efficiency problem. The main result is that dynamic efficiency implies that the growth rate of the marginal productivity of the resource in the capital good production sector must be equal to the net marginal productivity of capital in the capital good production sector. This is a result which cannot be isolated in any model in which the consumption good and the capital good production sectors are merged together. This is also the kind of result which is difficult to isolate in fully disaggregated models like the Dixit *et al.* (1980) model in which sectors do not exist.

With respect to the Hartwick rule, we first consider the form of the rule having to be satisfied along any efficient path sustaining a constant consumption rate. Taking as current *numéraire* at each point of time the non renewable resource to express the shadow current value of the other asset, that is the current shadow value of the capital stock, we show that the value of the capital accumulation must balance the value of the resource exhaustion rate. Next adapting the very powerful proof strategy of Michel (1982), we show that the converse of the rule should hold. We derive this result without relying upon any transversality condition about the asymptotic shadow value of the capital stock at infinity as frequently invoked in the literature.

Last we turn to the optimality issue. We first prove that any efficient path may be seen as an optimal path where the objective function of the program is the sum of the consumption rates weighted by the inverse of the marginal productivity of the resource in the consumption good production sector, that is the marginal rate of transformation of the resource into the consumption good. Next we consider the standard criterion of maximizing the sum of discounted utilities at some given constant and positive discount rate. An optimal policy having to be efficient, it should be evident that the discounted marginal utility has to be equal at each point of time to the inverse of the marginal productivity of the resource in the consumption good production sector. This is, in the present setting, the equivalent expression of the Hotelling rule of the D.H model. The growth rate of the discounted marginal utility of consumption has to be equalized to the growth rate of the marginal productivity of the resource in the sole consumption good production sector. This is also a result which cannot be isolated in a one man made production good economy.

We last consider the generalized Hartwick rule as defined in the Dixit *et al.* (1980) paper. We show first that in our model the converse of the rule should hold, that is along any optimal constant utility path, the instantaneous change in asset endowments, valued in discounted marginal utility terms, should be nil at each date. As for the rule itself, we show that under our primitive assumptions concerning the production processes in the two sectors of the economy, the smoothness assumptions of Dixit *et al.* are satisfied by any optimal, and hence efficient, path. It follows immediately that the generalized Hartwick rule in the Dixit *et al.* sense should imply a constant consumption level along an optimal path.

The paper is organized as follows. Section 2 describes the bisectoral model. Efficiency is studied in Section 3 while optimality is considered in Section 4. Section 5 concludes.

2 THE MODEL

We consider an economy in which the labor supply is inelastic and constant through time. Let l be the amount of labor available at each point of time. The economy is producing two goods, a consumption good and a capital good. Each good is produced from capital, labor and a non renewable resource.

Let us denote by c both the production function of the consumption good sector and its production level:

$$c = c(K^c, l^c, s^c) \quad ,$$

where K^c, l^c and s^c are the amounts of capital, labor and resource allocated to the sector.

Assumptions $C.1$ and $C.2$ are both standard assumptions distinguished for analytical reasons:

Assumption C.1: $c : \mathcal{R}^3_+ \to \mathcal{R}_+$ is a function of class $\mathcal{C}^2$ strictly increasing, strictly quasi-concave, satisfying the Inada conditions, that is[2]:

$$\lim_{K^c \downarrow 0} c_K = \lim_{l^c \downarrow 0} c_l = \lim_{s^c \downarrow 0} c_s = +\infty \quad ,$$

Furthermore each input is essential: $c(K^c, l^c, s^c) = 0$ whatever the input which is equal to 0 and whatever the quantities of the two other factors.

A more stringent condition is that c is homogeneous:

Assumption C.2: c satisfies $C.1$ and:

$$c(K^c, l^c, s^c) = c_K K^c + c_l l^c + c_s s^c \ , \quad \forall (K^c, l^c, s^c) \in \mathcal{R}^3_{++} \quad .$$

When used in the consumption good sector, wear and tear is depreciating the capital allocated to the sector according to some radio-active law. We denote by δ its proportional decay rate.

Let k be both the net output of the capital good production sector and its *net* production function:

$$k = k(K^k, l^k, s^k) \quad ,$$

where K^k, l^k and s^k are the amounts of capital, labor and resource allocated to the sector.

[2]For any function $f(x, y, z, ...)$ we denote by f_x, f_y, f_z, ... its partial derivatives with respect to x, y, z, ... For any limit concerning any argument, the other arguments are held constant.

In order that the solution of the static efficiency problem of the below subsection 3.1 be unambiguously determined we must assume that k is concave. The concavity of k and the strict quasi-concavity of c are together sufficient conditions for the unicity of the solution of the efficiency problem which has to be solved at each point of time along any efficient path of the economy.

Assumption K.1: $k : \mathcal{R}^3_+ \to \mathcal{R}_+$ is a function of class $\mathcal{C}^2$ strictly increasing and concave, satisfying the Inada conditions, that is:

$$\lim_{K^k \downarrow 0} k_K = \lim_{l^k \downarrow 0} k_l = \lim_{s^k \downarrow 0} k_s = +\infty \quad .$$

Furthermore each input is essential.

As for c, a more stringent condition for k is that the function be homogenous.

Assumption K.2: k satisfies $K.1$ and:

$$k(K^k, l^k, s^k) = k_K K^k + k_l l^k + k_s s^k \quad , \quad \forall (K^k, l^k, s^k) \in \mathcal{R}^3_{++}.$$

The capital can be costlessly and instantaneously reallocated from any production sector to the other one. Thus denoting by $K(t)$ the amount of capital which is available in the economy at time t, we must have at each point of time:

$$K(t) \equiv K^c(t) + K^k(t)$$

We denote by K^0 the initial capital endowment: $K(0) = K^0$. Because capital is an essential input in each sector we assume that $K^0 > 0$ to avoid trivialities.

Since k is the net output of the capital good production sector, then the capital instantaneous increase rate is given by:

$$\dot{K}(t) = k(K^k(t), l^k(t), s^k(t)) - \delta K^c(t) \quad .$$

The labor can be costlessly and instantaneously reallocated from any production sector to the other one, so that the only constraints having to be satisfied are:

$$l - l^c(t) - l^k(t) \geq 0 \quad , \quad l^c(t) \geq 0 \text{ and } l^k(t) \geq 0 \ , \ t \geq 0. \tag{2.1}$$

Let $S(t)$ be the stock of the non renewable resource available at time t, then:

$$\dot{S}(t) \;=\; -s(t) \;\; ,$$

where: $s(t) \equiv s^c(t) + s^k(t)$. We denote by S^0 the initial resource endowment: $S^0 > 0$. Extraction costs are neglected.

Although this paper is mainly focusing upon efficient paths, the optimality problem cannot be escaped and it is worth checking what the standard criterion of maximizing the sum of discounted utilities is implying.

Assumption W : The welfare W is the sum of the discounted instantaneous utilities at some constant positive social rate of discount $\rho > 0$:

$$W \;\equiv\; \int_0^\infty u(c(t))e^{-\rho t}dt \;\; ,$$

where $u : \mathcal{R}_{++} \to \mathcal{R}$ is a function of class $\mathcal{C}^2$ strictly increasing, strictly concave, satisfying the Inada condition: $\lim_{c\downarrow 0} u'(c) = +\infty$, where $u'(c) \equiv du(c)/dc$.

A policy $\mathcal{P}$ is a path $\{(K^i(t), l^i(t), s^i(t), i = c, k\ , c(t)), t \geq 0\}$. To avoid trivialities $K^i(t)$, $l^i(t)$, $s^i(t)$, $i = c, k$ and $c(t)$, $t \geq 0$, are all assumed to be non negative. Such a policy is *potentially feasible* starting from K^0 provided that at each point of time $c(t)$ is attainable and that the cumulated use of the non renewable resource is finite. Thus leaving aside the non negativity constraints, such a potentially feasible path must satisfy:

1. The consumption attainability condition, that is:

   $$c(K^c(t), l^c(t), s^c(t)) - c(t) \;\; \geq 0 \qquad \forall t \geq 0 \; ,$$

   together with:

   $$K^0 + \int_0^t \left[k(K^k(\tau), l^k(\tau), s^k(\tau)) - \delta K^c(\tau)\right] d\tau - K^c(t) - K^k(t) \;\; \geq 0 \quad , \; \forall t \geq 0 \; ,$$

   and (2.1).

2. The finiteness of the cumulated resource extraction condition:

$$\int_0^t \left[s^c(\tau) + s^k(\tau) \right] d\tau < \infty \qquad , \ \forall t \geq 0 \ .$$

We denote by $\mathcal{P}(K^0)$ a policy which is potentially feasible starting from K^0.

Given some consumption path $\{c(t), t \geq 0\}$ and some initial capital stock $K^0 > 0$, the potentially feasible policy $\mathcal{P}(K^0)$ is said to be *sustaining* this consumption path iff its consumption path is the consumption path at stake.

3 EFFICIENCY

For the sake of simplicity, we shall restrict the attention to consumption paths $\{c(t), t \geq 0\}$ which are strictly positive, $c(t) > 0$, $t \geq 0$, and are continuously differentiable time functions. Consider such a path $\{c^*(t), t \geq 0\}$ and let $\mathcal{P}^*(K^0)$ be the potentially feasible policy starting from K^0 and sustaining this path. Denote by S^* the cumulated use of the resource that this policy is requiring: $S^* = \int_0^\infty [s^{c*}(t) + s^{k*}(t)]dt$. According to the usual definition of efficiency the policy $\mathcal{P}^*(K^0)$ is efficient if it does not exist any alternative feasible policy $\mathcal{P}(K^0)$ such that $c(t) \geq c^*(t)$, $t \geq 0$, with the strict inequality over some non degenerate time interval. Under $C.1$ and $K.1$ and the assumption $c^*(t) > 0$, this definition is equivalent to the following one. $\mathcal{P}^*$ is efficient if, first for any time interval $[t_1, t_2]$, $0 \leq t_1 < t_2$, $[t_1, t_2] \subseteq [0, \infty)$, possibly infinite, the restriction of the policy to the interval is minimizing the cumulated use of the resource amongst the set of subpolicies which are securing a consumption rate $c(t) \geq c^*(t)$ over the whole interval, when starting from $K(t_1) = K^*(t_1)$ and ending at $K(t_2) \geq K^*(t_2)$.

