WORKING PAPERS

N° 17-828

July 2017

# “Variance stochastic orders”

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# Variance stochastic orders

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July 6, 2017

**Abstract**

Suppose that the decision-maker is uncertain about the variance of the payoff of a gamble, and that this uncertainty comes from not knowing the number of zero-mean i.i.d. risks attached to the gamble. In this context, we show that any n-th degree increase in this variance risk reduces expected utility if and only if the sign of the 2n-th derivative of the utility function $u$ is $(-1)^{n+1}$. Moreover, increasing the statistical concordance between the mean payoff of the gamble and the n-th degree riskiness of its variance reduces expected utility if and only if the sign of the $2n+1$ derivative of $u$ is $(-1)^{n+1}$. These results generalize the theory of risk apportionment developed by Eeckhoudt and Schlesinger (2006), and is useful to better understand the impact of stochastic volatility on welfare and asset prices.

*Keywords*: Long-run risk, stochastic dominance, prudence, temperance, stochastic volatility, risk apportionment.

*JEL codes*: D81

*Acknowledgement:* I acknowledge funding from the chair SCOR and FDIR at TSE. I also gratefully acknowledge the hospitality of the Economics Department of the University College London where the first version of this paper was written.

# 1 Introduction

Suppose that the variance of the payoff of an asset is uncertain. How does this uncertainty affect the attitude towards this asset? More generally, how does a shift in the distribution of this variance influence expected utility? In this paper, we build a theory of stochastic dominance on variance that is based on the seminal work of Eeckhoudt and Schlesinger (2006) who raised the following question: Do you prefer to bear a zero-mean risk for sure, or two independent and identically distributed zero-mean risks with probability 1/2? They showed that shifting from the first risk context to the second one is an example of fourth degree risk increase as defined by Ekern (1980), in the sense that it is perceived as undesirable by any von Neumann-Morgenstern individual with a negative fourth derivative of the utility function, a condition coined as "temperance" in decision theory (Kimball (1993), Gollier and Pratt (1996)). We generalize this result by showing that any Rothschild-Stiglitz increase in risk in the number of zero-mean risks attached to the gamble reduces expected utility if and only if $u^{(4))}$ is negative. In other words, in this context of additive i.i.d. risks, temperant people dislike increasing variance risk.

The role of putting risk on risk has emerged has an important research object over the last decade or so. For example, Weitzman (2007) has shown that the uncertainty surrounding the variance of the growth rate of consumption can have a first-order impact on welfare and asset prices. Using a Bayesian approach, he assumed an inverted gamma posterior distribution for the variance of the growth rate and consumption, which implies a Student-$t$ distribution for log consumption. This yields an unbounded risk premium at equilibrium, under constant relative risk aversion (CRRA). This is an extreme illustration of our result, since constant relative aversion implies temperance, and the Student-$t$ has fatter tails, a necessary condition for a 4th degree risk increase. This is also related to the literature on long-run risk pioneered by Bansal and Yaron (2004), in which the variance of the growth rate of consumption is subject to persistent stochastic shocks. In the discounted expected utility model, this positively affects the systematic long-term risk premium under temperance, as shown in Gollier (2017).

In the spirit of Eeckhoudt and Schlesinger (2006), Eeckhoudt et al. (2009), Crainich et al. (2013), Ebert (2013) and Deck and Schlesinger (2014), we systematize the risk apportionment approach by considering other classes of changes in distribution of variance. Following Ekern (1980), we say that a random variable $\tilde{x}$ undergoes a n-th degree increase in risk if and only if it reduces the expectation of $f(\tilde{x})$ for all real-valued functions $f$ such that $(-1)^n f^{(n)}$ is negative. Cases $n = 1$ and $n = 2$ correspond respectively to first-degree stochastic dominance and Rothschild-Stiglitz increases in risk. A n-th degree risk increase in $\tilde{x}$ does noth affect its $(n-1)$ first moments. It raises its n-th moment if $n$ is even, and it reduces it when $n$ is odd. We show that a n-th degree increase in the variance risk generates a (2n)th degree increase in consumption risk if $n$ is even. The opposite result holds when $n$ is an odd number. For example, an increase in downside (i.e., third degree) risk in variance yields a sixth degree reduction in consumption risk, which increases expected utility if the sixth derivative of the utility function is negative, as in the CRRA case.

An interesting feature of this property comes from the possibility to use it recursively. For example, suppose that the uncertainty affecting the variance $\tilde{v}$ of a gamble is measured by the variance of $\tilde{v}$, and that this object is itself uncertain. Performing a second degree risk increase on the variance of $\tilde{v}$ generates a fourth degree increase in the risk affecting $\tilde{v}$, and thereby a eighth degree increase in consumption risk. This "vol-of-vol" type of model exists in the asset pricing literature. For example, the standard long-run risk model with an AR(1) stochastic volatility generates a deterministic term structure of risk premia, which is counterfactual. To solve this problem, Bollerslev et al. (2009), Tauchen (2011) and Drechsler and Yaron (2011) introduced some uncertainty on the volatility of the volatility to generate a time-varying variance premium as observed on financial markets. Our result indicates that on top of these time variations of the equilibrium price of risk, this new ingredient generates the additional result to raise the expected premium if and only if the eighth derivative of the utility function of the representative agent is negative, which is the case under constant relative risk aversion.

