Optimal illusions and decisions under risk

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Abstract

As Brunnermeier and Parker (2003), we examine a static one-riskfree-one-risky asset portfolio choice when the investor's well-being is affected by the anticipatory feelings associated to potential capital gains and losses. These feelings can be manipulated by the choice of subjective beliefs on the distribution of returns. However, the bias of these endogeneous subjective beliefs induces the choice of a portfolio that is suboptimal with respect to the objective expected utility of final wealth. We characterize the structure of these optimal beliefs. We first show that optimal subjective beliefs must be degenerate with only two possible returns. Moreover, under some weak conditions on the utility function, these two atoms are at the lower and upper bounds of the objectively feasible returns, as suggested in the cumulative prospect theory. When the intensity of anticipatory feelings is small, the formation of beliefs must be biased in favor of optimism, which implies an increase in the equilibrium demand for the risky asset. We also show that the optimal beliefs are approximately independent of the investor's degree of risk aversion.

Keywords: anticipatory feelings, portfolio choice, overconfidence, positive thinking, endogeneous beliefs, cumulative prospect theory.

1 Introduction

In the late XIXth century Emile Coué, a french psychologist at the University of Nancy, promoted the idea that learning to control our thoughts can do much to improve well-being. Positive thinking improves the quality of life of patients with a life-threatening disease by inducing them to reduce their subjective probability of dying. The so-called "method Coué" has however an important undesirable effect. By artificially downgrading the risk, the patient may spend less effort to fight the illness. Psychotherapists are well aware of the problem, as most of them forcefully claim that the method does never replace the medical treatment.

Learning to play with our beliefs is important but dangerous for our well-being. In his famous book entitled "The Gambler", Dostoevsky describes a young middle-class man who dreams that he will become wealthy by gambling one day at the casino. However, he perfectly knows that the odds at the casino are unfair, as he forcefully advises other people not to gamble. This illustrates what Sigmund Freud will describe sixty years later as illusions, i.e., beliefs that establish themselves by the will of our desires. The gambler's optimism allows him to survive in a world of pretentious wealthy Russian expatriates. As stated by Glaeser (2004), "consumers will be more likely to accept false beliefs when those beliefs make them happier". However, relying on his subjective beliefs that he knows to be optimistic compared to the objective chances, the gambler eventually decides to take a chance, and loses everything.

In this paper, we want to apply these ideas to other choice problems under uncertainty. In particular, we examine the portfolio choice problem of risk-averse consumers.[1] In order to fit with the ideas developed by Dostoevsky, Coué and Freud, among others, we use a model introduced recently by Brunnermeier and Parker (2003). Following their approach, we recognize that current felicity is affected by the anticipation of future pleasures and displeasures. As a consequence, controlling our thoughts about the likelihood

[1] Alternative interpretations of our choice problem can be found in insurance economics and in the theory of investment. A consumer faces a risk of loss for which there exists an insurance market offering proportional insurance contracts with an actuarially unfair tariff. The problem of the consumer is to select the rate of insurance coverage for the risk. In the theory of investment, a risk-averse entrepreneur with a linear technology must determine the optimal capacity of production under uncertainty about the output price.

of these events has a direct effect on welfare. In a portfolio context, positive thinking implies a mental manipulation of the objective probability distribution of assets' returns. If the investor has a positive demand for stocks, method Coué means increasing the subjective probability of a positive excess return. The undesirable effect of positive thinking is that this manipulation of beliefs is likely to affect the asset allocation of the investor. This in turn affects negatively the investor's future felicity. We assume, as in Brunnermeier and Parker (2003), that the investor selects subjective beliefs in order to maximize his lifetime well-being which is an increasing function of both current and future felicities. Because positive thinking raises current felicity but reduces future felicity, the problem of method Coué is to determine the best compromise between these two opposite forces.

This work departs from the long tradition in economics to measure an individual's lifetime utility as a discounted sum of the flow of felicity generated by direct consumption, as described for example by Samuelson (1937). This tradition is incompatible with the idea that happiness is extracted not only from the immediate consumption of goods and services, but also from thoughts. This is particularly the case for thoughts related to savoring the possibility of future pleasant events, or to fearing the consequences of adverse ones. Anticipatory feelings have been incorporated in preferences by Caplin and Leahy (2001) who considered belief-dependent felicity functions. In the economic literature, Akerlof and Dickens (1982) were the first to assume that subjective beliefs are derived from a welfare-maximizing process.

The distortion of beliefs affects the individual decision process in a complex manner. There is an important literature on the effect of a change in the perceived distribution of risk on the optimal exposure to it. In the case of the one-riskfree-one-risky portfolio choice problem that we examine in this paper, Rothschild and Stiglitz (1971) have shown that a mean-preserving spread in the distribution of returns of the risky asset does not necessarily reduce the demand for the risky asset. In the same fashion, Fishburn and Porter (1976) have shown that a first-order stochastically favorable shift in this distribution can reduce the demand for the risky asset by some risk-averse investors. Gollier (1995) characterizes the stochastic dominance order that yields a reduction of the demand for the risky asset by all risk-averse agents. More recently, Abel (2002) considered the effect of distorted beliefs on the equilibrium asset prices. Abel defined optimism by using very specific first-order stochastic dominant shifts in the subjective distribution of the risky asset's payoffs. He showed that optimism raises the demand for this asset by all risk-averse investors, thereby reducing the equity premium. This observation is particularly important in our framework as we will show that risk-averse agents optimally distort the distribution of the risky asset in an optimistic way.