The problem of minimizing the cumulated extraction is best understood when conceived as a two stages optimization problem. The first stage is a standard static optimization problem which has to be solved at each point of time. At any date, given the available capital and given that the available labor has to be wholly used since it is not storable, there exists some static efficiency frontier in the three dimensional space: Consumption, capital accumulation and resource use, leaving aside the labor dimension since the labor supply is assumed to be inelastic and constant through time. This frontier may be described as some function denoted by κ, giving the maximum instantaneous capital stock net increase which can be obtained from some available capital K and resource use s, assuming that a given consumption rate c has to be achieved:

$$\dot{K}(t) \quad = \quad \kappa(K(t), s(t), c(t)) \quad .$$

The second stage is the truly dynamical problem. For a given consumption path $c^*(t)$ to be sustained, the tradeoff at each point of time is between accumulating capital at a higher rate today but at the cost of a higher present use of the resource, allowing to save the resource in the future, versus saving the resource today but at the cost of a lower capital accumulation inducing a higher use of the resource in the future. Using the function κ, this second stage arbitrage problem may be formulated as a problem in which the only command variable is the global resource extraction rate $s(t)$.

3.1 Solving the static optimization problem

Delete the time index and let K, l, be the capital and labor available at some time t and s some given resource extraction rate at the same time. The maximum consumption rate which can be expected is attained when all the inputs are allocated to the consumption good production sector. Let us denote by $\bar{c}$ this maximum consumption rate. Because l is constant through time, we shall write simply $\bar{c}$ as a function of K and s: $\bar{c} \equiv \bar{c}(K, s)$.

Assume that some consumption rate c, $0 < c \leq \bar{c}$, has to be produced. The problem is now to allocate K, l, s amongst the two sectors so as to maximize the capital increase $\dot{K}$, that is to solve the following static efficiency program $(S.E)$:

$$(S.E) \quad \max_{(K^k, l^k, s^k)} \quad k(K^k, l^k, s^k) - \delta(K - K^k)$$

$$\text{s.t} \quad c(K - K^k, l - l^k, s - s^k) - c \geq 0 \tag{3.1}$$

$$K - K^k \geq 0 \quad \text{and} \quad K^k \geq 0 \tag{3.2}$$

$$l - l^k \geq 0 \quad \text{and} \quad l^k \geq 0 \tag{3.3}$$

$$s - s^k \geq 0 \quad \text{and} \quad s^k \geq 0 \tag{3.4}$$

Proposition 1 *Under C.1 and K.1 the solution of the program* $(S.E)$ *is unique. Furthermore for* $c \in (0, \bar{c})$ *the constraints (3.2)-(3.4) are not binding so that the first order conditions reduce to:*

$$k_K + \delta = \gamma c_K \ , \ k_l = \gamma c_l \quad \textit{and} \quad k_s = \gamma c_s \ , \tag{3.5}$$

where γ *is the shadow marginal cost of consumption in terms of capital accumulation.*

Proof. For $c = \bar{c}$, all the available inputs must be allocated to the consumption good production sector: $K^k = l^k = s^k = 0$ and unicity is trivial. For $c \in (0, \bar{c})$ the proof of unicity is given in Appendix A.1. In this later case, because $c > 0$, we must have $c(K - K^k, l - l^k, s - s^k) > 0$ and since all the inputs are essential, then $K - K^k > 0$, $l - l^k > 0$ and $s - s^k > 0$. Next, because $c < \bar{c}$, some new capital can be produced and again by the essentiality argument, we must have $K^k > 0$, $l^k > 0$ and $s^k > 0$. Hence the constraints (3.2)-(3.4) are not tight and the Lagrangian of the problem may be written as:

$$\mathcal{L} \ = \ k(K^k, l^k, s^k) - \delta(K - K^k) + \gamma[c(K - K^k, l - l^k, s - s^k) - c] \ ,$$

from which we deduce (3.5).

The conditions (3.5) are implying that:

$$\frac{k_K + \delta}{k_l} = \frac{c_K}{c_l} \quad , \quad \frac{k_K + \delta}{k_s} = \frac{c_K}{c_s} \ \text{ and } \ \frac{k_l}{k_s} = \frac{c_l}{c_s} \quad . \tag{3.6}$$

These conditions are the usual ones according to which the marginal rates of transformation between any pair of inputs must be the same in the both sectors of the economy. But here the two sectors are the consumption good production sector and the capital accumulation sector and not the mere capital good production sector. This is the reason why the term $k_K + \delta$ instead of k_K is appearing in the ratios involving the allocation of capital.

The conditions (3.5) also imply that:

$$\frac{1}{\gamma} = \frac{c_K}{k_K + \delta} = \frac{c_l}{k_l} = \frac{c_s}{k_s} \quad . \tag{3.7}$$

Equations (3.7) mean that the marginal cost of capital accumulation in terms of the consumption good must be the same whatever the factor (capital, labor or resource) which is marginally diverted from the consumption good production sector towards the capital accumulation sector.

Let us denote by $(\tilde{K}^k, \tilde{l}^k, \tilde{s}^k)$ the unique solution of the problem $(S.E)$. $\tilde{K}^k$, $\tilde{l}^k$ and $\tilde{s}^k$ are functions of K, l, s and c, $c \leq \bar{c}(K,s)$. Again because l is constant through time we may delete the argument and write more simply:

$$\tilde{K}^k = \tilde{K}^k(K,s,c) \ , \ \tilde{l}^k = \tilde{l}^k(K,s,c) \ \text{ and } \tilde{s}^k = \tilde{s}^k(K,s,c) \ .$$

Then the maximum capital accumulation function κ may be defined as:

$$\kappa(K,s,c) \equiv k(\tilde{K}^k(K,s,c), \tilde{l}^k(K,s,c), \tilde{s}^k(K,s,c)) - \delta[K - \tilde{K}^k(K,s,c)]$$

By the envelope theorem:

$$\kappa_K \equiv \frac{\partial \kappa}{\partial K} = -\delta + \gamma c_K \ , \ \kappa_s \equiv \frac{\partial \kappa}{\partial s} = \gamma c_s \ \text{ and } \ \kappa_c \equiv \frac{\partial \kappa}{\partial c} = -\gamma \ . \tag{3.8}$$

Substituting for γ as given by (3.7) results in:

$$\kappa_K = k_K \ , \ \kappa_s = k_s \ \text{ and } \ \kappa_c = -\frac{k_K + \delta}{c_K} = -\frac{k_l}{c_l} = -\frac{k_s}{c_s} \ . \tag{3.9}$$

Furthermore, as a function of c, for $K > 0$ and $s > 0$ given, the extreme values of the range of κ are given by:

$$\lim_{c \uparrow \bar{c}(K,s)} \kappa(K,s,c) = -\delta K \ , \tag{3.10}$$

and:

$$\lim_{c \downarrow 0} \kappa(K,s,c) = k(K,l,s) \tag{3.11}$$

Before turning to the dynamical problem let us define a last boundary relationship which will happen to be useful later for characterizing the solution of the second stage problem. Consider some consumption rate c and assume that no new capital has to be produced, so that $K^c = K$. Then we must have:

$$\kappa(K,s,c) = -\delta K \ .$$

This equation may be solved for s as a function of K and c. Let us denote by $\underline{s}(K,c)$ its solution. $\underline{s}$ is the minimum rate of resource extraction necessary to achieve the consumption rate c when the available capital amounts to K. This minimum is attained when no new capital has to be built up in which case $\kappa = -\delta K$. Said in another way, $\underline{s}(K, c)$ is this value of s for which we would have $c = \bar{c}(K, l, s)$. Hence $\underline{s}(K, c)$ is nothing but than the solution of $c = \bar{c}(K, l, s)$ for a given K and the given constant labor supply l, which is deleted as an argument of $\underline{s}$ to clear the notation. Thus $\underline{s}$ is a decreasing function of K and an increasing function of c:

$$\underline{s}_K \equiv \frac{\partial \underline{s}}{\partial K} = -\frac{c_K}{c_s} < 0 \quad \text{and} \quad \underline{s}_c \equiv \frac{\partial \underline{s}}{\partial c} = \frac{1}{c_s} > 0 \ , \tag{3.12}$$

where the derivatives of c are valued at $K^c = K$, $l^c = l$ and $s^c = s$.

3.2 Solving the dynamical problem

Armed with the κ function we may focus the attention upon the proper dynamical aspect of the problem. Consider some efficient policy $\mathcal{P}^*(K^0)$ defined over the infinite time interval $[0, \infty)$.Given that $c^*(t)$, $t \in [t_1, t_2] \subseteq [0, \infty)$, has to be sustained, minimizing the cumulated extraction of the resource over any time subinterval $[t_1, t_2]$ may be formulated as the following problem (E) in which the only command variable is the instantaneous rate of the resource extraction $s(t)$:

$$
\begin{aligned}
(E) \qquad & \max_{\{s(t), t \in [t_1, t_2]\}} \ -\int_{t_1}^{t_2} s(t)dt \\
s.t. \quad & \dot{K}(t) = \kappa(K(t), s(t), c^*(t)) \ , \qquad t \in [t_1, t_2] \quad (3.13) \\
& K(t_1) = K^*(t_1) \text{ with } K(t_1) = K^0 \text{ if } t_1 = 0 \ , \quad (3.14) \\
& K(t_2) - K^*(t_2) \geq 0 \ , \quad (3.15) \\
& s(t) - \underline{s}(K(t), c^*(t)) \geq 0 \ . \qquad t \in [t_1, t_2] \quad (3.16)
\end{aligned}
$$

Note that if the constraint (3.16) is tight, then $s(t) = \underline{s}(K(t), c^*(t))$, so that $\dot{K}(t) = \kappa(K(t), \underline{s}(K(t), c^*(t)), c^*(t)) = -\delta K(t)$. No new capital is produced. The capital stock $K(t)$, wholly allocated to the consumption good production sector, is decreasing at the proportional rate δ.

Let $\mathcal{L}^E(t)$ be the Lagrangian of the program (E):

$$\mathcal{L}^E(t) \ = \ -s(t) + \nu^E(t)\kappa(K(t), s(t), c^*(t)) + \alpha^E(t)[s(t) - \underline{s}(K(t), c^*(t))] \ .$$

The first order condition is:

$$\frac{\partial \mathcal{L}^E}{\partial s} = 0 \quad \Longleftrightarrow \quad \nu^E(t)\kappa_s(t) = 1 - \alpha^E(t) \ , \tag{3.17}$$

$$\alpha^E(t) \geq 0 \ , \ s(t) - \underline{s}(K(t), c^*(t)) \geq 0 \quad \text{and } \alpha^E(t)[s(t) - \underline{s}(K(t), c^*(t))] = 0 \quad . \tag{3.18}$$

$\nu^E(t)$ is a continuous time function and its dynamics at any time t at which it is differentiable must satisfy:

$$\dot{\nu}^E(t) = -\frac{\partial \mathcal{L}^E}{\partial K} \quad \Longleftrightarrow \quad \dot{\nu}^E(t) = -\nu^E(t)\kappa_K(t) + \alpha^E(t)\underline{s}_K(t) \ . \tag{3.19}$$

Last if t_2 is finite, the transversality condition is:

$$\nu^E(t_2)[K(t_2) - K^*(t_2)] = 0 \quad \text{and} \quad \nu^E(t_2) \geq 0 \ , \tag{3.20}$$

and no condition if t_2 is infinite.

