Labor Adjustment Costs and Complex Eigenvalues

Xavier Fairise

University of Angers, GEAPE and EUREQua

Patrick Fève*

University of Toulouse (GREMAQ and IDEI)

Abstract

Aggregate fluctuations display both persistence and damped oscillations in response to transitory shocks. The standard Real Business Cycle model cannot explain these patterns, because its stable eigenvalues are positive and real. We demonstrate that this model with labor adjustment costs can yield complex eigenvalues. Numerical experiments illustrate this results but they suggest that the imaginary part of the eigenvalues remains insufficiently large compared to the real one. However, the paper shows that labor adjustment costs can potentially improve the dynamic properties of a standard RBC model.

Keywords : Labor adjustment costs, Business cycle model, Complex eigenvalues.

J.E.L. classification : C12, C52

*Address: Gremaq, Université de Toulouse I, Manufacture des Tabacs, Bâtiment F, 2ème étage, 21 Allée de Brienne, 31 000 Toulouse, France. e-mail: patrick.feve@univ-tlse1.fr. We thank P. Beaudry, J.P. Benassy, F. Collard, P. Flaschel, P.Y. Hénin, T. Kollintzas, F. Langot, F. Portier and W. Zemmler for fruitful comments. An earlier version of this paper has benefited from discussions at the workshop *Human Capital Mobility* (Louvain-la-Neuve, sept. 1996), T2M Conference (Montreal, may 1999), SED conference (Alghero ,june 1999), ESEM99 (Santiago, august 1999), GEAPE seminar (Angers, december 1999) and *"Microstructures, Macrodynamics and Macroeconometrics"* workshop (Bielefeld, january 2000). The usual disclaimers apply.

Introduction

Aggregate output fluctuations and related measures of economic activity display both persistence and damped oscillations in response to transitory shocks. For Azariadis, Bullard and Ohanian [2001], this pattern appears to be a robust empirical finding through both the roots of simple autoregressive and vector autoregressive representations of aggregate variables. The standard Real Business Cycle (RBC) model, in the sense of the one sector optimal growth model governed by a technological shock, cannot explain these stylized facts. This failure of the standard RBC model partly results in its inability to produce complex eigenvalues. As pointed out by Azariadis et al. [2001], building models in accordance with these business cycle facts is actually sensible.

This paper demonstrates that the standard RBC model with a slight modification is qualitatively able to produce complex eigenvalues. The extension concerns the labor input, which is now considered as a quasi–fixed factor. The standard model abstracts from employment lags. But, as suggested by Oi [1962], labor displays smooth adjustments along the business cycle, usually modeled by labor adjustment costs. Capital and labor are interrelated through the equilibrium factor prices, that depend on the households'preferences. Complex eigenvalues occur if capital and labor display both similar persistence and sufficiently antisymmetric behavior. For small costs, the model behaves as the standard model. Conversely, large costs imply a labor almost constant over time and thus the interrelations with the capital vanish. There exists an intermediate situation where the labor adjustment costs imply similar persistence for the two factors. It is worth noting that the assumptions on the utility function are central. If the intertemporal substitution of leisure is sufficiently large, complex eigenvalues occur. The paper shows that sufficiently conditions for complex eigenvalues are satisfied for most preferences specifications typically used in the RBC literature. Some quantitative experiments illustrate this property but suggest that the imaginary part remains insufficiently large compared to the real one.

The paper is organized as follows. A first section presents the model economy. Section 2 characterizes the local dynamic properties of the model and discusses the conditions under which complex eigenvalues occur. Section 3 presents some numerical experiments. A last section offers some concluding remarks. Proofs are given in appendix.

1 The model

There exists a single good both consumed and invested. The economy is populated by an infinite number of identical agents with infinite lifetime. Their preferences are described by a time separable utility function in consumption and leisure $u(C_t, L_t)$. Time endowment is normalized to one and hours worked are given by $N_t = 1 - L_t$. The utility function satisfies the following conditions:

Assumption: *(i)* The utility function $u(.)$: $\mathbb{R}_+^* \times]0,1[\rightarrow \mathbb{R}_+^*$ is strictly increasing and concave in C and $L \equiv 1 - N$, *(ii)* verifies the additional restrictions $u_{CL}u_L - u_{LL}u_C \geq 0$ and $u_{CL}u_C - u_{CC}u_L \geq 0$ with at least one strict inequality and *(iii)* satisfies the Inada conditions.

Condition *(i)* is rather standard, whereas condition *(ii)* imposes that consumption and leisure are normal goods. We will see later that this restriction is central for the saddle path property. Because the approximate solution is obtained through a log–linearization about the steady state, it is useful to express previous conditions in terms of elasticities of the marginal utilities:

$$\xi_{CC} = Cu_{CC}/u_C \qquad \xi_{CL} = Lu_{CL}/u_C \qquad \xi_{LC} = Cu_{CL}/u_L \qquad \xi_{LL} = Lu_{LL}/u_L$$

Using these elasticities, the condition (i) becomes $\xi_{CC}\xi_{LL} - \xi_{CL}\xi_{LC} \geq 0$ and (ii) $\xi_{CL} - \xi_{LL} \geq 0$ and $\xi_{LC} - \xi_{CC} \geq 0$ with at least one strict inequality. The condition *(iii)* also insures the existence and uniqueness of the steady state.