Our model is a two-date version of the dynamic model examined by Brunnermeier and Parker (2003), hereafter denoted BP. We assume that the consumer's lifetime utility is a weighted sum of the date-1 felicity extracted from savoring and of the date-2 felicity of consumption. The weight measures the intensity of anticipatory feelings, anxiety and savoring. This parameter can take any value between 0 and 1, whereas BP only consider the special case with equal weights. This will allow us to explore the effect of increasing anticipatory feelings on optimal beliefs and on the demand for the risky asset. Assuming without loss of generality that the objective expected excess return of the risky asset is positive, any risk-averse investor with a zero intensity of anticipatory feeling will have a positive demand α^* for the risky asset. One of the main results of BP is to show that risk-averse investors with anticipatory feelings will always distort beliefs, and that they will do so in such a way as to either increase their demand of the risky asset above α^*, or to go short on the risky asset. In this paper, we provide an in-depth description of the optimal subjective distribution of beliefs, and we show that it is not optimal to go short on the risky asset.

We first exploit the linearity of expected utility with respect to state probabilities to prove that the optimal subjective probability distribution must be degenerate with at most two atoms, i.e., optimal beliefs are binary. This result is true for any von Neumann-Morgenstern preference functional, any intensity of anticipatory feelings, and any objective distribution of the risky asset. In a second step, we show under weak restrictions on the utility function that investors select the two atoms that are at the bounds of the set of possible asset returns. In other words, optimally controlling thoughts leads the individual to believe that only the smallest possible return and the largest possible return can have a positive probability to occur. This strong result is compatible with the idea introduced by Tversky and Kahneman (1992) that the worst and best outcomes receive particular attention from decision makers. Cumulative prospect theory takes this into account by assuming an inverse S-shaped transformation function of the objective cumulative distribution function. This is equivalent to transferring the probability mass from the interior of the support of the distribution to its lower and upper bounds.[2] We claim that the transformation of probabilities described in cumulative prospect theory, rather than being a genetic characteristic of human beings, corresponds to a natural tendency of rational agents to optimize their intertemporal welfare.

Given the fact that optimal beliefs are degenerate at the extreme events, the only remaining problem is to determine the subjective probability of the best state. When the intensity of anticipatory feelings is small, we show that the demand for the risky asset is larger than the demand that is optimal under the objective distribution of excess returns. Thus, we eliminate the possibility allowed by BP that risk-averse investors go short on the risky asset. Moreover, we show that the optimal subjective probability of the large return and the demand for the risky asset are increasing in the intensity of anticipatory feelings.

Things are more complex when we allow for larger intensities of anticipatory feelings. Because the maximum is a convex operator, the maximum subjective expected utility of the investor is a convex function of his subjective probability distribution. For example, this explains why the value of information is always positive, or why refining the information structure à la Blackwell (1951) makes the decision-maker better off. The convexity of the felicity extracted from anticipatory feelings with respect to the subjective probability distribution alerts us about an important difficulty in the selection of optimal beliefs, since the objective function does not need anymore to be concave in the decision variables. In the extreme case where only anticipatory feelings matter for lifetime well-being, optimal beliefs degenerate to subjective certainty at either the worse or the best possible return, yielding an infinite demand for the risky asset and unbounded well-being. When the intensity of anticipatory feelings is smaller than unity, the Inada assumption that marginal utility tends to infinity when consumption tends to zero guarantees that the actual demand for the risky asset will be small enough to yield positive consumption in all states with a positive objective probability. This implies that optimal beliefs cannot degenerate to certainty.

[2]For more details, see for example Tversky and Wakker (1995) and Abdellaoui (2000).

2 The model

Our model is static, with a decision date $t = 0$ and a consumption date $t = 1$. At date 0, the consumer selects an asset portfolio. The portfolio is liquidated at date 1, and its value is consumed. We consider an economy with two assets. The first asset is riskfree and yields a return that is normalized to 0 over the period. The second asset is risky. It yields a random excess return $\widetilde{x}$ at date 1. It is assumed that the excess return of the risky asset is bounded below by $a < 0$ and above by $b > 0$. There is an objective cumulative probability distribution $Q \in X[a, b]$ for $\widetilde{x}$. $X[a, b]$ denotes the set of cumulative distribution functions whose support is in $[a, b]$:

$$X[a,b] = \left\{ F : [a,b] \to [0,1] \mid dF(x) \geq 0 \ \forall x \in [a,b], \quad \int_a^b dF(x) = 1 \right\}$$

The consumer has a von Neumann-Morgenstern utility function u that is assumed to be twice differentiable, increasing and concave. We assume that the Inada conditions are satisfied, with $\lim_{c \to 0_+} u'(c) = +\infty$ and $\lim_{c \to \infty} u'(c) = 0$. The decision problem of the agent at date $t = 0$ is to determine the size α of his investment in the risky asset. Because his initial wealth is w_0, he invests the remaining $w_0 - \alpha$ in the riskfree asset. His final wealth at date 1 is therefore equal to $w_0 + \alpha\widetilde{x}$. At decision date $t = 0$, the beliefs of the consumer are characterized by a subjective cumulative probability distribution $P \in X[a, b]$ that may differ from the objective probability distribution Q. Given these beliefs P, the consumer selects the portfolio $(\alpha, w_0 - \alpha)$ that maximizes his subjective future expected utility on consumption. We obtain the following decision problem:

$$S(P) = \max_{\alpha} \quad E_P \left[u(w_0 + \alpha\widetilde{x}) \right] = \int_a^b u(w_0 + \alpha x) dP(x). \tag{1}$$

The expectation operator E_P refers to the subjective probability distribution P. $S(P)$ measures the felicity at date $t = 0$ generated by anticipatory feelings. The optimal demand for the risky asset as a function of the beliefs is denoted $\alpha(P)$. It satisfies the following first-order condition:

$$E_P \left[\widetilde{x} u'(w_0 + \alpha(P)\widetilde{x}) \right] = 0. \tag{2}$$

Because $E_P \left[u(w_0 + \alpha\widetilde{x}) \right]$ is concave in α, this first-order condition is necessary and sufficient for optimality. By the Inada condition, it must be true that $w_0 + \alpha(P)x > 0$ for all x with a positive subjective probability $dP(x)$.

