Business cycle fluctuations and learning-by-doing externalities in a one-sector model[*]

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Abstract: *We consider a one-sector Ramsey-type growth model with inelastic labor and learning-by-doing externalities based on cumulative gross investment (cumulative production of capital goods), which is assumed, in accordance with Arrow [4], to be a better index of experience than the average capital stock. We prove that a slight memory effect characterizing the learning-by-doing process is enough to generate business cycle fluctuations through a Hopf bifurcation leading to stable periodic orbits. This is obtained for reasonable parameter values, notably for both the amount of externalities and the elasticity of intertemporal substitution. Hence, contrary to all the results available in the literature on aggregate models, we show that endogenous fluctuations are compatible with a low (in actual fact, zero) wage elasticity of the labor supply.*

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1 Introduction

Explaining the economic fluctuations associated with the business cycle is one of the main goals of modern macroeconomic theory. Two complementary explanations coexist in the literature: the endogenous cycle theory and the real business cycle theory based on exogenously-driven fluctuations. The common framework used in these two theories is given by the Ramsey [31] optimal growth model. The standard aggregate dynamics are characterized by a unique monotonically convergent equilibrium path and thus business cycle fluctuations can only be obtained if exogenous shocks on the fundamentals are introduced. Contrary to this, multisector optimal growth models easily exhibit endogenous fluctuations without any stochastic perturbation. However, depending on whether time is discrete or continuous, the number of goods matters. In a discrete-time model, the consideration of two sectors with both consumption goods and investment goods is sufficient to generate period-two cycles through a flip bifurcation as shown by Benhabib and Nishimura [11].[1] In a continuous-time model, Benhabib and Nishimura [10] show that at least three sectors with one consumption good and two capital goods need to be considered to generate endogenous fluctuations through a Hopf bifurcation.[2]

It is worth noting here that Hopf bifurcations explain endogenous fluctuations better than flip bifurcations. They are more likely to imply persistent, positively auto-correlated endogenous fluctuations (see Dufourt *et al.* [21]), while the latter generate several counterfactual characteristics in the simulated time series.

More recently, endogenous fluctuations through the existence of local indeterminacy and sunspot equilibria have been shown to occur even in one-sector models. Building on the work by Romer [31], Benhabib and Farmer [9] consider a Ramsey-type continuous-time aggregate model augmented to include economy-wide externalities in the production function measured by the aggregate stock of capital and total labor, which are assumed to be a proxy for some learning-by-doing process. It is indeed assumed that by us-

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[1]The consumption goods sector needs to be more capital intensive than the investment goods sector.

[2]The optimal path necessarily converges monotonically to the steady state in two-dimensional continuous-time models. See Hartl [25] for a proof of this result in general autonomous control problems with one state variable.

ing capital over time, agents increase their experience and are thus able to increase their productivity. Within such a framework, Benhabib and Farmer [9] show that local indeterminacy and fluctuations derived from agents' self-fulfilling expectations can occur can occur, but not through a Hopf bifurcation. Moreover, besides. However, besides external effects in production with large enough increasing returns at the social level, the basic model also has to be increased by the consideration of endogenous labor supply, whose wage elasticity is sufficiently high, i.e., close enough to infinity.[3] Since the elasticity of the aggregate labor supply is usually shown to be low,[4] it follows that the occurrence of local indeterminacy relies on parameter values that do not match empirical evidence.

In this paper, we consider a continuous-time aggregate model with inelastic labor and learning-by-doing externalities in the production process.[5] We depart significantly from most existing contributions in the literature and notably from Romer [31] in which the average level of capital is used as a proxy of experience. We assume, in accordance with Arrow [4], that cumulative gross investment (cumulative production of capital goods) is a better index of experience than the instantaneous average capital stock. More precisely, the learning-by-doing effects are measured by the whole gross investment process over some fixed period of time.[6] This assumption, which will importantly shape the equilibrium dynamics, represents a memory effect suggesting that investments made a long time ago do not have the same impact on the index of experience as recent ones. This can be justified, for instance, by the finite life expectancy of the workers.[7] For computational convenience, we have chosen a memory process similar to a one-hoss shay

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[3]Nishimura *et al.* [29] show that this is a generic condition for local indeterminacy in one-sector models.

[4]Most econometric analyses available in the literature conclude that the wage elasticity of labor lies in $(0, 0.3)$ for men and $(0.5, 1)$ for women (see Blundell and MaCurdy [13]).

[5]It is possible to consider a discrete time framework, but this would yield to a large scale dynamical system for which the analytical characterization is tricky.

[6]D'Autume and Michel [5] consider the original formulation by Arrow in which society's stock of knowledge, measured as the cumulative gross investment over an infinite past, acts as an externality in the production of all firms. They prove that endogenous growth can occur as a result of this.

[7]Nevertheless, in the present paper, we do consider an infinitely-lived representative individual. For the dynamics of an economy with learning-by-doing externalities and a continuum of finitely-lived individuals, see d'Albis and Augeraud-Véron [2].

depreciation: the weight of a given vintage in the index of experience is one during a given time interval, then zero. Our results extend to more general specifications.

Given this assumption, the equilibrium path is described by a system of functional differential equations. Some papers have already studied functional differential equations with Ramsey-type aggregate models, notably with vintage capital. As initially shown by Kalecki [27], some production lag is a possible source of aggregate fluctuations. Benhabib and Rustichini [12] and Boucekkine *et al.* [14] thus show that vintage capital leads to oscillatory dynamics governed by replacement echoes.[8] More recently, Bambi [6] considers an endogenous growth model based on some AK technology with time-to-build, and shows that damped fluctuations occur, but that persistent endogenous fluctuations through a Hopf bifurcation are ruled out. A similar result has been obtained by Bambi and Licandro [7] in the Benhabib and Farmer [9] model augmented to include time-to-build. However, once utility and production functions are non linear, Hopf bifurcations can occur (see Benhabib and Rustichini [12][9], Rustichini [32][10], Asea and Zak [3][11]).

The main difficulty that emerges in the course of the analytical resolution of vintage capital models is that optimality conditions are formulated as a system of mixed functional differential equations in which delayed (the capital stock) and advanced (the shadow price) terms are considered simultaneously. In our paper, introducing a lagged capital stock through a Romer-type externality leads to optimality conditions formulated as a system of delay functional differential equations. This simpler framework notably allows us to provide results on local stability and determinacy.

The main purpose of our paper is to show the existence of a Hopf bi-

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[8]See also Boucekkine *et al.* [16], Boucekkine *et al.* [17] and Fabbri and Gozzi [22].

[9]Benhabib and Rustichini [12] mention the possibility of a Hopf bifurcation in an aggregate model with non-linear utility and vintage capital but do not provide any formal proof of its existence and do not discuss the stability of the bifurcating solutions.

[10]Rustichini [32] considers a two-sector optimal growth model in which delays are introduced on both the control (investment and output) and state (capital stock) variables and derives a system of mixed functional differential equations. He shows that endogenous fluctuations can occur through a Hopf bifurcation.

[11]Asea and Zak [3] consider an exogenous growth model with time-to-build and claim that the steady state can exhibit Hopf cycles. However, their result is puzzling because a time-to-build assumption should lead to a system of mixed functional differential equations with both delay and advance, whereas they only consider delay in their model.

furcation and to establish the conditions under which the equilibrium path converges towards the periodic solution in an aggregate growth model with learning-by-doing externalities. In particular, we prove that persistent endogenous fluctuations can occur, first without considering endogenous labor and external effects coming from the labor supply, and second with standard CES preferences and technology characterized by small values for the elasticity of intertemporal substitution in consumption and a capital-labor elasticity of substitution in line with recent empirical estimates. We hence demonstrate that a simple aggregate model may generate business cycle fluctuations under plausible parameterization of the fundamentals and a low amount of externalities based on a Arrow-type learning-by-doing process. Moreover, we show that the equilibrium, be it convergent toward the steady state or the periodic cycle, is locally determinate, i.e. unique.

The economic intuition for such fluctuations is the following. Assume, for instance, that the initial level of experience is low. Then, private returns to capital are high as well as the level of investment. This increases the experience and reduces the returns to capital. Investments are then slowed down. However, due to the memory function, experience is reduced, which subsequently increases the return to capital. Permanent fluctuations are then possible, whereas they are ruled out with Romer [31]'s assumption.

Our paper also proposes an analysis of the local determinacy around the steady-state. Even though this exercise is intuitively similar to the ones that are performed with systems of ordinary differential equations, one has here to deal with a continuum of initial values and with a stable manifold whose dimension is infinite. We show that the steady-state cannot be indeterminate and give a condition for local determinacy. Non existence of equilibrium paths cannot be excluded in our framework.

The paper is organized as follows: Section 2 presents the model and defines the intertemporal equilibrium. Section 3 contains the main results on the existence of a Hopf bifurcation, the local stability properties of the periodic orbits, the local determinacy of the steady state, and presents a numerical example. Section 4 contains concluding comments and proofs are provided in a final Appendix.

2 The model

2.1 The production structure

Let us consider a perfectly competitive economy in which the final output is produced using capital K and labor L. Although production at the firm level takes place under constant return-to-scale, we assume that each of the many firms benefits from positive externalities due to learning-by-doing effects. We consider indeed that by using capital over time, agents increase their experience and are thus able to increase their productivity. Contrary to most contributions in the literature following Romer [31], in which the average level of capital is used as a proxy of experience, we assume, in accordance with Arrow [4], that cumulative gross investment (cumulative production of capital goods) is a better index of experience. "*Each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli*" (Arrow [4], page 157). However, like Romer [31], we consider that these learning-by-doing effects enter the production process as external effects. In what follow, we consider a model with no endogenous growth but our framework could be easily extended to constant return to scale in the accumulation factor.

The production function of a representative firm is thus $F(K, L, e)$, where $F(K, L, .)$ is homogeneous of degree one with respect to (K, L) and $e \geq 0$ represents the externalities. Denoting, for $L \neq 0$, $k = K/L$ the capital stock per labor unit, we consider a CES production function in intensive form such that

$$f(k, e) = A \left[\alpha k^{-\nu} + (1 - \alpha)e^{-\beta\nu} \right]^{-\frac{1}{\nu}} \tag{1}$$

with $\alpha \in (0, 1)$, $\beta \in (0, 1)$, $\nu > -1$ and $A > 0$. At the optimum of the firm, the interest rate $r(t)$ and the wage rate $w(t)$ then satisfy:

$$r(t) = f_1(k(t), e(t)) - \delta, \quad w(t) = f(k(t), e(t)) - k(t)f_1(k(t), e(t)) \tag{2}$$

with $\delta \geq 0$ the depreciation rate of capital.