The technology is described by a Cobb–Douglas production function with constant returns to scale

$$\overline{Y}_t = ZK_t^{1-\alpha}N_t^{\alpha} \tag{1}$$

with $0 < \alpha < 1$. K_t, N_t, $\overline{Y}_t$ and $Z > 0$ denote the capital stock, the labor input, the raw product and the level of the technology, respectively. Capital accumulation is described by the following law of motion

$$K_{t+1} = (1 - \delta)K_t + I_t \tag{2}$$

where $\delta \in]0,1[$ denotes the depreciation rate and I_t is the flow of investments. The employment evolves according to

$$N_{t+1} = (1 - \nu)N_t + H_t \tag{3}$$

where $\nu \in]0,1[$ is the quit rate and H_t represents the flow of hirings. Productive employment at time $t+1$ is hired at time t, implying some labor hoarding phenomenon (see Burnside, Eichenbaum and Rebelo [1993] and Fairise and Langot [1994]). Labor is a quasi-fixed factor. The adjustment costs function follows a standard quadratic specification:

$$\mathcal{G}(H_t, N_t) = \frac{b}{2}\frac{(H_t - \nu N_t)^2}{N_t}$$

with $b > 0$. This function satisfies convexity and is homogeneous of degree one. The decision rule on hirings is thus independent of the size of the economy and the hiring rate only depends on the marginal value of labor. At the steady state, this function satisfies $\mathcal{G}(.) = \mathcal{G}_H(.) = \mathcal{G}_N(.) = 0$ and $\mathcal{G}_{HH}(.) = b/N^*$, where N^* denotes the steady state employment. This implies that the steady state of the model does not differ from the one of the standard model. Adjustment costs only affect the convergence path toward the steady state. This allows us to concentrate on the dynamic implications of labor adjustment costs.
The aggregate resources constraint is given by :

$$ZF(K_t, N_t) - \mathcal{G}(H_t, N_t) = C_t + I_t \tag{4}$$

The central planer solves the following intertemporal problem :

$$\max_{I_t, H_t} \sum_{i=0}^{\infty} \beta^i u(C_{t+i}, 1 - N_{t+i})$$

subject to the period–by–period aggregate resources constraint (4), the laws of motion on capital (2) and employment (3) and for K_0, N_0 given and strictly positive. The parameter $\beta \in]0,1[$ denotes the constant discount factor. The first order conditions are:

$$p_t = u_C(t) \tag{5}$$
$$\lambda_t = u_C(t)\mathcal{G}_H(t) \tag{6}$$
$$p_t = \beta\left\{u_C(t+1)ZF_K(t+1) + (1-\delta)p_{t+1}\right\} \tag{7}$$
$$\lambda_t = \beta\left\{u_C(t+1)(ZF_N(t+1) - \mathcal{G}_N(t+1)) - u_L(t+1) + (1-\nu)\lambda_{t+1}\right\} \tag{8}$$

where p_t and λ_t are the implicit prices of capital and labor, respectively. These two implicit prices satisfy usual terminal conditions. The first order conditions (5)–(8), the aggregate resources constraint (4) and the laws of motion (2) and (3) define the optimal path of the economy.

2 Dynamic properties

This section establishes the dynamic properties of the model. We report in appendix A the linearized model, its transformation and some general results.[1] With our specification of the labor adjustment costs, the steady state corresponds exactly to the one of the standard RBC model. There exists an unique steady state $(I^\star, K^\star, H^\star, N^\star, p^\star, \lambda^\star, C^\star)$ that satisfies: $I^\star - \delta K^\star = 0$, $H^\star - \nu N^\star = 0$, $\beta[ZF_K(K^\star, N^\star)+1-\delta]-1 = 0$, $u_C(C^\star, 1-N^\star)ZF_N(K^\star, N^\star) - u_L(C^\star, 1-N^\star) = 0$, $p^\star = u_C(C^\star, 1-N^\star)$, $\lambda^\star = 0$ and $ZF(K^\star, N^\star) - C^\star - I^\star = 0$. Given these steady state values, we thus study the dynamic properties of the log–linearized system (2)–(8). We first establish the following property:

Proposition 1 *If the assumptions (i) and (ii) on the utility function hold, then there exists a unique convergence path toward the steady state.*

Proposition 1 shows that the introduction of labor adjustment costs does not alter the dynamic properties of the standard RBC model. Note that our assumptions on the utility function, *i.e.* consumption and leisure are normal goods, are sufficient to establish this result. Compared to the standard RBC model, we only add an additional restriction that insures the saddle path property, that is the convexity of the adjustment costs function. Given this result, we study in details other dynamic properties of the model. The following proposition raises the possibility for complex eigenvalues.

Proposition 2 *If the preferences satisfy the conditions :*

$$\xi_{CC} \geq -1 \tag{9}$$

$$\xi_{CC} - \xi_{LC} \leq -1 \tag{10}$$

then, there exists an interval $[\underline{b}, \overline{b}]$*, with* $0 < \underline{b} < \overline{b} < \infty$*, such that eigenvalues are (i) complex if* $b \in]\underline{b}, \overline{b}[$ *and (ii) real if* $b \in]0, \underline{b}] \cup [\overline{b}, +\infty[$.

