Stochastic volatility implies fourth-degree risk dominance: Applications to asset pricing

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Abstract

We demonstrate that increasing the risk surrounding the variance of future consumption generates a fourth-degree risk deterioration in future consumption, yielding an increase in its excess kurtosis. Its impact on the equilibrium risk premium is thus positive if only if the fourth derivative of the utility function is negative. Its impact on interest rates is negative only if its fifth derivative is positive. We also show that the persistence of shocks to the variance of the consumption growth rate, as assumed in long-run risk models, has no effect on the term structure of the variance ratio which remains flat in expectation, but it makes the term structure of the annualized fourth cumulant of log consumption increasing. It generates term structures of interest rates and risk premia that are respectively decreasing and increasing under constant relative risk aversion. Using recursive preferences does not qualitatively modify these results, which are counterfactual. However, the persistence of shocks to the variance of changes in log consumption is supported by the observation that their annualized 4th cumulant exhibits an increasing term structure over the period 1947Q1-2016Q4 in the United States.

Keywords: Long-run risks, fourth-degree risk dominance, temperance, edginess, recursive utility, kurtosis.

JEL codes: D81

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1 Introduction

This paper provides a theoretical analysis of the effect of stochastic volatility on intertemporal welfare and asset prices. Since the seminal work by Bansal and Yaron (2004), it has been demonstrated that introducing stochastic volatility in the process governing aggregate consumption can contribute to the resolution of the classical puzzles in finance when combined with other ingredients. Our main objective in this paper is more theoretical. Rather than showing that a realistic calibration of the parameters of the model can explain the observed asset prices if the model is rich enough, we focus on a single ingredient, i.e., stochastic volatility, and we examine its consequence on asset pricing from a theoretical point of view. Technically, the stochastic volatility contained in the Bansal-Yaron process produces an increase in risk in the variance of log consumption. It is often suggested that stochastic volatility adds a new layer of risk, which implies that risk-averse consumers should dislike it. This common wisdom is not true.

To show this, consider two lotteries. With lottery L_1, one loses or wins one monetary unit with equal probabilities. With lottery L_2, one loses or wins two monetary units with probability 1/8. What is the preferred lottery? Observe that the first three moments of these two lotteries are identical, with a zero first and third moments, and a unit second moment. This implies that all expected-utility-maximizers with a third-degree polynomial utility function will be indifferent between the two lotteries, independent of their degree of risk aversion and prudence. In particular, it is not true that L_2 is riskier than L_1 in the classical sense defined by Rothschild and Stiglitz (1970).[1] How is this observation related to stochastic volatility? In Figure 1, we represented L_2 as two alternative compound lotteries, L_2' and L_2''. Lottery L_2' compounds a sure payoff of $x_0 = 0$ with probability 3/4 and a zero-mean lottery $x_1' \sim (-2, 1/2; +2, 1/2)$ with probability 1/4. Because the variance of x_1' equals 4, L_1 differs from L_2' by the fact that the sure variance of 1 is replaced by an uncertain variance that is distributed as $(0, 3/4; 4, 1/4)$. This is a mean-preserving spread in variance. This is also the case for the compound lottery L_2'', where the conditional variance is distributed as $(0, 1/2; 2, 1/2)$. In other words, determining the preference order between lotteries L_1 and L_2 is equivalent to determining the welfare impact of an uncertain variance. The cornerstone of our analysis is our Theorem 1 which states that any mean-preserving spread in the distribution of the variance of a random variable x generates a fourth-degree risk increase of x in the sense of Ekern (1980). In other words, it reduces (increases) $Ef(x)$ if and only if the fourth derivative of f is negative (positive). A necessary (but not sufficient) condition is that the kurtosis of x is increased. Thus, we conclude that, under expected utility, increasing the risk on the variance of final consumption is disliked if and only if the fourth derivative of the utility function is negative. Under this condition, lottery L_1 is preferred to lottery L_2.

The negativeness of the fourth derivative of the utility function is called "temperance"

[1] In fact, L_2 is obtained from L_1 through a sequence of two Mean-Preserving Spreads (MPS) and one Mean-Preserving Contraction (MPC). In the first MPS, the outcome 1 of L_1 is replaced by $(+2, 3/4; -2, 1/4)$. In the second MPS, the outcome -1 of L_1 is replaced by $(+2, 1/4; -2, 3/4)$. Finally, the MPC takes the form of displacing probability masses of 3/8 from respectively -2 and 2 to 0.

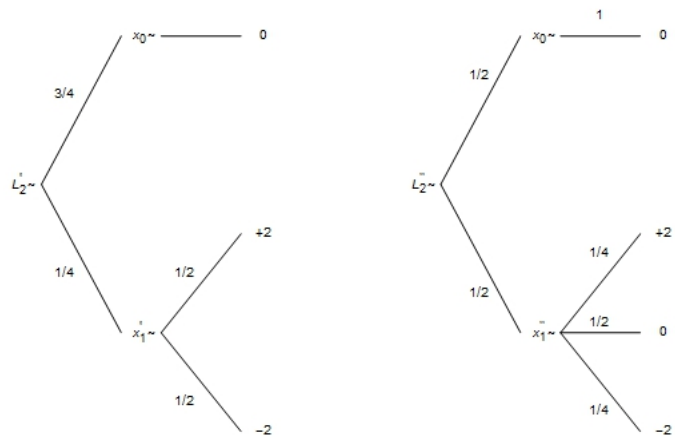

Figure 1: Two ways of representing lottery $L_2 \sim (-2, 1/8; 0, 3/4; +2, 1/8)$ as a compound lottery with stochastic variance.

in expected utility theory. Eeckhoudt and Schlesinger (2006) have shown that temperance is necessary and sufficient for consumers to prefer compound lottery $(x_1, 1/2; x_2, 1/2)$ over compound lottery $(0, 1/2; x_1 + x_2, 1/2)$, for any pair (x_1, x_2) of zero-mean independent lotteries. This is another form of aversion to a stochastic variance. In fact, the preference of L_1 over L_2'' can be represented in this way, with $L_1 \sim x_1 \sim x_2$. Gollier and Pratt (1996) showed that temperance is necessary for any zero-mean background risk to raise the aversion to any other independent risk. Under Constant Relative Risk Aversion (CRRA), temperance goes together with risk aversion since the successive derivatives of the utility function alternate in sign.

Under Discounted Expected Utility (DEU), the interest rate is decreasing in the risk surrounding future consumption if and only if it raises the expected marginal utility of future consumption. Since Leland (1968), Drèze and Modigliani (1972) and Kimball (1990), it is well-known that, by Jensen's inequality, this is the case if and only if the representative agent is prudent, i.e., if the third derivative of the utility function is positive. Because it increases the fourth-degree risk (kurtosis), the stochastic nature of the variance of consumption also reduces the interest rate under DEU if and only if the fifth derivative of the utility function is positive. In a two-period model with recursive preferences à la Kreps and Porteus (1978) and Selden (1978), we characterize a necessary and sufficient condition in the small that combines the fourth and fifth derivatives of the risk utility function with the difference between risk aversion and fluctuation aversion.

The long-run risks literature focused on the prediction of the short-term interest rate and of the price of equity. Up to our knowledge, there has been no analysis of the term structure of zero-coupon bond and equity returns, with the exception of Beeler and Campbell (2012) who examined interest rates. Bansal and Yaron (2004) considered a stochastic process of log consumption in which the variance itself is governed by an independent stochastic process.

This is useful to explain why the equity premium on financial markets is time varying. Because it raises the kurtosis of future log consumption, it reduces the interest rate and it raises the equity premium, which is helpful to solve the financial puzzles. Moreover, the shocks on variance are highly persistent in this literature, with an half-life around 35 years in the most recent calibration of the model by Bansal et al. (2016). This persistence implies that the term structure of the annualized kurtosis of log consumption is increasing. This magnifies the effect of stochastic volatility at higher maturities. This means that, under DEU, the term structures of interest rates and risk premia are respectively decreasing and increasing. These theoretical predictions of this model are contradicted by asset prices observed on financial markets. In particular, recent findings document the fact that dividend strip risk premia have a decreasing term structure (Binsbergen et al. (2012), Binsbergen and Koijen (2016), Belo et al. (2015), and Marfè (2016)).[2] The traditional combination of stochastic volatility with persistent shocks to the expected growth rate will make the problem even more puzzling, since it will generate an increasing term structure of the annualized variance of log consumption.[3] Because of prudence, this will make the term structure of interest rates more decreasing (Campbell (1986), Gollier (2008)). Because of risk aversion, it makes the term structure of risk premia more increasing. And the traditional combination of stochastic volatility with recursive preferences does not solve the problem either. This is shown in this paper by deriving analytically the term structures from the stochastic volatility model extracted from Bansal and Yaron (2004). Because of the high persistence of the shocks, it takes many centuries of duration for the interest rates and risk premia to converge to their asymptotic value. For example, a dividend strip generated by a diversified portfolio of equity has a risk premium 1.71% for a one-month maturity, and it goes up to 3.17% when the maturity tends to infinity. However, this equity premium is only 1.86% for a 10-year maturity, and 2.28% for a 50-year maturity.

The paper is organized as follows. In Section 2, we provide some generic results linking stochastic volatility, fourth-degree stochastic dominance and kurtosis. We apply these findings in a two-period Kreps-Porteus preferences in Section 3. In the next two sections, we explore the term structures of asset prices under the standard long-run risk specification yielding persistent shocks to the variance of log consumption. We do that in the DEU framework in Section 4, and in the case of Epstein-Zin-Weil preferences in Section 5. In Section 6, we test whether the term structure of the 4th cumulant of changes in log consumption is upward-sloping due to the persistence of shocks to the variance. We provide some concluding remarks in the last section.

[2] These findings are for maturities up to 10 years. For longer maturities, Giglio et al. (2015) and Giglio et al. (2016) provide evidence for real estate assets (leasehold contracts) with maturities measured in decades and centuries.

[3] Beeler and Campbell (2012) show evidence of mean-reversion rather than persistence in U.S. consumption growth in the period since 1930. On the contrary, mean-reversion makes the aggregate risk in the longer run relatively smaller and can thus explain why interest rates and risk premia are respectively increasing and decreasing in maturity, contrary to what the persistence in volatility shocks implies.

2 Stochastic volatility and stochastic dominance

This section is devoted to the analysis of the impact of the stochastic variance of a random variable x on $Ef(x)$, where f is a four time differentiable real-valued function. The risk structure of x is described by the following model:

$$x = \overline{x} + \sigma\eta \tag{1}$$
$$\sigma^2 = \overline{\sigma}^2 + w, \tag{2}$$

where (η, w) is a pair of independent random variables with a zero mean so that $\overline{x}$ and $\overline{\sigma}^2$ are the mean of respectively x and σ^2.[4] If we assume that $E\eta^2 = 1$, then σ^2 measures the variance of x conditional to σ. Finally, we also assume that $E\eta^3 = 0$. This assumption guarantees that the unconditional third moment of x is zero, independently of the distribution of σ.