We also compute the share of capital in total income:

$$s(k, e) = \frac{k f_1(k,e)}{f(k,e)} = \frac{\alpha}{\alpha + (1-\alpha)e^{-\beta\nu}k^{\nu}} \in (0, 1) \tag{3}$$

the elasticity of capital-labor substitution $\sigma = 1/(1 + \nu)$, and the following share and elasticity related to the externalities e:

$$
\begin{aligned}
\varepsilon_e(k,e) &= \frac{ef_2(k,e)}{f(k,e)} = \beta(1 - s(k,e)), \\
\varepsilon_{ke}(k,e) &= \frac{ef_{12}(k,e)}{f_1(k,e)} = \frac{\beta(1 - s(k,e))}{\sigma}
\end{aligned}
\tag{4}
$$

The share $\varepsilon_e(k,e)$ provides a measure of the size of the externalities and $\varepsilon_{ke}(k,e)$ is the elasticity of the rental rate of capital with respect to e. Note that the restriction $\beta \in (0,1)$ implies that the externalities are small enough to be compatible with a demand for capital which is decreasing with respect to the rental rate and that the marginal productivity of capital is an increasing function of the externalities. We will also assume in the following that $\nu > 0$, i.e. $\sigma \in (0,1)$. This restriction ensures that over the business cycles, the labor share is countercyclical while the capital share is pro-cyclical.[12]

Assuming a stationary population, the dynamics of capital per capita writes:
$$
\dot{k}(t) = f(k(t), e(t)) - \delta k(t) - c(t)
\tag{5}
$$
where $c(t)$ denotes the consumption per capita. As explained previously, we assume that the externalities are generated by a learning-by-doing process in the sense described by Arrow [4], and correspond to the per capita cumulative gross investment:

Assumption 1. $e(t) = \int_{t-\tau}^{t} \left(\dot{k}(s) + \delta k(s) \right) ds \geq 0$, *with* $t \geq \tau \geq 0$.

The parameter τ is exogenous and represents a memory effect. We assume indeed that workers improve their experience by using capital over time but their memory is bounded in the sense that after some time τ, experiences that are too old are forgotten. It is worth noting that a different formulation for the depreciation of memory could be considered. For instance Boucekkine *et al.* [15] assume an exponential depreciation rate.

Remark 1. *The formula in Assumption 1 encompasses the Ramsey [30] model when $\delta = \tau = 0$ and the Romer [31] model when $\delta = 0$, $\tau \to +\infty$ and $\lim_{t \to -\infty} k(t) = 0$ as particular cases.*

2.2 Preferences and intertemporal equilibrium

The representative infinitely-lived individual supplies a fixed amount of labor $l = 1$ and derives utility from consumption c according to a CRRA function

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[12]These properties are shown to match empirical evidences from the US economy over the period 1948-2004 (see Guo and Lansing [24]).

such that

$$u(c) = \frac{c^{1-\frac{1}{\epsilon_c}}}{1-\frac{1}{\epsilon_c}} \tag{6}$$

with $\epsilon_c \in (0, +\infty)$ that denotes the elasticity of intertemporal substitution in consumption.

The intertemporal maximization program of the representative agent is given by:

$$
\begin{aligned}
\max_{c(t),k(t)} \quad & \int_{t=0}^{+\infty} e^{-\rho t} u(c(t)) dt \\
s.t. \quad & \dot{k}(t) = f(k(t), e(t)) - \delta k(t) - c(t) \\
& k(t) = k_0(t) \text{ for } t \in [-\tau, 0] \text{ and } \{e(t)\}_{t \geq 0} \text{ given}
\end{aligned} \tag{7}
$$

where $\rho > 0$ denotes the discount factor. By substituting $c(t)$ from the capital accumulation equation into the utility function we obtain a problem of calculus of variations

$$
\begin{aligned}
\max_{k(t)} \quad & \int_{t=0}^{+\infty} e^{-\rho t} u \left(f(k(t), e(t)) - \delta k(t) - \dot{k}(t) \right) dt \\
s.t. \quad & (k(t), \dot{k}(t)) \in \mathcal{D}(\{e(t)\}_{t \geq 0}) \\
& k(t) = k_0(t) \text{ for } t \in [-\tau, 0] \text{ and } \{e(t)\}_{t \geq 0} \text{ given}
\end{aligned} \tag{8}
$$

with

$$\mathcal{D}(\{e(t)\}_{t \geq 0}) = \left\{ (k(t), \dot{k}(t)) \in \mathbb{R}_+ \times \mathbb{R} | f(k(t), e(t)) - \delta k(t) - \dot{k}(t) \geq 0, \ \forall e(t) \geq 0 \right\}$$

being the convex set of admissible paths. An interior solution to problem (8) satisfies the Euler equation

$$
\begin{aligned}
\left[(f_1(k(t), e(t)) - \delta) \dot{k}(t) + f_2(k(t), e(t)) \dot{e}(t) - \ddot{k}(t) \right] u''(c(t)) \quad & + \\
\left[f_1(k(t), e(t)) - \delta - \rho \right] u'(c(t) \quad & = 0
\end{aligned} \tag{9}
$$

and the transversality condition

$$\lim_{t \to +\infty} u'(c(t)) k(t) e^{-\rho t} = 0 \tag{10}$$

for all given $e(t)$. At the individual level, a solution of the Euler equation (9) is thus a path of capital stock parameterized by a given path of externalities, namely $k(t, \{e(s)\}_{s \geq 0})$. At the aggregate level, as the externalities are defined according to Assumption 1, an equilibrium path is the solution of a fixed-point problem defined such that

$$e(t) = \int_{t-\tau}^{t} \left(\dot{k}(s, \{e(z)\}_{z \geq 0}) + \delta k(s, \{e(z)\}_{z \geq 0}) \right) ds \tag{11}$$

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for all $t \geq 0$. Assuming that such a fixed-point problem has a solution,[13] the capital dynamics are characterized by the following non-linear functional differential equation with distributed delays. By deriving equations (5) and (9) as well as $e(t)$ with respect to time, and replacing the elasticity of intertemporal substitution by the parameter defined above, we obtain:

$$
\begin{aligned}
\ddot{k}(t) = {}& \left[f_1 \left(k(t), k(t) - k(t - \tau) + \delta \int_{t-\tau}^{t} k(s)\,ds \right) - \delta \right] \dot{k}(t) \\
& + f_2 \left(k(t), k(t) - k(t - \tau) + \delta \int_{t-\tau}^{t} k(s)\,ds \right) \\
& \times \left[\dot{k}(t) - \dot{k}(t - \tau) + \delta \left[k(t) - k(t - \tau) \right] \right] \\
& - \epsilon_c \left(f \left(k(t), k(t) - k(t - \tau) + \delta \int_{t-\tau}^{t} k(s)\,ds \right) - \delta k(t) - \dot{k}(t) \right) \\
& \times \left[f_1 \left(k(t), k(t) - k(t - \tau) + \delta \int_{t-\tau}^{t} k(s)\,ds \right) - \delta - \rho \right]
\end{aligned}
\tag{12}
$$

together with the transversality condition (10).

We now study the existence of an interior steady state in the neighborhood of which an equilibrium path exists by continuity.

2.3 Steady state and characteristic equation

We consider the dynamics in the neighborhood of the steady state. Along a stationary path $k(t) = \bar{k}$ for any $t \geq 0$, Assumption 1 implies $e(t) = \bar{e} = \delta \tau \bar{k}$ provided that $\tau \ll +\infty$. An interior steady state is thus a $\bar{k}$ that solves:

$$
f_1 \left(\bar{k}, \delta \tau \bar{k} \right) = \delta + \rho
\tag{13}
$$

and the corresponding stationary consumption level is

$$
\bar{c} = f \left(\bar{k}, \delta \tau \bar{k} \right) - \delta \bar{k} > 0
\tag{14}
$$

We immediately obtain:

Proposition 1. *Under Assumption 1, there exists a unique steady state*

$$
\bar{k} = (\delta \tau)^{\frac{\beta}{1-\beta}} \left(\frac{\left(\frac{\alpha A}{\delta + \rho} \right)^{\frac{\nu}{1+\nu}} - \alpha}{1 - \alpha} \right)^{\frac{1}{(1-\beta)\nu}}
\tag{15}
$$

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[13]In a continuous-time framework, the existence of a solution of this kind of fixed-point problem is a difficult issue. Simple cases in endogenous growth settings have nevertheless been studied by Romer [31] and d'Albis and Le Van [1].

We easily derive from (3) and (4) that at the steady state:

$$s = \alpha \left(\frac{\delta+\rho}{\alpha A} \right)^{\frac{\nu}{1+\nu}}, \quad \varepsilon_e = \beta(1-s), \quad \varepsilon_{ke} = \frac{\beta(1-s)}{\sigma} \qquad (16)$$

The elasticity of intertemporal substitution in consumption ϵ_c will be used as the bifurcation parameter. Let us now establish the characteristic equation.

Lemma 1. *The characteristic equation is* $D(\lambda) = 0$ *with*

$$
\begin{aligned}
D(\lambda) &= \lambda^2 - \rho\lambda - \frac{\epsilon_c(1-s)(\delta+\rho)[\delta(1-s)+\rho]}{s\sigma} \\
&\quad - \frac{\beta(1-s)(\delta+\rho)}{s\tau} \left[\frac{\lambda^2}{\delta} + \left(1 - \frac{\epsilon_c[\delta(1-s)+\rho]}{\sigma\delta} \right)\lambda \right. \\
&\quad \left. - \frac{\epsilon_c[\delta(1-s)+\rho]}{\sigma} \right] \int_{-\tau}^{0} e^{\lambda s} ds = 0
\end{aligned}
\qquad (17)
$$

When $\tau \in (0, +\infty)$, the characteristic equation is transcendental and there exist an infinite number of roots, some of them being complex with negative real part. When $\tau \notin (0, +\infty)$, one may in some cases derive characteristic polynomials that are easy to study. For instance, when $\tau = 0$, there is no externality and the characteristic equation becomes:

$$D(\lambda) = \lambda^2 - \rho\lambda - \frac{\epsilon_c(1-s)(\delta+\rho)[\delta(1-s)+\rho]}{s\sigma}$$

There are two real roots of opposite sign and the steady-state is saddle-point stable. Similarly, if $\delta = 0$, $\tau \to +\infty$ and $\lim_{t\to-\infty} k(t) = 0$, the characteristic equation becomes

$$D(\lambda) = \lambda^2 - \rho\lambda - \frac{\epsilon_c(1-s)\rho^2}{s\sigma}$$

and again there are two real roots of opposite sign.

When there is no externality ($\tau = 0$), the formulation corresponds to a standard optimal growth model with one state variable, the capital stock, and one forward variable, the associated implicit price, or equivalently the time derivative of the capital stock $\dot{k}(t)$. One negative, i.e. stable, characteristic root is then associated to the state variable, while one positive, i.e. unstable, characteristic roots is associated with the forward variable. Starting from a given initial value for the capital stock $k(0)$, the unique optimal path is obtained from the existence of a unique value for the implicit price which allows to select the unique converging path.