The existence of a complex eigenvalues imposes restrictions on preferences. The elasticity ξ_{CC} appears in both conditions. This shows that the specification of the utility function matters for the dynamic properties of the model economy. Conversely, none of the structural parameters that characterize the technology and the accumulation process enters in the

[1] More details are avialable from the authors upon request.

sufficient conditions Few parameters enter in the sufficient conditions (9) and (10). Complex eigenvalues can therefore be easily checked. The following examples illustrates the proposition.

Example 1 Consider the isoelastic utility function:

$$u(C_t, 1-N_t) = \frac{1}{1-\sigma}[C_t^{\theta}(1-N_t)^{1-\theta}]^{1-\sigma}$$

with $\theta \in]0,1[$ and $\sigma \in]0,1[\cup]1,\infty[$. It is for instance the one used by Kydland and Prescott [1982]. We have $\xi_{CC} = \theta(1-\sigma) - 1$ and $\xi_{CC} - \xi_{LC} = -1$. Condition (10) is always satisfied and condition (9) hold if $\sigma \leq 1$. The standard case of logarithmic and separable utility function satisfies these conditions. In this case, $\sigma = 1$, $\xi_{CC} = -1$ and $\xi_{LC} = 0$.

Example 2 Consider the utility function with indivisible labor supply proposed by Hansen [1985] and Rogerson [1988]:

$$u(C_t, 1-N_t) = \log(C_t) + \theta(1-N_t)$$

We directly deduce that $\xi_{CC} = -1$ and $\xi_{CC} - \xi_{LC} = -1$ and conditions (9) and (10) are satisfied.

Example 3 Consider the class of utility functions that produces static labor supply:

$$\log\left(C_t - \psi_0 \frac{N_t^{1+\psi}}{1+\psi}\right)$$

with $\psi, \psi_0 > 0$. This function, used by Hercowitz and Sampson [1991] among others, implies that the income effect on leisure is zero. It follows that $\xi_{CL} - \xi_{LL} = \psi L^*/(1-L^*) > 0$ and $\xi_{CC} - \xi_{LC} = 0$. The condition (10) is thus not verified.
In example 1, the condition (9) is not verified if $\sigma > 1$. Nevertheless, a less restrictive condition can be obtained from the very plausible assumption that the labor share exceeds the depreciation rate of the capital.

Proposition 3 *If the preferences satisfy the conditions :*

$$\xi_{CC} \geq -(1+\alpha) \tag{11}$$

$$\xi_{CC} - \xi_{LC} \leq -1 \tag{12}$$

and if $\alpha > \delta$*, then, there exists an interval* $[\underline{b}, \overline{b}]$*, with* $0 < \underline{b} < \overline{b} < \infty$*, such that the eigenvalues are (i) complex if* $b \in]\underline{b}, \overline{b}[$ *and (ii) real if* $b \in]0, \underline{b}] \cup [\overline{b}, +\infty[$*.*

We immediately see that the condition (11) is less restrictive than the condition (9). For instance, in example 1, for $\sigma = 1.5$, $\theta = 1/3$, $\alpha = 0.64$ and $\delta = 0.025$ as in Kydland and Prescott [1982], complex eigenvalues can occur. The condition (12) in proposition 3 is exactly the same than condition (10) in proposition 2. It follows that example 3 does not verify condition (12).

3 Numerical experiments

Following example 2, we choose a utility function with indivisible labor supply.[2] From our assumptions on the structure of the labor adjustment costs, the steady state of the model is the same than the one of the standard RBC model. This allows to set the values of the structural parameters in accordance to previous calibrations and thus to use freely the parameter b of labor adjustment costs. The parameter α corresponds to a labor share of 64% at steady state. The parameter β is set in order to imply a 4% annual subjective discount rate. The depreciation rate δ is equal to 2.5% per quarter. The quit rate ν is fixed in order to roughly match the average destruction rate in the US manufacturing sector over the period 1972–1993.[3] The time spent to productive activity is equal to 40%. The value of θ is thus deduced from the steady state conditions. Finally, the parameter of the production function Z is set to scale the adjustment costs parameter. So, in what follows, the value of b must be interpreted with respect to the scale parameter Z. All these values are reported in table 1. Figure 1 presents the modulus, the real part and the imaginary