Assume that the solution is an interior solution, i.e. (3.16) is not effective so that $\alpha^E(t) = 0$. An abrupt upward shift of $K(t)$ is unfeasible. A sudden drop down of the capital stock by capital destruction could not help reduce the cumulated use of the resource, and thus would be inefficient. Hence $K(t)$ is a continuous function of time along an efficient path. Since $c^*(t)$ is assumed to be a continuous function of time, $\nu^E(t)$ is also a continuous function of time along any efficient path. Taking into account (3.17), we conclude that the command variable $s(t)$ is a continuous time function along an efficient interior path. This in turn implies that $\dot{K}(t) = \kappa(K(t), s(t), c(t))$ has to be a continuous time function along efficient interior paths.

Because $c^*(t)$ has been assumed to be a $\mathcal{C}^1$ time function, then $\partial\kappa(K, s, c^*(t))/\partial t = \kappa_c\dot{c}(t)$ is also a $\mathcal{C}^1$ function of time. Using this property and the fact that $s(t)$ is a continuous time function, we may resort to a standard optimal control result[3] showing that $\nu^E(t)$ is differentiable at any time so that (3.19) is verified at any $t \in [t_1, t_2]$ along an efficient path.

Because the costate variable $\nu^E(t)$ is a continuous and continuously differentiable function of time, we may differentiate (3.17) w.r.t. time, resulting in:

$$\dot{\nu}^E(t) \quad = \quad -\frac{1}{\kappa_s^2}[\kappa_{sK}\dot{K}(t) + \kappa_{ss}\dot{s}(t) + \kappa_{sc}\dot{c}(t)] \ .$$

[3]See Seierstad and Sydsæter, 1987, Theorem 2, p 85.

The direct and cross second derivatives of κ are continuous functions of (K, s, c) under $K.1$ and $C.1$. Hence $s(t)$ is also a differentiable function of time along any efficient interior path. Note that this implies in turn that the Hamiltonian $\mathcal{H}(t) \equiv -s(t) + \nu^E(t)\kappa(K(t), s(t)c^*(t))$ is a continuous and differentiable function of time along such a path. We shall use this property later when applying the dynamic envelope theorem to the Hamiltonian function.

Last through a logarithmic time differentiation of (3.17) and making use of (3.19), we obtain the below relationship (3.21) characterizing the efficient interior paths in terms of the κ function, and, using (3.7) and (3.9) the equations (3.22)-(3.23) characterizing the efficient interior paths in terms of the k and c functions.

Proposition 2 *Under $C.1$ and $K.1$, along any dynamically efficient interior path:*

$$\frac{\dot{\kappa}_s(t)}{\kappa_s(t)} = -\frac{\dot{\nu}^E(t)}{\nu^E(t)} = \kappa_K(t) \ , \tag{3.21}$$

that is by (3.9):

$$\frac{\dot{k}_s(t)}{k_s(t)} = -\frac{\dot{\nu}^E(t)}{\nu^E(t)} = -k_K(t) \ , \tag{3.22}$$

and by (3.7):

$$\frac{\dot{k}_s(t)}{k_s(t)} = k_l(t)\frac{c_K(t)}{c_l(t)} - \delta \quad \textit{and} \quad \frac{\dot{k}_s(t)}{k_s(t)} = k_s(t)\frac{c_K(t)}{c_s(t)} - \delta \ . \tag{3.23}$$

The conditions (3.21)-(3.23) are conditions warranting that all the arbitrage opportunities are locally exhausted. Any trade-off, either direct or indirect, between some increase of the resource extraction rate and some simultaneous decrease of the investment rate today followed later by a higher investment rate in the near future and a simultaneous decrease of the extraction rate, while maintaining the consumption level c^*, cannot reduce the cumulative resource extraction.

3.3 Remarks about the Dasgupta and Heal (1974) canonical model

The DH (1974) model is not explicitly framed as a two sectors model. But it can be understood as such a model in which first, the production function of the capital good sector takes a one to one form, second, either the working life of capital goods is infinite ($\delta = 0$) or the aggregate production function is defined in net terms, and third, the capital accumulation process is perfectly reversible, that is the capital can be instantaneously and freely transformed back into the consumption good and consumed. The same kind of framework is also found in Mitra (1978) and Dasgupta and Mitra (1983), although in a slightly more general form and in a discrete time model.

The production core of the DH model may be written as follows, leaving aside the reversibility option, that is the case in which some part of the consumption would be supplied by a decrease of the capital stock:

$$g = g(K, l, s)\ , c = g^c \quad \text{and} \quad k = g^k$$

where g is both the production function and the production level of the unique man-made good, g^c is this part of the production which is consumed and g^k this part which is allocated to the capital accumulation. The function κ takes here the following simple form:

$$\kappa(K, s, c) = g(K, l, s) - c\ ,$$

which may negative and (3.21) results in:

$$\frac{\dot{g}_s(t)}{g_s(t)} \;=\; g_K(t)\ ,$$

which is nothing but than the well known efficiency condition of the DH model.

3.4 Efficiency and Hartwick's rule

The Hartwick rule is basically a value relationship which must hold along any efficient path sustaining a constant consumption rate. Thus the rule should be stated in terms of shadow efficiency prices.

Let us define a *global efficiency problem* (GE) as any problem (E) where $[t_1, t_2) = [0, \infty)$, and a *uniform global efficiency* $(GE.u)$ problem as a (GE) problem in which the consumption rate having to be sustained is constant through time and strictly positive, $c(t) = c > 0$, $t \geq 0$.

Proposition 3 *Assume that* $C.1$ *and* $K.1$ *hold and consider some constant consumption path* $c^*(t) = c^* > 0$ *which, given* $K^0 > 0$*, is potentially feasible. Let* $\{s^*(t), t \geq 0\}$ *be some continuous path of resource use such that* $\int_0^\infty s^*(t)dt < \infty$*. Denote by* $K^*(t)$ *the solution of (3.13) for* $c^*(t) = c^*$, $s(t) = s^*(t), t \geq 0\}$ *and* $K(0) = K^0$*. Assume that for* $\{(s^*(t), K^*(t)), t \geq 0\}$*, (3.16) is satisfied as a strict inequality. If* $\{s^*(t), t \geq 0\}$ *is solving this* $(GE.u)$ *problem, then it is an interior solution and there exists some* $\mathcal{C}^1$ *function* $\{\nu^{E*}(t), t \geq 0\}$*, the costate variable of* $K^*(t)$*, such that:*

$$\nu^{E*}(t)\dot{K}^*(t) \quad = \quad s^*(t) \quad , \quad t \in [0, \infty) \ . \tag{3.24}$$

Note that in this version of the rule, $\nu^{E*}(t)$, the shadow marginal value of the capital stock, is a current efficiency price in terms of the cumulated resource use given the objective function of the problem (GE). Thus $\nu^{E*}(t)$ is the amount of resource which could be marginally saved were the stock of capital $K^*(t)$ be marginally higher at time t. In such a context, the current marginal valuation of the resource is equal to 1 at any time t. Thus what (3.24) is stating is that the value of the instantaneous change in asset endowment[4] at any time t, at prices $(\nu^{E*}(t), 1)$, that is $\nu^{E*}(t)\dot{K}^*(t) - s^*(t)$, must be nil.

The proof is running as follows. Let $\mathcal{H}(t)$ be the Hamiltonian of the $(GE.u)$ problem:

$$\mathcal{H}(t) \quad = \quad -s(t) + \nu^E(t)\kappa(K(t), s(t), c^*) \ .$$

[4] Not to be confused with the instantaneous change of the endowment value which amounts to $\dot{\nu}^{E*}(t)K^*(t) + \nu^{E*}(t)\dot{K}^*(t) - s^*(t)$.

By the dynamic envelope theorem[5], we must have:

$$\frac{d\mathcal{H}(t)}{dt} = \frac{\partial \mathcal{H}(t)}{\partial t}$$

Thanks to the fact that $c^*(t)$ is constant through time, $\partial\mathcal{H}/\partial t = 0$ so that $d\mathcal{H}/dt = 0$, implying that:

$$\mathcal{H}(t) = h \iff \nu^E(t)\kappa(K(t), s(t), c^*) - s(t) = h$$

where h is some constant. Thus:

$$\nu^E(t)\dot{K}(t) - s(t) = h$$

To prove that $h = 0$, we follow the general strategy developed in Michel (1982) with due care to the fact that here there is no discounting. The idea of the proof, formally developed in Appendix A.2, is to convert the problem $(GE.u)$ into a Bolza problem of the form:

$$\max_{\{s(t), t\in[0,T)\}} \int_0^T (-s(t))dt + R(T)$$

where T is any finite time and $R(T) \equiv \int_T^\infty (-s^*(t))dt$, where $\{s^*(t), t \in [T, \infty)\}$ is an efficient path followed from T onwards starting from the efficient level of the capital stock, $K^*(T)$ at time T. Remark that since $\{s^*(t), t \in [T, \infty)\}$ has been assumed to be efficient and hence feasible, one should have:

$$-R(T) = \int_T^\infty s^*(t)dt < \infty \ .$$

Thus $R(T)$ is a well defined integral bounded from below. Using the same assumptions as imposed by Michel (1982)[6], it is possible to derive the limit properties of an efficient solution letting $T \to \infty$. For a constant consumption path having to be sustained, this will result in $\lim_{T\uparrow\infty} \mathcal{H}(T) = 0$, a

[5]For a standard formulation of the theorem, see for example Seierstad and Sydsæter, 1987, Chap 2, Note 3, p 61. Since $c^*(t) = c^* > 0$, $t \geq 0$, $c^*(t)$ is trivially a continuous and continuously differentiable time function implying that the Hamiltonian function is a continuous and continuously differentiable time function along an efficient interior path, as shown in subsection 3.2.

[6]In particular, Michel's proof does not require that the Hamiltonian of the Bolza problem be strictly concave in the vector of state and control variables, an assumption sometimes made to derive transversality conditions in infinite time horizon problems, see Seierstad and Sydsæter, 1987, Chap 3, Theorem 13, p 235 for an example. In the present case, since we want to maximize a linear criterion, strict concavity is a true issue and our proof should not depend upon such an assumption.

generalization over an infinite time horizon of a well known transversality condition for a finite free endpoint problem terminating at T. But since the Hamiltonian must be constant along a path solving the $(GE.u)$ problem, this in turn implies that $\mathcal{H}(t) = 0$, $t \geq 0$, that is $h = 0$ which is nothing but than the Hartwick's rule.

4 OPTIMALITY

Optimality is clearly dependent upon the way the different consumption paths are valued, the amount of initially available capital K^0 and the amount of the initial resource endowment S^0.

Before turning to the implications of the strong assumption W, we first show that any efficient policy may appear as an optimal policy provided that the consumption levels at different dates be appropriately weighted and that the initial resource endowment of the economy, S^0, be the precise cumulated use of the resource required to implement the efficient policy. Because higher instantaneous consumption rates are more highly valued than lower ones in all the non pathological welfare criteria, whatever the way the consumption rates are weighted, the static efficiency conditions have to be satisfied. Thus we may resort to the κ function to frame the optimality problem.