Observe that all those findings provide a new characterization of the (2n)th derivatives of the utility function, leaving odd derivatives aside. Eeckhoudt and Schlesinger (2006) explored a road to characterize odd derivatives by combining zero-mean risks with sure losses. For example, they showed that shifting a zero-mean risk from a low income state to an equally likely larger income state raises expected utility when the third derivative of the utility function is positive, i.e., when the individual is prudent. To generalize this result, we use the concept of increasing concordance between two random variables, as introduced in economics by Epstein and Tanny (1980) and Tchen (1980). This concept is stronger than the linear concept of increasing correlation, and it preserves the marginal distributions of the two random variables. In this paper, we show that increasing the concordance between the background income and the number of zero-mean risks of the gamble increases expected utility if and only if the individual is prudent. In other words, it generates a third degree reduction in the consumption risk. This is linked to the result by Tinang (2017) who introduced a negative correlation between the shock on the trend and the shock on the volatility of consumption growth. From our analysis, the third degree risk increase that it generates should raise the risk premium, because CRRA individuals are prudent.[1]

We generalize this finding by showing that increasing concordance between background income and the n-th degree riskiness of the variance of the gamble yields a $(2n+1)$th degree change in consumption risk. Therefore, we obtain a complete characterization of all even and odd derivatives of the utility function by considering various changes in the distribution of the variance of the lottery under consideration and in its correlation with the background income.

The paper is structured as follows. In Section 2, we characterize the impact of increasing the n-th degree riskiness of the variance of consumption on expected utility. This impact is univocally linked to the sign of the successive even derivatives of the utility function. We characterize the impact of increasing the statistical relationship between the mean and the n-th degree riskiness of the variance of consumption in Section 3. This impact is univocally linked to the sign of the successive odd derivatives of the utility function. Section 4 is devoted to the implications of these results for asset pricing in a simple two-date Lucas tree

[1]The important contribution of Tinang (2017) is to show that the persistence of these correlated shocks makes the term structure of equity premia increasing, as observed on financial markets. This persistence generates a concordance between mean consumption and its variance that is increasing with maturity. Thus, the positive effect of the positive concordance is magnified for longer maturity, thereby contributing to the upward slope of the term structure. We emphasize that this effect is possible only because of the fact that power utility functions that are concave all have a positive third derivative.

economy. In the last section, we warn the reader that our results cannot easily be extended to a multiplicative framework for the variance risk.

## 2 N-th degree risk increase in variance

Let us first define a n-th degree risk increase. Consider a pair $(\tilde{v}_1, \tilde{v}_2)$ of random variables characterized by cumulative distribution functions $(P_1, P_2)$ whose supports are bounded above by $V$.To any cdf $P_i = P_i^0$, we can associate a set of functions $(P_i^2, P_i^3, ...)$ that are defined recursively as follows: $\forall v \leq V$:

$$P_i^n(v) = \int^v P_i^{n-1}(t)dt. \tag{1}$$

**Definition 1.** *(Ekern (1980)) $\tilde{v}_2$ has more n-th degree risk than $\tilde{v}_1$ if and only if*

$$P_2^k(V) = P_1^k(V) \quad \text{for } k = 1, 2, ..., n, \tag{2}$$

$$P_2^n(v) \geq P_1^n(v) \quad \text{for all } v \leq V. \tag{3}$$

A n-th degree risk reduction is defined in the same way, with a reversed inequality in (3). Equation (2) means that the first $n-1$ moments of $\tilde{v}$ are unaffected by the change in distribution, whereas equation (3) implies that the $n$th moment of $\tilde{v}$ is increased (decreased) if $n$ is even (odd). Ekern (1980) demonstrated that $\tilde{v}_2$ has more n-th degree risk than $\tilde{v}_1$ if and only if $Ef(\tilde{v}_2)$ is smaller than $Ef(\tilde{v}_1)$ for all functions $f$ such that $(-1)^n f^{(n)} \leq 0$, where $f^{(n)}$ denotes the n-th order derivative of $f$.

We consider an individual whose preferences under risk can be represented by a von Neumann-Morgenstern utility function $u$. The final consumption of the agent is given by

$$\tilde{c} = \tilde{c}^* + \tilde{x}_{\tilde{v}}, \tag{4}$$

where $\tilde{x}_{\tilde{v}}$ is a zero-mean risk. We assume that, conditional to $v$, $\tilde{x}_v$ is the sum of $v$ independent risks:

$$\tilde{x}_v = \sum_{i=1}^{v} \tilde{\varepsilon}_i. \tag{5}$$

We assume that $(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, ...)$ is a set of i.i.d. zero-mean random variables that are distributed as $\tilde{\varepsilon}$. We assume that $E\tilde{\varepsilon} = 0$ and that $E\tilde{\varepsilon}^2 = 1$, so that $v$ can be interpreted as the variance of risk $\tilde{x}_v$. Random variable $\tilde{v}$ has its support in $\mathbb{N}$, and that it is bounded above by $V$. In this section, we also assume that $\tilde{c}^*$, $\tilde{\varepsilon}$ and $\tilde{v}$ are statistically independent. In this section, we are interested in determining the impact of a change in distribution of variance $\tilde{v}$ on expected utility. The key result of this section is given in the next proposition, whose proof is relegated to the Appendix.

**Proposition 1.** *Suppose that $u$ is $2n$ times differentiable and that $\tilde{c}^*$ and $\tilde{v}$ are independent. Any n-th degree risk increase in the variance $\tilde{v}$ of lottery $\tilde{x}_{\tilde{v}}$ reduces (raises) expected utility if and only if $(-1)^n u^{(2n)}$ is negative (positive). It yields a 2n-th degree risk increase (reduction) in $\tilde{c}$ if $n$ is even (odd).*

The simplest illustration of Proposition 1 is obtained for $n = 1$. A first degree increase in risk of $\tilde{v}$ transfers some probability mass from high variance levels to lower ones. Proposition 1 states that risk-averse individuals like any such first degree increase in risk in variance $\tilde{v}$, because it yields a second degree risk reduction in consumption.