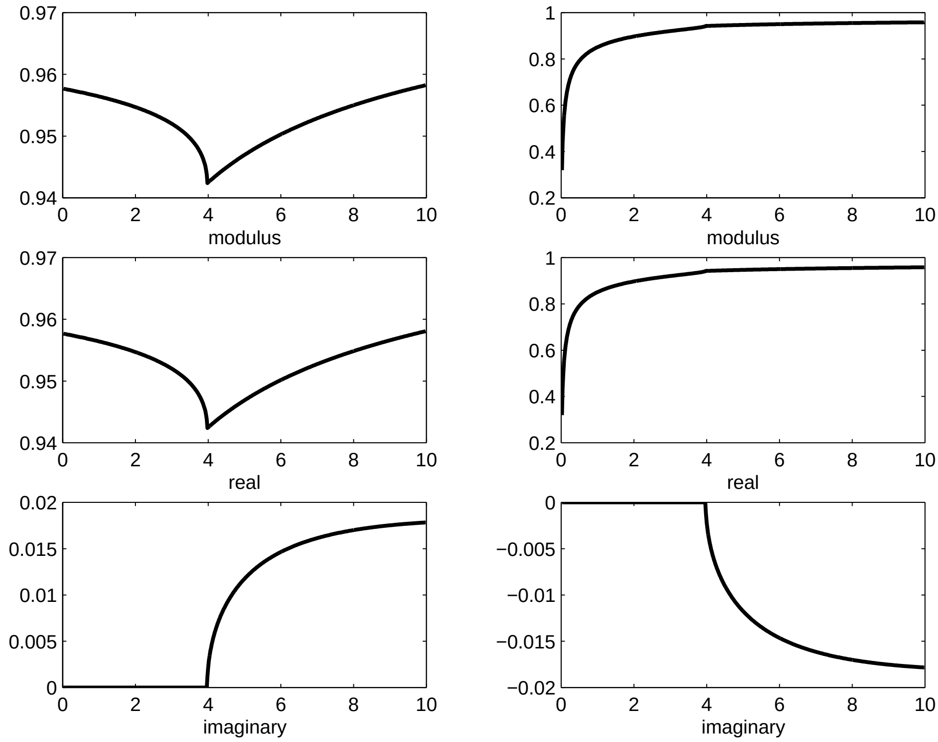
Table 1: Values of the structural parameters

Technology		Preferences	
α	0.640	β	0.99
δ	0.025	$N^\star$	0.40
ν	0.015		

[2] A similar exercice have been performed with isoelastic utility function. The results are quite similar, despite a lower imaginary part of the eigenvalues.

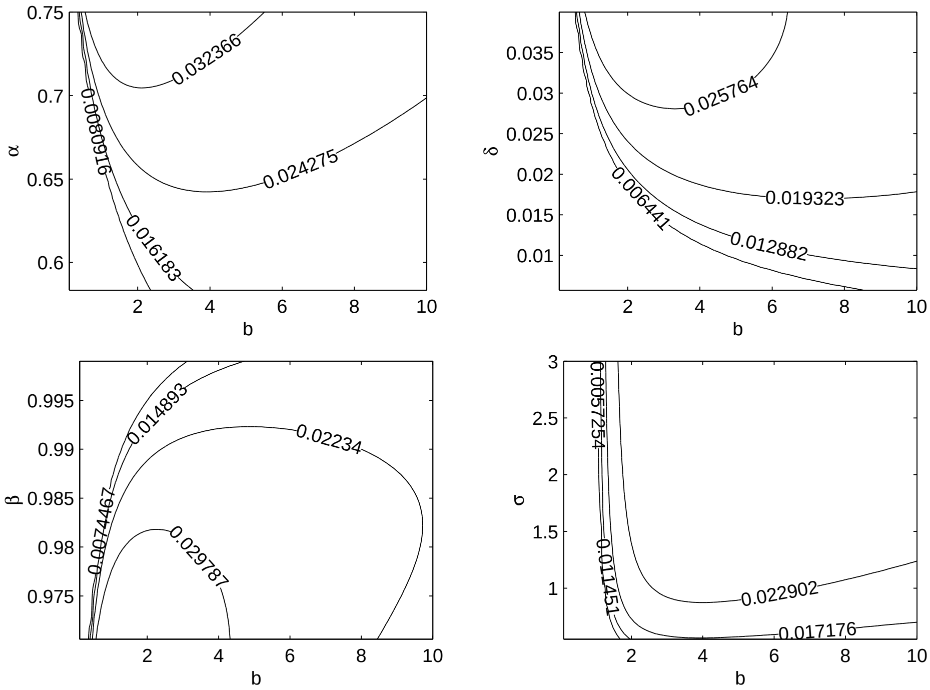
[3] If N_t should be interpreted as hours rather than employment, the calibration of ν should be adjusted accordingly. Nevertheless, our numerical results has appeared unsensitive to various values of this parameter.

Figure 1: Roots with labor adjustment costs



part of the two eigenvalues with respect to the adjustment costs parameter b. For b small, the two eigenvalues are real. As b increases, the modulus of these two eigenvalues becomes closer and then complex conjugate. However, for b large (not reported in figure 1), the imaginary part is zero. The imaginary part remains insufficiently large – it does not exceed 0.025 – compared to the real part – it is close to 0.95–. This result suggests the model may face some difficulties to generate aggregate damped oscillations in response to transitory shocks. We further explore the quantitative effects of other structural parameters changes on aggregate dynamics. We compute the imaginary part of the eigenvalue with respect to the adjustment costs parameter b and a selected structural parameter. We keep a utility function linear in leisure, but we consider that the elasticity ξ_{CC} can differ from minus unity. The four structural parameters are the steady state labor share α, the depreciation rate δ, the discount factor β and the curvature of the utility function with respect to consumption $\sigma = -\xi_{CC}$. The range for $\alpha \in [0.58; 0.75]$ reflects on how proprietors' income is treated, *i.e.* the share of total output paid to capital varies between 0.25 and 0.42. The range for $\delta \in [0.005; 0.040]$ is selected because it is commonly set to 0.025 and previous estimates lie within the selected range. The range for $\beta \in [0.970; 0.999]$ implies the annual subjective