When externalities are present ($\tau > 0$), while there is still one state variable, we have now to define a continuum of initial values for the capital stock, i.e. $k(t) = k_0(t)$ for $t \in [-\tau, 0]$. Moreover, as we will show in the

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next section, the initial condition for the forward variable is no longer a real number but is now a continuously differentiable function of time t that has also to be defined on $t \in [-\tau, 0]$.

Because of the delays, the spectral decomposition features an infinite number of characteristic roots with negative real parts. Let us now establish an important result:

Lemma 2. $D(\lambda) = 0$ *has at least a positive real root.*

The existence of characteristic roots with positive real parts implies that any path is unstable in the initial state of continuously differentiable functions on $[-\tau, 0]$. As the transversality condition rules out divergent paths,[14] the initial conditions for the forward variable should be chosen such that the equilibrium path belongs to the space associated with the stable characteristic roots.[15] As it will be shown below, the restrictions imposed on this choice will then depend on the (finite) number of unstable characteristic roots.

3 Endogenous business cycle fluctuations

The occurrence of business cycle fluctuations is obtained through the existence of a Hopf bifurcation, which generates periodic cycles. The analysis is conducted in two steps: first, we study the existence of a Hopf bifurcation and provide conditions for the occurrence of locally stable periodic cycles. Second, we analyze the local determinacy property of the steady state and the bifurcating cycles.

3.1 Hopf bifurcation: existence

In this first part of the analysis, we propose conditions that ensure the existence of a critical value $\epsilon_c^H > 0$ for the elasticity of intertemporal substitution in consumption such that when $\epsilon_c = \epsilon_c^H$, a pair of purely imaginary roots is the solution of the characteristic equation. Let us then consider the following lower bound:

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[14]This property can be shown using standard methods.

[15]From a mathematical point of view, the initial conditions have to be chosen to belong to the direct sum of the stable space and the center space.

$$\underline{\epsilon}_c = \frac{\delta\sigma\beta}{(1-s)(\delta+\rho)\left(\beta-\frac{\delta\tau}{2}\right)} \tag{18}$$

that is positive if $\tau < 2\beta/\delta$. We introduce the following restrictions:

Assumption 2. $\epsilon_c > \underline{\epsilon}_c > 0$.

We provide the following result:

Theorem 1. *Under Assumptions 1-2, there exists a critical value ϵ_c^H such that when $\epsilon_c = \epsilon_c^H$ a Hopf bifurcation occurs generically.*

We prove that endogenous business cycle fluctuations are compatible with small externalities as ε_{ke} is bounded above by $(1-s)/\sigma$, but cannot be obtained for arbitrarily low elasticities of intertemporal substitution in consumption. In Section 3.3, we nevertheless show that the critical value ϵ_c^H can remain compatible with plausible values. It is also worth noting that this result still applies to endogenous labor as long as the wage elasticity of the labor supply remains low enough.

We are now interested in the stability and the direction of the periodic orbit. Indeed, two different cases may occur: the periodic orbit may arise in the right neighborhood of ϵ_c^H and be stable, or may arise in the left neighborhood of ϵ_c^H and be unstable.

We use the methodology of Hassard *et al.* [26] to compute coefficients that determine the Hopf bifurcation direction and the stability properties of the bifurcating periodic solution. Our strategy can be described as follows: We write our system of delay functional differential equations as a system of ordinary differential equations but on a particular space (of functions $C^1([-\tau, 0], \mathbb{R}^2)$), on which we define a bilinear form. We look for the tangent space of the central manifold. We project the solution of the delay functional differential equations system on this tangent space and look at the dynamics that are described by an ordinary differential equation. Some coefficients of the Taylor approximation of this ordinary differential equation give the conditions for stability.

Let $y(t) = k(t) - \bar{k}$ and let us write equation (12) by considering the variable y instead of k. The resulting dynamic system admits 0 as a steady state. Let $\varphi = (\varphi_1, \varphi_2)^t$ with $y(t) = \varphi_2(t)$ and $dy(t)/dt = \varphi_1(t)$. System (12) becomes:

$$\frac{d\varphi_1(t)}{dt} = \left[f_1\left(\varphi_2(t)+\overline{k}, X(t)\right)-\delta\right]\varphi_1(t) + f_2\left(\varphi_2(t)+\overline{k}, X(t)\right)$$
$$\times \quad \left[\varphi_1(t)-\varphi_1(t-\tau)+\delta\left[\varphi_2(t)-\varphi_2(t-\tau)\right]\right]$$
$$- \quad \epsilon_c\left(f\left(\varphi_2(t)+\overline{k}, X(t)\right)-\delta\varphi_2(t)-\delta\overline{k}-\varphi_1(t)\right) \qquad (19)$$
$$\times \quad \left[f_1\left(\varphi_2(t)+\overline{k}, X(t)\right)-\delta-\rho\right]$$
$$\frac{d\varphi_2(t)}{dt} = \varphi_1(t)$$

where

$$X(t) = \varphi_2(t) - \varphi_2(t-\tau) + \delta\tau\overline{k} + \delta\int_{t-\tau}^{t}\varphi_2(s)\,ds$$

Let $\epsilon_c = \epsilon_c^H + \varepsilon$. System (19) can be written as:

$$\dot\varphi(t) = G(\varepsilon, \varphi_t) \qquad (20)$$

A Taylor expansion in the neighborhood of the steady state allows us to split this system into linear and non linear parts. Let $z(t) = \varphi_t$ be a solution of equation (20) when $\varepsilon = 0$. Following Hassard *et al.* [26], the projection of this dynamical system on the center manifold gives the following equation

$$\dot z(t) = i\omega_0 + g(z, \overline{z}) \qquad (21)$$

which allows to discuss the stability of the limit cycle. Let us indeed consider a Taylor expansion of $g(z, \overline{z})$ such that: $g(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + h.o.t.$ From the terms $(g_{02}, g_{11}, g_{20}, g_{21})$, applying the main results of Hassard *et al.* [26], we can compute three coefficients μ_1, μ_2 and μ_3 which will allow to characterize the bifurcation.

Theorem 2. *There exist three coefficients μ_1, μ_2 and μ_3 such that the stability of the periodic solution is determined as follows:*

1. μ_1 determines the direction of the Hopf bifurcation. If $\mu_1 > 0$ then the Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for $\epsilon_c > \epsilon_c^H$.

2. μ_2 determines the stability of the bifurcating periodic solutions. If $\mu_2 < 0$ then the bifurcating periodic solutions are stable.

3. Period T of the periodic solutions is given by

$$T = \frac{2\pi}{\omega_0}\left[1 + \mu_3(\epsilon_c - \epsilon_c^H)^2 + O\left((\epsilon_c - \epsilon_c^H)^4\right)\right]$$

In the Appendix 5.4, we show how to compute these coefficients explicitly and provide, in the section 3.3, some numerical illustrations.

3.2 Local determinacy

Externalities often generate local indeterminacy, i.e. the existence of a continuum of equilibrium paths from a given initial stock of capital. On a more formal basis, this is obtained when the number of stable characteristic roots is larger than the number of pre-determined variables, or equivalently, when the number of unstable characteristic roots is lower than the number of forward variables.

The main difference of our framework with the the standard optimal growth model hinges on the initial condition associated with the forward variable, i.e. the time derivative of the capital stock $\dot{k}(t)$. Contrary to the framework with ordinary differential equations, the initial value in presence of delays is a continuously differentiable function $\dot{k}_0(t)$ that is defined over the interval $[-\tau, 0]$. Therefore, $\dot{k}_0(t)$ has to be compatible with the fact that the initial value for the capital stock is given by $k(t) = k_0(t)$ for $t \in [-\tau, 0]$. As a result the only degree of freedom to compute the initial condition is to define a value for $\dot{k}_0(0)$.

Now, using the result obtained in Lemma 2 such that there exists at least one positive real root, we define the equilibrium path on a particular sub-space. Denoting by $n \geq 1$ the (finite) number of unstable characteristic roots, it can be shown that the function $\dot{k}_0(t)$ also has to satisfy n constraints. We then get the following result:

Proposition 2. *If $n = 1$, the steady state is locally determinate, i.e. there exists a unique equilibrium path, while there does not exist any equilibrium path when $n > 1$.*

Under Assumptions 1-2, the existence of the critical value ϵ_c^H allows now to discuss more precisely the local determinacy property of the steady state. Indeed, we can provide simple conditions on the share of capital s and the delay τ to get $n = 1$ when $\epsilon_c \in [\underline{\epsilon}_c, \epsilon_c^H)$.

Assumption 3. $s \in (0, 1/2)$

Theorem 3. *Under Assumptions 1-3, there exists $\overline{\tau} \in (0, 2\beta/\delta]$ such that when $\tau \in (0, \overline{\tau})$ and $\epsilon_c \in [\underline{\epsilon}_c, \epsilon_c^H)$, the steady state is locally determinate.*

Theorem 3 is based on the fact that, when $\tau \in (0, \overline{\tau})$ and $\epsilon_c \in [\underline{\epsilon}_c, \epsilon_c^H)$, the characteristic equation admits exactly one root with positive real part,

i.e. $n = 1$. As a result, there does not exist any fluctuations based of self-fulfilling expectations although external effects occur into the production process.

The local determinacy of periodic cycles can be also characterized. When ϵ_c is in a right neighborhood of ϵ_c^H, we get $n = 3$ and the steady state is now unstable. But using similar arguments, we conclude that if $\mu_1 > 0$ and $\mu_2 < 0$, the periodic cycle occurs when $\epsilon_c > \epsilon_c^H$ and is locally determinate, in the sense that there exist a unique initial value for the forward variable such that the equilibrium converges toward the limit cycle.

3.3 A numerical illustration

Considering a yearly calibration, we assume that the fundamental parameters are set to the following values: $\nu = 1$, $\alpha = 0.5$, $\delta = 0.1$, $\rho = 0.0808$, and $\tau = 0.1$.[16] It follows that the share of capital is, as usual, $s = 0.3$ and the elasticity of capital-labor substitution is $\sigma = 0.5$. Such a value for σ is in line with recent empirical estimates which show that σ is in the range of $0.4 - 0.6$.[17] The size of externalities in all the following simulations is contained between 0.15 and 0.20, an interval which is in line with the estimations of Basu and Fernald [8], although the latter consider a different production function. We also note that the learning-by-doing process is based on a rather small memory lag τ. But, in the following, we show that this small departure from the standard Ramsey model is enough to generate endogenous business cycle fluctuations.[18]

Recent empirical estimates of the elasticity of intertemporal substitution in consumption provide divergent views. Some authors like Campbell [18] and Vissing-Jorgensen [33] argue that it is definitely below one, while others, like Mulligan [28] and Gruber [23] repeatedly obtained estimates above the unity. However, all acknowledge that the elasticity should belong to the range 0.2-2.2. In the table that follows, we present the Hopf bifurcation value ϵ_c^H and the period of the cycle T computed for various combinations of parameters β and ε_e.