The case $n = 2$ is more interesting. The simplest illustration of this case is when $\tilde{v}$ is initially certain, and then becomes uncertain with the same mean. Because this is obviously a second degree risk increase in variance, Proposition 1 states that expected utility is reduced by this shift in the distribution of $\tilde{v}$ if and only if the fourth derivative of the utility function is negative. Eeckhoudt and Schlesinger (2006) provide some illustrations of this result in their section on temperance and on risk apportionment of order 4, by assuming that $\tilde{v}$ can take value in set $\{0, 1, 2\}$, with some stringent restrictions on probabilities. Our proposition generalizes this result to any second degree risk increase in the distribution of variance $\tilde{v}$. Suppose for example that the distribution of $\tilde{v}$ shifts from $(1, 1/2; 2, 1/2)$ to $(1, 3/4; 3, 1/4)$, which is an example of a second degree risk increase in variance that is not covered by Eeckhoudt and Schlesinger (2006). If $\tilde{\varepsilon}$ takes value $-1$ or $+1$ with equal probabilities and if $\tilde{c}^* = 3$ with certainty, Table 1 describes the initial and final distributions of consumption. The first three moments of $\tilde{c}$ are unaffected by the change in distribution, but the fourth moment is increased. Expected utility is decreased (increased) by it if $u^{(4)}$ is negative (positive).

We now turn to the case $n = 3$. As explained by Eeckhoudt and Schlesinger (2006), a third degree risk increase can be obtained by transferring a zero-mean risk $\tilde{\eta}$ from a high outcome to a lower outcome with the same probability. Suppose for example that the variance $\tilde{v}$ is initially distributed as $(1, 1/2; 2 + \tilde{\eta}, 1/2)$. Shifting this white noise from the high variance level to the lower one yields the new distribution $\tilde{v} \sim (1 + \tilde{\eta}, 1/2; 2, 1/2)$. This shift does not change the mean and the variance of $\tilde{v}$, but it reduces its skewness. This is an example of third degree risk increase in the variance. For example, if $\tilde{\eta}$ takes value $-1$ or $+1$ with equal probabilities, the probability distribution of the number $\tilde{v}$ of zero-mean risks in $\tilde{x}_{\tilde{v}}$ shifts from $(1, 3/4; 3, 1/4)$ to $(0, 1/4; 2, 3/4)$. Proposition 1 tells us that this third degree increase in variance risk generates a sixth degree reduction in consumption risk. Therefore, it raises expected utility when the sixth derivative of the utility function is negative. Table 2 describes the initial and final distributions of consumption when $\tilde{c}^* = 3$ and $\tilde{\varepsilon}$ takes value $-1$ and $+1$ with equal probabilities.

We can pursue this exploration to the fourth order, this time by using Proposition 1 recursively. This can be done by assuming that the variance $\tilde{v}$ is itself the sum of an uncertain number of i.i.d. zero-mean risks. More precisely, suppose that $\tilde{v}$ is distributed as

$$\tilde{v} = \tilde{v}^* + \tilde{y}_{\tilde{\theta}}, \qquad \text{with } \tilde{y}_\theta = \sum_{i=1}^{\theta} \tilde{\eta}_i, \tag{6}$$

where $(\tilde{\eta}, \tilde{\eta}_1, \tilde{\eta}_2, ...)$ are i.i.d. with $E\tilde{\eta} = 0$ and $\tilde{\eta}^2 = 1$, $(\tilde{c}^*, \tilde{v}^*, \tilde{\varepsilon}, \tilde{\eta}, \tilde{\theta})$ are independent random variables, and $\tilde{v}$ and $\tilde{\theta}$ have their support in $\mathbb{N}$. Suppose that $n$ is an even number. Then, applying Proposition 1 to $\tilde{v}$ rather than to $\tilde{c}$ implies that a n-th degree risk increase in the variance $\tilde{\theta}$ of the variance $\tilde{v}$ of lottery $\tilde{x}_{\tilde{v}}$ yields a 2n-th degree risk increase in $\tilde{v}$. Applying Proposition 1 again yields a 4n-th degree risk increase in lottery $\tilde{x}_{\tilde{v}}$. This has a negative impact on expected utility if $u^{(4n)}$ is negative. This result is restated as follows.

**Proposition 2.** *Suppose that consumption satisfies equations (4), (5) and (6), and that $u$ is $4n$ times differentiable. Any n-th degree risk increase in the variance $\tilde{\eta}$ of the variance $\tilde{v}$ of $\tilde{c}$*

*reduces (raises) the expected utility $Eu(\tilde{c})$ if and only if $(-1)^n u^{(4n)}$ is negative (positive). It yields a 4n-th degree risk increase (reduction) in $\tilde{c}$ if $n$ is even (odd).*

We hereafter illustrate this result in the case of $n = 2$, which corresponds to the case of increasing the volatility of the volatility. The role of this "vol-of-vol" in asset pricing is examined by Bollerslev et al. (2009), Tauchen (2011) and Drechsler and Yaron (2011). Suppose that both $\tilde{\varepsilon}$ and $\tilde{\eta}$ take value $-1$ or $+1$ with equal probabilities, and that $\tilde{c}^*$ and $\tilde{v}^*$ take respectively value 4 and 2 with certainty. Finally, suppose that the initial distribution of $\tilde{\theta}$ is degenerated at 1, and that it undergoes a second degree risk increase to $(0, 1/2; 2, 1/2)$. This implies that the distribution of $\tilde{v}$ shifts from $(1, 1/2; 3, 1/2)$ to $(0, 1/8; 2, 3/4; 4, 1/8)$, which is indeed a fourth degree risk increase in the variance of $\tilde{c}$. Table 3 describes the 8th degree increase in risk in consumption $\tilde{c}$ that it generates. It reduces expected utility when the eighth derivative of the utility function is negative. The first seven moments of the two distributions are identical, but the 8th moment is increased by the shift in distribution.