Figure 2: Imaginary part



discount rate lies within [0.4%;10.3%]. Finally, the range for the curvature of the utility function $[0.5; 3]$ roughly corresponds to previous estimates. In each case, one of the structural parameter varies within the range, whereas the others are fixed to their reference values (see table 1). We report in figure 2 the contours of the 3-D function that express the imaginary part of the eigenvalue as a function of b and $\{\alpha, \delta, \beta, \sigma\}$. Figure 2 shows that the imaginary part remains too small compared to the real part. Indeed, the real part in these experiments (not reported here) always exceeds 0.95, whereas the imaginary part never exceeds 0.035.

4 Concluding remarks

This paper studies the ability of a standard RBC model with labor adjustment costs to produce complex eigenvalues. The paper establishes sufficient conditions for complex eigenvalues and illustrates these properties using numerical experiments. However, the paper shows that labor adjustment costs can potentially improve the dynamic properties of a standard RBC model. Further research must therefore explore the dynamic and quantitative properties of equilibrium models when labor adjustment costs are combined with suitable assumptions on good and labor market arrangements.

Appendix

A Notations and the linearized model

This appendix derives the main dynamic properties of our model economy. We first introduce some notations: (1) elasticities of the adjustment cost functions: $\omega_{HH} = H^*\mathcal{G}_{HH}/ZF_N$, $\omega_{HN} = N^*\mathcal{G}_{HN}/ZF_N$, $\omega_{NH} = H^*\mathcal{G}_{NH}/ZF_N$ and $\omega_{NN} = N^*\mathcal{G}_{NN}/ZF_N$ with $N\mathcal{G}_{HN} + H\mathcal{G}_{HH} = 0$; (2) elasticities of the marginal utilities :$\xi_{CC} = C^*u_{CC}/u_C$, $\xi_{CL} = L^*u_{CL}/u_C$, $\xi_{LC} = C^*u_{CL}/u_L$ and $\xi_{LL} = L^*u_{LL}/u_L$; (3) elasticity of the marginal product of capital $\eta_K = -\alpha(1-\beta(1-\delta))$; (4) consumption share $s_C = C^*/Y^* = (1-\beta(1-\alpha\delta))(1-\beta(1-\delta))^{-1}$; (5) investment share $s_I \equiv 1 - s_c = ((1-\alpha)\delta\beta)(1-\beta(1-\delta))^{-1}$; (6) others: $\phi = 1/\delta$, $\psi = 1/\nu$. Let x, y and u the state variables (K, N). After some algebra, the linearized dynamical system formed by (2)–(8) takes the following form :

$$\Delta\widehat{x}_{t+2} + \Gamma\widehat{x}_{t+1} + \beta^{-1}\Delta'\widehat{x}_t = 0 \tag{A.1}$$

where the elements of the matrices Δ and Γ are:

$$\delta_{11} = \frac{K^*}{\beta}\left[-\frac{\phi s_I}{s_C}\xi_{CC}\right] \quad \delta_{12} = 0 \quad \delta_{21} = \frac{K^*}{\beta}\left[-\beta\frac{\alpha}{s_C}(\xi_{CC} - \xi_{LC})\right] \quad \delta_{22} = \frac{K^*}{\beta}\frac{\alpha\beta}{\phi_{sI}}\left[\psi\omega_{HH}\right]$$

$$\gamma_{11} = \frac{K^*}{\beta}\left[\eta_K + \left(1+\frac{1}{\beta}\right)\frac{\phi s_I}{s_C}\xi_{CC}\right] \quad \gamma_{12} = \frac{K^*}{\beta}\left[-\eta_K + \frac{\alpha}{s_C}(\xi_{CC} - \xi_{LC})\right]$$

$$\gamma_{22} = \frac{K^*}{\beta}\left[\frac{\alpha\beta}{\phi s_I}\left(-\psi\omega_{HH} - \frac{1}{\beta}\psi\omega_{HH}\right) + \eta_K + \frac{\alpha\beta}{\phi s_I}\left(\frac{\alpha}{s_C}(\xi_{CC} - \xi_{LC}) + \frac{N^*}{1-N^*}(\xi_{LL} - \xi_{CL})\right)\right]$$

For practical reasons, we transform equation (A.1) in a canonical form by the mean of a diagonalization. We follow an idea of Magill [1979] adapted by Cassing and Kollintzas [1991] to the case of a discrete time model. Such a method allows to highlight the symmetric and asymmetric characteristics of the dynamic system. We define the variable $\widehat{w}_t$ such that $\widehat{x}_t = \left(\beta^{-1/2}\right)^t \widehat{w}_t$ and (A.1) becomes :

$$\Delta\widehat{w}_{t+2} + \Gamma\beta^{1/2}\widehat{w}_{t+1} + \Delta'\widehat{w}_t = 0 \tag{A.2}$$