Consider any real-valued function f that is at least twice differentiable. We want to characterize the impact of the stochastic variance of x on $Ef(x)$. By the law of iterated expectations, we have that

$$Ef(x) = E\left[E[f(x) \mid \sigma^2]\right] = Eh(\sigma^2), \tag{3}$$

where function h is derived from function f in such a way that $h(\sigma^2)$ equals $E[f(x) \mid \sigma^2]$ for all σ. An increase in risk of variance is defined as a sequence of Rothschild-Stiglitz Mean-Preserving Spreads (MPS) in the distribution of the variance σ^2 of x. From Rothschild and Stiglitz (1970), it is also defined as a change in the distribution of σ^2 that reduces the expectation of any concave function of σ^2. This implies that, from equation (3), an increase in risk on variance reduces Ef if and only if h is concave in σ^2. The following theorem states the necessary and sufficient condition for this to be true independent of the distribution of the zero-mean random variable η.

Theorem 1. *Suppose that $x = \overline{x} + \sigma\eta$, where $\overline{x}$ is a constant and σ and η are two independent random variables with $E\eta = E\eta^3 = 0$. Any increase in risk in the variance σ^2 of x reduces (raises) $Ef(x)$ if and only if f'' is concave (convex).*

Proof: See the appendix.

This result is related to the theory of stochastic dominance orders. Following Ekern (1980), a random variable undergoes a n-th degree risk deterioration if and only if this change in distribution reduces the expectation of any function $g \in G_n$ of that random variable, where G_n is the set of all real-valued functions g such that $(-1)^n g^{(n)} \leq 0$, where $g^{(n)}$ denotes the n-th derivative of g. For example, the case $n = 1$ corresponds to the concept of first-order stochastic dominance, whereas the case $n = 2$ corresponds to the Rothschild-Stiglitz's notion of an increase in risk. Theorem 1 means that raising the uncertainty affecting the variance of

[4] Notice that $\overline{\sigma}$ is not the mean of σ in this model. By Jensen's inequality, the uncertainty affecting σ^2 has a negative impact on the expected value of σ.

x in the sense of Rothschild and Stiglitz (1970) deteriorates random variable x in the sense of fourth-degree risk.

Because $f(x) = (x - \overline{x})^i$ has a linear second derivative for $i = 1, 2$ and 3, an immediate consequence of Theorem 1 is that the mean, the variance and the skewness of x is unaffected by an increase in risk in σ^2. In the same vein, the fourth centered moment of x is increased by it. More specifically, we have that the excess kurtosis of x equals

$$\text{Kurt}[x] = \frac{E(x-\overline{x})^4}{(E(x-\overline{x})^2)^2} - 3 = \frac{E\sigma^4}{(E\sigma^2)^2}\frac{E\eta^4}{(E\eta^2)^2} - 3. \tag{4}$$

Because function σ^4 is convex in σ^2, the kurtosis of x is increased by any increase in risk in σ^2. If we assume that η is Normal, then the above equality simplifies to

$$\text{Kurt}[x] = 3\left(\frac{E\sigma^4}{(E\sigma^2)^2} - 1\right) = 3\frac{Var[\sigma^2]}{(E\sigma^2)^2} = 3\frac{\sigma_w^2}{\overline{\sigma}^4}. \tag{5}$$

The excess kurtosis of x is proportional to the variance of the conditional variance of x in that case. Thus, the variance of σ^2 should be interpreted as a measure of the excess kurtosis of x. An extreme illustration of this phenomenon has been proposed by Weitzman (2007). Suppose that η is $N(0, 1)$ and that the precision $p = \sigma^{-2}$ has a Gamma distribution. Then, as is well-known, the unconditional distribution of x is a Student-t, which has fatter tails than the Normal distribution with the same expected variance. The moment-generating function of the Student-t is undefined, which means that the expectation of $\exp(kx)$ is unbounded, for all $k \in \mathbb{R}$. This is an extreme illustration of Theorem 1 in which moving from a sure σ^2 to a risky one with the same mean makes the expectation of f undefined.[5]

It is useful to measure a "stochastic volatility premium" associated to function f which is defined as the sure reduction π_f in x that has the same impact on Ef as the uncertainty affecting σ^2. Technically, π_f satisfies the following condition:

$$Ef(\overline{x} + \overline{\sigma}\eta - \pi_f) = Ef(\overline{x} + \sigma\eta), \tag{6}$$

where $\overline{\sigma}^2$ is the expectation of the uncertain variance σ^2. Using Taylor expansions for both sides of the above equality, it is easy to show that the stochastic volatility premium satisfies the following property when $\eta = k\epsilon$ with $k \in \mathbb{R}$:

$$\pi_f = \frac{1}{4!}\psi_f(\overline{x})Var[\sigma^2]E\eta^4 + O(k^5), \tag{7}$$

where $\psi_f(\overline{x}) = -f^{(4)}(\overline{x})/f'(\overline{x})$ is an index of concavity of f''. When η has a standard Normal distribution, the above approximation simplifies to

$$\pi_f \simeq 0.125\psi_f(\overline{x})\sigma_w^2. \tag{8}$$

In the remainder of this section, we suppose that function f is exponential with $f(x) = \exp(-Ax)$ for some $A \in \mathbb{R}$, in which case $\psi_f = A^3$. In that case, we have that

$$Ef(x) = \chi(-A, \overline{x} + \sigma\eta), \tag{9}$$

[5] Gollier (2016) examined risk profiles for σ that generate a bounded solution.

where $\chi(\alpha, y) = \log(E\exp(\alpha y))$ is the Cumulant-Generating Function (CGF) of random variable y.[6] If η is standard normal, this can be rewritten as follows:

$$Ef(x) = -A\overline{x} + \chi(0.5A^2, \sigma^2). \tag{10}$$

Using the properties of the CGF function, this implies that

$$Ef(x) - f(\overline{x}) = \sum_{n=1}^{+\infty} \frac{A^{2n}}{2^n n!} \kappa_n^{\sigma^2}, \tag{11}$$

where $\kappa_n^{\sigma^2}$ is the nth cumulant of σ^2.[7] The first term in the right-hand side of this equality is $0.5A^2\overline{\sigma}^2$, which corresponds to the standard Arrow-Pratt risk premium $0.5A\overline{\sigma}^2$. The second term is $A\pi_f = 0.125A^4 Var[\sigma^2]$, which corresponds to the stochastic volatility premium approximated in equation (8).

Two special cases are useful to examine when f is exponential. In line with the literature on long-run risks pioneered by Bansal and Yaron (2004), suppose first that σ^2 has a Normal distribution. This implies that all cumulants of σ^2 of order larger than 2, as they appear in equation (11), are zero. This implies that approximation (8) is exact in that case. Although this specification of the model is ubiquitous in the long-run risks literature, it is problematic because of the positive probability of a negative variance.[8] An alternative specification which has a more satisfactory theoretical foundation is obtained when assuming that σ^2 has a Gamma distribution. We show in the appendix that this implies the following analytical characterization of the stochastic volatility premium:

$$\pi_f = -\frac{\overline{\sigma}^4}{A\sigma_w^2} \log\left(1 - \frac{1}{2}A^2\frac{\sigma_w^2}{\overline{\sigma}^2}\right) - \frac{1}{2}A\overline{\sigma}^2. \tag{12}$$

Let us apply these results to the case of an agent who has a constant relative risk aversion γ and whose log consumption next year conditional to σ is $x \sim N(\mu, \sigma^2)$. This agent's expected utility next period is thus given by $Ef(x)$, where $f(x)$ is proportional to $\exp((1-\gamma)x)$. The above formulas allow us to compute the stochastic volatility premium, i.e., the sure reduction in the growth rate of consumption that has the same impact on welfare than the uncertainty affecting its variance, in the Normal and Gamma cases. In Figure 2, we assume $\gamma = 10$. Suppose also that the expected annual volatility is $\overline{\sigma} = 3\%$. The dashed curve corresponds to the stochastic volatility premium as a function of the standard deviation of the variance σ^2 when σ^2 is normally distributed. As explained earlier, this premium is measured exactly by equation (8) with $\psi_f = (\gamma-1)^3$. The plain curve measures that function in the alternative case in which σ^2 has a Gamma distribution with the same first two cumulants. This premium

[6]Martin (2013) uses the properties of the CGF function to derive analytical solutions for interest rates and risk premia under Epstein-Zin-Weil preferences with i.i.d. growth rates.

[7]The nth cumulant of y is defined as $\kappa_n^y = \chi^{(n)}(0, y)$. For example, the 4th cumulant of y is equal to $\mu_4 - 3\mu_2^2$, where μ_n is the nth centered moment of y.

[8]Under this specification, the cumulants of x computed with formula $\kappa_n^x = \chi^{(n)}(0, x)$ are equal to $\overline{x}$, $\overline{\sigma}^2$ and $3\sigma_w^2$ respectively for the first, second and fourth orders. All other cumulants are zero. In fact, there is no random variable with a well-defined cumulative distribution function having such series of cumulants.

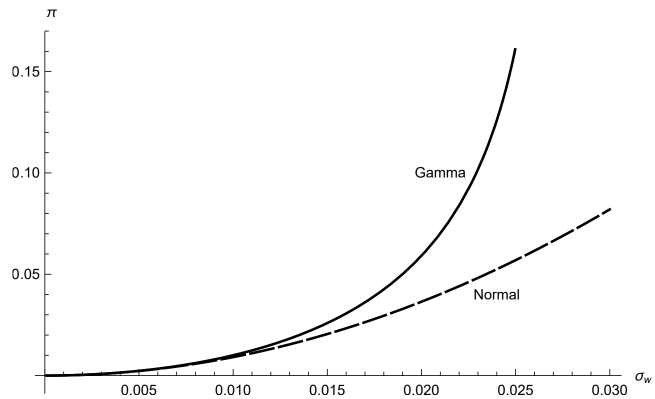

Figure 2: The relative stochastic volatility premium as a function of the degree of uncertainty σ_w affecting the variance of log consumption. The dashed curve corresponds to $\sigma^2 \sim N(3\%, \sigma_w^2)$. The plain curve corresponds to σ^2 having a Gamma distribution with the same mean and variance. We assume a CRRA of 10 and $\eta \sim N(0,1)$.

is given by equation (12) with $A = \gamma - 1$. It is easy to verify from these two functions are identical up to the fourth order of σ_w, so that the two curves coincide for low levels of uncertainty. In the long-run risks literature, σ_w is in the order of magnitude of $10^{-5}...10^{-6}$ on an annual basis (Bansal et al. (2016)). Figure 2 thus suggests that the Normal approximation is acceptable, at least for small maturities.[9]

3 Asset pricing in the Kreps-Porteus model

In this section, we explore the pricing implications of stochastic volatility in the simple 2-period model proposed by Kreps and Porteus (1978) and Selden (1978):

$$W_0 = u(c_0) + e^{-\delta} u(e_1) \tag{13}$$
$$v(e_1) = Ev(c_1). \tag{14}$$

The intertemporal welfare W_0 is a discounted sum of the current utility extracted from current consumption c_0 and of the future utility extracted from the certainty equivalent e_1 of future consumption c_1. Utility functions u and v are the time and risk aggregators, respectively. They are assumed to be increasing and five times differentiable. Parameter δ is the rate of pure preference for the present. Future consumption is affected by stochastic variance structured as follows:

$$c_1 = \bar{c}_1 + \sigma\eta \tag{15}$$
$$\sigma^2 = \overline{\sigma}^2 + w, \tag{16}$$

where η and w are independent and have a zero mean, with $E\eta^3 = 0$.