# 3 More concordance between income and the n-th degree riskiness in variance

In the previous section, we assumed that the conditional mean income $\tilde{c}^*$ is independent of the conditional variance $\tilde{v}$ of $\tilde{x}_{\tilde{v}}$. In this section, we allow for some statistical dependence between $\tilde{c}^*$ and $\tilde{v}$. More precisely, we are interested in determining the impact on expected utility of an increase in concordance between $\tilde{c}^*$ and the n-th degree riskiness of $\tilde{v}$. We show that signing this effect is linked to the sign of the odd derivatives of $u$. Because the previous section characterized the even derivatives of $u$, this section completes the characterization of the sign of the successive derivatives of the utility function.

As in the previous section, suppose that consumption $\tilde{c}$ is governed by equations (4) and (5), but we relax the assumption that income $\tilde{c}^*$ and variance $\tilde{v}$ are independent. Suppose that $\tilde{v}$ is related to $\tilde{c}^*$ only through some random variable $\tilde{z}$. Suppose also that $\tilde{v}|z$ can be ordered according to the n-th degree risk order, for some $n \in \mathbb{N}_0$. This means that for all $z_1$ and $z_2$ in the support of $\tilde{z}$, we have that

$$z_2 > z_1 \qquad \Rightarrow \qquad \tilde{v}|z_2 \text{ is a n-th degree risk increase of } \tilde{v}|z_1. \tag{7}$$

Thus, $z$ can be interpreted as an index of n-th degree riskiness of $\tilde{v}$. In the following definition, we use the concept of increasing concordance introduced in economics by Epstein and Tanny (1980), Tchen (1980) and Atkinson and Bourguignon (1982). If $F$ and $G$ represent respectively the initial and final joint probability distribution of $(\tilde{c}^*, \tilde{z})$, an increase in concordance between these two random variables prevails if and only if $G(c^*, z)$ is larger than $F(c^*, z)$ for all $(c^*, z)$ in the support of $(\tilde{c}^*, \tilde{z})$, assuming that the marginal distributions are unchanged. More intuitively, any increase in concordance between $\tilde{c}^*$ and $\tilde{z}$ can be constructed by a sequence of simple marginal-preserving transfers of probability masses as described in Figure 1.

**Definition 2.** *We say that the concordance between mean income $\tilde{c}^*$ and the n-th degree risk in variance $\tilde{v}$ is increased if condition (7) is satisfied and $(\tilde{c}^*, \tilde{z})$ becomes more concordant.*

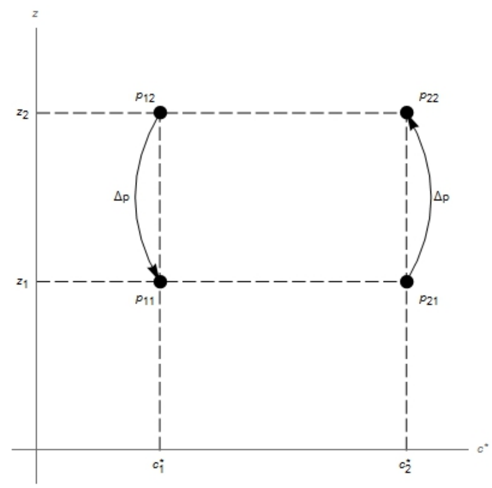


Figure 1: A simple increase in concordance in mean consumption $\tilde{c}^*$ and the index $\tilde{z}$ of the n-th degree riskiness of variance $\tilde{v}$ expressed as two symmetric transfers of probability mass in the support of this pair of random variables.

Let us now define function $m$ as follows:

$$m(c^*, z) = E\left[u\left(c^* + \tilde{x}_{\tilde{v}}\right)|z\right]. \tag{8}$$

This implies that

$$Eu\left(\tilde{c}^* + \tilde{x}_{\tilde{v}}\right) = Em\left(\tilde{c}^*, \tilde{z}\right). \tag{9}$$

As is intuitive from the way simple increases in concordance are built, we know from Epstein and Tanny (1980) that any increase in concordance in $(\tilde{c}^*, \tilde{z})$ raises the expectation of $m$ if and only if $m$ is supermodular. This is true iff

$$m'(c^*, z) = E\left[u'\left(c^* + \tilde{x}_{\tilde{v}}\right)|z\right]. \tag{10}$$

is increasing in $z$. Thus, the problem simplifies to determining the condition under which a n-th degree risk increase in the variance $\tilde{v}$ raises expected marginal utility. Proposition 1 tells us that this is the case if and only if the $(-1)^n u^{(2n+1)}$ is positive. This demonstrates the following result.

**Proposition 3.** *Suppose that $u$ is $2n+1$ times differentiable. Any increase in concordance between the mean income $\tilde{c}^*$ and the n-th degree risk in variance $\tilde{v}$ raises (reduces) the expected utility if and only if $(-1)^n u^{(2n+1)}$ is positive (negative). It yields a (2n+1)-th degree risk reduction (increase) in $\tilde{c}$ if $n$ is even (odd).*

The case $n = 1$ arises when states with larger mean incomes $c^*$ are associated to first-degree stochastically deteriorated distributions of variance $\tilde{v}$. A simple illustration of this case is the risk apportionment of order 3, as defined by Eeckhoudt and Schlesinger (2006).[2] Using Figure 1, this is a case in which $c_1^*$ and $c_2^* > c_1^*$ are the only two possible mean consumption levels and are equally likely. Moreover, the distribution of the conditional variance $\tilde{v} \mid z$ takes

[2]This case was examined earlier by Eeckhoudt et al. (1995).

value $1-z$ with certainty. This obviously implies that an increase in $z$ yields a first-degree stochastic deterioration in the distribution of $\tilde{v} \mid z$. Initially, both $\tilde{z} \mid c_1^*$ and $\tilde{z} \mid c_2^*$ are degenerated, respectively at values $z=1$ and $z=0$. These two degenerated distributions are switched under the new joint distribution, which indeed illustrates an increased concordance. In words, this corresponds to the transfer of a zero-mean risk from a high income level $c_2^*$ to the lower income level $c_1^*$. Proposition 3 tells us that this generates a third degree increase in consumption risk, which is disliked if the third derivative of $u$ is positive.