Let us define the matrices $A = (1/2)(\Delta + \Delta')$ and $B = (1/2)(\Delta - \Delta')$. A is a symmetric matrix whereas B is a skew matrix. We have the following useful lemma:

Lemma 1 *Let α_1 and α_2 the real eigenvalues of the matrix $(\beta^{1/2}\Gamma)^{-1}(-A)$ and t_1 and t_2 the associated eigenvectors. The matrix $T = \begin{bmatrix} t_1 & t_2 \end{bmatrix}$ can be choosen such that $T'(-\beta^{1/2}\Gamma)T = I_2$ and $T'AT = diag(\alpha_1, \alpha_2)$*

The skew matrix B implies:

$$T'BT = \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix}$$

We define $\widehat{w}_t = T\widehat{z}_t$ and (A.2) becomes :

$$(T'\Delta T)\widehat{z}_{t+2} + T'(\beta^{1/2}\Gamma)T\widehat{z}_{t+1} + T'\Delta'T\widehat{z}_t = 0$$

From Lemma 1, we have:

$$\begin{bmatrix} \alpha_1 & d \\ -d & \alpha_2 \end{bmatrix}\widehat{z}_{t+2} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\widehat{z}_{t+1} + \begin{bmatrix} \alpha_1 & -d \\ d & \alpha_2 \end{bmatrix}\widehat{z}_t = 0 \tag{A.3}$$

The parameters α_1, α_2 and d are function of the structural parameters. The characteristic roots of equation (A.3) are solution of :

$$(\alpha_1\alpha_2 + d^2)\lambda^4 - (\alpha_1 + \alpha_2)\lambda^3 + (2\alpha_1\alpha_2 + 1 - 2d^2)\lambda^2 - (\alpha_1 + \alpha_2)\lambda + \alpha_1\alpha_2 + d^2 = 0$$

This equation can be solved using $\mu = \lambda + \frac{1}{\lambda}$ and $(\alpha_1\alpha_2 + d^2)\mu^2 - (\alpha_1 + \alpha_2)\mu + (1 - 4d^2) = 0$. Now consider the discriminant

$$\kappa = (\alpha_1 + \alpha_2)^2 - 4(1 - 4d^2)(\alpha_1\alpha_2 + d^2)$$

In order to determine the roots of (A.3), one must consider two cases:

$$\lambda_j + \frac{1}{\lambda_j} = \frac{\alpha_1 + \alpha_2 \pm \sqrt{\kappa}}{2(\alpha_1\alpha_2 + d^2)} \text{ if } \kappa > 0 \quad \text{and} \quad \lambda_j + \frac{1}{\lambda_j} = \frac{\alpha_1 + \alpha_2 \pm i\sqrt{-\kappa}}{2(\alpha_1\alpha_2 + d^2)} \text{ if } \kappa < 0$$

for $j = 1, 2$. Note that the previous expressions define second order equations, whose coefficients are not necessarily real, *i.e.* the discriminant κ can be negative. The eigenvalues of equation (A.1) are deduced using $\rho = \lambda/\sqrt{\beta}$.

Lemma 2 *Let denote $\varphi_1 = (1 - 4d^2)$, $\varphi_2 = (\alpha_1\alpha_2 + d^2)$, $\varphi_3 = (\alpha_1 + \alpha_2)$, $\overline{\beta} = \sqrt{\beta} + (1/\sqrt{\beta})$ and $\underline{\beta} = \sqrt{\beta} - (1/\sqrt{\beta})$. Consider the dynamic system described by equation (A.1). The stationary equilibrium is a saddle path and its convergence path is (i) cyclical iff $4\varphi_1\varphi_2 > \varphi_3^2$ and $(\varphi_1/\varphi_2)\overline{\beta} > (\varphi_3/\varphi_2)^2 + \overline{\beta}^2\underline{\beta}^2$ and (ii) monotone iff $4\varphi_1\varphi_2 < \varphi_3^2$, $(\varphi_3/\varphi_2) > 2\overline{\beta}$ and $\overline{\beta}^2 - (\varphi_3/\varphi_2)\overline{\beta} + (\varphi_1/\varphi_2) > 0$.*

Lemma 2 presents two types of convergence path toward the steady state. The first one is cyclical because the eigenvalues have no zero imaginary part. In the second case, the eigenvalues are real and the convergence is monotone. Lemma 2 presents only two cases. There exists also two other cases which are not discussed here: a case where the eigenvalues are both negative and a case where there exists both positive and negative eigenvalues. We will not discuss these two last cases, because negative eigenvalues cannot occur in our model.