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[16]We have also checked that in a neighborhood of those parameters, the fixed point as defined by (11) exists and satisfies $k(t) > 0$ and $e(t) > 0$.

[17]See Chirinko [19] for a review of the many studies that have attempted to estimate the elasticity of capital-labor substitution using various econometric methods.

[18]Similar results can be obtained with larger values of τ as long as $\tau < 0.7$.

β	ε_e	ϵ_c^H	T
0.286	20%	0.6	12.35
0.245	17.2%	1	4.03
0.243	17%	1.2	3.3
0.24	16.8%	2	2.17

In each configuration, the bifurcating periodic orbit solutions exist when $\epsilon_c > \epsilon_c^H$ (i.e. $\mu_1 > 0$) and are orbitally stable (i.e. $\mu_2 < 0$). Moreover, for any ϵ_c in the right neighborhood of ϵ_c^H, the period of the bifurcating solutions is proportional to T. Note that the lower the amount of externalities ε_e, the higher the Hopf bifurcation value ϵ_c^H and the lower the period of the cycle T.

These numerical illustrations prove that with standard values of the fundamental parameters, persistent endogenous fluctuations easily arise with an elasticity of intertemporal substitution in consumption that is sufficiently high but that still remains compatible with the recent empirical estimates of Campbell [18]. It is also worth noting that similar results still apply to bifurcation values such that $\epsilon_c^H \in (0.5, 2.1)$ when different sizes of externalities are considered, or with endogenous labor as long as the wage elasticity of the labor supply remains low enough, a property that is compatible with the empirical studies of the labor market.[19]

4   Concluding comments

We have considered a one-sector Ramsey-type growth model with inelastic labor and learning-by-doing externalities based on cumulative gross investment, which is assumed, in accordance with Arrow [4], to be a better index of experience than the Romer-type formulation based on the instantaneous aggregate capital stock. We have proven that a slight memory effect characterizing the learning-by-doing process and a small amount of externality are enough to generate business cycle fluctuations through a Hopf bifurcation if the elasticity of intertemporal substitution is high enough but remains within limits compatible with recent empirical estimates. Moreover, contrary to all the results available in the literature on aggregate models, we have shown that endogenous fluctuations are compatible with a zero (or at

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[19]See, for instance, Blundell and MaCurdy [13].

least low enough) wage elasticity of the labor supply. Finally, we have shown that the equilibrium, be it convergent toward the steady state or the limit cycle, is always locally unique.

5 Appendix

5.1 Proof of Lemma 1

Let us use the following notations:
$$f = f\left(\overline{k}, \delta\tau\overline{k}\right), \quad f_1 = f_1\left(\overline{k}, \delta\tau\overline{k}\right), \quad f_2 = f_2\left(\overline{k}, \delta\tau\overline{k}\right),$$
$$f_{11} = f_{11}\left(\overline{k}, \delta\tau\overline{k}\right), \quad f_{12} = f_{12}\left(\overline{k}, \delta\tau\overline{k}\right)$$

Linearizing system (12) around the steady state $\overline{k}$ and defining $k(t) = \overline{k} + \varepsilon x(t)$ leads to

$$\ddot{x}(t) = [f_1 - \delta + f_2]\dot{x}(t) - f_2\dot{x}(t-\tau) + [f_2\delta - \epsilon_c\overline{c}(f_{11} + f_{12})]x(t) \\ - [f_2\delta - \epsilon_c\overline{c}f_{12}]x(t-\tau) - \epsilon_c\overline{c}f_{12}\delta\int_{t-\tau}^{t} x(s)\,ds \tag{22}$$

The characteristic equation $D(\lambda) = 0$ is obtained by replacing $x(t) = x(0)e^{\lambda t}$ and rearranging using $f_1 = \delta + \rho$ together with the shares and elasticities (3), (4). $\qquad\square$

5.2 Proof of Lemma 2

From the definition of $D(\lambda)$, we get $\lim_{\lambda\to\infty} D(\lambda) = +\infty$, and $D(0) < 0$. The result follows. $\qquad\square$

5.3 Proof of Theorem 1

Adding the extra root $\lambda = 0$ and letting $\Delta(\lambda) = \lambda D(\lambda)$, the characteristic polynomial can then be written as a third-order quasi-polynomial

$$\Delta(\lambda) = P(\lambda) + Q(\lambda)e^{-\lambda\tau} \tag{23}$$

with

$$P(\lambda) = \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0, \quad Q(\lambda) = q_2\lambda^2 + q_1\lambda + q_0$$

and

$$q_0 = -\frac{\epsilon_c\beta(1-s)(\delta+\rho)[\delta(1-s)+\rho]}{s\tau\sigma} = -p_0$$

$$q_1 = \frac{\beta(1-s)(\delta+\rho)}{s\tau}\left[1 - \frac{\epsilon_c[\delta(1-s)+\rho]}{\delta\sigma}\right]$$

$$q_2 = \frac{\beta(1-s)(\delta+\rho)}{s\delta\tau} = -(\rho + p_2)$$

$$p_1 = -\left(\frac{u'}{u''}f_{11} + q_1\right) = -\left(\frac{\epsilon_c(1-s)(\delta+\rho)[\delta(1-s)+\rho]}{s\sigma} + q_1\right)$$

(24)

The proof of Theorem 1 is given through the next three lemmas.

Lemma 5.1. *Under Assumption 2, there exists $q > 0$ such that $|Q(iq)| = 0$.*

Proof: We study the occurrence of imaginary roots of the characteristic equation. Let $\lambda = p + i\omega$ and then rewrite equation $\Delta(\lambda) = 0$ such that:

$$-i\omega^3 - p_2\omega^2 + i\omega p_1 + p_0 + \left(-q_2\omega^2 + iq_1\omega + q_0\right)(\cos(\omega\tau) - i\sin(\omega\tau)) = 0$$

We are looking for $\omega_0 > 0$ such that $Q(i\omega_0) = 0$. Separating real and imaginary parts, we have

$$p_2\omega_0 - p_0 = \left(q_0 - q_2\omega_0^2\right)\cos(\omega_0\tau) + q_1\omega_0\sin(\omega_0\tau)$$

$$-\omega_0^3 + p_1\omega_0 = \left(q_0 - q_2\omega_0^2\right)\sin(\omega_0\tau) - q_1\omega\cos(\omega_0\tau)$$

Squaring both sides of the previous equation yields to

$$\omega_0^4 + \left(p_2^2 - 2p_1 - q_2^2\right)\omega_0^2 + \left(p_1^2 + 2p_0\rho - q_1^2\right) = 0$$

which rewrites:

$$x^2 + 2\eta x + \psi = 0 \qquad (25)$$

where $x = \omega_0^2$ and, using the shares and elasticities (3)-(4):

$$\eta = \epsilon_c\frac{(1-s)(\delta+\rho)[\delta(1-s)+\rho]}{\sigma s\delta\tau}(\delta\tau - \beta) + \frac{\beta(1-s)(\delta+\rho)^2}{s\delta\tau} + \frac{\rho^2}{2}$$

$$\psi = \frac{\epsilon_c(1-s)^2(\delta+\rho)^2[\delta(1-s)+\rho]^2(\delta\tau-2\beta)\left(\epsilon_c - \frac{\delta\beta\sigma}{(1-s)(\delta+\rho)\left(\beta-\frac{\delta\tau}{2}\right)}\right)}{s^2\sigma^2\delta\tau}$$

The discriminant of (25) is $\Delta = \eta^2 - \psi$ and the roots are $x_{1,2} = -\eta \pm \sqrt{\eta^2 - \psi}$. A first condition for the existence of a real root is $\Delta \geq 0$. Then there are two cases depending on the sign of ψ:

   - if $\psi < 0$ then $\Delta \geq 0$ and there exists a unique positive real root for any sign of η.

   - if $\psi \geq 0$, then the existence of a positive real root requires $\eta < 0$ and $\eta^2 \geq \psi$.

Assumption 2 implies $\psi < 0$. It follows that the positive root of $x^2 + 2\eta x +$

$\psi = 0$ is $x_1 = -\eta + \sqrt{\eta^2 - \psi}$. As η and ψ are functions of ϵ_c, let us denote $\omega_0 = \omega(\epsilon_c) = \sqrt{x_1}$. We also have

$$
\begin{aligned}
\cos(\omega_0 \tau) &= \frac{\omega_0^4 (q_1 - p_2 q_2) + \omega_0^2 (p_0 q_2 - p_1 q_1 + q_0 p_2) - p_0 q_0}{\left((q_1 \omega_0)^2 + \left(q_0 - q_2 \omega_0^2\right)^2\right)} \\
\sin(\omega_0 \tau) &= \frac{\omega_0^5 q_2 + \omega_0^3 (p_2 q_1 - p_1 q_2 - q_0) + \omega_0 (p_1 q_0 - p_0 q_1)}{\left((q_1 \omega_0)^2 + \left(q_0 - q_2 \omega_0^2\right)^2\right)}
\end{aligned}
\tag{26}
$$

It follows that the bifurcation value ϵ_c^H is obtained as the value of ϵ_c that solves the following equation:

$$
\cos\left(\tau \sqrt{x_1}\right) \equiv G_1(\epsilon_c) = \frac{x_1^2 (q_1 - p_2 q_2) + x_1 (p_0 q_2 - p_1 q_1 + q_0 p_2) - p_0 q_0}{\left(q_1^2 x_1 + (q_0 - q_2 x_1)^2\right)} \equiv G_2(\epsilon_c) \tag{27}
$$

Recall from Assumption 2 that $\epsilon_c \in (\underline{\epsilon_c}, +\infty)$. We can easily show that

$$
\lim_{\epsilon_c \to \underline{\epsilon_c}} G_1(\epsilon_c) = \lim_{\epsilon_c \to \underline{\epsilon_c}} G_2(\epsilon_c) = 1
$$

We can also compute a series expansion of $G_2(\epsilon_c)$ in order to compute the limit when $\epsilon_c \to +\infty$. We obtain:

$$
G_2(\epsilon_c) = -1 + \frac{\delta s \tau}{\epsilon_c} \frac{\left[2q_2 \left(\varepsilon_{ke} - \frac{(1-s)\delta \tau}{2\sigma}\right) + \delta \varepsilon_{ke} + \rho \left(\varepsilon_{ke} - \frac{(1-s)\tau}{2\sigma}\right)\right]^2}{\varepsilon_{ke}^2 (\delta + \rho)[\delta(1-s) + \rho]\left(\varepsilon_{ke} - \frac{(1-s)\delta \tau}{2\sigma}\right)} + o\left(\frac{1}{\epsilon_c^2}\right)
$$

It follows that under Assumption 2, $\lim_{\epsilon_c \to +\infty} G_2(\epsilon_c) = -1^+$. Moreover, as ϵ_c increases from $\underline{\epsilon_c}$ to $+\infty$, the function $G_1(\epsilon_c)$ oscillates continuously between 1 and -1. It follows that there necessarily exist an infinite number of solutions to equation (27) for ϵ_c large enough. $\qquad \square$

Let us then consider the smallest solution of equation (27), denoted ϵ_c^H, which corresponds to the Hopf bifurcation value such that $\pm i \omega_0$ is an imaginary root of (23).