Because of the potential bias in the subjective beliefs, the objective expected utility of the consumer at date 1 may differ from $S(P)$. The objective expected utility of a consumer with subjective beliefs P equals

$$O(P) = E_Q\left[u(w_0 + \alpha(P)\widetilde{x})\right] = \int_a^b u(w_0 + \alpha(P)x)dQ(x). \tag{3}$$

It is important to observe that the consumer's objective expected utility depends upon the subjective probability distribution P only through the choice of the portfolio allocation induced by P.

We now specify the lifetime well-being of the consumer with subjective beliefs P. At date $t = 0$, the consumer savors his subjective future utility, yielding savoring felicity $S(P)$ at that date. At date $t = 1$, the agent extracts felicity $O(P)$ from consuming his terminal wealth. His lifetime well-being W is assumed to be a convex combination of his felicity at these two dates:

$$W(P) = kS(P) + (1-k)O(P). \tag{4}$$

Parameter $k \in [0, 1]$ measures the intensity of anticipatory feelings in lifetime utility. When $k = 0$, the consumer has no anticipatory feeling at date 0. When $k = 1$, he extracts felicity just from savoring future consumption flows. Brunnermeier and Parker (2003) consider the special case with $k = 1/2$.

As justified in the introduction, we assume that prior to date $t = 0$, the agent controls his thoughts. He selects the beliefs P that maximizes his lifetime well-being:

$$P^* = \arg\max_{P \in X[a,b]} W(P). \tag{5}$$

The optimal demand for the risky asset is $\alpha^* = \alpha(P^*)$. The main objective of the paper is to compare P^* to Q, and α^* to $\alpha(Q)$.

3 Some basic properties of optimal beliefs

As stated before, date-1 felicity depends upon beliefs P only through its effect on the choice of the optimal portfolio $\alpha = \alpha(P)$ at date $t = 0$. In general, there are more than one probability distribution that yield the same optimal portfolio α. Let $B(\alpha) \subset X[a, b]$ be the set of subjective cumulative probability distributions that yield the same optimal portfolio choice α:

$$B(\alpha) = \left\{P \in X[a, b] \mid \alpha(P) = \alpha\right\}. \tag{6}$$

It implies that $O(P) = O(P')$ for all P and P' in $B(\alpha)$.

This observation has an important consequence on the structure of optimal beliefs. Consider the optimal demand $\alpha^* = \alpha(P^*)$ that is induced by the optimal subjective beliefs P^*. From the various subjective probability distributions P that yield this demand α^*, the one that is selected by the consumer prior to date 0 must maximize the date-0 anticipatory felicity $S(P)$, since they all yield the same date-1 felicity $O(P^*)$. In other words, it must be true that

$$P^* \in \arg\max_{P \in B(\alpha^*)} S(P). \tag{7}$$

Observe that this property of optimal beliefs holds independent of the characteristics of the objective probability distribution Q. It allows us to derive the following useful properties of optimal beliefs.

3.1 Optimal beliefs must be binary

Proposition 1 *The optimal subjective probability distribution P^* has at most two atoms: $\exists(x_-, x_+) \in [a, 0] \times [0, b]$ such that $dP^*(x) = 0$ for all $x \in [a, b]$ except at x_- and x_+.*

Proof: We can rewrite problem (7) as follows:

$$dP^* \in \arg\max_{dP} \quad \int_a^b u(w_0 + \alpha^* x) dP(x) \tag{8}$$

$$\begin{aligned} s.t. \quad \int_a^b x u'(w_0 + \alpha^* x) dP(x) &= 0 \\ \int_a^b dP(x) &= 1 \\ dP(x) &\geq 0 \quad \forall x \in [a, b]. \end{aligned}$$

The first constraint states that P belongs to $B(\alpha^*)$, i.e., that beliefs P yield the optimal risk exposure α^*. The other two constraints define a cumulative probability distribution with support in $[a, b]$. Because the feasible set is compact, this problem has a solution. Observe that the above program is a linear programming problem on a compact set with two equality constraints.

As is well known, its solution has at most two atoms.[3] In order to satisfy the first-order condition, it must be that x_- and x_+ alternate in sign. ■

Thus, we conclude from this proposition that the optimal subjective beliefs take the form $P^* = (x_-, 1-p^*; x_{+,}p^*)$ for some pair (x_-, x_+) and some scalar p^* such that $a \leq x_- < 0 < x_+ \leq b$ and $p^* \in [0,1]$. Beliefs are linked to the optimal risk exposure α^* by the following rewriting of the first-order condition:

$$p^* x_+ u'(w_0 + \alpha^* x_+) + (1-p^*) x_- u'(w_0 + \alpha^* x_-) = 0 \tag{9}$$

Proposition 1 is useful because it replaces the problem of finding a probability distribution in the infinite dimensional space $X[a,b]$ into a problem of finding a triplet (x_-, x_+, p) that maximizes $W(P)$. From the technique presented above, we can easily derive the following property of optimal beliefs: when there are n independent assets in the economy, there must be at most n states with a positive optimal subjective probability.