B Proof of proposition 1

For α_1, α_2 and d, we have the following expressions :

$$\begin{aligned}
\alpha_1 + \alpha_2 &= \frac{\beta^{1/2}}{\beta(\gamma_{11}\gamma_{22} - \gamma_{12}^2)}\left[\gamma_{12}(\delta_{12} + \delta_{21}) - \delta_{11}\gamma_{22} - \delta_{22}\gamma_{11}\right] \\
\alpha_1\alpha_2 &= = \frac{\left[\delta_{11}\delta_{22} - 1/4(\delta_{12} + \delta_{21})^2\right]}{\beta(\gamma_{11}\gamma_{22} - \gamma_{12}^2)} \\
d^2 &= = \frac{1/4(\delta_{12} - \delta_{21})^2}{\beta(\gamma_{11}\gamma_{22} - \gamma_{12}^2)}
\end{aligned}$$

From Lemma 2, we have a saddle path if the following inequalities are satisfied :

$$\begin{aligned}
\frac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2 + d^2} &> 2(\sqrt{\beta} + 1/\sqrt{\beta}) \\
(\sqrt{\beta} + 1/\sqrt{\beta})^2 - \frac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2 + d^2}(\sqrt{\beta} + 1/\sqrt{\beta}) + \frac{1 - 4d^2}{\alpha_1\alpha_2 + d^2} &> 0
\end{aligned}$$

These two inequalities can be expressed with respect to the structural parameters:

$$\frac{K^{*2}}{\beta^2}\left[-\eta_K\left(-\frac{\phi s_I}{s_C}\xi_{CC} - \beta\frac{\alpha}{s_C}(\xi_{CC} - \xi_{LC}) + \frac{\alpha\beta}{\phi s_I}\phi\omega_{HH}\right) + \frac{\alpha\beta}{s_C}\frac{N}{1-N}(\xi_{CC}\xi_{LL} - \xi_{CL}\xi_{LC})\right] > 0$$

and

$$\frac{K^{*2}}{\beta^2}\eta_K\left[(1-\beta)\frac{\alpha\beta}{s_C}(\xi_{CC}-\xi_{LC})+\frac{\alpha\beta}{\phi s_I}\left(\frac{\alpha}{s_C}(\xi_{CC}-\xi_{LC})+\frac{N}{1-N}(\xi_{LL}-\xi_{CL})\right)\right]>0$$

From the assumptions that consumption and leisure are normal goods and that $u(.)$ is concave, we have $\xi_{CC}-\xi_{LC}\leq 0$, $\xi_{LL}-\xi_{CL}\leq 0$ and $\xi_{CC}\xi_{LL}-\xi_{CL}\xi_{LC}>0$. Moreover, $\eta_K<0$ and $\omega_{HH}>0$. It follows that the two inequalities are satisfied. This completes the proof. □

C Proof of proposition 2

From proposition 1, the stationary equilibrium is a saddle path. To determine the nature of the adjustment path, we have to determine the sign of $(\alpha_1+\alpha_2)^2-4(1-4d^2)(\alpha_1\alpha_2+d^2)$. From $\psi\omega_{HH}=\frac{b}{ZF_N}\equiv\frac{b}{W}$, the previous expression can be expressed as a second order polynomial in b

$$f(b) \;=\; \frac{1}{W^2}\zeta_1 b^2+\frac{1}{W}\zeta_2 b+\zeta_3 \qquad \text{(C.1)}$$

where

$$\begin{aligned}
\zeta_1 &= \beta\left(\frac{\alpha\beta}{\phi s_I}\right)^2\eta_K^2\\
\zeta_2 &= \beta\Bigg[4\beta(1-\beta)\frac{\alpha}{s_C}\xi_{CC}\frac{\alpha}{s_C}(\xi_{CC}-\xi_{LC})\eta_K-2\frac{\alpha\beta}{\phi s_I}\frac{\alpha\beta}{s_C}\frac{N}{1-N}(\xi_{CC}\xi_{LL}-\xi_{CL}\xi_{LC})\eta_K\\
&+ 4\frac{\alpha}{s_C}\xi_{CC}\frac{\alpha\beta^2}{\phi s_I}\left(\frac{\alpha}{s_C}(\xi_{CC}-\xi_{LC})+\frac{N}{1-N}(\xi_{LL}-\xi_{CL})\right)\eta_K\\
&- 2\frac{\alpha\beta}{s_I\phi}\left(\beta\frac{\alpha}{s_C}(\xi_{CC}-\xi_{LC})+\frac{\phi s_I}{s_C}\xi_{CC}\right)\eta_K^2\Bigg]\\
\zeta_3 &= \beta\left[\left(\beta\frac{\alpha}{s_C}(\xi_{CC}-\xi_{LC})+\frac{\phi s_I}{s_C}\xi_{CC}\right)\eta_K+\frac{\alpha\beta}{s_C}\frac{N}{1-N}(\xi_{CC}\xi_{LL}-\xi_{CL}\xi_{LC})\right]^2
\end{aligned}$$