In order to illustrate the generality of Proposition 3, let us consider a more sophisticated application of the case $n=1$ where $\tilde{c}^*$ is not uniformly distributed and $\tilde{z} \mid c^*$ is not degenerated as in the risk apportionment of order 3. Suppose that the marginal distributions of $\tilde{c}^*$ and $\tilde{z}$ are respectively $(1, 1/3; 2, 2/3)$ and $(0, 1/2; 1, 1/2)$. Suppose also that $\tilde{v} \mid z = 1-z$ as in the previous example. Initially, $\tilde{c}^*$ and $\tilde{z}$ are independent. They are then made positively concordant by shifting a probability mass $\Delta p = 1/6$ as in Figure 1. This increase in concordance between mean income and the first degree riskiness of variance $\tilde{v}$ also implies a third degree increase in consumption risk, as described in Table 4.

We now provide an illustration of Proposition 3 in case $n=2$. This is a case in which higher mean income levels are associated with a riskier variance in the sense of Rothschild and Stiglitz. Suppose for example that $\tilde{c}^*$ and $\tilde{z}$ can take two values, 2 or 3, with equal probabilities. Suppose also that $\tilde{v} \mid z=2$ equals 1 with certainty, whereas $\tilde{v} \mid z=3$ takes value 0 or 2 with equal probabilities. Initially, the mean income $\tilde{c}^*$ and the index $\tilde{z}$ of riskiness of the variance are independent. This joint distribution is shifted to make $\tilde{c}^*$ and $\tilde{z}$ perfectly correlated, an example of increased concordance between mean income and the second degree riskiness of the variance. This shift in distribution is described in Figure 5. From Proposition 3, we know that this implies a fifth degree reduction in consumption risk, as described in Table 5. In fact, the four first moment of $\tilde{c}$ are unaffected by the increased concordance, but the fifth moment is increased. People with a positive fifth derivative of their utility function like this.

# 4 Variance stochastic orders and asset pricing

Our results have useful immediate consequences for asset pricing in the discounted expected utility framework, as we show in this section. Consider a simple two-date Lucas tree economy with a representative agent who consumes $c_0$ today and $\tilde{c} = \tilde{c}^* + \tilde{x}_{\tilde{v}}$ in the future. This yields a lifetime utility equaling

$$u_0(c_0) + Eu_1(\tilde{c}), \tag{11}$$

where $u_0$ and $u_1$ are two increasing utility functions. An investment opportunity arises today that would generate a net benefit $\tilde{b} = \tilde{\xi} g(\tilde{c})$ dollars in the future per dollar invested today. We assume that $\tilde{\xi}$ and $\tilde{c}$ are independent, that $E\tilde{\xi}$ equals one, and that $g$ is a real-valued function. The marginal willingness to pay for this investment equals

$$P(g) = \frac{E\tilde{b}u_1'(\tilde{c})}{u_0'(c_0)} = \frac{Ef(\tilde{c})}{u_0'(c_0)}, \tag{12}$$

with

$$f(c) = g(c)u_1'(c). \tag{13}$$

$P(g)$ can also be interpreted as the equilibrium price of the asset that delivers $\tilde{b} = \tilde{\xi}g(\tilde{c})$ in the future. Equation (12) is the standard asset pricing formula of the consumption-based CAPM, with a price kernel $u_1'(c)/u_0'(c_0)$.

We are interested in determining the effect of the uncertain variance – which is usually referred to as stochastic volatility in the asset pricing literature – of future consumption on the equilibrium price of assets. The results presented in this section are direct consequences of propositions 1 and 3 when applied to equation (12). Let first examine the case of the risk-free asset that is characterized by $\tilde{b} = g(.) = 1$. The interest rate is defined as $r_f = -\log(P(1))$, and is inversely related to the equilibrium price $P$.

**Proposition 4.** *Suppose that the future consumption $\tilde{c}$ of the representative agent satisfies the additive structure (4) and (5). Suppose also that the successive derivatives of the utility function $u_1$ alternate in sign in the relevant domain of future consumption. If $n$ is even (odd), then*

- *Assuming that $\tilde{c}$ and $\tilde{v}$ are independent, a n-th degree risk increase in the variance $\tilde{v}$ of future consumption reduces (raises) the equilibrium interest rate;*
- *An increase in concordance between the mean income $\tilde{c}^*$ and the n-th degree risk in variance $\tilde{v}$ reduces (raises) the equilibrium interest rate.*

For example, a second degree risk increase in the variance of future consumption reduces the interest rate, as does a reduction in concordance between the mean and the variance of future consumption. As an alternative illustration, we can also examine the price of a claim on future consumption in order to evaluate the systematic risk premium in the economy. This corresponds to a function $f$ such that $f(c) = cu_1'(c)$ for all $c$. The equilibrium expected return of equity $\rho$ is defined as $-\log(P(c)/E\tilde{c})$, and is decreasing in $P(c)$.

**Proposition 5.** *Suppose that the future consumption $\tilde{c}$ of the representative agent satisfies the additive structure (4) and (5). Suppose also that the successive derivatives of function $f(c) = cu_1'(c)$ alternate in sign in the relevant domain of future consumption, starting with a negative first derivative. If $n$ is even (odd), then*

- *Assuming that $\tilde{c}$ and $\tilde{v}$ are independent, a n-th degree risk increase in the variance $\tilde{v}$ of future consumption raises (reduces) the equilibrium expected return of equity;*
- *An increase in concordance between the mean income $\tilde{c}^*$ and the n-th degree risk in variance $\tilde{v}$ raises (reduces) the equilibrium equilibrium expected return of equity.*

A special case of this result is obtained when $u_1$ exhibits a constant relative risk aversion larger than unity.[3] In that case, a second degree risk increase in the variance of future consumption or a reduction in concordance between the mean and the variance of future consumption raises the equilibrium expected return of equity.