4.1 Efficient programs as optimal programs

Let $\{\pi(t), t \geq 0\}$ be a time profile of consumption rate weights and consider the problem of maximizing the sum of the weighted consumption rates, that is the following problem (D):

$$
\begin{array}{llll}
(D) & \max_{\{(c(t),s(t)),t\geq 0)\}} \int_0^\infty \pi(t)c(t)dt & & \\
s.t. & c(t) \geq 0 & t \in [0,\infty) & (4.1) \\
 & \dot{S}(t) = -s(t) & t \in [0,\infty) & (4.2) \\
 & S(0) = S^0 > 0 \text{ given, and } S(t) \geq 0 \ , & t \in [0,\infty) & (4.3) \\
 & s(t) \geq 0 \text{ and } s(t) - \underline{s}(K(t),c(t)) \geq 0 & t \in [0,\infty) & (4.4) \\
 & \dot{K}(t) = \kappa(K(t),s(t),c(t)) & t \in [0,\infty) & (4.5) \\
 & K(0) = K^0 > 0 \text{ given, and } K(t) \geq 0 \ , & t \in [0,\infty) \ . & (4.6)
\end{array}
$$

Let $\mathcal{L}^D(t)$ be the Lagrangian of the problem:

$$
\begin{aligned}
\mathcal{L}^D \ = \ & \pi(t)c(t) + \alpha_c^D(t)c(t) - \lambda^D(t)s(t) + \alpha_S^D(t)S(t) + \alpha_s^D(t)s(t) \\
& + \underline{\alpha}_s^D(t)[s(t) - \underline{s}(K(t),c(t))] + \nu^D(t)\kappa(K(t),s(t),c(t)) + \alpha_K^D(t)K(t)
\end{aligned}
$$

The first order conditions are:

$$
\begin{aligned}
\frac{\partial \mathcal{L}^D}{\partial c} = 0 \iff & \pi(t) = -\nu^D(t)\kappa_c(t) + \underline{\alpha}_s^D(t)\underline{s}_c(t) - \alpha_c^D(t) & (4.7) \\
\frac{\partial \mathcal{L}^D}{\partial s} = 0 \iff & \lambda^D(t) = \nu^D(t)\kappa_s(t) + \underline{\alpha}_s^D(t) + \alpha_s^D(t) & (4.8) \\
& \alpha_c^D(t) \geq 0 \ , \ c(t) \geq 0 \ \text{ and } \ \alpha_c^D(t)c(t) = 0 & (4.9) \\
& \alpha_s^D(t) \geq 0 \ , \ s(t) \geq 0 \ \text{ and } \ \alpha_s^D(t)s(t) = 0 & (4.10) \\
& \underline{\alpha}_s^D(t) \geq 0 \ , \ s(t) - \underline{s}(K(t),c(t)) \geq 0 & \\
& \text{and } \ \underline{\alpha}_s^D(t)[s(t) - \underline{s}(K(t),c(t))] = 0 \ . & (4.11)
\end{aligned}
$$

Assuming that $\pi(t)$ is a function of class $\mathcal{C}^1$, the dynamics of the costate variables must satisfy:

$$
\begin{aligned}
\dot{\lambda^D}(t) = -\frac{\partial \mathcal{L}^D}{\partial S} \iff & \dot{\lambda}^D(t) = -\alpha_S^D(t) & (4.12) \\
\dot{\nu}^D(t) = -\frac{\partial \mathcal{L}^D}{\partial K} \iff & \dot{\nu}^D(t) = -\nu^D(t)\kappa_K(t) + \underline{\alpha}_s^D(t)\underline{s}_K(t) & \\
& -\alpha_K^D(t) \ . & (4.13) \\
\alpha_S^D(t) \geq 0 \quad , \quad & S(t) \geq 0 \ \text{ and } \alpha_S^D(t)S(t) = 0 & (4.14) \\
\alpha_K^D(t) \geq 0 \quad , \quad & K(t) \geq 0 \ \text{ and } \alpha_K^D(t)K(t) = 0 & (4.15)
\end{aligned}
$$

Because the objective criterion of the problem (D) is not a strictly concave function and may be unbounded in infinite time, it is well known that 'transversality' conditions for the asymptotic behavior of the shadow values of the capital stock may not exist[7]. Since the resource stock is finite, optimal management of the natural resource should result in $\lim_{t\uparrow\infty} S(t) = 0$. But no equivalent condition can be derived for the shadow value of the capital stock when time goes to infinity without imposing specific restrictions over the weights sequence $\{\pi(t), t \geq 0\}$.

Consider now some consumption path $\{c^*(t), t \geq 0\}$ assumed to be a strictly positive and continuously differentiable time function as usual. Assume that for the initial value K^0 of the above problem (D) there exist potentially feasible policies sustaining this consumption path. Let $\mathcal{P}^*(K^0)$ be an efficient potentially feasible policy and assume that the solution of the corresponding efficiency problem (E) is an interior solution as characterized in Proposition 2. Last denote by S^* the cumulated use of the resource required for the effectiveness of the potentially feasible policy $\mathcal{P}^*(K^0)$: $S^* = \int_0^\infty s^*(t)dt$ where $s^*(t)$ is the resource use component of $\mathcal{P}^*(K^0)$.

Let us show that $\{(c^*(t), s^*(t)), t \geq 0\}$ is a solution of the problem (D) provided that $S^0 = S^*$ and $\pi(t) = -\kappa_c^*(t)/\kappa_s^*(t)$, $t \geq 0$, where $\kappa_c^*(t)$ and $\kappa_s^*(t)$ are evaluated along the efficient path $\mathcal{P}^*(K^0)$. The proof consists in checking that for $\lambda^D(t) = 1$, $t \geq 0$, and $\nu^D(t) = \nu^E(t)$, $t \geq 0$, where $\nu^E(t)$ is the costate variable of $K(t)$ in the efficiency problem (E), all the optimality conditions (4.7)-(4.15) are satisfied. Furthermore the optimized value of the objective function must be well defined.

First note that all the multipliers α_c^D, α_s^D, $\underline{\alpha}_s^D$, α_S^D and α_K^D must be equal to 0 at any date because the efficient policy is an interior path. Thus (4.9)-(4.11), (4.14) and (4.15) are satisfied.

Given that all these multipliers are equal to zero, we get:

(4.12) $\Longrightarrow \dot{\lambda}^D(t) = 0$, which is trivially satisfied by setting $\lambda^D(t) = 1$, $t \geq 0$.

(4.13) $\Longrightarrow \dot{\nu}^D(t) = -\nu^D(t)\kappa_K^*(t)$, $t \geq 0$, which is satisfied provided that

[7] See Michel (1990) for a comprehensive account of the difficulty to derive transversality conditions at infinity with such optimality criterions.

$\nu^D(t) = \nu^E(t)$ according to the second equality of (3.21) in Proposition 2.

(4.8) $\Longrightarrow 1 = \nu^D(t)\kappa_s^*(t)$ which is satisfied if $\nu^D(t) = \nu^E(t)$ according to the efficiency condition (3.17) (with $\alpha^E(t) = 0$ for an interior path)

Note also that this above equality is implying that:

$$-\frac{\dot{\nu}^D(t)}{\nu^D(t)} = \frac{\dot{\kappa}_s^*(t)}{\kappa_s^*(t)} ,$$

which is nothing but than the first equality of (3.21) when $\nu^D(t) = \nu^E(t)$.

(4.7) $\Longrightarrow \pi(t) = -\nu^D(t)\kappa_c^*(t)$, a condition which is clearly satisfied when $\pi(t) = -\kappa_c^*(t)/\kappa_s^*(t)$ as assumed and when $\nu^D(t) = 1/\kappa_s^*(t)$ according to the above condition (4.8) when $\nu^D(t) = \nu^E(t)$.

The last point having to be checked is that the optimized value of the objective function of the program (D) be well defined. Given that $\pi(t) = -\kappa_c^*(t)/\kappa_s^*(t)$ the static efficiency conditions (3.9) imply that $\pi(t) = 1/c_s^*(t)$. Thus the efficient program sustaining the consumption path may appear as an optimal program provided that $\int_0^\infty [c^*(t)/c_s^*(t)]dt < \infty$ where $c_s^*(t)$ is valued along the efficient path. Hence we may conclude as follows:

Proposition 4 *Let $\{c^*(t), t \geq 0\}$ be some positive and continuously differentiable consumption path. Assume that both $C.1$ and $K.1$ hold and that there exists an efficient potentially feasible policy $\mathcal{P}^*(K^0)$ sustaining the consumption path starting from $K^0 > 0$ and requiring a cumulated resource use S^**[8]*. Assume also that this policy is an interior solution of the efficiency problem (E). Then $\{(c^*(t), s^*(t)), t \geq 0\}$, where $\{s^*(t), t \geq 0\}$ is the resource use component of $\mathcal{P}^*(K^0)$, is solving the optimality problem (D) provided that:*

- *The resource endowment of the economy be equal to the resource stock required to implement the policy $\mathcal{P}^*(K^0)$, that is:*

$$S^0 = S^*$$

[8]Note that assuming that the policy $\mathcal{P}^*(K^0)$ is potentially feasible implies that $S^* < \infty$.

- *The consumption weights be equal to:*

$$\pi(t) = \frac{1}{c_s^*(t)} \quad , \ t \geq 0$$

where $c_s^*(t)$ is the value of $c_s(t)$ along the efficient path $\mathcal{P}^*(K^0)$ sustaining $\{c^*(t), t \geq 0\}$,

- *and:*

$$\int_0^\infty \frac{c^*(t)}{c_s^*(t)} dt < \infty$$

An immediate implication of the above proposition and the Proposition 3 is the following. If a positive constant path $\{c^*(t) = c^* > 0, t \geq 0\}$ can be sustained by an efficient policy $\mathcal{P}^*(K^0)$, solution of (E) for $[t_1, t_2) = [0, \infty)$, it is also a solution of the program (D) provided that $S^0 = S^*$, $\pi(t) = 1/c_s^*(t)$ and $\int_0^\infty [1/c_s^*(t)] dt < \infty$. Along such an optimal path, the Hartwick rule (3.24) should be verified.