[9]However, the high persistence of the volatility shocks magnifies the variance of log consumption at very long maturities.

An immediate application of Theorem 1 is that stochastic volatility reduces welfare if and only if $v^{(4)}$ is negative. Following Eeckhoudt and Schlesinger (2006), this condition is referred to as "temperance". Gollier and Pratt (1996) showed that temperance is necessary for any zero-mean background risk to raises the aversion to any other independent risk. Eeckhoudt and Schlesinger (2006) showed that temperance is necessary and sufficient for an individual to prefer a 50-50 compound lottery yielding either x_1 or x_2 over another 50-50 compound lottery yielding either 0 or $x_1 + x_2$, where x_1 and x_2 are two zero-mean independent lotteries.

We now turn to the analysis of asset prices. As in Campbell (1986), Abel (1999) and Martin (2013) for example, let $P(\phi)$ denote the price today of an asset that generates a payoff distributed as c_1^ϕ in the future. The risk-free asset corresponds to $\phi = 0$, and a claim on aggregate consumption corresponds to $\phi = 1$. In a Lucas tree economy with a representative agent whose preferences are represented by equations (13) and (14), and where c_0 and c_1 are the fruits endowment at date 0 and 1, the equilibrium price $P(\phi)$ is characterized by the following pricing equation:

$$P(\phi) = e^{-\delta} \frac{Ec_1^\phi v'(c_1)}{v'(e_1)} \frac{u'(e_1)}{u'(c_0)}. \tag{17}$$

The continuously compounded expected return $r(\phi)$ of the asset is equal to the logarithm of $Ec_1^\phi / P(\phi)$. For example, the interest rate r^f is equal to $-\log(P(0))$. The risk premium of asset ϕ is the difference between the expected return of that asset and of the risk-free asset. It is equal to

$$\pi(\phi) = \log \left(\frac{Ec_1^\phi Ev'(c_1)}{Ec_1^\phi v'(c_1)} \right). \tag{18}$$

We first examine the impact of stochastic volatility on the risk-free rate. We obtain a clearcut result in the special case of Discounted Expected Utility (DEU) where u and v are identical. In the DEU case, equation (17) directly implies that the price $P(0)$ of a risk-free asset is increased by stochastic volatility if and only if it raises $Ev'(c_1)$. The following proposition is thus another immediate application of Theorem 1.

Proposition 1. *Suppose $u \equiv v$. Any increase in risk in the variance of future consumption reduces the interest rate if and only if v''' is convex.*

This is the consequence of the fact that increasing risk surrounding the variance of future consumption generates a 4th-degree risk deterioration of consumption. By definition, this raises Ev' if the fourth derivative of v' is positive. This is in line with earlier results by Eeckhoudt and Schlesinger (2008) who demonstrated that, in the DEU model, any nth-degree increase in future income risk increases optimal savings if and only if $sgn[v^{(n+1)}] = (-1)^n$. Notice that condition $v^{(5)} \geq 0$ is referred to as "edginess" by Lajeri-Chaherli (2004) and Eeckhoudt and Schlesinger (2008). Deck and Schlesinger (2014) and Deck and Schlesinger (2016) tested the sign of up to the fifth derivative of the utility function in the laboratory, showing some evidence of edginess.[10]

[10]Theoretical results relating asset prices to the fifth derivative of the utility function are scarce. Gollier (2001) showed that wealth inequality reduces the equilibrium interest rate if $-v''/v'''$ is concave.

With recursive preferences, the impact of stochastic volatility on the interest rate is also affected by the marginal rate of transformation $u'(e_1)/v'(e_1)$. This rate tells us how a marginal increase in risk utility $v(e_1)$ generated by an increase in e_1 translates into an increase in temporal utility $u(e_1)$. Because stochastic volatility affects the certainty equivalent consumption e_1, it also affects this marginal rate of transformation. It is increasing in e_1 if and only if $-v''(c)/v'(c)$ is larger than $-u''(c)/u'(c)$ for all c. If $v^{(4)}$ is negative, we know that e_1 is reduced by stochastic volatility. This implies that stochastic volatility reduces the marginal rate of transformation $u'(e_1)/v'(e_1)$ when v is more concave than u. This counterbalances the direct effect of stochastic volatility on $Ev'(c_1)$ when $v^{(5)}$ is positive, implying an ambiguous effect. This is summarized in the following proposition, which also characterizes the risk premium $\pi(1)$ on aggregate consumption.

Proposition 2. *Suppose that the risk on the variance of future consumption is small in the sense that η is distributed as $k\epsilon$ with k small. Any increase in risk in the variance of future consumption reduces the interest rate if and only if*

$$v^{(5)}(\bar{c}_1) + v^{(4)}(\bar{c}_1)\left(\frac{-v''(\bar{c}_1)}{v'(\bar{c}_1)} - \frac{-u''(\bar{c}_1)}{u'(\bar{c}_1)}\right) \geq 0. \tag{19}$$

It raises the risk premium $\pi(1)$ on aggregate consumption if and only if $v^{(4)}(\bar{c}_1)$ is negative.

Proof: See the appendix.

If we assume that $v^{(4)}$ is negative, then condition (20) can be rewritten as follows:

$$\frac{v^{(5)}(\bar{c}_1)}{-v^{(4)}(\bar{c}_1)} \geq \frac{-v''(\bar{c}_1)}{v'(\bar{c}_1)} - \frac{-u''(\bar{c}_1)}{u'(\bar{c}_1)}. \tag{20}$$

Because u is concave, a sufficient condition for an increase in risk in the variance of consumption to reduce the interest rate is that the index of edginess $-v^{(5)}/v^{(4)}$ be larger than the index of risk aversion $-v''/v'$.[11] This condition, which is necessary and sufficient in the small when u is linear, is satisfied for example when v is a power or an exponential function.

Notice also that these results hold only in the small. As already shown by Gollier (1995) and Abel (2002), an increase in consumption risk does not necessarily raise the risk premium at equilibrium under risk aversion. Similarly, a 4th-degree increase in consumption risk does not necessarily raise it either under condition $v^{(4)} \leq 0$.

In the long-run risks literature, the uncertainty affects the variance of log consumption rather than consumption itself. This means that equations (15) and (16) are replaced by the

[11] This condition is parallel to the condition by Kimball and Weil (2009) who showed that an increase in future income risk raises savings – and thus reduces the interest rate at equilibrium – if the index of prudence $-v^{(3)}/v^{(2)}$ is larger than the index of risk aversion $-v''/v'$. This condition is equivalent decreasing absolute risk aversion. Bostian and Heinzel (2016) examine the impact of a nth-degree risk affecting future income on optimal saving in the large within the Kreps-Porteus framework.

following specification:

$$x = \log\left(\frac{c_1}{c_0}\right) = \mu + \sigma\eta \tag{21}$$

$$\sigma^2 = \overline{\sigma}^2 + w, \tag{22}$$

where η and w are independent and have a zero mean, with $E\eta^3 = 0$. In this framework, although the uncertainty surrounding σ does not affect the first three moments of x, it increases all moments of c_1, since function $f(x) = exp(nx)$ has a positive fourth derivative for all $n \in \mathbb{N}_0$. The problem is much simplified if one assumes that the utility function v exhibits constant relative risk aversion $\gamma \geq 0$, i.e., $v(c) = c^{1-\gamma}/(1-\gamma)$, so that $Ev(c_1)$ is equal to $Ef(x)$ with

$$f(x) = -\frac{\exp\left(-(\gamma-1)x\right)}{\gamma - 1}. \tag{23}$$

This implies that function f exhibits constant absolute risk aversion $A = \gamma - 1$, yielding $f^{(4)}(x) = -(\gamma-1)^3\exp(-(\gamma-1)x)$. Theorem 1 implies that the stochastic volatility of the consumption growth rate reduces welfare in the Kreps-Porteus model with constant relative risk aversion if and only if relative risk aversion is larger than unity. When η is standard Normal, the relative stochastic volatility premium is approximately equal to $0.125(\gamma-1)^3\sigma_w^2$. This approximation is exact when the variance of log consumption is normally distributed.

We now turn to the analysis of the impact of the uncertain variance of log consumption on the risk-free rate and the risk premium. Following Epstein and Zin (1989) and many others after them, we hereafter assume that $v(c) = c^{1-\gamma}/(1-\gamma)$ and $u(c) = c^{1-\rho}/(1-\rho)$. Parameter ρ is the relative aversion to consumption fluctuations over time. It is the inverse of the elasticity of intertemporal substitution. The following proposition is a direct consequence of using equation (8) (which is exact under our specification) to estimate the expectations that appear in equations (17) and (18).

Proposition 3. *Suppose that* $\log(c_1/c_0)$ *is distributed as* $\mu + \sigma\eta$ *where* σ *and* η *are independent,* η *has a standard Normal distribution, and* σ^2 *is* $N(\overline{\sigma}^2, \sigma_w^2)$*. Suppose also that relative risk aversion* γ *and the relative aversion to fluctuations* ρ *are constant. The interest rate* r^f *and the risk premium* $\pi(\phi)$ *associated to an asset whose future payoff is* c_1^ϕ *satisfy the following conditions:*

$$r^f = \delta + \rho\mu - \frac{1}{2}\Big(\gamma^2 - (\gamma-\rho)(\gamma-1)\Big)\overline{\sigma}^2 - \frac{1}{8}\Big(\gamma^4 - (\gamma-\rho)(\gamma-1)^3\Big)\sigma_w^2 \tag{24}$$

$$\pi(\phi) = \phi\gamma\overline{\sigma}^2 + \frac{1}{2}\phi\gamma\left(\gamma^2 - \frac{3}{2}\phi\gamma + \phi^2\right)\sigma_w^2. \tag{25}$$

These results are reminiscent of earlier results by Martin (2013) who characterized the interest rate and risk premia with power functions for u and v when changes in log consumption are i.i.d. but not Normal. The coefficients of the last term in equations (24) and (25) correspond to those obtained by Martin for the impact of the 4th cumulant of x on the interest rate and the risk premia. This is the consequence of the fact that the 4th cumulant of x equals $3\sigma_w^2$ in this framework.