Lemma 5.2. *$\pm i \omega_0$ is generically a simple root.*

Proof: If we suppose by contradiction that it is not a simple root, we have

$$
P'(i\omega_0) + \left(Q'(i\omega_0) - \tau Q(i\omega_0)\right) e^{-i\omega_0 \tau} = 0
$$

separating imaginary and real part, and squaring each member, we have:

$$
\omega_0^4 \left(\tau^2 q_2^2 - 9\right) + \omega_0^2 \left(6 p_1 - 4 p_2^2 + 2 \tau q_2 (q_1 - \tau q_0) + (2 q_2 - \tau q_1)^2\right) + (q_1 - \tau q_0)^2 - p_1^2 = 0
$$

As it is also a root of the characteristic equation, we also have

$$
\omega_0^4 + \left(p_2^2 - 2 p_1 - q_2^2\right) \omega_0^2 + \left(p_1^2 + 2 p_0 \rho - q_1^2\right) = 0
$$

That implies

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$$\left(+\sqrt{\left(p_2^2 - 2p_1 - q_2^2\right)^2 - 4\left(p_1^2 + 2p_0\rho - q_1^2\right)} \right)$$

$$= \left(\begin{array}{c} -\left(6p_1 - 4p_2^2 + 2\tau q_2\left(q_1 - \tau q_0\right) + \left(2q_2 - \tau q_1\right)^2\right) \\ -\sqrt{\begin{array}{c}\left(6p_1 - 4p_2^2 + 2\tau q_2\left(q_1 - \tau q_0\right) + \left(2q_2 - \tau q_1\right)^2\right)^2 \\ -4\left(\left(q_1 - \tau q_0\right)^2 - p_1^2\right)\left(\tau^2 q_2^2 - 9\right)\end{array}} \end{array} \right)$$

Such equality is non generic. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let ϵ_c^H be the value for ϵ_c for which we have an imaginary root.

Lemma 5.3. $Re\left(\frac{d\lambda(\epsilon_c)}{d\epsilon_c}\right)_{|\epsilon_c=\epsilon_c^H} \neq 0$

Proof: Let us differentiate the following equation according to ϵ_c:

$$\Delta\left(\lambda, \epsilon_c\right) = \left(\lambda^3 + p_2\lambda^2 + p_1\left(\epsilon_c\right)\lambda + p_0\left(\epsilon_c\right)\right) + \left(q_2\lambda^2 + q_1\left(\epsilon_c\right)\lambda + q_0\left(\epsilon_c\right)\right)e^{-\lambda\tau} = 0$$

As $i\omega_0$ is a simple root, we can use the implicit function theorem

$$\left(\frac{d\lambda}{d\epsilon_c}\right)_{|\epsilon_c=\epsilon_c^H} = -\frac{\left(\left(p_1'\left(\epsilon_c^H\right)\lambda + p_0'\left(\epsilon_c^H\right)\right) + \left(q_1'\left(\epsilon_c^H\right)\lambda + q_0'\left(\epsilon_c^H\right)\right)e^{-\lambda\tau}\right)}{\left(P'(\lambda) + \left(Q'(\lambda) - \tau Q(\lambda)\right)e^{-\lambda\tau}\right)}$$

$$q'\left(\epsilon_c^H\right) = -\frac{\varepsilon_{ke}(\delta+\rho)(\delta(1-s)+\rho)}{s\delta\tau} = \frac{q_1}{\epsilon_c^H} - \frac{\varepsilon_e(\delta+\rho)}{\epsilon_c^H s\tau}$$

$$p'\left(\epsilon_c^H\right) = -\left(\frac{\left((1-s)(\delta+\rho)(\delta(1-s)+\rho)\right)}{s\sigma} + \frac{q_1}{\epsilon_c^H} - \frac{\varepsilon_e(\delta+\rho)}{\epsilon_c^H s\tau}\right)$$

$$p'\left(\epsilon_c^H\right) = \frac{p_1}{\epsilon_c^H} + \frac{\varepsilon_e(\delta+\rho)}{\epsilon_c^H s\tau}$$

$$q_0' = \frac{q_0}{\epsilon_c^H} = -p_0' = \frac{-p_0}{\epsilon_c^H}$$

$$\left(p_1'\left(\epsilon_c^H\right)\lambda + p_0'\left(\epsilon_c^H\right)\right) + \left(q_1'\left(\epsilon_c^H\right)\lambda + q_0'\left(\epsilon_c^H\right)\right)e^{-\lambda\tau}$$

$$= \left(\frac{p_1}{\epsilon_c^H} + \frac{\varepsilon_e(\delta+\rho)}{\epsilon_c^H s\tau}\right)\lambda + \frac{p_0}{\epsilon_c^H} + \left(\left(\frac{q_1}{\epsilon_c^H} - \frac{\varepsilon_e(\delta+\rho)}{\epsilon_c^H s\tau}\right)\lambda + \frac{q_0}{\epsilon_c^H}\right)e^{-\lambda\tau}$$

$$= \frac{p_1}{\epsilon_c^H}\lambda + \frac{p_0}{\epsilon_c^H} + \left(\frac{q_1}{\epsilon_c^H}\lambda + \frac{q_0}{\epsilon_c^H}\right)e^{-\lambda\tau} + \left(1 - e^{-\lambda\tau}\right)\frac{\varepsilon_e(\delta+\rho)}{\epsilon_c^H s\tau}\lambda = \frac{-\lambda^2\left(\lambda + p_2 + q_2 e^{-\lambda\tau} - \frac{\left(1-e^{-\lambda\tau}\right)}{\lambda}\delta q_2\right)}{\epsilon_c^H}$$

Substituting $\lambda = i\omega_0$, we get:

$$\left(\begin{array}{c} \left(p_1'\left(\epsilon_c^H\right)\lambda + p_0'\left(\epsilon_c^H\right)\right) \\ +\left(q_1'\left(\epsilon_c^H\right)\lambda + q_0'\left(\epsilon_c^H\right)\right)e^{-\lambda\tau} \end{array} \right)_{|\lambda=i\omega_0} = \frac{\omega^2}{\epsilon_c^H}\left(\begin{array}{c} p_2 + q_2\cos\left(\omega_0\tau\right) - \frac{\sin(\omega_0\tau)}{\omega}\delta q_2 \\ +i\left(\omega_0 - q_2\sin\left(\omega_0\tau\right) + \frac{1-\cos(\omega_0\tau)}{\omega_0}\delta q_2\right) \end{array} \right)$$

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Moreover

$$P^{'}\left(\lambda\right)+\left(Q^{'}\left(\lambda\right)-\tau Q\left(\lambda\right)\right)e^{-\lambda\tau} = P^{'}\left(\lambda\right)+Q^{'}\left(\lambda\right)e^{-\lambda\tau}+\tau P\left(\lambda\right)$$
$$= \tau\lambda^3+\left(3+\tau p_2\right)\lambda^2+\left(2p_2+\tau p_1\right)\lambda+\left(p_1+\tau p_0\right)$$
$$+ e^{-\lambda\tau}\left(2q_2\lambda+q_1\right)$$

$$\left(P^{'}\left(\lambda\right)+\left(Q^{'}\left(\lambda\right)-\tau Q\left(\lambda\right)\right)e^{-\lambda\tau}\right)_{|\lambda=i\omega_0} = \begin{pmatrix} -\left(3+\tau p_2\right)\omega_0^2+\left(p_1+\tau p_0\right) \\ +\cos\left(\omega_0\tau\right)q_1 \\ +2q_2\omega\sin\left(\omega_0\tau\right) \\ +i\begin{pmatrix} -\tau\omega_0^3+\left(2p_2+\tau p_1\right)\omega_0 \\ -\sin\left(\omega_0\tau\right)q_1+2q_2\omega_0\cos\left(\omega_0\tau\right) \end{pmatrix} \end{pmatrix}$$

Let us then consider

$$Re\left(\frac{\begin{pmatrix} -\left(3+\tau p_2\right)\omega_0^2+\left(p_1+\tau p_0\right) \\ +\cos\left(\omega_0\tau\right)q_1 \\ +2q_2\omega_0\sin\left(\omega_0\tau\right) \end{pmatrix}+i\begin{pmatrix} -\tau\omega_0^3+\left(2p_2+\tau p_1\right)\omega_0 \\ -\sin\left(\omega_0\tau\right)q_1+2q_2\omega_0\cos\left(\omega_0\tau\right) \end{pmatrix}}{p_2+q_2\cos(\omega_0\tau)-\frac{\sin(\omega_0\tau)}{\omega_0}\delta q_2+i\left(\omega_0-q_2\sin(\omega_0\tau)+\frac{1-\cos(\omega_0\tau)}{\omega_0}\delta q_2\right)}\right)$$

$$= \begin{pmatrix} -\left(3+\tau p_2\right)\omega_0^2+\left(p_1+\tau p_0\right) \\ +\cos\left(\omega_0\tau\right)q_1 \\ +2q_2\omega_0\sin\left(\omega_0\tau\right) \end{pmatrix}\left(p_2+q_2\cos\left(\omega_0\tau\right)-\frac{\sin(\omega_0\tau)}{\omega_0}\delta q_2\right)$$

$$+ \left(\begin{pmatrix} -\tau\omega_0^3+\left(2p_2+\tau p_1\right)\omega_0 \\ -\sin\left(\omega_0\tau\right)q_1+2q_2\omega_0\cos\left(\omega_0\tau\right) \end{pmatrix}\left(\omega_0-q_2\sin\left(\omega_0\tau\right)+\frac{1-\cos(\omega_0\tau)}{\omega_0}\delta q_2\right)\right)$$

$$= p_1p_2+q_1q_2+\tau p_0p_2+2\delta p_2q_2+\tau\delta p_1q_2-\tau\omega_0^4-\omega_0^2p_2+\tau\omega_0^2p_1-\tau\delta\omega_0^2q_2-\tau\omega_0^2p_2^2-2\delta q_2^2$$

$$+ \begin{pmatrix} 3\delta\omega_0q_2-\omega_0q_1-\tau\omega_0p_1q_2+\tau\delta\omega_0p_2q_2+\tau\omega_0^3q_2 \\ -\frac{\delta}{\omega_0}p_1q_2-\frac{\delta}{\omega_0}q_1q_2-\tau\frac{\delta}{\omega_0}p_0q_2 \end{pmatrix}\sin\tau\omega_0$$

$$+ \begin{pmatrix} p_1q_2+p_2q_1+\tau p_0q_2-2\delta p_2q_2-\tau\delta p_1q_2 \\ +2\delta q_2^2-\omega_0^2q_2+\tau\delta\omega_0^2q_2-\tau\omega_0^2p_2q_2 \end{pmatrix}\cos\tau\omega_0 = H\left(\omega_0\right)$$