This result is quite robust. It is independent of individual preferences. Throughout this paper, we assume that beliefs can be distorted without any constraint, so that the only constraints to subjective probabilities are that $dP(x) \geq 0$ for all x in $[a,b]$. A more realistic would be to assume that only a proportion $1-\kappa$ of the probability mass can be shifted. The logic of the above proposition can easily be extended to this model where, for all x, $dP(x)$ is constrained to be larger or equal to $\kappa dQ(x)$ rather than to zero as in our model. Under this constrained model, the optimal beliefs must be such that $dP^*(x) = \kappa dQ(x)$ for all x except for at most two values of x. Notice finally that the assumption that instantaneous welfare can be measured by an expected utility functional is essential to derive this class of results.

3.2 Only the extreme returns may have a positive subjective probability

In this section, we first show that at least one of the two subjectively possible returns must be at the bounds of interval $[a,b]$. We define $A(z) = -u''(z)/u'(z)$ as the Arrow-Pratt index of absolute risk aversion.

[3] See Gollier and Kimball (1997) and Gollier (2001, chapter 6) for a proof and other applications of this central result.

Proposition 2 *The optimal subjective distribution $P^* = (x_-, 1-p^*; x_+, p^*) \in X[a,b]$ is such that either $x_- = a$ or $x_+ = b$.*

Proof: Suppose by contradiction that $x_- > a$ and $x_+ < b$. Consider a marginal change in P such that a marginal increase in x_+ is compensated by a marginal reduction in x_- in such a way that α^* is unaffected. Totally differentiating condition (9) yields

$$\left.\frac{dx_-}{dx_+}\right|_{\alpha^*} = -\frac{p^* u'(w_0 + \alpha^* x_+)}{(1-p^*)u'(w_0 + \alpha^* x_-)} \frac{1 - \alpha^* x_+ A(w_0 + \alpha^* x_+)}{1 - \alpha^* x_- A(w_0 + \alpha^* x_-)}.$$

The subjective expected utility using optimal beliefs equals

$$S = p^* u(w_0 + \alpha^* x_+) + (1-p^*)u(w_0 + \alpha^* x_-). \tag{10}$$

Totally differentiating this equality yields

$$\left.\frac{dS}{dx_+}\right|_{\alpha^*} = p^* \alpha^* u'(w_0 + \alpha^* x_+) + (1-p^*)\alpha^* u'(w_0 + \alpha^* x_-) \left.\frac{dx_-}{dx_+}\right|_{\alpha^*},$$

or equivalently,

$$\left.\frac{dS}{dx_+}\right|_{\alpha^*} = p^* \alpha^{*2} u'(w_0 + \alpha^* x_+) \frac{x_+ A(w_0 + \alpha^* x_+) - x_- A(w_0 + \alpha^* x_-)}{1 - \alpha^* x_- A(w_0 + \alpha^* x_-)}.$$

Because $x_- < 0 < x_+$ and $A(.) > 0$, this is unambiguously positive. This change in beliefs increases the lifetime well-being of the consumer, which is a contradiction. ■

This result states that at least one of the two possible returns must be an extreme return a or b. In the next proposition, we claim that the two subjectively possible returns are extreme under some mild additional assumptions on the utility function. Let $R(z) = zA(z) = -zu''(z)/u'(z)$ be the relative risk aversion of u at z. It is weakly increasing if $R'(z)$ is non-negative for all $z > 0$.

Proposition 3 *Suppose that absolute risk aversion is strictly decreasing (DARA) and that relative risk aversion is weakly increasing (IRRA). Then, the optimal subjective distribution of returns has support $\{a, b\}$: $\exists p^* \in [0,1]$ such that $P^* = (a, 1-p^*; b, p^*)$.*

Proof: Suppose by contradiction that $x_- > a$ or $x_+ < b$. Suppose for example that x_+ is less than b. We consider a marginal increase in x_+ that is compensated by a change in p^* in such a way that α^* be unaffected by the change. Totally differentiating the definition of subjective utility using optimal beliefs, we have that

$$\left.\frac{dp^*}{dx_+}\right|_{\alpha^*} = -\frac{p^* u'(w_0 + \alpha^* x_+)\left[1 - \alpha^* x_+ A(w_0 + \alpha^* x_+)\right]}{x_+ u'(w_0 + \alpha^* x_+) - x_- u'(w_0 + \alpha^* x_-)}. \tag{11}$$

By definition of the subjective expected utility, we have that

$$\left.\frac{dS}{dx_+}\right|_{\alpha^*} = p^* \alpha^* u'(w_0 + \alpha^* x_+) + \left[u(w_0 + \alpha^* x_+) - u(w_0 + \alpha^* x_-)\right] \left.\frac{dp^*}{dx_+}\right|_{\alpha^*}.$$

Using (11), dS/dx_+ is positive if

$$\begin{aligned} K(x_+, x_-) &= \alpha^* x_+ u'(w_0 + \alpha^* x_+) - \alpha^* x_- u'(w_0 + \alpha^* x_-) \\ &\quad - \left[1 - \alpha^* x_+ A(w_0 + \alpha^* x_+)\right]\left[u(w_0 + \alpha^* x_+) - u(w_0 + \alpha^* x_-)\right] \end{aligned}$$

is positive. Observe that, by risk aversion,

$$K(0, x_-) = u(w_0 + \alpha^* x_-) - \alpha^* x_- u'(w_0 + \alpha^* x_-) - u(w_0)$$

is positive for all x_-. Notice also that

$$\begin{aligned} \left.\frac{dK}{dx_+}\right|_{(x_+, x_-)} &= \alpha^* \left[u(w_0 + \alpha^* x_+) - u(w_0 + \alpha^* x_-)\right]\left[A(w_0 + \alpha^* x_+) + \alpha^* x_+ A'(w_0 + \alpha^* x_+)\right] \\ &= \alpha^* \left[u(w_0 + \alpha^* x_+) - u(w_0 + \alpha^* x_-)\right]\left[R'(w_0 + \alpha^* x_+) - w_0 A'(w_0 + \alpha^* x_+)\right]. \end{aligned}$$