Increasing the risk on the variance of log consumption always raises the risk premium associated to any asset with $\phi > 0$ since the last term in equation (25) has the same sign as ϕ. In particular, it increases the risk premium associated to a claim on aggregate consumption ($\phi = 1$). Increasing the risk on the variance of log consumption reduces the interest rate if γ^4 is larger than $(\gamma-\rho)(\gamma-1)^3$. The intuition of this result is similar to the one of condition (20): The increased kurtosis of log consumption raises $Ev'(c_1)$ proportionally to γ^4. This "precautionary effect" tends to reduce the interest rate. It also reduces the certainty equivalent growth rate proportionally to $(\gamma-1)^3$. This implies in turn a reduction in the marginal rate of transformation $u'(e_1)/v'(e_1)$ proportionally to $(\gamma-\rho)(\gamma-1)^3$. This "income effect" tends to raise the interest rate when γ is larger than ρ. Globally, the stochastic volatility affecting log consumption reduces the interest rate if and only if the precautionary effect dominates the income effect. This is the case for example when both γ and ρ are larger than unity. When ρ equals 1, it requires that γ be larger than 1/2.

We now explore the impact of stochastic volatility on wealth, which is here defined as the equilibrium price at date 0 of a claim on the future aggregate consumption c_1. It is equal to Ec_1 discounted at the risk-adjusted discount rate $r^f + \pi(1)$. Using Proposition 3, we obtain that

$$Wealth = c_0 \exp\left(-\delta - (\rho-1)\mu + \frac{1}{2}(\gamma-1)(\rho-1)\overline{\sigma}^2 + \frac{1}{8}(\gamma-1)^3(\rho-1)\sigma_w^2\right). \tag{26}$$

As is well-known, in the DEU model ($\rho = \gamma$), an increase in uncertainty affecting growth, as measured by $\overline{\sigma}^2$, raises wealth in the economy. We obtain the same result for an increase in the uncertainty affecting volatility, as measured by σ_w^2. One of the benefits of the recursive utility model is to reverse the sign of these impacts when $\rho \leq 1 \leq \gamma$. In this case, an increase in risk on growth or on volatility reduces wealth in the economy.

4 Long-run stochastic volatility under Discounted Expected Utility

In the remainder of this paper, we characterize the term structures of the stochastic volatility premia for intertemporal welfare, interest rates and risk premia in the context examined by Bansal and Yaron (2004) in which the variance of the growth rate of consumption follows an autoregressive process of order 1:

$$\log\left(\frac{c_{t+1}}{c_t}\right) = \mu + \sigma_t \eta_{t+1}, \tag{27}$$

with

$$\sigma_{t+1}^2 = \sigma^2 + \nu\left(\sigma_t^2 - \sigma^2\right) + \sigma_w w_{t+1}, \tag{28}$$

where σ^2 is the unconditional variance, $\nu \in [0,1[$ is the coefficient of persistence of shocks on variance, σ_w is the standard deviation of these shocks, and the two shocks η and w are assumed to be i.i.d. standard Normal.[12] This stochastic process has three interesting

[12] Bansal and Yaron (2004) consider a more general model in which the expected growth rate μ also follows an autoregressive process. In this paper, we focus on the impact of the uncertain variance of growth.

features. First, the volatility σ_t is stochastic. Second, the shocks to volatility exhibit some persistence. Third, the volatility is known one period in advance. Notice also that an asset ϕ that generates a cash flow $D_t = c_t^{\phi}$ is governed by the following stochastic process:

$$\log\left(\frac{D_{t+1}}{D_t}\right) = \mu_D + \phi\sigma_t\eta_{t+1}, \tag{29}$$

with $\mu_D = \phi\mu$.[13] As in Bansal and Yaron (2004), the stochastic volatility of dividend growth is proportional to the stochastic volatility of consumption growth.

Under equations (27) and (28), the variance of log-consumption t periods ahead is as follows:

$$v_t = Var\left[\log\left(\frac{c_t}{c_0}\right)\middle| w_1, ...w_{t-1}, \sigma_0\right] = t\sigma^2 + \frac{1-\nu^t}{1-\nu}\left(\sigma_0^2 - \sigma^2\right) + \sigma_w \sum_{\tau=1}^{t} \frac{1-\nu^{t-\tau}}{1-\nu} w_\tau. \tag{30}$$

The last term in this equality characterizes the stochastic nature of the volatility of long-term growth. It is normally distributed. This means that the stochastic process (27)-(28) yields the same stochastic structure of log consumption as the one described by equations (21)-(22), with maturity-specific parameters. The following lemma characterizes the nature of the stochastic variance of log-consumption at different maturities in this long-run risk context.

Lemma 1. *Suppose that log-consumption is governed by process (27)-(28). Then, for any maturity $t \in \mathbb{N}_0$, $x_{0,t} = \log(c_t/c_0)$ conditional to σ_0 is distributed as $\mu t + \sqrt{v_t}\varepsilon$, where v_t and ε are independent, ε is $N(0,1)$, and v_t is normally distributed with annualized mean*

$$\frac{E_0[v_t]}{t} = \sigma^2 + \frac{1-\nu^t}{t(1-\nu)}\left(\sigma_0^2 - \sigma^2\right) \tag{31}$$

and annualized variance

$$\frac{Var_0[v_t]}{t} = \frac{1}{3}\frac{\kappa_4^{x_{0,t}}}{t} = \frac{\sigma_w^2}{(1-\nu)^2}\left(1 - 2\frac{1-\nu^t}{t(1-\nu)} + \frac{1-\nu^{2t}}{t(1-\nu^2)}\right), \tag{32}$$

where $\kappa_4^{x_{0,t}}$ is the fourth cumulant of log consumption $x_{0,t}$.

Notice that $E_0 v_t$ measures the unconditional variance of $\log(c_t/c_0)$. If one divides this measure by $t\sigma_0$, one obtains the "variance ratio" that has already been examined by Cochrane (1988), Beeler and Campbell (2012) and many others. Equation (31) tells us that this variance ratio has a flat term structure in expectation, i.e., when σ_0 equals σ. This is consistent with the recent findings obtained by Marfè (2016) who used postwar U.S. output.[14] As explained earlier, stochastic volatility does not increase the risk as measured by the variance

[13] This constraint on the expected growth of dividends is irrelevant for our analysis. It affects the price of the asset, but not its risk premium.

[14] In fact, Marfè (2016) obtained term structures of the variance ratio for output, salary and dividend that are respectively flat, increasing and decreasing. He convincingly argues that this comes from the fact that firms provide short-term insurance to their employees against the transitory fluctuations of their labor productivity, in line with the theory of implicit labor contract.

of log consumption. Similarly, it does not affect its skewness. Turning to equation (32), the annualized variance of v_t goes from 0 for a one-period maturity up to $\sigma_w^2/(1-\nu)^2$ for very large maturities. This upward sloping term structure is due to the combination of the fact that the variance is known one period in advance and of the persistence of shocks to volatility. These two features of the stochastic process of growth magnify the long-term risk on variance. As noticed in Section 2, in this Gaussian framework, the fourth cumulant of $x_{0,t}$ is equal to three times the variance of v_t. It implies that the annualized fourth cumulant of log consumption has an increasing term structure too. We test this hypothesis in Section 6.

As a benchmark, consider a Lucas tree economy with a representative agent that maximizes the discounted expected utility of the flow of aggregate consumption:

$$W_0 = E\left[\sum_{t=0}^{+\infty} \exp(-\delta t) v(c_t)\right]. \tag{33}$$

We hereafter assume that W_0 is bounded. Using the same approach as in the previous section adapted to the case with $u(c) = v(c) = c^{1-\gamma}/(1-\gamma)$ and with the maturity-varying parameters of the stochastic volatility process described in Lemma 1, we obtain the following proposition.

Proposition 4. *Suppose that the growth process is governed by equations (27)-(28). Under the DEU model with constant relative risk aversion γ, the term structures at date 0 of interest rates and the risk premia satisfy the following conditions:*

$$r_t^f = \delta + \gamma\mu - \frac{1}{2}\gamma^2 \frac{E_0\left[v_t\right]}{t} - \frac{1}{8}\gamma^4 \frac{Var_0\left[v_t\right]}{t} \tag{34}$$

$$\pi_t(\phi) = \phi\gamma \frac{E_0\left[v_t\right]}{t} + \frac{1}{2}\phi\gamma\left(\gamma^2 - \frac{3}{2}\phi\gamma + \phi^2\right) \frac{Var_0\left[v_t\right]}{t}, \tag{35}$$

where $E_0[v_t]$ and $Var_0[v_t]$ are given by Lemma 1.

The first two terms in the right-hand side of equation (34) correspond to the Ramsey rule (Ramsey (1928)). The third term measures the impact of the risk affecting economic growth on the interest rate in the absence of stochastic volatility. Its term structure is flat when the current variance σ_0^2 equals its historical mean σ^2. This is because stochastic volatility does not increase the long-run risk measured by the annualized unconditional variance, which is equal to σ^2 at all maturities in that case. The last term measures the stochastic volatility premium for interest rates. Because the volatility of the growth rate is known one period in advance in the Bansal-Yaron model, this premium is zero for a one-period maturity. Because of the persistence of the shocks to volatility, the annualized variance of the conditional variance of log consumption defined as $Var_0\left(v_t\right)/t$ is increasing with maturity, thereby magnifying the kurtosis of the distant log consumption. This makes the term structure of interest rates decreasing in expectation.

The first term in the right-hand side of equation (35) is the classical CCAPM risk premium, which is the product of three elements: the CCAPM beta of the asset, the relative

	$r^f_{+\infty} - r^f_1$	$\pi_{+\infty}(1) - \pi_1(1)$	$\pi_{+\infty}(3) - \pi_1(3)$
$\gamma = \rho = 1$	-2.63×10^{-4}	5.27×10^{-4}	1.74×10^{-2}
$\gamma = \rho = 10$	-2.63	0.91	2.02
$\gamma = \rho = 20$	-42.13	7.82	20.16

Table 1: Expected term spreads of interest rates, risk premia on aggregate consumption and equity premia under neutral expectations ($\sigma_0 = \sigma$). As in Bansal et al. (2016), we assume $\sigma_w = 2.12 \times 10^{-6}$, and $\nu = 0.9984$ on a monthly basis. Spreads are expressed in percents per year.

risk aversion of the representative agent, and the expected annualized variance of the growth rate until maturity t. This latter element has a flat term structure when the current variance σ_0^2 is equal to its historical mean σ^2. An important feature of this model is that short-term risk premia are time-varying, in parallel to changes in expectation about the short-term volatility. However, the long-term risk premium of any asset ϕ is fixed. The second term measures the impact of stochastic volatility. It is always positive and proportional to the annualized variance of the conditional variance of $\log(c_t/c_0)$. Because this element has an increasing term structure, we can conclude that, on average, risk premia have an increasing term structure due to the persistence of the shocks to volatility. This magnifies the kurtosis of the distribution of long term log consumption, together with the contribution of any risky asset to this kurtosis. This explain why the term structure of risk premia must be increasing in expectation in this model with stochastic volatility. This raises new concern about the empirical test of this model. Indeed, it has recently been shown that the term structure of equity premia is decreasing (Binsbergen et al. (2012), Binsbergen and Koijen (2016), Giglio et al. (2015), and Giglio et al. (2016)).