It is noteworthy that when constant relative risk aversion is larger than unity, reducing the concordance between the mean and the variance of future consumption reduces the risk-free rate and raises the expected rate of return of equity, so that it also raises the equity

[3]When constant relative risk aversion is less than one, the results of Proposition 5 are reversed.

premium. This is linked to a recent observation by Tinang (2017). This author proposes a generalization of the long-run risk model of Bansal and Yaron (2004) in which the shocks on the trend of consumption growth and the shock on its volatility are negatively correlated. It generates a third degree increase in consumption risk (negative skewness), yielding the above mentioned consequences on asset prices. This can contribute to solve the standard asset pricing puzzle.

## 5 A final remark

In this paper, we have characterized the impact of the uncertainty surrounding the variance of consumption on expected utility and asset prices. A crucial element of our analysis is the additive structure of the lottery, which is assumed to be the sum of independent and identically distributed zero-mean risks. The uncertainty is about the number $v$ of risks contained in this lottery. The additive structure of the uncertainty affecting the variance implies that that a n-th risk increase in $\tilde{v}$ does not affect the $2n-1$ moments of the lottery.[4] For example, It is easy to verify that, given (5), we have

$$E\tilde{x}_v^3 = vE\tilde{\varepsilon}^3. \tag{14}$$

Because this conditional expectation is linear in $v$, it implies that a second degree risk increase in $\tilde{v}$ does not affect the third unconditional moment of $\tilde{x}_{\tilde{v}}$.

The structure of the problem is different if we would consider a multiplicative version of the lottery where equation (5) would be replaced by the following specification:

$$\tilde{x}_v = \sqrt{v}\tilde{\varepsilon}, \tag{15}$$

with $E\tilde{\varepsilon} = 0$ and $E\tilde{\varepsilon}^2 = 1$, and where $\tilde{v}$ and $\tilde{\varepsilon}$ are independent. This multiplicative specification is standard in the long-run risk literature initiated by Bansal and Yaron (2004) for example. But it is not true in general that a n-th degree risk increase in $\tilde{v}$ preserves the $2n-1$ first moments of $\tilde{x}_{\tilde{v}}$. For example, it is immediate that in the multiplicative specification (15), we have that

$$E\tilde{x}_v^3 = v^{3/2}E\tilde{\varepsilon}^3. \tag{16}$$

Because this is a convex function of $v$, a second degree risk increase in $v$ yields an increase (reduction) in the skewness of $\tilde{x}_{\tilde{v}}$ if $\tilde{\varepsilon}$ is positively (negatively) skewness. It is thus generally not true in general that a Rothschild-Stiglitz increase in risk in the variance of a random variable yields a 2n-th degree risk increase in that random variable. This illustrates the key role of the additive risk specification (5) of the results presented in this paper.

[4] It is a direct consequence of Proposition 1. It can also be demonstrated from the main result in Packwood (2012) who provides an analytical formula to evaluate the moments of the sum of i.i.d. random variables.

**Appendix**

**Appendix 1: Proof of Theorem 1**

Suppose that the initial probability distribution of $\tilde{v}$ is given by $(p_0, p_1, ..., p_V)$, with $p_i = Pr[\tilde{v} = v_i]$, whereas the final distribution is $(q_0, q_1, ..., q_V)$. For all $v \in \{0, ..., V\}$, and for all $n \in \mathbb{N}_0$, let us define

$$P^n(v) = \sum_{i=0}^{v} P^{n-1}(i), \qquad Q^n(v) = \sum_{i=0}^{v} Q^{n-1}(i), \tag{17}$$

where $P^0(i)$ and $Q^0(i)$ are respectively equal to $p_i$ and $q_i$. Let us define function $h(.,0) : \mathbb{N} \rightarrow \mathbb{R}$ as follows:

$$h(v, 0) = Eu\left(\tilde{c}^* + \tilde{x}_v\right). \tag{18}$$

We can then define recursively functions $h(., n)$ for any integer $n$ in such a way that for all $v \in \mathbb{N}$,

$$h(v, n) = h(v + 1, n - 1) - h(v, n - 1). \tag{19}$$

In fact, the finite-difference function $f(., n)$ plays the same role as the n-th derivative of $f$ if $v$ would be a continuous variable. We are interested in signing the difference in expected utility generated by the shift in distribution of $\tilde{v}$ from $(p_0, ..., p_V)$ to $(q_0, ..., q_V)$. It is defined as follows:

$$\Delta EU = E_q u(\tilde{c}) - E_p u(\tilde{c}) = \sum_{v=0}^{V} (q_v - p_v) h(v, 0). \tag{20}$$

By definition of functions $h(., 1)$, $P^n$ and $Q^n$, this implies that

$$\Delta EU = h(V + 1, 0)\left(Q^1(V) - P^1(V)\right) - \sum_{v=0}^{V} h(v, 1)\left(Q^1(v) - P^1(v)\right). \tag{21}$$

proceeding by recursion, we obtain more generally that

$$\begin{aligned} \Delta EU \quad = \quad & \sum_{k=1}^{n} (-1)^{k-1} h(V + 1, k - 1)\left(Q^k(V) - P^k(V)\right) \\ & + (-1)^n \sum_{v=0}^{V} h(v, n)\left(Q^n(v) - P^n(v)\right). \end{aligned} \tag{22}$$

This equation is the discrete equivalent version of the traditional integration by part equation that is ubiquitous in stochastic dominance theory (see for example Ekern (1980)). The following lemma is a direct consequence of this equation combined with Definition 1.