We now study the sign of this polynomial with respect to b. Without ambiguity, $\zeta_1>0$ and $\zeta_3>0$. If $\zeta_2<0$ and $disc=\zeta_2^2-4\zeta_1\zeta_3>0$, the polynomial has two positive roots and it is negative if it is evaluated at values which lie between the two roots. The discriminant is given by $disc=T_1T_2$ with :

$$\begin{aligned}
T_1 &= \beta\Bigg[4\beta(1-\beta)\frac{\alpha}{s_C}\xi_{CC}\frac{\alpha}{s_C}(\xi_{CC}-\xi_{LC})\eta_K\\
&+ 4\frac{\alpha}{s_C}\xi_{CC}\frac{\alpha\beta^2}{\phi s_I}\left(\frac{\alpha}{s_C}(\xi_{CC}-\xi_{LC})+\frac{N}{1-N}(\xi_{LL}-\xi_{CL})\right)\eta_K\Bigg]\\
T_2 &= \Bigg[\zeta_2-\beta\Bigg(2\frac{\alpha\beta}{\phi s_I}\frac{\alpha\beta}{s_C}\frac{N}{1-N}(\xi_{CC}\xi_{LL}-\xi_{CL}\xi_{LC})\eta_K\\
&+ 2\frac{\alpha\beta}{s_I\phi}\left(\frac{\alpha\beta}{s_C}(\xi_{CC}-\xi_{LC})+\frac{\phi s_I}{s_C}\xi_{CC}\right)\eta_K^2\Bigg)\Bigg]
\end{aligned}$$

T_1 is without ambiguity negative. We thus have to determine the sign of T_2. We introduce the following useful notations $\xi_{CC}-\xi_{LC}=-X$ and $\xi_{LL}-\xi_{CL}=-Y$. Therefore, T_2 becomes:

$$T_2 \;=\; 4\beta\frac{\alpha\beta}{s_C}\frac{\alpha\beta}{\phi s_I}\eta_K\left[\frac{\alpha}{s_C}X^2+\left(\eta_K-(1-\beta)\frac{\alpha}{s_C}\frac{\phi s_I}{\alpha\beta}\xi_{CC}\right)X+(1-\alpha)\xi_{CC}\right]$$

Consider now the term in brackets :

$$g(X) \quad = \quad \frac{\alpha}{s_C}X^2 + \left(\eta_K - (1-\beta)\frac{\alpha}{s_C}\frac{\phi s_I}{\alpha\beta}\xi_{CC}\right)X + (1-\alpha)\xi_{CC}$$

As $(1-\alpha)\xi_{CC} < 0$ and $\frac{\alpha}{s_C} > 0$, the above polynomial has a positive discriminant. The two roots have opposite sign. For values of X greater than the positive root, the above expression is also positive. Consider now :

$$g(1) \quad = \quad \alpha\beta(1-\delta) + \frac{(1-\alpha)\beta\alpha\delta}{1-\beta(1-\alpha\delta)}(1+\xi_{CC}) \tag{C.2}$$

A sufficient condition for $g(1)$ be positive is $\xi_{CC} \geq -1$. Moreover, if $X = -(\xi_{CC} - \xi_{LC}) \geq 1$, then T_2 is negative and ζ_2 is also necessarily negative. To sum up, we have $disc = \zeta_2^2 - 4\zeta_1\zeta_2 > 0$ and $\zeta_2 < 0$ and equation (C.1) has two positive real roots. We conclude that there exists two positive real numbers $0 < \underline{b} < \overline{b} < +\infty$ such that for all $b \in]\underline{b}, \overline{b}[$, equation (C.1) is negative and complex eigenvalues occur. This completes the proof. $\square$

D Proof of proposition 3

The proof follows the one of C. Consider equation (C.2) and suppose that $\alpha > \delta$. We have :

$$g(1) \quad = \quad \frac{\alpha\beta}{1-\beta(1-\alpha\delta)}\left[(1-\beta(1-\delta))(1-\alpha\delta) + (1-\alpha)\delta\xi_{CC}\right]$$

It is then easy to verify that if $\alpha > \delta$, then $(1-\beta(1-\delta))(1-\alpha\delta)((1-\alpha)\delta))^{-1} > (1-\alpha\delta)(1-\alpha)^{-1} > 1+\alpha$. The end of the proof is then similar to the one of proposition 2. This completes the proof. $\square$

References

Azariadis, C., J. Bullard, and L. Ohanian, *Complex eigenvalues and trend–reverting fluctuations*, mimeo, Federal Reserve bank of Saint-Louis 2001.

Burnside, C., M. Eichenbaum, and S. Rebelo, Labor hoarding and the business cycle, *Journal of Political Economy*, april 1993, *101* (2), 245–273.

Cassing, S. and T. Kollintzas, Recursive factor of production interrelations and endogenous cycling, *International Economic Review*, 1991, *32* (2), 417–440.

Fairise, X. and T. Langot, Labor productivity and the business cycle: can RBC be saved?, *European Economic Review*, october 1994, *38* (8), 1581–1594.

Hansen, G.D., Indivisible Labor and the Business Cycles, *Journal of Monetary Economics*, November 1985, *16* (3), 309–327.

Hercowitz, Z. and M. Sampson, Output growth, the real wage and employment fluctuations, *American Economic Review*, 1991, *81*, 1215–1237.

Kydland, F. and E. Prescott, Time to build and aggregate fluctuations, *Econometrica*, 1982, *50*, 1345–1370.

Magill, M., The stability of equilibrium, *International Economic Review*, 1979, *20*, 577–597.

Oi, W., Labor as a quasi–fixed factor, *Journal of Political Economy*, 1962, *70.*

Rogerson, R., Indivisible labor, lotteries and equilibrium, *Journal of Monetary Economics*, 1988, *21* (1), 3–17.