Let us now quantify the impact of stochastic volatility on asset prices under the DEU model. We focus on the term spreads. Because volatility is known one period in advance, stochastic volatility has no impact on the short-term interest rate and risk premium. But it reduces the long-term interest rate and it raises the long-term risk premium. Bansal et al. (2016) assumed $\gamma = 9.67$, $\sigma_w = 2.12 \times 10^{-6}$, and $\nu = 0.9984$ on a monthly basis. This yields a term spread of interest rates of -2.30% on an annual basis. This number provides an upper bound of the impact of stochastic volatility on interest rate for finite maturities. Bansal et al. (2016) also calibrated the elasticity of the payoff of equity to aggregate consumption at $\phi = 3$. This yields a term spread of annualized equity premia of 1.80%. It should be stressed that these term spreads are very sensitive to relative risk aversion. This is particularly the case for the interest rate whose term spread is proportional to γ^4. In Table 1, we document this high sensitivity.

To sum up, the stochastic volatility of the growth rate raises the kurtosis of future log consumption. Because the 4th and 5th derivatives of v are respectively negative and positive, this tends to reduce interest rates and to raise equity premia in the DEU framework. The persistence of shocks to volatility magnifies these effects for longer maturities, thereby making the term structures of interest rates and equity premia respectively decreasing and increasing on average.

5 Long-run stochastic volatility under Recursive Utility

The use of recursive preferences is helpful to solve the equity premium puzzle and the risk-free rate puzzle. Can it also solve the puzzle of the decreasing term structure of risk premia? In the Epstein-Zin-Weil model, welfare V_t is obtained by backward induction:

$$\begin{aligned} V_t^{1-\rho} &= (1-\beta)c_t^{1-\rho} + \beta\left(E_t V_{t+1}^{1-\gamma}\right)^{\frac{1-\rho}{1-\gamma}} && \text{if } \rho \neq 1 && (36) \\ \log V_t &= (1-\beta)\log c_t + \beta \log\left(E_t V_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}} && \text{if } \rho = 1. && (37) \end{aligned}$$

where parameters γ and ρ are the indices of relative aversion to risk and to consumption fluctuations, respectively. Parameter $\beta = \exp(-\delta)$ is a discount factor. The equilibrium price at date t for an asset that generates a single payoff $D_\tau = c_\tau^\phi$ at date $\tau > t$ satisfies the following condition:

$$P_{t,\tau} = E_t\left[D_\tau \frac{S_\tau}{S_t}\right]. \tag{38}$$

The one-period-ahead stochastic discount factor $S_{\tau+1}/S_\tau$ to be used at date τ to value a payoff occurring at date $\tau+1$ is:

$$\frac{S_{\tau+1}}{S_\tau} = \beta\left(\frac{c_{\tau+1}}{c_\tau}\right)^{-\gamma} Z_{\tau+1}^{\rho-\gamma}\left(E_\tau\left(\frac{c_{\tau+1}}{c_\tau}Z_{\tau+1}\right)^{1-\gamma}\right)^{\frac{\gamma-\rho}{1-\gamma}}, \tag{39}$$

where $Z_\tau = V_\tau/c_\tau$ is the future expected utility per unit of current consumption. Proposition 5 describes an approximation of the term structures of interest rates and risk premia with recursive preferences and stochastic volatility.[15] These results are obtained by assuming that the $\log(Z_t)$ is linear in the state variable σ_t^2, which is the case when the EIS ρ^{-1} is equal to one. We show in the appendix that the coefficient

$$b = \left.\frac{d\log(Z_t)}{d\sigma_t^2}\right|_{\sigma_t^2=\sigma^2} \tag{40}$$

satisfies the following condition:

$$b = \left(\frac{1}{2}(1-\gamma) + b\nu\right)\beta\exp\left((1-\rho)\left(\mu + \frac{1}{2}(1-\gamma)\sigma^2 + \frac{1}{2}(1-\gamma)b^2\sigma_w^2\right)\right). \tag{41}$$

When ρ equals unity, b equals $0.5\beta(1-\gamma)/(1-\beta\nu)$. Notice that b is negative when γ is larger than unity, which implies that the intertemporal welfare Z is decreasing in the current volatility of the growth rate of consumption.

[15] As done in this literature, we will hereafter use equalities to refer to the approximations obtained when using this linearization.

Proposition 5. *Suppose that the growth process is governed by equations (27)-(28). Under the recursive utility model (36)-(37), the term structures at date 0 of interest rates and the risk premia are approximated by the following equations:*

$$\begin{aligned} r_t^f &= \delta + \rho\mu - \frac{1}{2}\left(\gamma^2 - (\gamma-\rho)(\gamma-1)\right)\frac{E_0\left[v_t\right]}{t} - \frac{1}{8}\left(\gamma^2 - (\gamma-\rho)(\gamma-1)\right)^2 \frac{Var_0\left[v_t\right]}{t} \\ &\quad - \frac{1}{2}(\gamma-\rho)(1-\rho)b^2\sigma_w^2 + \frac{1}{2}\frac{\gamma-\rho}{1-\nu}\left(\gamma^2 - (\gamma-1)(\gamma-\rho)\right) b \left(1 - \frac{1-\nu^t}{t(1-\nu)}\right)\sigma_w^2 \quad (42) \end{aligned}$$

$$\begin{aligned} \pi_t(\phi) &= \phi\gamma\frac{E_0\left[v_t\right]}{t} + \frac{1}{2}\phi\left(\gamma(\gamma-\rho+\gamma\rho) - \frac{1}{2}\phi(2\gamma^2+\gamma-\rho+\gamma\rho) + \gamma\phi^2\right)\frac{Var_0\left[v_t\right]}{t} \\ &\quad + \frac{1}{2}\sigma_w^2(\gamma-\rho)\phi b\frac{\phi-2\gamma}{1-\nu}\left(1 - \frac{1-\nu^t}{t(1-\nu)}\right), \quad (43) \end{aligned}$$

where $E_0[v_t]$ and $Var_0[v_t]$ are given by Lemma 1, and b solves equation (41). These approximations are exact when ρ is equal to unity.

Proof: See the appendix.

A direct consequence is that, in expectation ($\sigma_0 = \sigma$), the short-term interest rate simplifies to[16]

$$r_1^f = \delta + \rho\mu - \frac{1}{2}\left(\gamma^2 - (\gamma-\rho)(\gamma-1)\right)\sigma_0^2 - \frac{1}{2}(\gamma-\rho)(1-\rho)b^2\sigma_w^2. \quad (44)$$

Although the variance of growth is known one period in advance, the stochastic nature of future volatility reduces the short-term interest rate when the representative agent has a Preference for an Early Resolution of Uncertainty (PERU), i.e., when γ is larger than ρ, and ρ is smaller than unity. The representative agent will observe at date 1 the volatility σ_1 that will prevail in the second period, and this additional date-1 risk has an impact on the willingness to raise savings today for date-1 consumption.

The term structure of interest rates in this long-run risk model with recursive preferences combines features already discussed in the DEU model and a new element coming from PERU. Indeed, the first line in equation (42) is symmetric to the equation (34), with adapted coefficients for $E_0[v_t]$ and $Var[v_t]$ to account for the discrepancy between γ and ρ. The second line in equation (42) is new compared to the DEU framework. We have seen above that it tends to reduce the short-term interest rate. It is easy to check that the last term in this equation has a decreasing term structure under PERU and $\gamma \geq 1$ (so that b is negative). This means that PERU cannot reverse the tendency of the term structure of interest rates to be decreasing. This result parallels Beeler and Campbell (2012) who showed that when the persistence of shocks to the growth rate of consumption is added to the model as in Bansal

[16] Although they do not compute the term structure of interest rates, Bansal et al. (2016) characterize the short-term interest rate r_1^f, which corresponds to equation (44) with their constant of log-linearization κ_1 equaling $2b(1-\beta\nu)/(1-\gamma)$.

β	0.999	discount factor
γ	9.67	relative risk aversion
ρ^{-1}	2.18	elasticity of intertemporal substitution
μ	0.0016	expected growth rate of consumption
$\sigma_0 = \sigma$	0.007	current and expected volatility
σ_w	2.12×10^{-6}	standard deviation of shocks to volatility
ν	0.9984	coefficient of persistence

Table 2: Calibration parameter based on monthly data extracted from Bansal et al. (2016).

et al. (2012), then real interest rates have a decreasing term structure, and are negative for maturities exceeding 10 years.

The short-term risk premium $\pi_1(\phi) = \phi\gamma\sigma_0^2$ is not affected by stochastic volatility. This is because the volatility for the first period is known at date 0. The middle term in the right-hand side of equation (43) is similar to the last term of equation (35) in the DEU model. The persistence of shocks to volatility magnifies the long-term kurtosis of log consumption, thereby tending to make the term structure of risk premia increasing. The last term of equation (43) takes account of PERU in the recursive utility model. As long as ϕ is smaller than 2γ, this new term has an increasing term structure too. We can thus conclude that PERU cannot reverse the increasing nature of the term structure of risk premia already observed under DEU.

Consider again the calibration extracted from Bansal et al. (2016), as summarized in Table 2. Figure 3 describes the term structures under this calibration, respectively for the interest rates (r_t^f), the risk premia on aggregate consumption ($\pi_t(1)$), and the equity premia ($\pi_t(3)$). The benchmark case without stochastic volatility is obtained by selecting $\sigma_w = 0$. It is represented by the dashed lines in this figure. As in the DEU framework, although shocks to volatility are small, their high persistence has a strong impact on the pricing of long-dated assets. It reduces the long interest rate by around 0.4%, and it raises the long equity premium by almost 1.5%. Notice that because the half-life of the shocks to the variance of log consumption is around 36 years, it takes many centuries for these term structures to converge to these asymptotic values.