We show that the right-hand side of this equality is positive. Obviously, $\alpha^* \left[u(w_0 + \alpha^* x_+) - u(w_0 + \alpha^* x_-)\right]$ is positive. The second bracketed term in the right-hand side of the above equality is also positive since, by assumption, R' is non-negative and A' is negative. We conclude that K is positive for all nonnegative x_+. Therefore, this change in beliefs raises the lifetime well-being of the decision maker, a contradiction. A parallel proof can be made when x_- is larger than a. ■

The familiar set of power utility functions $u(z) = z^{1-\gamma}/(1-\gamma)$ exhibits constant relative risk aversion and decreasing absolute risk aversion. Therefore, it satisfies the condition of the above proposition. More generally, decreasing absolute risk aversion is commonly accepted by the profession as a reasonable assumption. Non-decreasing relative risk aversion is compatible with the observation that, conditional on holding a portfolio, wealthier consumers invest a smaller share of their wealth in stocks.[4]

In the remainder of the paper, we will assume that the optimal subjective probability distribution is of the form $(a, 1-p^*; b, p^*)$, where p^* denotes the probability of the state with the highest possible return $x = b$. This probability p^* is the only remaining degree of freedom to be determined. It depends upon the objective function Q and the utility function u.

3.3 Link with Prospect Theory

In Rank-Dependent Expected Utility (RDEU, Quiggin (1982)) and in Cumulative Prospect Theory (CPT, Tversky and Kahneman (1992)), it is assumed that agents maximize their expected utility by using a (subjective) probability distribution P that is a non-linear function f of the objective function Q: $P(x) = f(Q(x))$. By definition, it must be that $f(0) = 0$ and $f(1) = 1$. These theories make the assumption that the weighting function f is intrinsic to the preferences of the agents, exactly as is the utility function u. Together with Brunnermeier and Parker (2003), we take a different road by assuming that agents endogenously select the weighting function f in order to maximize their lifetime utility. At this stage, it is thus interesting to see whether the estimated function f of RDEU and CPT exhibits properties that are shared by the optimal beliefs characterized in this paper.

CPT is consistent with the psychological principle of diminishing sensitivity, the two endpoints of the support of the distribution of returns serving as reference points. It has been observed that increments near these endpoints have more impact than increments in the middle of the support. The transformation function estimated by Tversky and Kahneman (1992) in the gain domain, as depicted by the smooth inverse S-shaped curve in Figure 1, satisfies this property.[5] Compared to the 45° line corresponding to Expected

[4] See for example Guiso, Jappelli and Terlizzese (1996).

[5] Tversky and Kahneman (1992) used the following specification:

$$r(q) = 1 - \frac{(1-q)^\gamma}{\left[(1-q)^\gamma + q^\gamma\right]^{1/\gamma}}.$$

They estimated γ to be equal to 0.61 (as in Figure 1) . Abdellaoui (2000) obtained similar results by using a parameter-free method to elicitate the weigthing function.

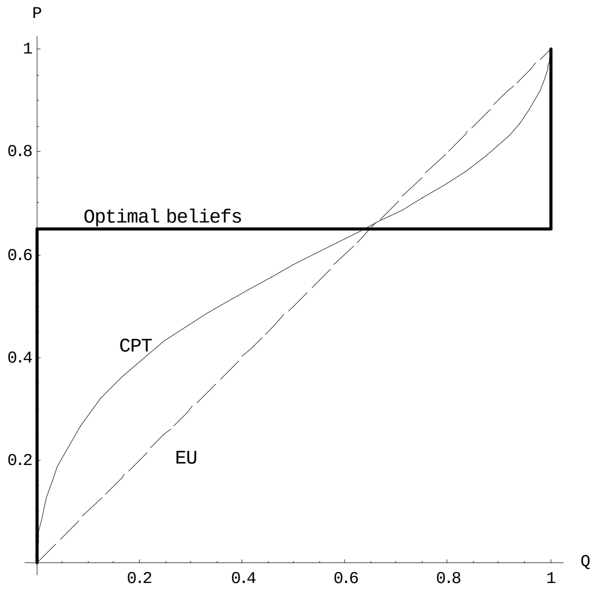

Figure 1: Typical transformation function of the cumulative probability function Q.

Utility, we see that the probability transformation function in CPT transfers much of the probability mass from the center of the support to the two extreme possible returns a and b. The optimal beliefs in our model push this kind of transformations to the limit by transferring the entire probability mass from the interior of the support to its two endpoints. This corresponds to the thick stepwise curve in Figure 1. Therefore, we claim that the transformation of probabilities described in Prospect Theory, rather than being a genetic characteristic of human beings, corresponds to a natural tendency of rational agents to optimize their intertemporal welfare.

4 Selection of the degree of optimism

We now turn to the choice of the subjective probability p of the highest possible return $x = b$. Using an intuitive shortcut in notation, let us define $S(p)$ and $O(p)$ respectively as the subjective expected utility

$$S(p) = pu(w_0 + \alpha(p)b) + (1-p)u(w_0 + \alpha(p)a),$$

and objective expected utility

$$O(p) = E_Q\left[u(w_0 + \alpha(p)\widetilde{x})\right],$$

as functions of p, where $\alpha(p)$ is the unique root of the following equation:

$$pbu'(w_0 + \alpha(p)b) + (1-p)au'(w_0 + \alpha(p)a) = 0. \qquad (12)$$

We can rewrite the problem of selecting subjective beliefs as

$$p^* \in \arg\max_p W(p;k) = kS(p) + (1-k)O(p).$$

Before proceeding to characterize the optimal subjective probability of the high state, it is useful to determine the effect of an increase in this probability on the optimal demand for the risky asset. In the next lemma, we show that an increase in the subjective probability of the high-return state raises the demand for the risky asset.