$H\left(\omega_0\right)$ rewrites

$$H\left(\omega_0\right)=H_0+H_2\omega_0^2+H_4\omega_0^4+H_6\omega_0^6$$

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where expressions of H_i can be computed. As $\omega_0^4 + \left(p_2^2 - 2p_1 - q_2^2\right)\omega_0^2 + \left(p_1^2 + 2p_0\rho - q_1^2\right) = 0$, $\widetilde{H}\left(\omega_0\right) = H\left(\omega_0\right)$ where $\widetilde{H}$ is defined by

$$
\begin{aligned}
\widetilde{H}\left(\omega\right) &= H\left(\omega\right) - \left(H_6\omega^2 + H_4 - \frac{H_6}{\left(p_2^2 - 2p_1 - q_2^2\right)}\right)\left(\omega^4 + \left(p_2^2 - 2p_1 - q_2^2\right)\omega^2\right. \\
&+ \left.\left(p_1^2 + 2p_0\rho - q_1^2\right)\right) = A_2\omega^2 + A_0
\end{aligned}
$$

So replacing ω_0^2 we can compute $Re\left(\left(\frac{d\lambda}{d\epsilon_c}\right)_{|\epsilon_c=\epsilon_c^H}\right)$ as a function of ϵ_c

$$
Re\left(\left(\frac{d\lambda}{d\epsilon_c}\right)_{|\epsilon_c=\epsilon_c^j}\right) = \varphi_0 + \varphi_1\epsilon_c^j + \varphi_2\left(\epsilon_c^j\right)^2
$$

where φ_0, φ_1, φ_2 are independent of ϵ_c^j. Then there exists ϵ_c^j such that $Re\left(\left(\frac{d\lambda}{d\epsilon_c}\right)_{|\epsilon_c=\epsilon_c^j}\right) \neq 0$. $\qquad\square$

5.4 Proof of Theorem 2

We follow the procedure described by Hassard *et al.* [26] and we provide various Lemmas to derive the dynamics of the system on the center manifold.

Lemma 5.4. *System (20) can be written as the following functional differential equation in C:*

$$
\dot{\varphi}\left(t\right) = \Lambda_\varepsilon\varphi_t + F\left(\varepsilon, \varphi_t\right) \tag{28}
$$

where $\varphi_t\left(\theta\right) = \varphi\left(t + \theta\right)$ and $\Lambda_\varepsilon : C \to \mathbb{R}^2$ is given by

$$
\Lambda_\varepsilon\varphi = L\left(\epsilon_c^H + \varepsilon\right)\varphi\left(0\right) + R\left(\epsilon_c^H + \varepsilon\right)\varphi\left(-\tau\right) + M\left(\epsilon_c^H + \varepsilon\right)\int_{-\tau}^0\varphi\left(u\right)du
$$

with

$$
L\left(\epsilon_c\right) = \begin{bmatrix} -p_2 & -p_1 \\ 1 & 0 \end{bmatrix}, R\left(\epsilon_c\right) = \begin{bmatrix} -q_2 & -q_1 \\ 0 & 0 \end{bmatrix}, M\left(\epsilon_c\right) = \begin{bmatrix} 0 & -p_0 \\ 0 & 0 \end{bmatrix}
$$

the coefficients p_i, q_i being defined in (24), and $F\left(\varepsilon, \varphi_t\right) = G\left(\varepsilon, \varphi_t\right) - \Lambda_\varepsilon\varphi_t$.

Proof: Equation in the y variable writes

$$
\begin{aligned}
\ddot{y}\left(t\right) &= \left[f_1\left(y\left(t\right) + \overline{k}, y\left(t\right) - y\left(t - \tau\right) + \delta\overline{k} + \delta\int_{t-\tau}^t y\left(s\right)ds\right) - \delta\right]\dot{y}\left(t\right) \\
&+ f_2\left(y\left(t\right) + \overline{k}, y\left(t\right) - y\left(t - \tau\right) + \delta\overline{k} + \delta\int_{t-\tau}^t y\left(s\right)ds\right) \\
&\times \left[\dot{y}\left(t\right) - \dot{y}\left(t - \tau\right) + \delta\left[y\left(t\right) - y\left(t - \tau\right)\right]\right] \\
&- \epsilon_c\left(f\left(y\left(t\right) + \overline{k}, y\left(t\right) - y\left(t - \tau\right) + \delta\overline{k} + \delta\int_{t-\tau}^t y\left(s\right)ds\right) - \delta\overline{k} - \delta y\left(t\right) - \dot{y}\left(t\right)\right) \\
&\times \left[f_1\left(y\left(t\right) + \overline{k}, y\left(t\right) - y\left(t - \tau\right) + \delta\overline{k} + \delta\int_{t-\tau}^t y\left(s\right)ds\right) - \delta - \rho\right]
\end{aligned}
$$

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The linearization of the system at $(0,0,0)$ is

$$
\begin{aligned}
\dot\varphi_1(t) &= [\rho + f_2]\,\varphi_1(t) - f_2\varphi_1(t-\tau) + [f_2\delta - \epsilon_c\bar c\,(f_{11}+f_{12})]\,\varphi_2(t) \\
&\quad - [f_2\delta - \epsilon_c\bar c f_{12}]\,\varphi_2(t-\tau) - \epsilon_c\bar c f_{12}\delta\,(\varphi_3(t) - \varphi_3(t-\tau)) \\
\dot\varphi_2(t) &= \varphi_1(t) \\
\dot\varphi_3(t) &= \varphi_2(t)
\end{aligned}
\tag{29}
$$

Let $F : \mathbb{R} \times C \to \mathbb{R}$ and denote the partial derivatives of f as $f_i = f_i\left(\bar k, \delta\tau\bar k\right)$, $f_{ij} = f_{ij}\left(\bar k, \delta\tau\bar k\right)$, $f_{ijk} = f_{ijk}\left(\bar k, \delta\tau\bar k\right)$, $f_{ijkl} = f_{ijkl}\left(\bar k, \delta\tau\bar k\right)$, $i,j,k,l = 1,2$. $\qquad\square$

The following Lemma gives a Taylor expansion up to order three of $F\left(\varepsilon, \varphi_{1t}(0), \varphi_{1t}(-\tau), \varphi_{2t}(0), \varphi_{2t}(-\tau), \int_{-\tau}^{0}\varphi_{2t}(u)\,du\right)$.

Lemma 5.5. *Let* $\left(\varphi_{1t}(0), \varphi_{1t}(-\tau), \varphi_{2t}(0), \varphi_{2t}(-\tau), \int_{-\tau}^{0}\varphi_{2t}(u)\,du\right) = (x_1, x_2, x_3, x_4, x_5)$. *Then*

$$
F\left(\varepsilon, \varphi_{1t}(0), \varphi_{1t}(-\tau), \varphi_{2t}(0), \varphi_{2t}(-\tau), \int_{-\tau}^{0}\varphi_{2t}(u)\,du\right)
$$

$$
= \sum_{i=1}^{5}\sum_{j=i}^{5} a_{ij}x_i x_j + \sum_{i=1}^{5}\sum_{j=i}^{5}\sum_{m=j}^{5} a_{ijm}x_i x_j x_m
$$

with a_{ij} and a_{ijm} some coefficients that depend on the second, third and fourth order derivatives of the production function evaluated at the steady state.[20]

We can also formulate $\Lambda_\varepsilon\varphi$ as follows:

$$
\Lambda_\varepsilon\varphi = \int_{-\tau}^{0} d\eta(u)\,\varphi(u)
$$

with

$$
d\eta(\epsilon_c, u) = L(\epsilon_c)\,\delta(u) + R(\epsilon_c)\,\delta(u+\tau) + M(\epsilon_c)\,du
$$

To determine the normal form, the projection method is used as in Hassard *et al.* [26]. We first need to compute the eigenvector relative to the eigenvalue $i\omega_0$. Instead of writing the delay dynamic system, we use the infinitesimal generator expression, as is usually done for delay functional differential equations. For $\varphi \in C^1\left([-\tau, 0], \mathbb{R}^2\right)$, let us define

$$
A(\varepsilon)\varphi = \begin{cases} \dfrac{d\varphi(\theta)}{d\theta}, & \text{if } \theta \in [-\tau, 0[\\[2mm] L\left(\epsilon_c^H + \varepsilon\right)\varphi(0) + R\left(\epsilon_c^H + \varepsilon\right)\varphi(-\tau) + M\left(\epsilon_c^H + \varepsilon\right)\int_{-\tau}^{0}\varphi(u)\,du, & \text{if } \theta = 0 \end{cases}
$$

---

[20]The expressions of these coefficients are available upon request.

and

$$G\left(\varepsilon\right)\varphi = \begin{cases} 0, \text{ if } \theta \in [-\tau, 0[\\ F\left(\varepsilon, \varphi\right), \text{ if } \theta = 0 \end{cases}$$

It follows that (28) is equivalent to

$$\dot{\varphi}_t = A\left(\varepsilon\right)\varphi_t + G\left(\varepsilon\right)\varphi_t \tag{30}$$

To compute the normal form on the center manifold, we use the projection method, which is based on the computation of the eigenvector relative to $i\omega_0$ and the corresponding adjoint eigenvector. The computation of the adjoint eigenvector requires the definition of the adjoint space and adjoint operator of $A\left(\varepsilon\right)$.

We define the adjoint space C^* of continuously differentiable functions $\chi : [0, \tau] \rightarrow \mathbb{R}^2$ with the adjoint operator $A^*\left(\varepsilon\right)$.

$$A^*\left(\varepsilon\right)\chi = \begin{cases} -\frac{d\chi(\sigma)}{d\sigma}, \text{ for } \sigma \in]0, \tau] \\ \int_{-\tau}^{0} d\overline{\eta}^t\left(\varepsilon, t\right)\chi\left(-t\right) \text{ for } \sigma = 0 \end{cases}$$

Remark 2. *As $d\eta\left(\varepsilon, t\right)$ is real, we have $d\eta^t\left(\varepsilon, t\right) = d\overline{\eta}^t\left(\varepsilon, t\right)$.*

We consider the bilinear form

$$\begin{aligned} (v, u) &= \overline{v}^t\left(0\right)u\left(0\right) - \int_{\theta=-\tau}^{0}\int_{\xi=0}^{\theta}\overline{v}^t\left(\xi - \theta\right)d\eta\left(\epsilon_c^H + \varepsilon, \theta\right)u\left(\xi\right)d\xi \\ &= \overline{v}^t\left(0\right)u\left(0\right) + \int_{\xi=-\tau}^{0}\overline{v}^t\left(\xi + \tau\right)R\left(\epsilon_c^H + \varepsilon\right)u\left(\xi\right)d\xi \\ &\quad - \int_{\theta=-\tau}^{0}\int_{\xi=0}^{\theta}\overline{v}^t\left(\xi - \theta\right)M\left(\epsilon_c^H + \varepsilon\right)u\left(\xi\right)d\xi d\theta \end{aligned}$$

The following Lemma now provides a basis for the eigenspace and adjoint eigenspace.