A comparative static analysis is summarized in Table 3. Long-term rates are much more affected by the unilateral change of a parameter than short-term rates. For example, consider an increase in the persistence parameter ν from its benchmark value of 0.9984 to 0.999. Because Bansal et al. (2016) estimated ν with a standard error of 0.0007, this change in ν cannot be excluded by their data.[17] This unilateral change has no effect on the risk premium

[17]In fact, $\nu = 0.999$ is the calibration used by Bansal et al. (2012). Notice that using the calibration with $\nu = 0.987$ as in Bansal and Yaron (2004) would yield the following equilibrium prices: $r_1^f = 1.68\%$, $r_{+\infty}^f = 1.67\%$, $\pi_1(3) = 1.71\%$ and $\pi_{+\infty}(3) = 1.73\%$. As noticed by Beeler and Campbell (2012), the stochastic nature of volatility has very little effect on asset prices in this initial calibration of the long-run risk model.

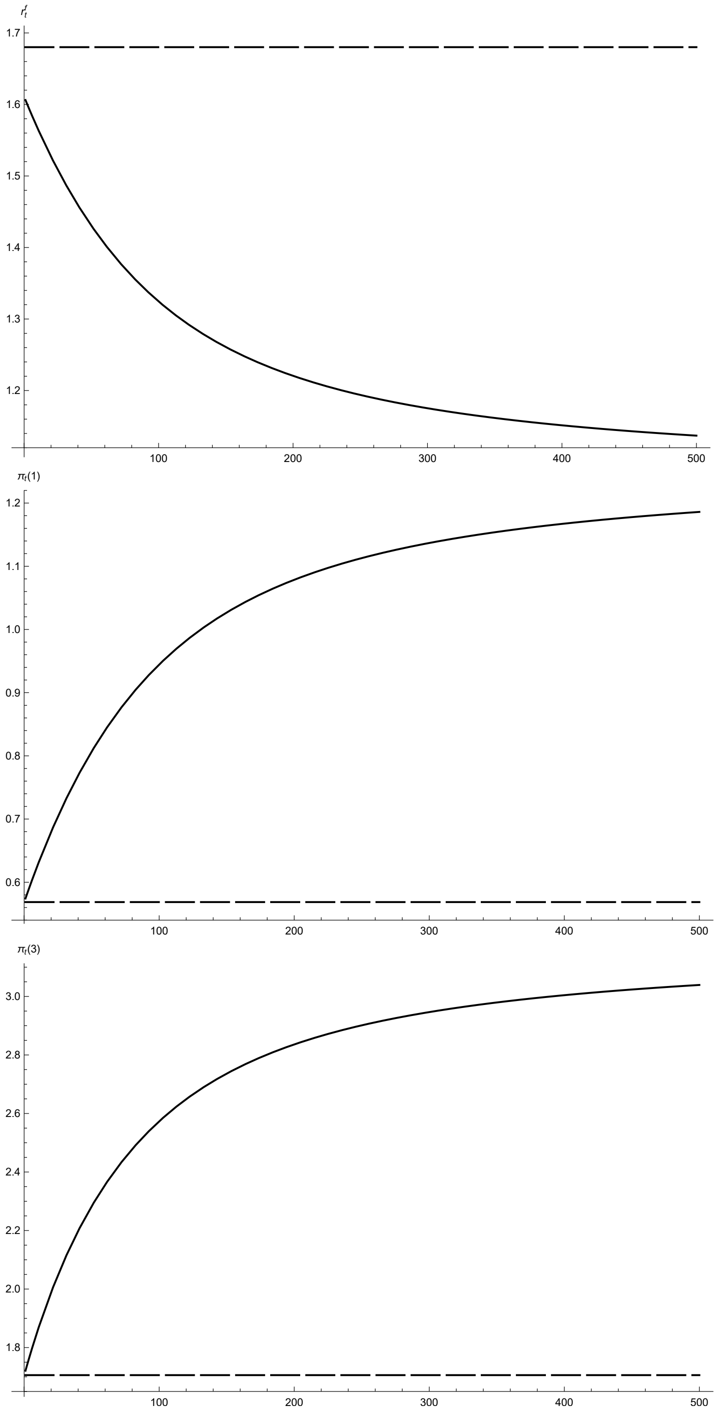

Figure 3: The term structures of interest rates (top), risk premia on aggregate consumption ($\phi = 1$, middle) and equity premia ($\phi = 3$, bottom) with recursive utility under the stochastic process (27)-(28). The calibration of the parameters is described in Table 2. The dashed curves are obtained by imposing $\sigma_w = 0$ (no stochastic volatility). Rates are in percent per year and durations are in years.

	r_1^f	$r_{+\infty}^f$	$\pi_1(3)$	$\pi_{+\infty}(3)$
Benchmark	1.61	1.08	1.71	3.17
$\gamma = 5$	1.87	1.80	0.88	1.04
$\rho = 1$	2.58	2.04	1.71	2.76
$\sigma_w = 4 \times 10^{-6}$	1.46	-0.32	1.71	6.57
$\nu = 0.999$	1.54	0.35	1.71	4.91
Fixed volatility ($\sigma_w = 0$)	1.68	1.68	1.71	1.71

Table 3: Comparative static analysis based on the benchmark described in Table 2, with $\gamma = 9.67$, $\rho = 1/2.18$, $\sigma_w = 2.12 \times 10^{-6}$, and $\nu = 0.9984$. The first column describes the modified parameter, everything else held unchanged. Rates are in percent per year.

on short-term equity, but it raises its term spread from 1.46% to 3.20%.

6 The term structure of the 4th cumulant of log consumption

The long-run risk specification (27)-(28) of stochastic volatility generates an increasing term structure of the uncertainty affecting the conditional variance of the growth rate. This does not change the annualized variance whose term structure remains flat. But it makes the term structure of the annualized fourth cumulant of log consumption increasing, as expressed in equation (32). This is the driving force behind the shapes of the term structures described in the previous two sections. It is useful to test whether consumption data generates an increasing term structure of the annualized fourth cumulant of log consumption. As Bansal et al. (2016) and many others, our consumption data are extracted from the NIPA Tables 2.33, 2.34 et 7.1 of the Bureau of Economic Analysis. We use quarterly data of per-capita real consumption expenditure on nondurables and services from 1947Q1 to 2016Q4. In Figure 4, we used plain circles to represent the term structure of the empirical annualized fourth moment of log consumption, which is computed as follows for each maturity $t \in \{1, ..., 40\}$:

$$\frac{K_4^{x_{0,t}}}{t} = \frac{1}{t}\left[(278-t)^{-1}\sum_{\tau=1}^{278-t}\left(x_{\tau,\tau+t}-\overline{x}_t\right)^4 - 3\left((278-t)^{-1}\sum_{\tau=1}^{278-t}\left(x_{\tau,\tau+t}-\overline{x}_t\right)^2\right)^2\right], \quad (45)$$

where $\overline{x}_t$ is the mean of the series $\left(x_{\tau,\tau+t}\right)_{\tau=1,...,278-t}$. The increasing term structure for $K_4^{x_{0,t}}/t$ observed in the data suggests the persistence of shocks to the variance and is supportive of decreasing interest rates and increasing risk premia. This is compatible with the long-run risk model (27)-(28) with a positive coefficient of persistence. The quarterly calibration proposed by Bansal et al. (2016) is also described in Figure 4, with $\nu = 0.9978$ and $\sigma_w = 6.07 \times 10^{-6}$. This is computed as follows:

$$\frac{\kappa_4^{x_{0,t}}}{t} = \frac{3\sigma_w^2}{(1-\nu)^2}\left(1 - 2\frac{1-\nu^t}{t(1-\nu)} + \frac{1-\nu^{2t}}{t(1-\nu^2)}\right). \quad (46)$$

This figure shows that this calibration tends to generate too much kurtosis for maturities exceeding one year. This is why we also calibrated the term structure of the annualized 4th

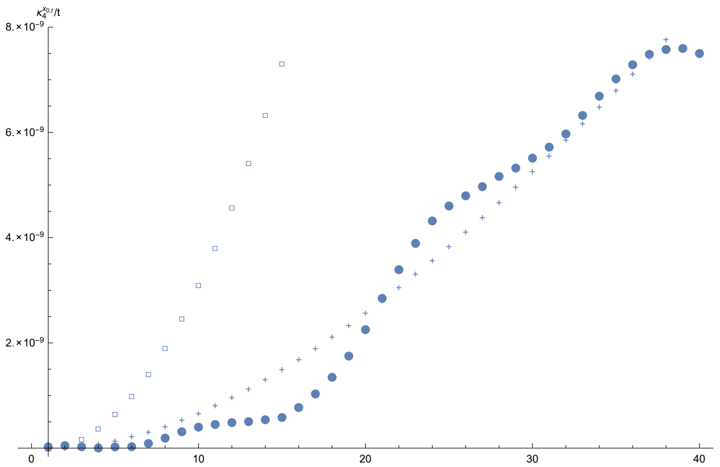

Figure 4: The term structure of the annualized 4th cumulant of log consumption $\kappa_4^{x_0,t}/t$, where the maturity t is measured in quarters. The bullets are obtained for maturities $t \in \{1, ..., 40\}$ by measuring the empirical 4th cumulant of the series $\left(x_{\tau,\tau+t}\right)_{\tau=1,...,278-t}$, using quarterly real U.S. consumption from 1947Q1 to 2016Q4. The squares correspond to the theoretical term structure of the 4th cumulant using equation (46) with the calibration $\nu = 0.9978$ and $\sigma_w = 6.07 \times 10^{-6}$ proposed by Bansal et al. (2016). The crosses correspond to an alternative calibration with $\nu = 0.9826$ and $\sigma_w = 2.96 \times 10^{-6}$.

cumulant with parameters $\nu = 0.9826$ and $\sigma_w = 2.96 \times 10^{-6}$ that better fit the data. In fact, these parameter values minimize the sum of the square of the differences between $K_4^{x_0,t}/t$ and $\kappa_4^{x_0,t}/t$ for maturities between 1 and 40 quarters. This calibration exhibits a smaller degree of persistence of shocks to the variance, and a smaller standard deviation for these shocks.

7 Concluding remarks

With power utility functions, risk aversion, prudence, temperance, edginess and higher degree risk attitudes are all summarized by one parameter usually referred to as risk aversion. The classical asset pricing theory, which heavily relies on this isoelastic specification, has therefore developed arguments based on the sole concept of risk aversion. We find that problematic for the development of this theory. For example, the negative impact of risk – as measured by the variance – on the interest rate is exclusively linked by the notion of prudence ($v''' \geq 0$), which is orthogonal to the concept of risk aversion ($v'' \leq 0$). In the same fashion, this paper demonstrates that stochastic volatility also reduces the interest rate if and only if the fifth derivative of the risk utility function v is positive. This condition is sometimes referred to as "edginess". In other words, departing from the power utility function would in theory allows us to disentangle these psychological traits of our preferences under risk. This agenda of research is in line with recent findings that are incompatible with constant relative risk aversion (see for example Ogaki and Zhang (2001), Guiso and Paiella (2008), and Deck and Schlesinger (2014)). This is also in line with the trend of behavioral finance in which the utility function is distorted by additive habit formation, by background risks, or by a reference point for example.