4.2 Optimality under Assumption W

Under the welfare criterion W the time profile of the consumption rates weights, $\{\pi(t), t \geq 0\}$, which is exogenously given in the problem (D) is now endogenously determined. Let (P) be the optimality problem under the Assumption W:

$$\begin{array}{ll} (P) & \max\limits_{\{(c(t),s(t)),t\geq 0)\}} \int_0^\infty u(c(t)) e^{-\rho t} dt \\ & s.t. \quad (4.1) - (4.6) \ . \end{array}$$

Under W, $c(t)$ must be positive at any time and the resource being an essential factor under $C.1$ and $K.1$, then $s(t)$ must be positive implying that $S(t)$ is also positive. The same argument applies for $K(t)$ which must be also positive. Thus we my leave aside the corresponding non negativity constraints and write the current value Lagrangian of the problem (P) as follows:

$$\begin{aligned} \mathcal{L}^P(t) &= u(c(t)) - \lambda(t)s(t) + \nu(t)\kappa(K(t), s(t), c(t)) \\ &\quad + \underline{\alpha}(t)[s(t) - \underline{s}(K(t), c(t))] \ . \end{aligned}$$

The first order conditions are:

$$\frac{\partial \mathcal{L}^P}{\partial c} = 0 \iff u'(c(t)) = -\nu(t)\kappa_c(t) + \underline{\alpha}(t)\underline{s}_c(t) \tag{4.16}$$

$$\frac{\partial \mathcal{L}^P}{\partial s} = 0 \iff \lambda(t) = \nu(t)\kappa_s(t) + \underline{\alpha}(t) \tag{4.17}$$

$$\underline{\alpha}(t) \geq 0 \ , \ s(t) - \underline{s}(K(t), c(t)) \geq 0$$

$$\text{and} \quad \underline{\alpha}(t)[s(t) - \underline{s}(K(t), c(t))] = 0 \ . \tag{4.18}$$

The dynamics of the costate variables must satisfy at any time t at which λ and ν are time differentiable:

$$\begin{aligned} \dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial \mathcal{L}^P}{\partial S} &\iff \dot{\lambda}(t) = \rho\lambda(t) \\ &\iff \lambda(t) = \lambda_0 e^{\rho t} \text{ where } \lambda_0 = \lambda(0) \end{aligned} \tag{4.19}$$

$$\begin{aligned} \dot{\nu}(t) = \rho\nu(t) - \frac{\partial \mathcal{L}^P}{\partial K} &\iff \dot{\nu}(t) = \rho\nu(t) - \nu(t)\kappa_K(t) \\ &\qquad\qquad + \alpha(t)\underline{s}_K(t) \ . \end{aligned} \tag{4.20}$$

Since the criterion in the problem (P) is strictly concave, we may resort to standard results (see Michel, 1982) to get the following transversality conditions:

$$\lim_{t\uparrow\infty} e^{-\rho t}\lambda(t)S(t) = \lambda_0 \lim_{t\uparrow\infty} S(t) = 0 \tag{4.21}$$

$$\lim_{t\uparrow\infty} e^{-\rho t}\nu(t)K(t) = 0 \ . \tag{4.22}$$

Let us check that under the assumptions $C.1$, $K.1$ and W, an interior optimal path, that is a path along which (4.4) is satisfied as a strict inequality at any time t, is a continuous and differentiable function of time over $t \in [0, \infty)$. Since $u(c)$ is a strictly concave functions under the assumption W, a standard arbitrage argument would show that a jump of the consumption path in any direction is suboptimal. Thus the optimal consumption path should be a continuous function of time. If the optimal path is interior, $\underline{\alpha}(t) = 0$. Since the costate variables $\lambda(t)$ and $\nu(t)$ are continuous functions of time, and since $K(t)$ is also a continuous time function, we conclude from (4.17) that $s(t)$ is a continuous function of time. This in turn implies that $\lambda(t)$ and $\nu(t)$ are differentiable functions of time along an interior optimal path.

Next consider a time interval $\mathcal{D}$ within which $c(t)$ and $s(t)$ are time differentiable functions. Time differentiating (4.17) and making use of (4.19) and (4.20) we get for $t \in \mathcal{D}$:

$$-\rho\lambda(t) + (\rho\nu(t) - \nu(t)\kappa_K(t))\kappa_s(t) + \nu(t)[\kappa_{sK}\dot{K}(t) + \kappa_{ss}\dot{s}(t) + \kappa_{sc}\dot{c}(t)] = 0$$

Using (4.17) and simplifying, this is equivalent to:

$$\kappa_{sK}\dot{K}(t) + \kappa_{ss}\dot{s}(t) + \kappa_{sc}\dot{c}(t) = \kappa_s\kappa_K \quad , \ t \in \mathcal{D}$$

from which we obtain the following expression of $\dot{s}(t)$, $t \in \mathcal{D}$:

$$\dot{s}(t) = \kappa_{ss}^{-1}[\kappa_s\kappa_K - \kappa_{sK}\dot{K}(t) - \kappa_{sc}\dot{c}(t)] \equiv \dot{s}(K, s, c, \dot{c}) \ ,$$

where $\dot{s}$ is a continuous function of $(K, s, c, \dot{c})$. Differentiating (4.16) with respect to time for $t \in \mathcal{D}$, we obtain, dropping the time index:

$$u''(c)\dot{c} + \dot{\nu}\kappa_c + \nu[\kappa_{cK}\dot{K} + \kappa_{cs}\dot{s} + \kappa_{cc}\dot{c}] = 0$$

Making use of the previous expression of $\dot{s}$ as a function of $\dot{c}$, this is equivalent to:

$$[u''(c) + \nu(-\kappa_{cs}\kappa_{ss}^{-1}\kappa_{sc} + \kappa_{cc})]\dot{c} = -\dot{\nu}\kappa_c - \nu[(\kappa_{cK} - \kappa_{cs}\kappa_{ss}^{-1}\kappa_{sK})\dot{K} + \kappa_{cs}\kappa_{ss}^{-1}\kappa_s\kappa_K]$$

Note that the functions between brackets appearing into the above relation should be continuous time functions since (K, s, c, λ, ν) is a vector of continuous functions of time. We conclude that a jump of $\dot{c}(t)$ at the junction of two adjacent intervals $\mathcal{D}$ and $\mathcal{D}'$ of time differentiability of the function $c(t)$ would violate this continuity requirement. Thus the consumption path must be a differentiable and continuous function of time along an optimal interior path. This implies in turn that $s(t)$ must also be a differentiable and continuous time function. Hence we may conclude as follows:

Proposition 5 *Let $\mathcal{P}^*$ be an optimal policy. Under $C.1$, $K.1$ and W, $c^*(t) > 0$, $t \geq 0$ and, if the solution $\{(K^*(t), c^*(t), s^*(t)), t \geq 0\}$ is an interior solution, then it is a time differentiable function and the Hamiltonian of the problem (P) valued along the optimal path is also a time differentiable function.*

Because $c^*(t)$ is time differentiable and strictly positive and because an optimal path must be an efficient path, Propositions 4 and 5 together imply that:

Proposition 6 *Assume that* $C.1$, $K.1$ *and* W *hold. Then for any optimal interior policy* $\mathcal{P}^*$ *we must have:*

$$u'(c^*(t))e^{-\rho t} \quad = \quad \frac{1}{c_s^*(t)} \; , \; t \geq 0 \; , \tag{4.23}$$

where $c_s^*(t)$ *is the value of* c_s *along the optimal path.*

Proof: Let $\mathcal{P}^*$ be an interior solution of the problem (P). This policy should satisfy the following system of necessary conditions for $t \geq 0$:

$$\begin{aligned} u'(c^*(t))e^{-\rho t} &= -\nu^*(t)\kappa_c^*(t) \\ \lambda^*(t) &= \nu^*(t)\kappa_s^*(t) \\ \dot{\lambda}^*(t) &= \rho\lambda^*(t) \\ \dot{\nu}^*(t) &= \rho\nu^*(t) - \nu^*(t)\kappa_K^*(t) \; , \end{aligned}$$

where the costate variables and $\kappa_x^*(t)$, $x = c, s, K$ are valued along the optimal path. Under $C.1$, $K.1$ and W, there exists a unique path of the costate variables $\{\nu^*(t), \lambda^*(t), t \geq 0\}$ solving the above system of necessary condition for a given optimal interior policy $\mathcal{P}^*$.

Furthermore, any interior solution of the program (D) should satisfy the following system of necessary conditions:

$$\begin{aligned} \pi(t) &= -\nu^D(t)\kappa_c(t) \\ \lambda^D(t) &= \nu^D(t)\kappa_s(t) \\ \dot{\lambda}(t) &= 0 \\ \dot{\nu}^D(t) &= -\nu^D(t)\kappa_K(t) \end{aligned}$$

Let $\theta^D(t) \equiv \lambda^D(t)e^{\rho t}$ and $\mu^D(t) \equiv \mu^D(t)e^{\rho t}$. Thus the above system is equivalent to:

$$\begin{aligned} \pi(t)e^{\rho t} &= -\mu^D(t)\kappa_c(t) \\ \theta^D(t) &= \mu^D(t)\kappa_s(t) \\ \dot{\theta}^D(t) &= \rho\theta^D(t) \\ \dot{\mu}^D(t) &= \rho\mu^D(t) - \mu^D(t)\kappa_K(t) \end{aligned}$$

Since the costate variables vector $\{\nu^*(t), \lambda^*(t), t \geq 0\}$ corresponding to the optimal interior policy $\mathcal{P}^*$ is unique, this policy is a solution of the program (D) only if $\nu^*(t) = \mu^D(t)$ and $\lambda^*(t) = \theta^D(t)$, $t \geq 0$. This implies that the weights vector $\pi(t)$ should satisfy:

$$\pi(t) \quad = \quad u'(c^*(t))e^{-\rho t} \ , \ t \geq 0 \ .$$

The optimal path $\mathcal{P}^*$ being an efficient path, (4.23) is an immediate implication of Proposition 4.

Assume that there exists an optimal path such that $c^*(t) = c^* > 0$. It results from the above Proposition 6 that:

$$u'(c^*) \quad = \quad \frac{e^{\rho t}}{c_s^*(t)} \qquad t \geq 0 \ .$$

This implies that $c_s^*(t)$ should increase exponentially at the rate ρ.

4.3 Hotelling rule

Assume that the constraint (4.4) is not effective so that $\underline{\alpha}(t) = 0$. Remember that an optimal interior path solution of the problem (P) is time differentiable. Denote by $\eta(c)$ the absolute value of the elasticity of marginal utility $-u''(c)c/u'(c)$. Then time differentiating (4.23), we obtain:

Proposition 7 *Under C.1, K.1 and W, along an interior optimal path:*

$$\eta(c)\frac{\dot{c}}{c} + \rho \quad = \quad \frac{\dot{c}_s}{c_s} \ . \tag{4.24}$$

The standard formulation of the Hotelling rule as appears in the D.H model (1979,p 291), is:

$$\eta(c)\frac{\dot{c}}{c} + \rho \quad = \quad \frac{\dot{g}_s}{g_s}$$

where g is the production function of the unique man-made good.

What Proposition 7 is showing is that in a two sectors model the derivatives of the production function involved in the right side of (4.24) must be the derivatives of the production function of the consumption good sector, that is c.