Lemma 5.6. *Let $q\left(\theta\right)$ be the eigenvector of A associated with eigenvalue $i\omega_0$, and $q^*\left(\sigma\right)$ be the eigenvector associated with $-i\omega_0$. Then*

$$q\left(\theta\right) = \begin{pmatrix} i\omega_0 \\ 1 \end{pmatrix}e^{i\omega_0\theta}, q^*\left(\theta\right) = u_1\begin{pmatrix} 1 \\ -\frac{\left(p_1 + q_1 e^{-i\omega_0\tau} - ip_0\frac{1-e^{-i\omega_0\tau}}{\omega_0}\right)}{i\omega_0} \end{pmatrix}e^{i\omega_0\theta}$$

with

$$\overline{u}_1 = \left[i\omega_0 + i\frac{\frac{\left(-p_1 - q_1 e^{i\omega_0\tau} + ip_0\frac{\left(-1 + e^{i\omega_0\tau}\right)}{\omega_0}\right)}{\omega_0}}{-p_0\frac{\left(-1 + i\omega_0\tau e^{-i\omega_0\tau} + e^{-i\omega_0\tau}\right)}{\omega_0^2}} - \tau(iq_2\omega_0 + q_1)e^{-i\omega_0\tau}\right]^{-1}$$

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Proof: As $q(\theta)$ is the eigenvector of A associated with eigenvalue $i\omega_0$, $q(\theta)$ solves, for $\theta \neq 0$

$$\frac{dq}{d\theta} = i\omega_0 q \Rightarrow q(\theta) = q(0)\, e^{i\omega_0\theta}$$

For $\theta = 0$, initial conditions write:

$$L\left(\epsilon_c^H\right) q(0) + R\left(\epsilon_c^H\right) q(-\tau) + M\left(\epsilon_c^H\right) \int_{-\tau}^0 q(u)\, du = i\omega_0 q(0)$$

Let $q(0) = v = (v_1, v_2)^t$. Replacing the expression we first obtained in the second equation yields to

$$L\left(\epsilon_c^H\right) q(0) + R\left(\epsilon_c^H\right) v e^{-i\omega_0\tau} + M\left(\epsilon_c^H\right) v \frac{1 - e^{-i\omega_0\tau}}{i\omega_0} = i\omega_0 v$$

that is

$$-\left[\left(p_2 + q_2 e^{-i\omega_0\tau}\right) v_1 + \left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \frac{1-e^{-i\omega_0\tau}}{i\omega_0}\right) v_2\right] = i\omega_0 v_1$$
$$v_1 = i\omega_0 v_2$$

Substituting v_1 obtained in the second equation as a function of v_2 we have

$$-v_2\left[-\omega_0^2 + \left(p_2 + q_2 e^{-i\omega_0\tau}\right) i\omega_0 + \left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \frac{1-e^{-i\omega_0\tau}}{i\omega_0}\right)\right] = 0$$
$$v_1 = i\omega_0 v_2$$

which rewrites

$$v_2\left[D\left(i\omega_0\right)\right] = 0$$
$$v_1 = i\omega_0 v_2$$

As $i\omega_0$ is a root of the characteristic equation, we can choose v_2 as we want (for example $v_2 = 1$), so v is completely determined. Similarly we obtain:

$$\eta_1(\sigma) = e^{i\omega_0\sigma}$$

with initial conditions:

$$L^t\left(\epsilon_c^H\right) \eta_1(0) + R^t\left(\epsilon_c^H\right) \eta_1(-\tau) + \int_{-\tau}^0 M^t\left(\epsilon_c^H\right) \eta_1(u)\, du = i\omega_0 \eta_1(0)$$

Let $\eta(0) = u = (u_1, u_2)^t$, the previous expression rewrites:

$$-u_1\left(p_2 + q_2 e^{-i\omega_0\tau}\right) + u_2 = i\omega_0 u_1$$

$$-u_1\left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \int_{-\tau}^0 e^{i\omega_0 u}\, du\right) = i\omega_0 u_2$$

Substituting u_2 in the first expression we have

$$-u_1\left(+\omega_0^2 + i\omega_0\left(p_2 + q_2 e^{-i\omega_0\tau}\right) + \left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \int_{-\tau}^0 e^{i\omega_0 u}\, du\right)\right) = 0$$

$$-u_1\left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \int_{-\tau}^0 e^{i\omega_0 u}\, du\right) = i\omega_0 u_2$$

$$u_1 D\left(i\omega_0\right) = 0$$

$$-u_1\left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \int_{-\tau}^0 e^{i\omega u}\, du\right) = i\omega_0 u_2$$

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As $i\omega_0$ is a root of the characteristic equation, we can choose u_1 as we want, so u rewrites.

$$u = u_1 \begin{pmatrix} 1 \\ i\frac{\left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \int_{-\tau}^0 e^{i\omega u} du\right)}{\omega_0} \end{pmatrix}$$

We now compute u_1 thanks to equation $(q^*, q) = 1$, which leads to:

$$\overline{u}_1 = \left[i\omega_0 + i\frac{\frac{(-p_1 - q_1 e^{i\omega_0\tau} + ip_0 \frac{(-1+e^{i\omega_0\tau})}{\omega_0})}{\omega_0}}{-p_0\frac{(-1+i\omega_0\tau e^{-i\omega_0\tau} + e^{-i\omega_0\tau})}{\omega_0^2}} - \tau(iq_2\omega_0 + q_1)e^{-i\omega_0\tau} \right]^{-1}$$

$\square$

Remark 3. *Computations lead to* $(q^*, \overline{q}) = 0$.

Let φ_t be a solution of equation (30) when $\varepsilon = 0$. We associate a pair (z, w) where

$$z(t) = (q^*, \varphi_t) \tag{31}$$

Solutions $\varphi_t(\theta)$ on the central manifold are given by

$$\varphi_t = w(z, \overline{z}, \theta) + z(t)q(\theta) + \overline{z}(t)\overline{q}(\theta) \tag{32}$$

Let us denote $w(t, \theta) = w(z, \overline{z}, \theta)$, where z and $\overline{z}$ are local coordinates for the center manifold in direction q^* and $\overline{q}^*$, and $F_0(z, \overline{z}) = F(0, w(z, \overline{z}, 0) + 2Re(z(t)q(0)))$. Hassard *et al.* [26] then show that the dynamics on the center manifold is given by

$$\dot{z}(t) = i\omega_0 + g(z, \overline{z})$$

and in the neighborhood of ϵ_c^H the complete dynamic of the system is given by

$$\dot{w}(t, \theta) = A(0)w(t, \theta) - 2Re(g(z, \overline{z})q(\theta)) \text{ if } \theta \in [-\tau, 0)$$
$$\dot{w}(t, \theta) = A(0)w(t, \theta) - 2Re(g(z, \overline{z})q(\theta)) + F_0(z, \overline{z}) \text{ if } \theta = 0$$

Let

$$w(z, \overline{z}, \theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\overline{z} + w_{02}(\theta)\frac{\overline{z}^2}{2} + h.o.t.$$

Lemma 5.7.

$$F_0(z, \overline{z}) = \begin{pmatrix} \left(\phi_{20}\frac{z^2}{2} + \phi_{11}z\overline{z} + \phi_{02}\frac{\overline{z}^2}{2} + \phi_{21}\frac{\overline{z}z^2}{2} + h.o.t\right) \\ 0 \end{pmatrix}$$

with $\phi_{20}, \phi_{11}, \phi_{02}, \phi_{21}$ *some complex functions of the coefficients* a_{ij} *and* a_{ijm} *derived in Lemma 5.4.*[21]

---

[21]The expressions of these functions are available upon request.

Proof: We know from the proof of lemma 5.4 that

$$F\left(\varepsilon, \varphi_{1t}\left(0\right), \varphi_{1t}\left(-\tau\right), \varphi_{2t}\left(0\right), \varphi_{2t}\left(-\tau\right), \int_{-\tau}^{0} \varphi_{2t}\left(u\right) du\right)$$

$$= \sum_{i=1}^{5} \sum_{j=i}^{5} a_{ij} x_i x_j + \sum_{i=1}^{5} \sum_{j=i}^{5} \sum_{m=j}^{5} a_{ijm} x_i x_j x_m \tag{33}$$

As on the central manifold we have:

$$\varphi_t\left(\theta\right) = w\left(t, \theta\right) + 2Re\left(q\left(\theta\right) z\left(t\right)\right)$$

and $q\left(\theta\right) = \left(i\omega_0, 1\right)^T e^{i\omega_0\theta}$, coefficients of the solution can be expressed as:

$$\varphi_{1t}\left(\theta\right) = w_{20}^1\left(\theta\right) \frac{z^2}{2} + w_{11}^1\left(\theta\right) z\overline{z} + w_{02}^1\left(\theta\right) \frac{\overline{z}^2}{2}$$
$$+ z\left(t\right) i\omega_0 e^{i\omega_0\theta} - \overline{z}\left(t\right) i\omega_0 e^{-i\omega_0\theta} + O\left(|z, \overline{z}|\right)$$

$$\varphi_{2t}\left(\theta\right) = w_{20}^2\left(\theta\right) \frac{z^2}{2} + w_{11}^2\left(\theta\right) z\overline{z} + w_{02}^2\left(\theta\right) \frac{\overline{z}^2}{2}$$
$$+ z\left(t\right) e^{i\omega_0\theta} + \overline{z}\left(t\right) e^{-i\omega_0\theta} + O\left(|z, \overline{z}|\right)$$

We use the preceding formula to compute $\left(\varphi_{jt}\left(.\right)\right)_{j=1..3}$. Then replacing $\left(\varphi_{jt}\left(.\right)\right)_{j=1..3}$ in (33) we obtain coefficients $\phi_{20}, \phi_{02}, \phi_{11}$ and ϕ_{21} as in the lemma. $\qquad\square$

It is worth noting that $\phi_{20}, \phi_{02}, \phi_{11}$ are obtained as constants, while ϕ_{21} depends on $w_{20}\left(.\right), w_{11}\left(.\right)$, which will be computed later on.

Lemma 5.8. $g_{20} = \overline{u}_1\phi_{20}$, $g_{11} = \overline{u}_1\phi_{11}$, $g_{02} = \overline{u}_1\phi_{02}$. and $g_{21} = \overline{u}_1\phi_{21}$.