The fact that increasing the risk surrounding the variance of log consumption yields a 4th-degree increase in the risk surrounding log consumption has important consequences for asset pricing. First, the proxy measure of 4th-degree risk is the kurtosis. This means that stochastic volatility generates a flat term structure of the variance ratio in expectation. This is in striking contrast with our theoretical result that the term structure of risk premia is decreasing. The explanation of this seemingly inconsistent results comes from the fact that the aggregate risk cannot be measured only by annualized variance of future log consumption alone. Outside the Gaussian world, the annualized kurtosis matters too, and the persistence of shocks to the variance of log consumption makes its term structure increasing. This explains the increasing nature of the term structure of risk premia. This suggests that the kurtosis ratio should be used in parallel to the variance ratio in order to interpret the term structures of asset prices.

Second, the impact of stochastic volatility on the aggregate risk premium is approximately proportional to the intensity of the aversion to kurtosis. In the isoelastic case, this intensity if equal to the third power of risk aversion. This implies that the link between the aggregate risk premium and stochastic volatility is highly sensitive to the choice of the index of risk aversion. This is even worse for the interest rate because its precautionary stochastic volatility premium is proportional to the fourth power of the index of relative risk aversion.

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Appendix

Appendix 1: Proof of Theorem 1

We have to prove that function h is concave, with

$$h(\sigma^2) = E\left[f(\overline{x} + \sigma\eta)|\,\sigma\right]. \tag{47}$$

It is easy to verify that

$$h''(\sigma^2) = \frac{\sigma^4}{4}E\left[\left(\sigma^2\eta^2 f''(\overline{x}+\sigma\eta) - \sigma\eta f'(\overline{x}+\sigma\eta)\right)\Big|\,\sigma\right].$$

For each σ, define random variable y_σ in such a way that y_σ/σ be distributed as η. This implies that $Ey_\sigma = 0$ and $Ey_\sigma^3 = 0$. Using this change of variables, the above equation can be rewritten as follows:

$$h''(\sigma^2) = \frac{\sigma^4}{4}E\left[y_\sigma^2 f''(\overline{x}+y_\sigma) - y_\sigma f'(\overline{x}+y_\sigma)\right].$$

The concavity of h requires the right-hand side of this equality to be non-positive for all random variable y_σ such that $Ey_\sigma = Ey_\sigma^3 = 0$. This condition is summarized as follows:

$$Ey = 0 \text{ and } Ey^3 = 0 \;\Rightarrow\; E\left[y^2 f''(\overline{x}+y) - yf'(\overline{x}+y)\right] \leq 0. \tag{48}$$

We will use the following lemma, which is a direct consequence of Theorem 3 in Gollier and Kimball (1996).

Lemma 2. *Consider three functions (f_1, f_2, f_3) from $\mathbb{R}$ to $\mathbb{R}$ such that $f_1(0) = f_2(0) = f_3(0) = 0$. The following two conditions are equivalent:*

- *For any random variables y such that $Ef_1(y) = 0$ and $Ef_2(y) = 0$, we have that $Ef_3(y) \leq 0$.*
- *There exists a pair $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that $f_3(y) \leq \lambda_1 f_1(y) + \lambda_2 f_2(y)$ for all $y \in \mathbb{R}$.*

Applying this lemma to condition (48) makes it equivalent to requiring that there exists a pair $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that

$$H(y) = y^2 f''(\overline{x}+y) - yf'(\overline{x}+y) - \lambda_1 y - \lambda_2 y^3 \leq 0 \tag{49}$$

for all y. Notice that $H(0) = 0$ and $H'(0) = -f'(\overline{x}) - \phi_1$. This implies that $\lambda_1 = -f'(0)$ is a necessary condition for H to be non-positive. We also have that

$$H''(y) = 3yf'''(\overline{x}+y) + y^2 f''''(\overline{x}+y) - 6\lambda_2 y,$$

which implies that $H''(0)$ also vanishes. We also get

$$H'''(y) = 3f'''(\overline{x}+y) + 5yf''''(\overline{x}+y) + y^2 f^{(5)}(\overline{x}+y) - 6\lambda_2.$$

This implies that $H'''(0) = 3f'''(\overline{x}) - 6\lambda_2$. A necessary condition for H to be non-positive is thus that $\lambda_2 = 0.5f'''(\overline{x})$. Because $H''''(0) = 8f''''(\overline{x})$, a necessary condition for an increase in risk in the variance v of x to reduces $Ef(x)$ is that f'''' be non-positive, or that f'' be concave.

We now show that this condition is also sufficient. Using the conditions on λ_1 and λ_2, we can rewrite H in such a way that for all y, $H(y) = yK(y)$ with

$$K(y) = yf''(\overline{x}+y) - f'(\overline{x}+y) + f'(0) - 0.5y^2 f'''(0). \tag{50}$$

We would be done if we could show that H is uniformly non-positive. This requires $K(y)$ to have a sign opposite to y. Because $K(0) = 0$ a sufficient condition for this to be true is that function K be decreasing. Observe now that

$$K'(y) = y\left[f'''(y) - f'''(0)\right]$$

is negative when f''' is decreasing, i.e., if f'' is concave. This demonstrates that this condition is necessary and sufficient for an increase in risk in variance to reduce Ef. ■

Appendix 2: Solution (12) when σ^2 has a Gamma distribution

Suppose that σ^2 has gamma distribution $\Gamma(a,b)$, so that its density function is

$$f(v;a,b) = v^{a-1}\frac{e^{-v/b}}{b^a\Gamma(a)}, \text{ for all } v > 0,$$

where a and b are two positive constants. This implies that the mean variance is $\overline{\sigma}^2 = ab$, and its variance is $\sigma_w^2 = ab^2$. It implies that

$$\begin{aligned} E\left[\exp(-A(\overline{x}+\sigma\eta)\right] &= E\left[\exp(-A\overline{x} + 0.5A^2\sigma^2\sigma_\eta^2)\right] \\ &= \int_0^{+\infty} \exp(-A\overline{x} + 0.5A^2 v\sigma_\eta^2) f(v;a,b)dv \\ &= \frac{\exp(-A\overline{x})}{b^a\Gamma(a)}\int_0^{+\infty} v^{a-1}\exp\left(v(0.5A^2\sigma_\eta^2 - b^{-1})\right)dv \\ &= \exp\left(-A\overline{x}\right)\left(1 - \frac{1}{2}A^2\sigma_\eta^2 b\right)^{-a}. \end{aligned} \tag{51}$$

The last equality is a consequence of the observation that[18]

$$\int_0^{+\infty} v^{a-1}e^{-v/k}dv = k^a\Gamma(a).$$

Similarly, we have that

$$E\left[\exp(-A(\overline{x}+\overline{\sigma}\eta - \pi_f)\right] = \exp(-A\overline{x} + 0.5A^2\overline{\sigma}^2\sigma_\eta^2 + A\pi). \tag{52}$$

[18]This is a direct consequence that the integral of the density f must be equal to 1.

Combining equations (6), (51) and (52) yields

$$\exp(0.5A^2\overline{\sigma}^2\sigma_\eta^2 + A\pi_f) = \left(1 - \frac{1}{2}A^2\sigma_\eta^2 b\right)^{-a}.$$

Using the fact that $a = \overline{\sigma}^4/\sigma_w^2$ and $b = \sigma_w^2/\overline{\sigma}^2$, this equation is equivalent to (12). ∎

Appendix 3: Proof of Proposition 2

It is useful to explore first the properties of the following function:

$$\begin{aligned} H_f(k) &= \log\left(Ef(\overline{x} + \sigma k\epsilon)\right) - \log\left(Ef(\overline{x} + \overline{\sigma}k\epsilon)\right) \\ &= \log\left(Ef(\overline{x} + \overline{\sigma}k\epsilon - \pi_f(k))\right) - \log\left(Ef(\overline{x} + \overline{\sigma}k\epsilon)\right) \end{aligned} \tag{53}$$

Function H_f measures the relative increase in the expectation of f which is due to the stochastic nature of the variance of x. Using the fact that the first three derivatives of function π_f evaluated at $k = 0$ are zero together with the assumption that $E\epsilon = E\epsilon^3 = 0$, it is straightforward to show that

$$H_f(k) = -\frac{1}{4!}\pi_f^{(4)}(0)\frac{f'(\overline{x})}{f(\overline{x})} + O(k^5).$$

Using equation (7), this implies that

$$H_f(k) = \frac{k^4}{4!}\frac{f^{(4)}(\overline{x})}{f(\overline{x})}Var(\sigma^2)E\epsilon^4 + O(k^5). \tag{54}$$

We first examine the risk-free rate. By equation (54), we have that

$$\log\left(Ev'(c_1)\right) = \log\left(Ev'(\overline{c}_1 + \overline{\sigma}\eta)\right) + \frac{k^4}{4!}\frac{v^{(5)}(\overline{c}_1)}{v'(\overline{c}_1)}Var(\sigma^2)E\epsilon^4 + O(k^5). \tag{55}$$

Define function ϕ such that $\phi(e) = u'(e)/v'(e)$ for all e, and define $e_1^\star(k)$ as the certainty equivalent of $\overline{c}_1 + \overline{\sigma}k\eta$. From equation (7), we have that

$$\begin{aligned} \log\left(\frac{u'(e_1)}{v'(e_1)}\right) &= \log\left(\phi(e_1)\right) \\ &= \log\left(\phi(e_1^\star)\right) + \frac{\phi'(\overline{c}_1)}{\phi(\overline{c}_1)}\frac{k}{4!}\frac{v^{(4)}(\overline{c}_1)}{v'(\overline{c}_1)}Var(\sigma^2)E\epsilon^4 + O(k^5) \\ &= \log\left(\phi(e_1^\star)\right) + \frac{v'(\overline{c}_1)}{u'(\overline{c}_1)}\frac{u''(\overline{c}_1)v'(\overline{c}_1) - u'(\overline{c}_1)v''(\overline{c}_1)}{(v'(\overline{c}_1))^2}\frac{k}{4!}\frac{v^{(4)}(\overline{c}_1)}{v'(\overline{c}_1)}Var(\sigma^2)E\epsilon^4 + O(k^5) \\ &= \log\left(\phi(e_1^\star)\right) + \left(\frac{u''(\overline{c}_1)}{u'(\overline{c}_1)} - \frac{v''(\overline{c}_1)}{v'(\overline{c}_1)}\right)\frac{k}{4!}\frac{v^{(4)}(\overline{c}_1)}{v'(\overline{c}_1)}Var(\sigma^2)E\epsilon^4 + O(k^5). \end{aligned} \tag{56}$$

From equation (17) and the above two equations, we see that the increase in log price due to stochastic volatility is equal to

$$\frac{k^4}{4!}\frac{v^{(5)}(\overline{c}_1)}{v'(\overline{c}_1)}Var(\sigma^2)E\epsilon^4 + \left(\frac{u''(\overline{c}_1)}{u'(\overline{c}_1)} - \frac{v''(\overline{c}_1)}{v'(\overline{c}_1)}\right)\frac{k}{4!}\frac{v^{(4)}(\overline{c}_1)}{v'(\overline{c}_1)}Var(\sigma^2)E\epsilon^4 + O(k^5).$$

If k is small enough, this is positive if and only if equation (20) is satisfied.