**Lemma 1.** *Any n-th degree risk increase in the variance $\tilde{v}$ increases (reduces) EU if and only if $(-1)^n h(., n)$ is positive (negative).*

Let us now define the sequence of functions $g(., v, k) : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$g(z, v, 0) \quad = \quad Eu\left(z + \tilde{c}^* + \tilde{x}_v\right) \tag{23}$$

$$g(z, v, k) \quad = \quad Eg(z + \tilde{\varepsilon}, v, k - 1) - g(z, v, k - 1), \qquad k = 1, 2, 3, ... \tag{24}$$

This implies that

$$g(0, v, k) = h(v, k), \tag{25}$$

for all $(v, k) \in \mathbb{N}^2$. Observe also that for all $k \in \mathbb{N}_0$, we have that

$$g^{(i)}(z, v, k) = Eg^{(i)}(z + \tilde{\varepsilon}, v, k-1) - g^{(i)}(z, v, k-1), \tag{26}$$

where $g^{(i)}$ is the ith order derivative of $g$ with respect to $z$. Now, observe that, by Jensen's inequality, $g^{(i)}(., v, k)$ is positive (negative) if and only if $g^{(i+2)}(., v, k-1)$ is positive (negative). By recursivity, this implies that $h(v, n) = g(0, v, n)$ is positive (negative) if and only if $g^{(2n)}(., v, 0)$ is positive (negative). By equation (23), this is true if and only if $u^{(2n)}(.)$ is positive (negative). This result is summarized in the following lemma.

**Lemma 2.** *Assume that $u$ is $2n$ times differentiable. Function $h(., n)$ is positive (negative) if and only if $u^{(2n)}(.)$ is positive (negative).*

Combining these two lemmata implies that any n-th degree risk increase in the variance $\tilde{v}$ increases (reduces) EU if and only if $(-1)^n u^{(2n)}(.)$ is positive (negative). From Ekern (1980), this means that a n-th degree risk increase in the variance $\tilde{v}$ generates a $2n$-th degree risk increase in consumption $\tilde{c}$ if $n$ is an even number, and it generates a $2n$-th degree risk reduction in consumption $\tilde{c}$ if $n$ is an odd number. This completes the proof of Proposition 1. ■

**Fourth degree increase in consumption risk**

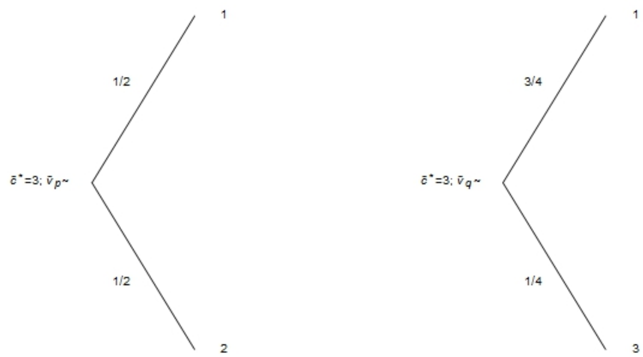


Figure 2: Example of a second degree increase in variance risk.

| consumption | initial probability | final probability |
|---|---|---|
| 0 | | 1/32 |
| 1 | 1/8 | |
| 2 | 1/4 | 15/32 |
| 3 | 1/4 | |
| 4 | 1/4 | 15/32 |
| 5 | 1/8 | |
| 6 | | 1/32 |

Table 1: The second degree increase in variance risk described in Figure 5 generates a 4th degree increase in consumption risk. Illustration of Proposition 1 with $\tilde{\varepsilon} \sim (-1, 1/2; +1, 1/2)$ and $\tilde{c}^* = 3$. The initial distribution of $\tilde{v}$ is $(1, 1/2; 2, 1/2)$. Its final distribution undergoes a second degree risk increase to $(1, 3/4; 3, 1/4)$. Individuals with $u^{(4)}$ negative dislike this shift in distribution.

**Sixth degree reduction in consumption risk**

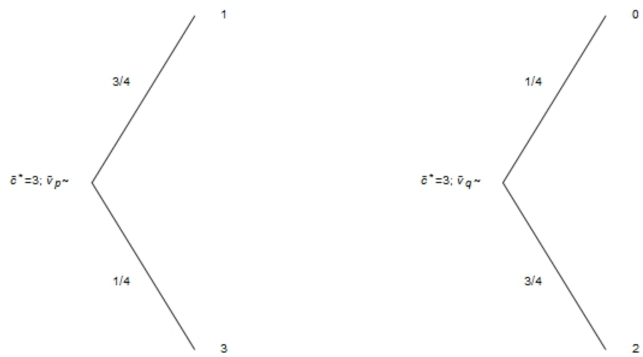


Figure 3: Example of a third degree increase in variance risk.

| consumption | initial probability | final probability |
|---|---|---|
| 0 | 1/32 | |
| 1 | | 3/16 |
| 2 | 15/32 | |
| 3 | | 10/16 |
| 4 | 15/32 | |
| 5 | | 3/16 |
| 6 | 1/32 | |

Table 2: The third degree increase in variance risk described in Figure 5 generates a 6th degree reduction in consumption risk. Illustration of Proposition 1 with $\tilde{\varepsilon} \sim (-1, 1/2; +1, 1/2)$ and $\tilde{c}^* = 3$. The initial distribution of $\tilde{v}$ is $(1, 3/4; 3, 1/4)$. Its final distribution undergoes a third degree risk increase to $(0, 1/4; 2, 3/4)$. Individuals with $u^{(6)}$ negative like this shift in distribution.