Lemma 1 *The demand for the risky asset is increasing in the subjective probability of the high-return state:* $\partial\alpha/\partial p \geq 0$.

Proof: Totally differentiating condition (12) yields

$$\frac{d\alpha}{dp} = \frac{au'(w_0 + \alpha a) - bu'(w_0 + \alpha b)}{pb^2 u''(w_0 + \alpha b) + (1 - p)a^2 u''(w_0 + \alpha a)}. \tag{13}$$

Both the numerator and the denominator are negative, which implies that $\partial\alpha/\partial p$ is positive. ■

This result is linked to the literature on the relationship between the probability distribution of returns and the optimal demand for the risky asset. Gollier (1995) provides the necessary and sufficient condition on a change in distribution to raise the demand for the risky asset by all risk-averse investors. The change in distribution considered in Lemma 1 is a special case of a stochastic order named monotone probability ratio order by Eeckhoudt and Gollier (1995) and Athey (2002).

4.1 Optimism is optimal when the intensity of anticipatory feelings is small

In general, the optimal beliefs depend upon the intensity of anticipatory feelings k. Let $p^*(k)$ denote the optimal subjective probability of the highest return as a function of k. In this section, we explore the special case of small intensities k of anticipatory feelings.

The benchmark case with no anticipatory feelings is easy to characterize. When k vanishes, the lifetime well-being $W(p; k = 0)$ equals the objective expected utility $O(p) = E_Q\left[u(w_0 + \alpha(p)\widetilde{x})\right]$. Because $E_Q\left[u(w_0 + \alpha\widetilde{x})\right]$ is concave in α, and because α is increasing in the subjective probability p of the high state as stated in Lemma 1, O is single-peaked in p. In spite of the fact that the agent has no anticipatory feelings, he still forms beliefs in such a way to satisfy (12). It is obvious in this case that the agent selects the subjective probability $p^*(0) = p_0^*$ yielding the demand for the risky asset that is optimal for the objective probability distribution:

$$p_0^* bu'(w_0 + \alpha(Q)b) + (1 - p_0^*)au'(w_0 + \alpha(Q)a) = 0. \tag{14}$$

We now examine the impact of introducing a small degree k of anticipatory feelings on the optimal subjective probability $p^*(k)$ of the high-return state. Because $W(p; k = 0) = O(p)$ is single-peaked in p, a simple continuity argument implies that $W(p; k)$ is also single-peaked in p for small values of k.[6] This implies that the first-order condition for p^* is necessary and sufficient when k is small. This first-order condition is written as

$$\begin{aligned} 0 &= \frac{\partial W}{\partial p}(p^*;k) = k\frac{\partial E_P u(w_0+\alpha\widetilde{x})}{\partial \alpha}\frac{d\alpha}{dp} \\ &\quad + k\left[u(w_0+\alpha b) - u(w_0+\alpha a)\right] \\ &\quad + (1-k)\frac{\partial E_Q u(w_0+\alpha\widetilde{x})}{\partial \alpha}\frac{\partial \alpha}{\partial p}. \end{aligned}$$

Because α maximizes $E_P\left[u(w_0+\alpha\widetilde{x})\right]$, the first term in the right-hand side of this equality is zero. Using equation (13), we can thus rewrite the first-order condition as follows:

$$0 = \frac{\partial W}{\partial p}(p^*;k) = k\left[u(w_0+\alpha b) - u(w_0+\alpha a)\right] \tag{15}$$

$$-(1-k)\frac{\left[bu'(w_0+\alpha b) - au'(w_0+\alpha a)\right] E_Q\widetilde{x}u'(w_0+\alpha\widetilde{x})}{p^*b^2u''(w_0+\alpha(p)b) + (1-p^*)a^2u''(w_0+\alpha(p)a)}. \tag{16}$$

When $k=0$, we verify that this condition simplifies to $E_Q\widetilde{x}u'(w_0+\alpha\widetilde{x}) = 0$, which is true only if $\alpha = \alpha(Q)$. This yields in turn $p^* = p_0^*$ as defined by (14). Because W is locally concave in p around p_0^*, the optimal subjective probability p^* is increasing in k around $k=0$ if and only if the cross-derivative of W is positive when evaluated at $(p_0^*; k=0)$. It is easy to show that

$$\frac{\partial^2 W}{\partial p \partial k}(p_0^*;0) = u(w_0+\alpha(Q)b) - u(w_0+\alpha(Q)a).$$

The right-hand side of this equality has the same sign as $\alpha(Q)$. Thus the sign of $\partial p^*/\partial k$ has the same sign as $\alpha(Q)$. Combining this result with Lemma 1 yields the next proposition. It relies on the degree of optimism which can be measured by the difference between the subjective probability and the objective probability of the state that is more favorable to the agent's wealth. When $\alpha(Q)$ is positive, that is, when $E_Q\widetilde{x}$ is positive, the favorable state is the high return state, and an increase in p represents an increase in

[6] The analysis of the solution is more complex when the intensity of anticipatory feelings is larger. Indeed, the objective function $W = kS + (1-k)O$ is usually not concave in p. This is due to the fact that $S(p)$ is the maximum of a sum of linear functions of p. Therefore, $S(p)$ is a convex function of the subjective probability p of the high return.

optimism. When $\alpha(Q)$ is negative, the investor goes short on the risky asset, and the favorable state is the low return state. The degree of optimism is inversely related to p in that case.