As pointed out in the above subsection 3.3, in the DH model, $\kappa_c = -1$, hence $\dot{\kappa}_c/\kappa_c = 0$, and (4.24) results in:

$$\eta(c)\frac{\dot{c}}{c} + \rho \;=\; g_K \;, \tag{4.25}$$

which is nothing but than the well known DH optimality condition [9]. This is basically the Ramsey-Keynes condition in the standard Ramsey-Solow optimal growth model. The DH model merges a Ramsey model, implying the same form of the arbitrage condition between savings and investment as expressed in (4.25), and a Hotelling model, characterized by an arbitrage condition between using the resource either today or in the future, a condition expressed in (4.17) in the present model. Time differentiating the Hotelling condition and identifying with the Ramsey Keynes condition in the DH model leads to the expression of the Hotelling rule:

$$\eta(c)\frac{\dot{c}}{c} + \rho \;=\; g_K = \frac{\dot{g}_s}{g_s}$$

The Hotelling rule is basically an efficiency requirement. In the present model, it takes the form of the local dynamic efficiency condition expressed by (3.21) in Proposition 2. An optimal path being efficient, the welfare expression (4.24) of the Hotelling rule appears then as a direct implication of (3.21), the necessary condition for the optimality of an efficient path (4.23) of Proposition 6 having to be verified. Note that while the dynamic efficiency condition appears to be different in a two sectors economy with respect to a one sector economy, the welfare expression of the Hotelling rule remains the same in the two economies as a consequence of static efficiency. Along the static efficiency frontier, the marginal rate of transformation of the resource into the consumption good $-\kappa_c/\kappa_s$ has to be equal to the marginal rate of transformation of the resource into consumption in the consumption good production sector $1/c_s$.

[9] Dasgupta and Heal (1979), Chapter 10, p 296, eq. (10.18).

4.4 National accounts

Under constant returns, that is under the additional assumptions $C.2$ and $K.2$, it is also possible to derive an interesting national accounting relationship. Note that these national accounts are expressed in net terms, in particular, the provision for wear and tear in the capital accumulation sector are included in the expression of the available product[10].

Proposition 8 *Under C.2, K.2 and W, for any optimal interior path:*

$$\begin{aligned}\int_0^\infty u'(c(t))c(t)e^{-\rho t}dt &= \nu(0)K^0 + \lambda_0 S^0 \\ &\quad + l\int_0^\infty u'(c(t))e^{-\rho t}c_l(K^c(t), l^c(t), s^c(t))dt \quad (4.26)\end{aligned}$$

Proof : The proof is given in Appendix A.3.

The left hand side of (4.26) is the sum of all the future consumption rates $c(t)$ valued at their discounted marginal utility $u'(c(t))e^{-\rho t}$. Absent any global economies or diseconomies of scale, the intuition suggests that the value of the optimized net output of the economy could be decomposed into the sum of the values of the components of the economy endowments. This is precisely what (4.26) is proving. Homogeneity of both c and k implies the homogeneity of the global production process. The endowments of the economy are its initial capital stock K^0, its initial stock of resource S^0 and last, the constant flow of labor l, that is the constant flow of a renewable resource. In (4.26) all these endowments are valued at their shadow prices at time $t = 0$: $\nu(0)$ and λ_0 for the capital and resource stocks respectively and for the labor flow its marginal productivity in the consumption good industry weighted by the discounted marginal utility of consumption, $u'(c(t)))e^{-\rho t}c_l(K^c(t), l^c(t), s^c(t))$.

[10] For a detailed treatment of accounts in gross and net terms, refer to Hartwick (2000).

4.5 Generalized Hartwick's rule

In their seminal paper, Dixit, Hammond and Hoel (1980) proved that a generalized version of the Hartwick's rule has to hold along any constant utility optimal path. It is easily checked that such a version of the Hartwick's rule holds also in our model.

The Hamiltonian in present value of the optimality problem (P) is:

$$\mathcal{H}(t) \quad = \quad u(c(t))e^{-\rho t} + \nu^d(t)\dot{K}(t) - \lambda^d(t)s(t)$$

where $\nu^d(t)$ and $\lambda^d(t)$ denote here the costate variables in discounted value. In the Dixit *et al.* formulation, $\nu^d(t)\dot{K}(t) - \lambda^d(t)s(t)$ is nothing but than the net present value at time t of investments in all the capital goods: the capital stock $K(t)$ and the resource stock $S(t)$. Because the optimized Hamiltonian function is a differentiable time function (cf Proposition 5) we can apply the dynamic envelope theorem and get:

$$\frac{d\mathcal{H}^*(t)}{dt} = \frac{\partial \mathcal{H}^*(t)}{\partial t} = -\rho u(c^*(t))e^{-\rho t} \ ,$$

where $c^*(t)$ is the optimal consumption level at time t. Integrating the above relation over $[t, \infty)$, we obtain:

$$\lim_{\tau \uparrow \infty} \mathcal{H}^*(\tau) - \mathcal{H}^*(t) \quad = \quad -\int_t^\infty \rho u(c^*(\tau))e^{-\rho\tau} d\tau \ .$$

Michel (1982) proved that in an optimality problem of this kind, we must have: $\lim_{\tau \uparrow \infty} \mathcal{H}^*(\tau) = 0$. This results in:

$$\mathcal{H}^*(t) \quad = \quad \int_t^\infty \rho u(c^*(\tau))e^{-\rho\tau} d\tau \ . \qquad (4.27)$$

Assume that the consumption level is constant along an optimal interior trajectory. Thus the instantaneous utility level should be constant. Let u^* be such a constant level, then (4.27) is equivalent to:

$$\begin{aligned} \mathcal{H}^*(t) \quad &= \quad u^* e^{-\rho t} + \left[\nu^d(t)\dot{K}^*(t) - \lambda^d(t)s^*(t)\right] = u^* e^{-\rho t} \\ &\Longrightarrow \quad \nu^d(t)\dot{K}^*(t) - \lambda^d(t)s^*(t) = 0 \ . \end{aligned}$$

The net present value of investments must be equal to zero if the optimal utility level is constant, that is the Hartwick rule must hold. Here the capital investment $\dot{K}^*(t)$ and the resource use $s^*(t)$ are both valued in terms of cumulative discounted utility, the objective function of the problem (P).

Conversely, consider an optimal path $\{(K^*(t), s^*(t), c^*(t)), t \geq 0\}$ satisfying the Hartwick rule at each time t. Denote by $u^*(t) \equiv u(c^*(t))$ the optimized value of the utility. Making use of (4.27) again, we obtain now:

$$\mathcal{H}^*(t) \;=\; u^*(t)e^{-\rho t} \;=\; \int_t^{\infty} \rho u^*(\tau)e^{-\rho\tau}d\tau \ .$$

Because $c^*(t)$ is a time differentiable function, $u^*(t)$ is also time differentiable over $[0, \infty)$. Differentiating through time we obtain:

$$\dot{u}^*(t)e^{-\rho t} - \rho u^*(t)e^{-\rho t} \;=\; -\rho u^*(t)e^{-\rho t} \implies \dot{u}^*(t)e^{-\rho t} = 0 \ .$$

Thus the utility level, hence the consumption level, should be constant along an optimal path satisfying the Hartwick rule at each time. This is the main result of Dixit *et al.* (1980). But note that Dixit *et al.* (Theorem 1, p 553) are assuming the smoothness of all the time functions along the optimal trajectory, an additional assumption which should have been deduced from their primitive regularity assumptions.

Whence the time continuity of the command variables along an optimal interior path has been proven using the second order properties of the utility function and the production functions, we may infer the time differentiability of the costate variables. Along an interior optimal path, proving the time differentiability of the command variables relies mainly upon proving that the system of short run optimality conditions (resulting from the "maximum principle") has only one solution for given instantaneous levels of the costate variables and the state variables at any time t. Once the time differentiability of the command variables has been demonstrated, making use of the time continuity of the dual variables and the time differentiability of the state variables resulting from the continuity of the command and state variables in an autonomous problem, the time differentiability of the hamiltonian function along the optimal path is trivially obtained. It is then possible to apply the dynamic envelope theorem to the optimized hamiltonian function along an interior path to derive interesting properties of an optimal path. This is the methodology we have used to prove the equivalence between the Hartwick's rule and a constant utility level along an optimal path in the present model. We conclude as follows:

Proposition 9 *Under* $C.1$*,* $K.1$ *and* W*, if along an interior optimal path* $\{(K^*(t), s^*(t), c^*(t)), t \geq 0\}$*, the current utility level is constant over time, then:*

$$\nu^d(t)\dot{K}^*(t) = \lambda^d(t)s^*(t) \quad , \; t \in [0, \infty) \tag{4.28}$$

where $\nu^d(t)$ *and* $\lambda^d(t)$ *are the costate variables of* $K^*(t)$ *and* $S^*(t)$ *respectively, both in terms of discounted utility. Reciprocally assume that (4.28) holds, then the current utility level is constant through time.*

Note that we get the generalized Hartwick's rule without invoking 'transversality' conditions about the limit of $\nu^d(t)$ as time increases up to infinity. A limit property of the optimized Hamiltonian, which can be shown to be a necessary condition for optimality (see Michel, 1982) with a constant discount rate, is just what is needed to obtain the rule along an optimal constant utility path.

5 CONCLUSION

The Dasgupta and Heal (1974) seminal contribution is the basic framework of numerous analysis of the long run sustainability issue through man made capital substitution to the use of an essential exhaustible resource. We depart from this framework by introducing a complete bisectoral model where the consumption good and the capital good are produced from labor, capital and an exhaustible resource. This is the minimum disaggregation allowing to isolate some fundamental relationships which are blurred in the Dasgupta and Heal model in which the two sectors are merged together.

We focus mainly upon efficiency issues, physical efficiency appearing as more fundamental for the sustainability of an economy submitted to an exhaustible resource depletion constraint. We show that local dynamic efficiency relates basically to the properties of the capital good production function. Our emphasis upon efficiency considerations proves also to be helpful in clarifying important aspects of the Hartwick's rule in resource models. With respect to optimality issues, we derive necessary conditions for an efficient path to be optimal. Making use of these conditions, we prove that the usual form of the Hotelling rule obtained in a one sector model remains valid in a bisectoral economy.

An important issue we do not consider is the existence of efficient or optimal positive constant consumption paths. It is clear that if the economy cannot sustain a constant consumption level through an efficient management of its scarce resources, it cannot do better than experiencing some declining to zero consumption level in the long run.

In our model, the economy is constrained both by the limited availability of an exhaustible resource and by a limited and constant amount of labor. It is also explicitly submitted to capital depreciation in the consumption good production sector when the productions processes are expressed in net terms. Most existence results of efficient plans sustaining some constant consumption level have been derived from monosectoral models of substitution between the exhaustible resource and a man made capital stock[11], and their counterparts in a bisectoral model remain an open question. These points are beyond the scope of the present study but are developed in a companion paper[12].