**Eight degree increase in consumption risk**

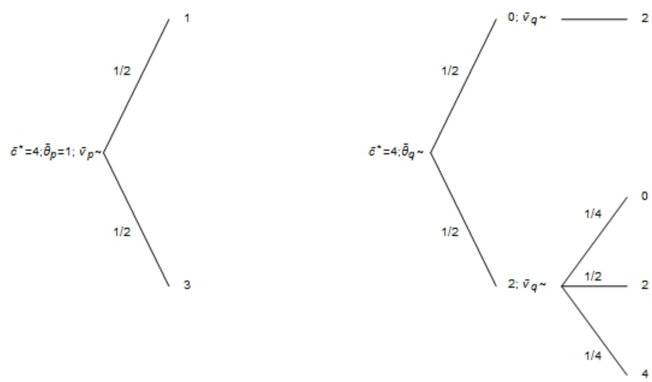


Figure 4: Example of a second degree risk increase in the variance $\tilde{\theta}$ of the variance $\tilde{v}$.

| consumption | initial probability | final probability |
|---|---|---|
| 0 | | 1/128 |
| 1 | 1/16 | |
| 2 | | 28/128 |
| 3 | 7/16 | |
| 4 | | 70/128 |
| 5 | 7/16 | |
| 6 | | 28/128 |
| 7 | 1/16 | |
| 8 | | 1/128 |

Table 3: A fourth degree increase in variance risk generates a 8th degree risk increase in consumption. It is obtained by a second degree risk increase in the variance $\tilde{\theta}$ of the variance $\tilde{v}$, as described in Figure 5. It is therefore also an illustration of Proposition 2 with $\tilde{\varepsilon} \sim \tilde{\eta} \sim (-1, 1/2; +1, 1/2)$, $\tilde{c}^* = 4$, and $\tilde{v}^* = 2$. The initial distribution of $\tilde{\theta}$ is degenerated at 1. Its final distribution undergoes a second degree risk increase to $(0, 1/2; 2, 1/2)$. Individuals with $u^{(8)}$ negative dislike this shift in distribution.

**Third degree increase in consumption risk**

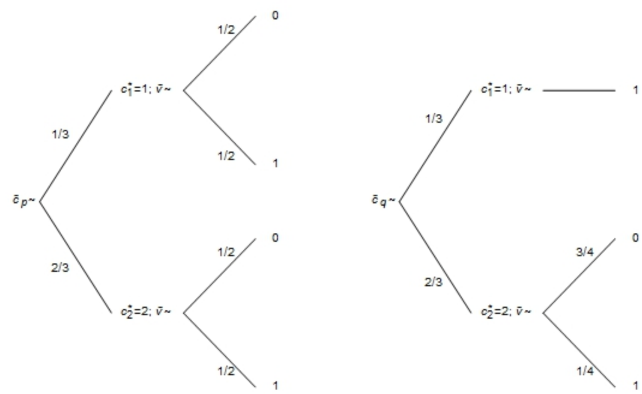


Figure 5: Example of an increase in concordance between the mean income and the first degree risk in variance. We assume that $\tilde{z} = 1 - \tilde{v}$. The initial and final distributions of $(\tilde{c}^*, \tilde{z})$ are as in Figure 1, with $c_1^* = 1$, $c_2^* = 2$, $z_1 = 0$, $z_2 = 1$, $p_{11} = p_{12} = 1/6$ , $p_{21} = p_{22} = 1/3$ and $\Delta p = 1/6$.

| consumption | initial probability | final probability |
|---|---|---|
| 0 | 1/12 | 2/12 |
| 1 | 4/12 | 1/12 |
| 2 | 5/12 | 8/12 |
| 3 | 2/12 | 1/12 |

Table 4: The increase in concordance between the mean income and the first degree risk in variance described in Figure 5 yields a third degree increase in consumption risk. We assume that $\tilde{\varepsilon} \sim (-1, 1/2; , +1, 1/2)$. Individuals with $u^{(3)}$ positive dislike this shift in distribution.

**Fifth degree reduction in consumption risk**

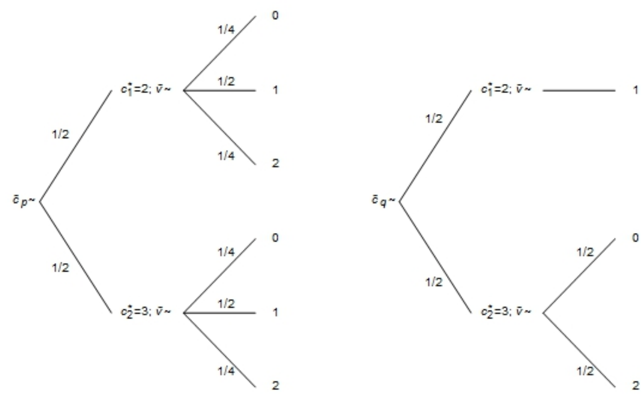


Figure 6: An increase in concordance between the mean income and the second degree risk in variance. We assume that $\tilde{v} \mid z_1$ equals 1 with certainty, and $\tilde{v} \mid z_2$ is distributed as $(0, 1/2; 2, 1/2)$. The initial and final distributions of $(\tilde{c}^*, \tilde{z})$ are as in Figure 1, with $c_1^* = 2$, $c_2^* = 3$, $p_{11} = p_{12} = p_{21} = p_{22} = 1/4$ , $\Delta p = 1/4$.

| consumption | initial probability | final probability |
|---|---|---|
| 0 | 1/32 | |
| 1 | 5/32 | 5/16 |
| 2 | 10/32 | |
| 3 | 10/32 | 10/16 |
| 4 | 5/32 | |
| 5 | 1/32 | 1/16 |

Table 5: The increase in concordance between the mean income and the second degree risk in variance described in Figure 5 yields a fifth degree reduction in consumption risk. We assume that $\tilde{\varepsilon}$ is distributed as $(-1, 1/2; , +1, 1/2)$. Individuals with $u^{(5)}$ positive like this shift in distribution.

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