Proposition 4 *Introducing small anticipatory feelings in the lifetime objective function of the consumer makes him more optimistic about his portfolio return:*

$$\alpha(Q) \left.\frac{dp^*}{dk}\right|_{k=0} \geq 0.$$

Moreover, it raises the optimal portfolio risk:

$$\alpha(Q) \left.\frac{d\alpha(p^*)}{dk}\right|_{k=0} \geq 0.$$

These inequalities are strict when the objective expected return $E_Q\widetilde{x}$ is not zero.

The intuition of this result is simple. Suppose that the objective expected return is positive, so that the optimal demand $\alpha(Q)$ for the risky asset is positive when there is no anticipatory feeling. It is sustained by the beliefs that the probability of the high return b is p_0^*. Consider a marginal increase in the subjective probability of that state. It marginally increases the demand for the risky asset. But, by the envelope theorem, this marginal increase in demand has no effect on the objective expected utility. To the contrary, it increases the subjective expected utility. Globally, when $k > 0$, it raises the lifetime well-being. This argument cannot be extended to consumers having a larger intensity of anticipatory feelings. Indeed, in this case, a marginal change in the subjective probability distribution would have an effect on the objective expected utility.

In Figures 2 and 3, we provide a numerical estimation of the problem in order to illustrate Proposition 4. We assume that the agent has a power utility function with constant relative risk aversion $\gamma = 3$. Moreover, the worst possible return is $a = -100\%$, whereas the best possible return is $b = +150\%$. The objective probability distribution is $Q \sim (-1, 1/2; +1.5, 1/2)$, yielding a positive expected excess return. In Figure 2, we have drawn the optimal subjective probability of the high return as a function of the intensity k of anticipatory feelings. In Figure 3, we depicted the relationship between k and the optimal share of wealth invested in the risky asset. As stated in

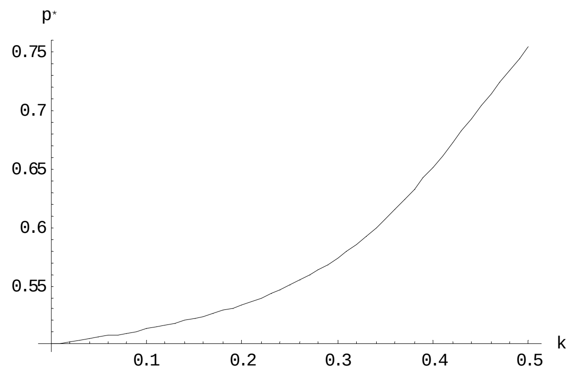

Figure 2: Optimal probability of the high return state, as a function of the intensity of anticipatory feelings. Parameter values: $\gamma = 3$, $Q \sim (-1, 1/2; +1.5, 1/2)$.

Proposition 4, we get upward sloping curves. When there is no anticipatory feeling, the optimal share of wealth invested in the risky asset equals 5.5%. When anticipatory feelings count as much as the objective future felicity ($k = 1/2$), this optimal share goes up to 21.0%.

4.2 The effect of risk aversion on optimal optimism

We now determine the impact of risk aversion on the optimal subjective probability distribution. In particular, we want to know whether less risk-averse agents are more optimistic. To explore this question, suppose first that $\mid \alpha \mid$ is small. It implies that we can approximate $u'(w_0 + \alpha x)$ by $u'(w_0) + \alpha x u''(w_0)$, which is equal to $u'(w_0)(1 - \alpha x A_0)$, where $A_0 = A(w_0)$. First-order condition (12) is thus approximated as

$$[pb + (1-p)a] - \alpha A_0 \left[pb^2 + (1-p)a^2\right] \simeq 0,$$

which implies that

$$\alpha(p) \simeq \frac{1}{A_0} \frac{pb + (1-p)a}{pb^2 + (1-p)a^2}. \tag{17}$$

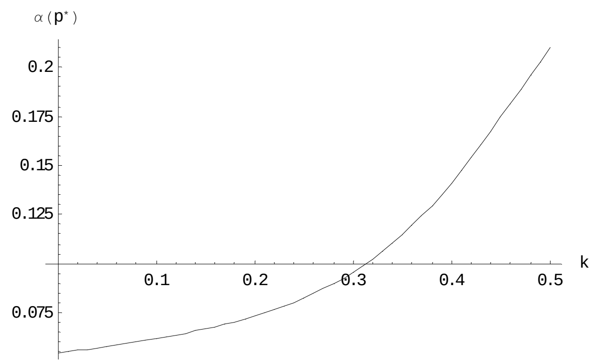

Figure 3: The demand for the risky asset, as a function of the intensity of anticipatory feelings. Parameter values: $\gamma = 3$, $Q \sim (-1, 1/2; +1, 1/2)$.