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APPENDIX

A.1 Appendix A.1: Proof of Proposition 1

The proof runs as follows:

1. There exists at least a solution
2. The set of vectors (K^k, l^k, s^k) satisfying the constraints (3.1)-(3.4) is convex
3. Assuming that the program could have different solutions implies a contradiction.

1. Because the objective function is continuous the only point having to be checked is that the set of vectors (K^k, l^k, s^k), satisfying the constraints (3.1)-(3.4), is compact that is bounded and closed, which is clearly the case. Hence there exists at least one solution.
2. Let $(K_i^k, l_i^k, s_i^k), i = 1, 2$ be two vectors satisfying (3.1)-(3.4), and consider any linear convex combination of these vectors:

   $$\alpha(K_1^k, l_1^k, s_1^k) + (1-\alpha)(K_2^k, l_2^k, s_2^k), \ \alpha \in (0, 1).$$

   Clearly (3.2)-(3.4) are satisfied. The only point having to be checked is (3.1). Because c is quasi-concave:

   $$\begin{aligned} & c(\alpha(K - K_1^k) + (1-\alpha)(K - K_2^k), \\ \alpha(l - l_1^k) + (1-\alpha)(l - l_2^k), & \alpha(s - s_1^k) + (1-\alpha)(s - s_2^k)) \\ & = c(K - (\alpha K_1^k + (1-\alpha)K_2^k), l - (\alpha l_1^k + (1-\alpha)l_2^k), \\ & \qquad s - (\alpha s_1^k + (1-\alpha)s_2^k)) \\ & \geq min\{c(K - K_i^k, l - l_i^k, s - s_i^k), i = 1, 2\} \ \geq \ c \qquad , \end{aligned}$$

   the first inequality resulting from the quasi-concavity of the function c and the second inequality from the assumption according to each $(K_i^k, l_i^k, s_i^k), i = 1, 2$ is satisfying (3.1).

3. Assume that $c < \bar{c}$ and let $(K_i^k, l_i^k, s_i^k), i = 1, 2$ be two solutions of the $(S.E)$ program:

$$\begin{aligned} & k(K_1^k, l_1^k, s_1^k) - \delta[K - K_1^k] = k(K_2^k, l_2^k, s_2^k) - \delta[K - K_2^k] \\ > \ & k(K^k, l^k, s^k) - \delta[K - K^k] \qquad \text{(A.1.1)} \end{aligned}$$

for any (K^k, l^k, s^k) satisfying (3.1)-(3.4), and

$$(K_1^k, l_1^k, s_1^k) \neq (K_2^k, l_2^k, s_2^k).$$

Because each vector "i" is within the domain and the domain is convex, then any convex linear combination of these vectors is also within the domain defined by (3.1)-(3.4).

Because the function k is concave, then for any $\alpha \in (0, 1)$:

$$\begin{aligned} & k(\alpha K_1^k + (1-\alpha)K_2^k, \alpha l_1^k + (1-\alpha)l_2^k, \alpha s_1^k + (1-\alpha)s_2^k) \\ & \quad +\delta[\alpha K_1^k + (1-\alpha)K_2^k] \\ \geq \ & \alpha[k(K_1^k, l_1^k, s_1^k) + \delta K_1^k] + (1-\alpha)[k(K_2^k, l_2^k, s_2^k) + \delta K_2^k] \\ = \ & k(K_i^k, l_i^k s_i^k) \ , \ i = 1, 2 \ , \qquad \text{(A.1.2)} \end{aligned}$$

the last equality being an immediate implication of (A.1.1). Note that we need the concavity of k to write the first inequality. The mere quasi-concavity would not be sufficient.

Because first, the function c is strictly quasi-concave, second $(K_1^k, l_1^k, s_1^k) \neq (K_2^k, l_2^k, s_2^k)$ and both are satisfying (3.1), then:

$$\begin{aligned} & c(\alpha(K - K_1^k) + (1-\alpha)(K - K_2^k), \\ & \alpha(l - l_1^k) + (1-\alpha)(l - l_2^k), \alpha(s - s_1^k) + (1-\alpha)(s - s_2^k)) \\ = \ & c(K - (\alpha K_1^k + (1-\alpha)K_2^k), l - (\alpha l_1^k + (1-\alpha)l_2^k), \\ & s - (\alpha s_1^k + (1-\alpha)s_2^k)) \\ > \ & min\{c(K - K_i^k, l - l_i^k, s - s_i^k), i = 1, 2\} \ \geq \ c \ . \qquad \text{(A.1.3)} \end{aligned}$$

Since the both vectors $(K_i^k, l_i^k, s_i^k), i = 1, 2$, are strictly positive then their linear combination is also strictly positive. Thus there exists a strictly positive vector $(\epsilon_K, \epsilon_l, \epsilon_s)$, sufficiently small, such that:

$$\begin{aligned} \alpha[K - K_1^k] + (1-\alpha)[K - K_2^k] - \epsilon_K &= K - (\alpha K_1^k + (1-\alpha)K_2^k + \epsilon_K) > 0 \\ \alpha[l - l_1^k] + (1-\alpha)[l - l_2^k] - \epsilon_l &= l - (\alpha l_1^k + (1-\alpha)l_2^k + \epsilon_l) > 0 \\ \alpha[s - s_1^k] + (1-\alpha)[s - s_2^k] - \epsilon_s &= s - (\alpha s_1^k + (1-\alpha)s_2^k + \epsilon_s) > 0 \ , \end{aligned}$$

and:

$$c(K - (\alpha K_1^k + (1-\alpha)K_2^k + \epsilon_K), l - (\alpha l_1^k + (1-\alpha)l_2^k + \epsilon_l),$$
$$s - (\alpha s_1^k + (1-\alpha)s_2^k + \epsilon_s)) \;\geq\; c \; . \tag{A.1.4}$$

Since trivially $\alpha(K_1^k, l_1^k, s_1^k) + (1-\alpha)(K_2^k, l_2^k, s_2^k) + (\epsilon_K, \epsilon_l, \epsilon_s)$ is strictly positive, then this vector is satisfying the whole set of constraints (3.1)-(3.4). Note that here the strict quasi-concavity of c is required for (A.1.3) permitting to obtain (A.1.4).

Because k is a strictly increasing function:

$$\begin{aligned}
& k(\alpha K_1^k + (1-\alpha)K_2^k + \epsilon_K, \alpha l_1^k + (1-\alpha)l_2^k + \epsilon_l, \\
& \alpha s_1^k + (1-\alpha)s_2^k + \epsilon_s) \;+\; \delta[\alpha K_1^k + (1-\alpha)K_2^k + \epsilon_K] \\
> \;& k(\alpha K_1^k + (1-\alpha)K_2^k, \alpha l_1^k + (1-\alpha)l_2^k, \\
& \alpha s_1^k + (1-\alpha)s_2^k) \;+\; \delta[\alpha K_1^k + (1-\alpha)K_2^k] \\
\geq \;& k(K_i^k, l_i^k, s_i^k) + \delta K_i^k \;,\; i = 1, 2 \;,
\end{aligned} \tag{A.1.5}$$

the last inequality resulting from (A.1.2).

The strict inequality (A.1.5) implies that (K_i^k, l_i^k, s_i^k), $i = 1, 2$, are not solving the program $(S.E)$, hence a contradiction.

A.2 Appendix A.2: Proof of Proposition 3

We adapt the proof strategy of Michel (1982) to the problem $(GE.u)$ of Proposition 3 in which, contrary to Michel's assumption, there is no discounting.

Denote by $\{(s^*(t), K^*(t)), t \in [0, \infty)\}$ the solution of the problem $(GE.u)$ of the Proposition 3 defined by $c^*(t) = c^* > 0$, $t \geq 0$. Let us define the new time variable τ as $\tau \equiv t - x$ so that $d\tau/dt = 1$ and $s^*(\tau) = s^*(t-x)$. For any given $T > 0$ and $x \geq 0$, define $R^*(T, x)$ as minus the cumulated extraction over the time interval $[T + x, \infty)$, the time being measured by τ:

$$R^*(T, x) \equiv \int_{T+x}^{\infty} (-s^*(\tau)) d\tau$$

Note that by construction $\partial R^*/\partial x = 0$.

Consider the following auxiliary problem (P_T) with the non negative state variables $Y(t)$ and $Z(t)$ and the control variables $r(t)$, $r(t) \in \Re_+$, and $v(t)$, $v(t) \in \Re_{++}$:

$$\begin{aligned} P_T \quad : & \quad \max_{\{(r(t),v(t)),t\in[0,T)\}} \int_0^T v(t)(-r(t))dt + R^*(T, Z(T) - T) \\ s.t. & \quad \dot{Y}(t) = v(t)f(Y(t), r(t)) \ \ Y(0) = K^0 \ \ Y(T) = K^*(T) \\ & \quad \dot{Z}(t) = v(t) \ \text{ and } Z(0) = 0 \end{aligned}$$

where $f(Y(t), r(t)) \equiv \kappa(Y(t), r(t), c^*) - \delta Y(t)$. It is proved in Michel (1982) that the states $(Y(t), Z(t)) = (K^*(t), t)$ and the controls $(r(t), v(t)) = (s^*(t), 1)$, for $t \in [0, T)\}$ are solving the auxiliary problem P_T (Michel, 1982, Lemma, p 977).

Let $H_T(t)$ be the Hamiltonian of the auxiliary problem (P_T):

$$H_T(t) \quad = \quad a_T v(t)(-r(t)) + \nu_T(t)v(t)f(Y(t), r(t)) + \vartheta_T(t)v(t)$$

Note that we explicitly introduce the scalar a_T, usually implicitly assumed to be equal to one, into the expression of the Hamiltonian.

As proved by Michel (1982, p 983), the necessary optimality conditions for the problem (P_T) are as follows.