We now turn to the risk premium. We have to prove that the following inequality holds for small k if and only if $v^{(4)}$ is negative:

$$\log\left(\frac{E(\bar{c}_1+\sigma k\epsilon)Ev'(\bar{c}_1+\sigma k\epsilon)}{E(\bar{c}_1+\sigma k\epsilon)v'(\bar{c}_1+\sigma k\epsilon)}\right) \geq \log\left(\frac{E(\bar{c}_1+\overline{\sigma} k\epsilon)Ev'(\bar{c}_1+\overline{\sigma} k\epsilon)}{E(\bar{c}_1+\overline{\sigma} k\epsilon)v'(\bar{c}_1+\overline{\sigma} k\epsilon)}\right) \tag{57}$$

Notice that

$$E(\bar{c}_1+\sigma k\epsilon) = E(\bar{c}_1+\overline{\sigma} k\epsilon)$$

because the expectation of ϵ is zero. This implies that we can rewrite the above inequality as follows:

$$\begin{aligned} &\log\left(Ev'(\bar{c}_1+\sigma k\epsilon)\right) &- &\log\left(Ev'(\bar{c}_1+\overline{\sigma} k\epsilon)\right) \\ \geq &\log\left(E(\bar{c}_1+\sigma k\epsilon)v'(\bar{c}_1+\sigma k\epsilon)\right) &- &\log\left(E(\bar{c}_1+\overline{\sigma} k\epsilon)v'(\bar{c}_1+\overline{\sigma} k\epsilon)\right). \end{aligned} \tag{58}$$

Let $H_f(k)$ be defined by equation (53), and let function f_1 and f_2 be respectively defined by $f_1(c) = v'(c)$ and $f_2(c) = cv'(c)$. This implies that the above inequality can be rewritten as follows:

$$H_{f_1}(k) \geq H_{f_2}(k). \tag{59}$$

Using equation (54) twice, this inequality holds for k small if and only if

$$\frac{f_1^{(4)}(\bar{c}_1)}{f_1(\bar{c}_1)} \geq \frac{f_2^{(4)}(\bar{c}_1)}{f_2(\bar{c}_1)}. \tag{60}$$

This can be rewritten as

$$\frac{v^{(5)}(\bar{c}_1)}{v'(\bar{c}_1)} \geq \frac{4v^{(4)}(\bar{c}_1)+\bar{c}_1 v^{(5)}(\bar{c}_1)}{\bar{c}_1 v'(\bar{c}_1)}.$$

This is true if and only if $v^{(4)}(\bar{c}_1)$ is negative. ■

Appendix 4: Proof of Proposition 5

Let us define the growth rate of consumption as $x_{t,\tau} = \log c_\tau/c_t$ and the annualized log return as $r_{t,\tau} = (\tau-t)^{-1}\log(c_\tau^\phi/P_{t,\tau})$, $\tau > t$. Let us also define the log SDF $s_t = \log S_t$, and the log future normalized utility $z_t = \log Z_t$. This allows us to rewrite the pricing equations (38) and (39) as follows:

$$0 = \chi_0\left(tr_{0,t}+s_t-s_0\right), \tag{61}$$

$$s_{\tau+1}-s_\tau = -\delta-\gamma x_{\tau,\tau+1}+(\rho-\gamma)z_{\tau+1}+\frac{\gamma-\rho}{1-\gamma}\chi_\tau\left((1-\gamma)(x_{\tau,\tau+1}+z_{\tau+1})\right), \tag{62}$$

where operator χ_t is defined as

$$\chi_\tau(x) = \log\left(E_\tau\exp(x)\right). \tag{63}$$

One can obtain z_τ by backward induction from equations (36)-(37) which are rewritten as follows:

$$z_\tau = \begin{cases} (1-\rho)^{-1}\log\left(1-\beta+\beta\exp\left(\frac{1-\rho}{1-\gamma}\chi_\tau((1-\gamma)(x_{\tau,\tau+1}+z_{\tau+1}))\right)\right) & \text{if } \rho\neq 1 \\ \frac{\beta}{1-\gamma}\chi_\tau((1-\gamma)(x_{\tau,\tau+1}+z_{\tau+1})) & \text{if } \rho = 1, \end{cases} \tag{64}$$

Suppose that the growth process is governed by equations (27)-(28). In case $\rho = 1$, it is easy to check that the guess solution

$$z_t = a_t + b\sigma_t^2 \tag{65}$$

satisfies equation (64) with

$$b = \frac{\beta(1-\gamma)}{2(1-\beta\nu)}. \tag{66}$$

When $\rho \neq 1$, the linearization (65) is a first-order approximation of the exact link between z_t and the state variable σ_t^2 around the steady state with $\sigma_t^2 = \sigma^2$ if b satisfies the following condition:

$$b = \left(\frac{1}{2}(1-\gamma)+b\nu\right)\beta\exp\left((1-\rho)\left(\mu+\frac{1}{2}(1-\gamma)\sigma^2+\frac{1}{2}(1-\gamma)b^2\sigma_w^2\right)\right). \tag{67}$$

Using this linearization, equation (62) can be rewritten as

$$s_{\tau+1}-s_\tau = -\delta-\rho\mu+\frac{1}{2}(\gamma-\rho)(1-\gamma)\left(\sigma_\tau^2+b^2\sigma_w^2\right)-\gamma\sigma_\tau\eta_{\tau+1}-(\gamma-\rho)b\sigma_w w_{\tau+1}. \tag{68}$$

By definition of $r_{0,t}$, we have that

$$tr_{0,t} = tE_0 r_{0,t}+\phi(x_{0,t}-E_0x_{0,t}) = tE_0r_{0,t}+\phi\sum_{\tau=0}^{t-1}\sigma_\tau\eta_{\tau+1}. \tag{69}$$

Combining this with equation (68), one can rewrite the pricing equation (61) as follows:

$$0 = \chi_0\left(Q\right), \tag{70}$$

with

$$\begin{aligned} Q &= tE_0r_{0,t}+\frac{1}{2}(\phi-\gamma)^2\sum_{\tau=0}^{t-1}\sigma_\tau^2-\delta t-\rho\mu t \\ &+ \frac{1}{2}(\gamma-\rho)(1-\gamma)\left(\sum_{\tau=0}^{t-1}\sigma_\tau^2+b^2\sigma_w^2t\right)-(\gamma-\rho)b\sigma_w\sum_{\tau=0}^{t-1}w_{\tau+1}. \end{aligned} \tag{71}$$

This is rewritten as

$$\begin{aligned} Q &= tE_0r_{0,t}-\delta t-\rho\mu t+\frac{1}{2}(\gamma-\rho)(1-\gamma)b^2\sigma_w^2t-(\gamma-\rho)b\sigma_w\sum_{\tau=0}^{t-1}w_{\tau+1} \\ &+ \frac{1}{2}\left((\phi-\gamma)^2+(\gamma-\rho)(1-\gamma)\right)\sum_{\tau=0}^{t-1}\sigma_\tau^2. \end{aligned} \tag{72}$$

Using equation (30), this can be rewritten as follows:

$$\begin{aligned} Q &= tE_0 r_{0,t} - \delta t - \rho\mu t + \frac{1}{2}(\gamma-\rho)(1-\gamma)b^2\sigma_w^2 t + \frac{1}{2}\left((\phi-\gamma)^2 + (\gamma-\rho)(1-\gamma)\right)E_0 v_t \\ &+ \sigma_w \sum_{\tau=1}^{t}\left(\frac{1}{2}\left((\phi-\gamma)^2 + (\gamma-\rho)(1-\gamma)\right)\frac{1-\nu^{t-\tau}}{1-\nu} - (\gamma-\rho)b\right) w_\tau. \end{aligned} \tag{73}$$

Because Q is normally distributed, the pricing equation (70) implies that

$$\begin{aligned} 0 &= tE_0 r_{0,t} - \delta t - \rho\mu t + \frac{1}{2}(\gamma-\rho)(1-\gamma)b^2\sigma_w^2 t + \frac{1}{2}\left((\phi-\gamma)^2 + (\gamma-\rho)(1-\gamma)\right)E_0 v_t \\ &+ \frac{1}{2}\sigma_w^2 \sum_{\tau=1}^{t}\left(\frac{1}{2}\left((\phi-\gamma)^2 + (\gamma-\rho)(1-\gamma)\right)\frac{1-\nu^{t-\tau}}{1-\nu} - (\gamma-\rho)b\right)^2. \end{aligned} \tag{74}$$

The return of the strategy in which one purchases at price P_0 at date 0 an asset that delivers payoff c_t^ϕ at date t is $R_{0,t} = c_t^\phi / P_0$. The expected return expressed as an annualized rate is

$$t^{-1}\log\left(E_0 R_{0,t}\right) = E_0 r_{0,t} + \frac{1}{2}\phi^2 \frac{E_0 v_t}{t} + \frac{1}{8}\phi^4 \frac{Var[v_t]}{t}. \tag{75}$$

Combining the above two equations implies that

$$\begin{aligned} t^{-1}\log\left(E_0 R_{0,t}\right) &= \delta + \rho\mu - \frac{1}{2}\left((\phi-\gamma)^2 - (\gamma-\rho)(\gamma-1) - \phi^2\right)\frac{E_0 v_t}{t} \\ &- \frac{1}{8}\left(\left((\phi-\gamma)^2 - (\gamma-\rho)(\gamma-1)\right)^2 - \phi^4\right)\frac{Var[v_t]}{t} - \frac{1}{2}(\gamma-\rho)(1-\rho)b^2\sigma_w^2 \\ &+ \frac{1}{2}\frac{\sigma_w^2 b(\gamma-\rho)}{1-\nu}\left((\phi-\gamma)^2 - (\gamma-\rho)(\gamma-1)\right)\left(1 - \frac{1-\nu^t}{t(1-\nu)}\right). \end{aligned} \tag{76}$$

The interest rate is obtained by replacing ϕ by zero, thereby generating equation (42). The risk premium is obtained by subtracting the interest rate from the expected return described above, which yields equation (43). This concludes the proof. ■