Using second-order Taylor approximations for $u(w_0 + \alpha x)$ yields in turn that

$$S(p) \simeq u(w_0) + 0.5 \frac{u'(w_0)}{A(w_0)} \frac{[pb + (1-p)a]^2}{pb^2 + (1-p)a^2}.$$

Let $m_i = E_Q \widetilde{x}^i$ denote the objective moment of order i of $\widetilde{x}$. Using again second-order Taylor approximations yields

$$O(p) \simeq u(w_0) + 0.5 \frac{u'(w_0)}{A(w_0)} \frac{pb + (1-p)a}{pb^2 + (1-p)a^2} \left[2m_1 - \frac{pb + (1-p)a}{pb^2 + (1-p)a^2} m_2 \right].$$

Combining these two observations implies that

$$W(p) \simeq u(w_0) + 0.5 \frac{u'(w_0)}{A(w_0)} F(p), \tag{18}$$

with

$$F(p) = \frac{pb + (1-p)a}{pb^2 + (1-p)a^2} \left\{ \frac{pb + (1-p)a}{pb^2 + (1-p)a^2} \left(k \left[pb^2 + (1-p)a^2 \right] - (1-k)m_2 \right) + 2(1-k)m_1 \right\}. \tag{19}$$

It is noteworthy that this approximation is exact when u is quadratic. We thus obtain the following interesting insight.

Proposition 5 *When u is quadratic in the relevant domain of wealth, the optimal subjective probability is independent of the consumer's attitude towards risk. It maximizes function F defined by (19), where m_1 and m_2 are the objective first two moments of the excess return of the risky asset.*

The first-order condition associated to the maximization of $F(p)$ is equivalent to finding the roots of a third-degree polynomial. We check that in the special case with no anticipatory feeling ($k = 0$), F is concave in p with a maximum p_0^* such that

$$\frac{p_0^* b + (1 - p_0^*)a}{p_0^* b^2 + (1 - p_0^*)a^2} = \frac{m_1}{m_2}.$$

This means that the subjective probability p_0^* is selected in such a way that the objective and subjective Sharpe ratios be the same. It yields the same optimal portfolio as the one that is optimal under rational expectations.

An important question is to determine whether the heterogeneity in risk aversion can explain the heterogeneity of subjective beliefs in the population. When preferences belong to the quadratic class, the optimal subjective probability distribution is independent of the degree of risk aversion of the investor. When the utility function is not quadratic, optimal beliefs are generally not independent of risk preferences. Brunnermeier and Parker (2003) conclude that the heterogeneity of risk aversion in the population could explain the heterogeneity of subjective beliefs. However, because smooth functions can always be well approximated by a quadratic utility function in a small domain, we should not expect to generate a lot of heterogeneity on beliefs in an economy with small portfolio risks at equilibrium. The assumption of small portfolio risks is compatible with the general tone of the literature on the equity premium puzzle. The puzzle is based on the observation that actual portfolio risks are very small compared to the optimal risk computed on the basis of the large objective risk premium on financial markets. We illustrate the low sensitivity of optimal beliefs to changes in risk aversion by considering again the numerical example used above. We examine in particular the effect of a change in the relative risk aversion γ on the optimal subjective probability of the high state. This relationship is described in Figure 4. The most striking aspect of this figure is the range of the vertical axis: as relative risk aversion varies from 0.5 to 10, the optimal subjective probability of the high state varies within the interval $[0.5126, 0.5131]$!

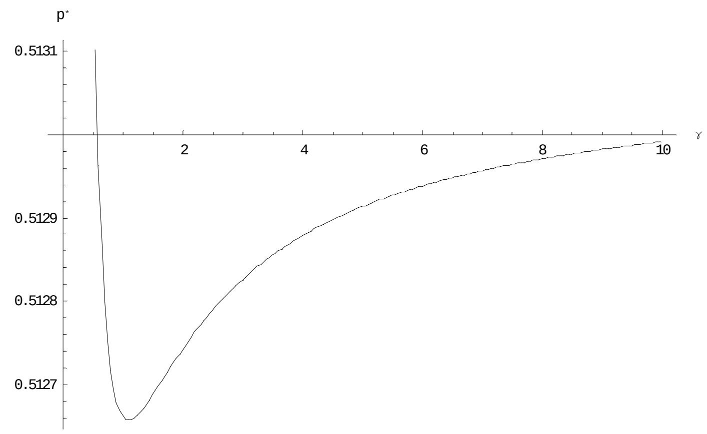

Figure 4: The impact of risk aversion on optimal beliefs. Parameter values: $Q \sim (-1, 1/2; +1.5, 1/2)$, $k = 0.1$.

5 Concluding remarks

We have shown that the selection of optimal beliefs in the one-riskfree-one-risky-asset portfolio problem is governed by very precise rules. First, we have shown that these beliefs must be degenerate at the worst and best possible returns. This is compatible with the observation that subjects in experimental studies tend to distort probabilities in favor of extreme events, as suggested for example by the cumulative prospect theory. Second, when the intensity of anticipatory feelings is small, the problem of selecting beliefs is well-behaved (single-peaked), yielding a unique optimal subjective probability distribution. Except in the case of a zero objective expected excess return, these optimal beliefs always yield an increase in the optimal risk exposure when compared to the one that is optimal under the objective probability distribution.

Moreover, investors with a larger intensity of anticipatory feelings have a larger subjective probability of the good state together with a larger optimal risk exposure. Because the mental process of distorting beliefs in favor of savoring the prospect of large capital gains, the induced optimism of investors will not be helpful to solve the equity premium puzzle, quite the contrary. When the optimal portfolio risk is small, we also showed that optimal beliefs are almost insensitive to the degree of risk aversion of the investor.

This work calls for further investigation in several directions. First, it would be interesting to examine a more general model in which more risk-taking opportunities are available. This would be useful in order to examine the effect of anticipatory feelings on the optimal diversification of individual asset portfolios. Second, the current model does not take into account the adverse effect of disappointment of the optimally optimistic investors when they will eventually be forced to recognize the objective performance of their asset portfolio. Third, this work suggests that delegating the selection of the individual asset portfolios to an independent agent can be efficient. This would neutralize the negative effect on portfolio choices of distorting individual beliefs.

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