First, there must exist a non negative real number a_T, a real number n_T, and continuous functions of time $\nu_T(t)$ and $\vartheta_T(t)$ such that:

$$(a_T, n_T) \neq (0, 0) \tag{A.2.1}$$

$$\dot{\nu}_T(t) = -\frac{\partial H_T}{\partial Y} = -\nu_T(t)v(t)\frac{\partial f}{\partial Y}$$

$$\Longrightarrow \dot{\nu}_T(t) = -\nu_T(t)\frac{\partial f}{\partial K}(K^*(t), s^*(t)) \ , \ t \in [0, T) \tag{A.2.2}$$

$$\nu_T(T) = n_T \tag{A.2.3}$$

$$\dot{\vartheta}(t) = -\frac{\partial H_T}{\partial Z} = 0 \tag{A.2.4}$$

$$\vartheta_T(T) = a_T\frac{\partial R^*}{\partial x}\frac{\partial x}{\partial Z} = a_T\frac{\partial R^*}{\partial x}(T, 0) = 0 \tag{A.2.5}$$

Second, the Hamiltonian must be maximized with respect to the control variables. Concerning $v(t)$, since the Hamiltonian is linear in $v(t)$, in the case $v(t) = 1 \neq 0$, this is implying that:

$$\nu_T(t) f(Y(t), r(t)) + \vartheta_T(t) \quad = \quad a_T r(t) \quad t \in [0, T) \tag{A.2.6}$$

Concerning $r(t)$, we obtain, for $v(t) = 1$:

$$\nu_T(t) \frac{\partial f}{\partial r}(Y(t), r(t)) \quad = \quad a_T \quad t \in [0, T) \tag{A.2.7}$$

Let us show now that both $a_T \neq 0$ and $\nu_T(0) \neq 0$. Consider the above condition (A.2.7) at time $t = 0$:

$$\nu_T(0) \kappa_s(Y(0), r(0)) \quad = \quad a_T \tag{A.2.8}$$

Under the assumptions $C.1$ and $K.1$, $\kappa_s(Y(0), r(0)) > 0$, hence:

$$\nu_T(0) = 0 \implies a_T = 0 \text{ and } a_T = 0 \Longrightarrow \nu_T(0) = 0 \ . \tag{A.2.9}$$

Thus:

- Either both $\nu_T(0) = 0$ and $a_T = 0$,
- Or $\nu_T(0) \neq 0$ and $a_T \neq 0$.

Assume that $\nu_T(0) = 0$, then by (A.2.2) and $\partial f / \partial K > 0$:

- Either $\nu_T(t) = 0$, $t \in [0, T)$, implying that first $\nu_T(T) = 0$ hence by (A.2.3) $n_T = 0$, and by (A.2.9) $a_T = 0$ because $\nu_T(0)$, thus $(a_T, n_T) = (0, 0)$ contradicting (A.2.1).
- Or $\nu_T(t) \neq 0$ over some first interval (t_1, t_2), $0 \leq t_1 < t_2 \leq T$ after having been equal to 0 over the interval $[0, t_1)$ (possibly degenerate). Because $\partial f / \partial K > 0$ then by (A.2.2) this is possible iff $\nu_T(t)$ is jumping either upwards or downwards at t_1 which is contradicting the continuity of $\nu_T(t)$ which must be equal to 0 over $[0, t_1)$, hence again a contradiction.

We conclude that $a_T > 0$ and $\nu_T(0) \neq 0$[13].

[13]Note that $\nu_T(0) \neq 0$ implies that $\nu_T(0) > 0$ under the assumptions of Proposition 3 according to which the efficient path is an interior path, that is (3.16) is satisfied as a strict inequality.

Multiplying side to side (A.2.2), (A.2.3), (A.2.6), (A.2.7) by a constant $\theta > 0$, while taking into account (A.2.4) and (A.2.5) which imply together that $\vartheta(t) = 0$, $t \in [0, T)$, we get:

$$\begin{aligned}
\theta\dot{\nu}_T(t) &= -\theta\nu_T(t)\frac{\partial f}{\partial K} \\
\theta\nu_T(T) &= \theta n_T \\
\theta\nu_T(t)f(Y(t), r(t)) &= \theta a_T r(t) \\
\theta\nu_T(t)\frac{\partial f}{\partial r} &= \theta a_T
\end{aligned}$$

By letting $a'_T \equiv \theta a_T$ and $\nu'_T(0) \equiv \theta\nu_T(0)$, we can choose a value of θ such that $\|a'_T, \nu'_T(0)\| = 1$ without changing the solution of the problem (P_T). Thus we can renormalize a_T and $\nu_T(0)$ in such a way that $(a_T, \nu_T(0))$ lies into the unit simplex, that is a compact set.

Since $(a_T, \nu_T(0))$ is of unit norm, there exists a sequence $(a_{T_n}, \nu_{T_n}(0))$ such that $lim_{T_n\to\infty}(a_{T_n}, \nu_{T_n}(0)) = (a, \nu^0)$ with $a > 0$ and $\nu^0 > 0$. Since $\lim_{T_n\to\infty} a_{T_n} = a$ and $\lim_{T_n\to\infty} \nu_{T_n}(0) = \nu^0$, we can define $\nu(t) = \lim_{T_n\to\infty} \nu_{T_n}(t)$ and $\vartheta(t) = \lim_{T_n\to\infty} \vartheta_{T_n}(t)$. Remembering that $\{K^*(t), s^*(t)\}_0^T$ should be a solution of the problem (P_T), $(\nu(t), \vartheta(t))$ should be a solution of:

$$\begin{aligned}
\dot{\nu}(t) &= -\nu(t)\frac{\partial f}{\partial K}(K^*(t), s^*(t)) \quad \nu(0) = \nu^0 \\
\dot{\vartheta}(t) &= 0 \\
\vartheta(t) &= a \lim_{x\uparrow\infty} \frac{\partial R^*}{\partial x}(0) = 0
\end{aligned}$$

The asymptotic properties of ϑ show that first $\dot{\vartheta}(t) = 0$, that is $\vartheta(t)$ should be constant, and second $\vartheta(t) = 0$. Since (a, ν^0) is of unit norm, we get also:

$$-s^*(t) + a^{-1}\nu(t)f(k^*(t), s^*(t)) = 0 \implies H^*(t) = -s^*(t) + \nu(t)\dot{K}^*(t) = 0$$

which is nothing but than the Hartwick's rule (3.24).

A.3 Appendix A.3: Proof of Proposition 8

Let us assume that the optimal path is an interior path ($\underline{\alpha}(t) = 0$). To simplify the exposition we denote by $\pi(t)$ the discounted marginal utility of consumption, $\pi(t) \equiv e^{-\rho t}u'(c(t))$, and by $\mu(t)$ the discounted value of the shadow price of capital, $\pi(t) \equiv e^{-\rho t}\nu(t)$.

Multiplying the both sides of (4.16) and (4.17) by $e^{-\rho t}$, we get:

$$\pi(t) \quad = \quad -\mu(t)\kappa_c(t) \tag{A.3.1}$$

and:

$$\lambda_0 \quad = \quad \mu(t)\kappa_s(t) \tag{A.3.2}$$

Multiplying the both sides of (4.20) by $e^{-\rho t}$, taking care that $\dot{\mu}(t) = -\rho e^{-\rho t}\nu(t) + e^{-\rho t}\dot{\nu}(t)$, we obtain:

$$\dot{\mu}(t) \quad = \quad -\mu(t)\kappa_K(t) \tag{A.3.3}$$

Next by $(C.2)$: $c = c_K K^c + c_l l^c + c_s s^c$, hence:

$$\pi c \quad = \quad \pi c_k K^c + \pi c_l l^c + \pi c_s s^c \tag{A.3.4}$$

Because $\kappa_c = -k_s/c_s$ (c.f (3.9)), then (A.3.1) may be rewritten as $\pi = \mu k_s/c_s$, and because $\kappa_s = k_s$ (by (3.9)), then (A.3.2) may be written as $\lambda_0 = \mu k_s$, so that:

$$\frac{\pi}{\lambda_0} \quad = \quad \frac{1}{c_s} \Longrightarrow \pi c_s = \lambda_0$$

Substituting for πc_s in (A.3.4), we get:

$$\pi c \quad = \quad \pi c_K K^c + \pi c_l l^c + \lambda_0 s^c \ . \tag{A.3.5}$$

Next by $(K.2)$: $k = k_K K^k + k_l l^k + k_s s^k$, hence:

$$\pi k \quad = \quad \pi k_K K^k + \pi k_l^k + \pi k_s s^k \tag{A.3.6}$$

As pointed out earlier $\lambda_0 = \mu k_s$, so that $\pi k_s = \pi\lambda_0/\mu$. Substituting for πk_s in (A.3.6) and multiplying both sides by μ results in:

$$\pi\mu k \quad = \quad \pi\mu k_K K^k + \pi\mu k_l l^k + \pi\lambda_0 s^k \tag{A.3.7}$$

Next $\kappa_K = k_K$ (c.f (3.9)) so that (A.3.3) may be rewritten as $\mu k_K = -\dot{\mu}$. Substituting for μk_K in (A.3.7), we get:

$$\pi\mu k \quad = \quad -\dot{\mu}K^k + \pi\mu k_l l^k + \lambda_0 s^k \ . \tag{A.3.8}$$

Remembering that $\mu = -\pi/\kappa_c$ (c.f (A.3.1)), we get:

$$-\frac{\pi}{\kappa_c}k \;=\; -\dot{\mu}K^k - \frac{\pi}{\kappa_c}k_l l^k + \lambda_0 s^k \ .$$

By (3.9), $\kappa_c = -k_l/c_l$ hence $-\pi k_l l^k/\kappa_c = \pi c_l$, so that:

$$-\frac{\pi}{\kappa_c}k \;=\; -\dot{\mu}K^k + \pi c_l l^k + \lambda_0 s^k \ . \tag{A.3.9}$$

Summing up (A.3.5) and (A.3.9), we obtain:

$$\pi c - \frac{\pi}{\kappa_c}k \;=\; \pi c_K K^c - \dot{\mu}K^k + \pi c_l[l^c + l^k] + \lambda_0[s^c + s^k]$$

Note that $k = \dot{K} + \delta K^c$, $l^c + l^k = l$ and $s^c + s^k = s$, hence:

$$\pi c \;=\; \frac{\pi}{\kappa_c}[\dot{K} + \delta K^c] + \pi c_K K^c - \dot{\mu}K^k + \pi c_l l + \lambda_0 s \ . \tag{A.3.10}$$

By (3.9): $\kappa_c = -(k_K + \delta)/c_K$. Substituting for κ_c in (A.3.10) results in:

$$\begin{aligned}
\pi c \;&=\; -\frac{\pi c_K}{k_K + \delta}[\dot{K} + \delta K^c] + \pi c_K K^c - \dot{\mu}K^k + \pi c_l l + \lambda_0 s \\
&=\; -\frac{\pi c_K}{k_K + \delta}\dot{K} + \frac{\pi c_K k_K}{k_K + \delta}K^c - \dot{\mu}K^k + \pi c_l l + \lambda_0 s \ . \qquad (A.3.11)
\end{aligned}$$

By (A.3.3) and $\kappa_K = k_K$ (c.f (3.9)) we have $\dot{\mu} = -\mu k_K$, and by (A.3.1) and $\kappa_c = -(k_K + \delta)/c_K$ (c.f (3.9)), we have:

$$\mu = \frac{\pi c_K}{k_K + \delta} \quad \text{and} \quad \dot{\mu} = -\frac{\pi c_K k_K}{k_K + \delta} \ .$$

Thus taking into account that $K^c + K^k = K$, (A.3.11) may be rewritten as:

$$\pi c \;=\; -\mu\dot{K} - \dot{\mu}K + \pi c_l l + \lambda_0 s = -\dot{(\mu K)} + \pi c_l l + \lambda_0 s \ .$$

Integrating over $[0, \infty)$ and using the transversality conditions (4.21) and (4.22) we obtain (4.26).