Proof: As

$$\overline{q}^*\left(0\right) = \overline{u_1}\left(1, i\overline{\frac{\left(p_1 + q_1 e^{-i\omega_0\tau} + p_0 \int_{-\tau}^{0} e^{i\omega u} du\right)}{\omega_0}}\right)$$

and using $g\left(z, \overline{z}\right) = \overline{q}^*\left(0\right) F_0\left(z, \overline{z}\right)$ we obtained easily the result of this lemma. $\qquad\square$

To end the computation of coefficients $\left(g_{02}, g_{11}, g_{20}, g_{21}\right)$, we need to compute $w_{11}\left(\theta\right)$ and $w_{20}\left(\theta\right)$.

Lemma 5.9.

$$w_{20}\left(\theta\right) = E_1 e^{2i\omega_0\theta} + \frac{ig_{20}}{q_0} q\left(0\right) e^{i\omega_0\theta} + \frac{i\overline{g_{02}}}{3q_0}\overline{q}\left(0\right) e^{-i\omega_0\theta}$$

$$w_{11}\left(\theta\right) = \frac{-ig_{11}}{q_0} q\left(0\right) e^{i\omega_0\theta} + \frac{i\overline{g_{11}}}{q_0}\overline{q}\left(0\right) e^{-i\omega_0\theta} + E_2$$

with

$$E_1 = \begin{pmatrix} 2i\omega_0 \frac{\phi_{20}}{[\Delta(2i\omega_0)]} \\ \frac{\phi_{20}}{[\Delta(2i\omega_0)]} \end{pmatrix} \quad and \quad E_2 = \begin{pmatrix} \frac{-\phi_{11}}{(p_2+q_2 e^{-2i\tau\omega_0})} \\ 0 \end{pmatrix}$$

Proof: We rewrite

$$\begin{aligned} \dot{w}(t,\theta) &= Aw - 2Re\left(g(z,\overline{z})q(\theta)\right) \text{ if } \theta \in [-\tau,0) \\ \dot{w}(t,0) &= Aw - 2Re\left(g(z,\overline{z})q(0)\right) + F_0(z,\overline{z}) \text{ if } \theta = 0 \end{aligned}$$

as

$$\dot{w} = Aw + H(z,\overline{z},\theta) \tag{34}$$

And we consider a Taylor expansion of $H(z,\overline{z},\theta) = H_{20}\frac{z^2}{2} + H_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} +$ h.o.t. As

$$H(z,\overline{z},\theta) = -g(z,\overline{z})q(\theta) - \overline{g(z,\overline{z})}\overline{q}(\theta) \text{ if } \theta \in [-\tau,0)$$

we have

$$\begin{aligned} H_{20}(\theta) &= -g_{20}q(\theta) - \overline{g_{02}\overline{q}}(\theta) \text{ if } \theta \in [-\tau,0) \\ H_{11}(\theta) &= -g_{11}q(\theta) - \overline{g_{11}\overline{q}}(\theta) \text{ if } \theta \in [-\tau,0) \end{aligned} \tag{35}$$

Moreover, we have on the central manifold

$$w(z,\overline{z},\theta) = w_{20}\frac{z^2}{2} + w_{11}z\overline{z} + w_{02}\frac{\overline{z}^2}{2} + h.o.t$$

which implies that

$$\frac{dw(z,\overline{z},\theta)}{dt} = 2w_{20}z\dot{z} + w_{11}\left(\dot{z}\overline{z} + z\dot{\overline{z}}\right) + w_{02}\overline{z}\dot{\overline{z}} + h.o.t$$

This rewrites, as $\dot{z} = i\omega_0 z + g(z,\overline{z})$

$$\begin{aligned} \frac{dw(z,\overline{z},\theta)}{dt} &= 2w_{20}z\left(i\omega_0 z + g(z,\overline{z})\right) \\ &+ w_{11}\left(\left(i\omega_0 z + g(z,\overline{z})\right)\overline{z} + z\left(-i\omega_0\overline{z} + \overline{g(z,\overline{z})}\right)\right) \\ &+ w_{02}\overline{z}\left(-i\omega_0\overline{z} + \overline{g(z,\overline{z})}\right) + ... \\ &= 2i\omega_0 w_{20}z^2 - i\omega_0\overline{z}^2 w_{02} + ... \end{aligned} \tag{36}$$

Coefficient identification in (34) and (36) leads to

$$\begin{aligned} (2i\omega_0 - A)w_{20}(\theta) &= H_{20}(\theta) \\ Aw_{11}(\theta) &= -H_{11}(\theta) \\ (2i\omega_0 + A)w_{02}(\theta) &= -H_{02}(\theta) \end{aligned}$$

Comparing (35) with the last expression, we have

$$\begin{aligned} (2i\omega_0 - A)w_{20}(\theta) &= -g_{20}q(\theta) - \overline{g_{02}\overline{q}}(\theta) \\ Aw_{11}(\theta) &= g_{11}q(\theta) + \overline{g_{11}\overline{q}}(\theta) \end{aligned}$$

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which rewrites, using definition of operator A,

$$\dot{w}_{20}(\theta) = 2i\omega_0 w_{20}(\theta) + g_{20}q(\theta) + \overline{g_{02}}\overline{q}(\theta) \text{ if } \theta \in [-\tau, 0)$$

Solving this equation, we obtain

$$w_{20}(\theta) = E_1 e^{2i\omega_0\theta} + \frac{ig_{20}}{q_0}q(0)e^{i\omega_0\theta} + \frac{i\overline{g_{02}}}{3q_0}\overline{q}(0)e^{-i\omega_0\theta}$$

In a similar way, we have

$$\dot{w}_{11}(\theta) = g_{11}q(\theta) + \overline{g_{11}}\overline{q}(\theta) \text{ if } \theta \in [-\tau, 0)$$

which implies

$$w_{11}(\theta) = \frac{-ig_{11}}{q_0}q(0)e^{i\omega_0\theta} + \frac{i\overline{g_{11}}}{q_0}\overline{q}(0)e^{-i\omega_0\theta} + E_2$$

where E_1 and E_2 can be determined with initial conditions

$$H(z, \overline{z}, 0) = -2Re\left(g(z, \overline{z})q(0)\right) + f_0(z, \overline{z})$$

that is

$$H_{20}(0) = -g_{20}q(0) - \overline{g_{02}}\overline{q}(0) + \begin{pmatrix} \phi_{20} \\ 0 \end{pmatrix}$$

$$H_{11}(0) = -g_{11}q(0) - \overline{g_{11}}\overline{q}(0) + \begin{pmatrix} \phi_{11} \\ 0 \end{pmatrix}$$

Remembering the definition of A and

$$(2iq_0 - A)w_{20}(\theta) = -g_{20}q(\theta) - \overline{g_{02}}\overline{q}(\theta)$$
$$Aw_{11}(\theta) = g_{11}q(\theta) + \overline{g_{11}}\overline{q}(\theta)$$

we have

$$2i\omega_0 w_{20}(0) + g_{20}q(0) + \overline{g_{02}}\overline{q}(0) = L(\epsilon_c)w_{20}(0) + R(\epsilon_c)w_{20}(-\tau)$$

$$+ M(\epsilon_c)\int_{-\tau}^{0}w_{20}(u)\,du + \begin{pmatrix} \phi_{20} \\ 0 \end{pmatrix}$$

$$w_{20}(0) = E_1 + \frac{ig_{20}}{\omega_0}q(0) + \frac{i\overline{g_{02}}}{3\omega_0}\overline{q}(0)$$

$$w_{20}(-\tau) = E_1 e^{-\tau 2i\omega_0} + \frac{ig_{20}}{\omega_0}q(0)e^{-\tau i\omega_0} + \frac{i\overline{g_{02}}}{3\omega_0}\overline{q}(0)e^{-\tau i\omega_0}$$

$$w_{20}(u) = E_1 e^{2i\omega_0 u} + \frac{ig_{20}}{\omega_0}q(0)e^{i\omega_0 u} + \frac{i\overline{g_{02}}}{3\omega_0}\overline{q}(0)e^{-i\omega_0 u}$$

Using the fact that $L\left(\epsilon_c^H\right)q(0) - R\left(\epsilon_c^H\right)q(-\tau) - M\left(\epsilon_c^H\right)\int_{-\tau}^{0}q(u)\,du = i\omega_0 q(\theta)$ and that $q(\theta)$ is the eigenvector of A according to $i\omega_0$, we derive

$$\left(2i\omega_0 - L\left(\epsilon_c^H\right) - R\left(\epsilon_c^H\right)e^{-\tau 2i\omega_0} - M\left(\epsilon_c^H\right)\int_{-\tau}^{0}e^{2i\omega_0 u}du\right)E_1 = \begin{pmatrix} \phi_{20} \\ 0 \end{pmatrix}$$

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which implies

$$\left[\left(2i\omega_0 + p_2 + q_2 e^{-2i\tau\omega_0}\right) E_1^1 + \left[p_1 + q_1 e^{-2i\tau\omega_0} + p_0 \int_{-\tau}^0 e^{2i\omega_0 u} du\right] E_1^2\right] = \phi_{20}$$
$$2i\omega_0 E_1^2 - E_1^1 = 0$$

and thus

$$E_1^2 = \frac{\phi_{20}}{[\Delta(2i\omega_0)]}$$
$$E_1^1 = 2i\omega_0 \frac{\phi_{20}}{[\Delta(2i\omega_0)]}$$

Similarly, we have

$$\begin{pmatrix} -\left[\left(p_2 + q_2 e^{-2i\tau\omega_0}\right) E_2^1 + \left[p_1 + q_1 e^{-2i\tau\omega_0} + p_0 \int_{-\tau}^0 e^{2i\omega_0 u} du\right] E_2^2\right] \\ E_2^2 \end{pmatrix} = \begin{pmatrix} \phi_{11} \\ 0 \end{pmatrix}$$

which implies

$$E_2^1 = \frac{-\phi_{11}}{\left(p_2 + q_2 e^{-2i\tau\omega_0}\right)}$$
$$E_2^2 = 0$$

Consider Lemma 5.3 and let us denote $\lambda'\left(\epsilon_c^H\right) = (d\lambda/d\epsilon_c)_{|\epsilon_c = \epsilon_c^H}$. From all these computations we derive the formula

$$C_1 = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \quad \mu_1 = -\frac{Re(C_1)}{Re\left(\lambda'\left(\epsilon_c^H\right)\right)}$$

$$\mu_3 = -\frac{Im(C_1) + \mu_2 Im\left(\lambda'\left(\epsilon_c^H\right)\right)}{\omega_0}, \qquad\qquad \mu_2 = 2Re(C_1) \tag{37}$$

and the result follows from Hassard *et al.* [26]. $\qquad\qquad\square$

5.5 Proof of Proposition 2

Let us linearise the equation (12) round the steady state and rewrite the dynamics as

$$\begin{cases} \dot{k}(t) = y(t) \\ \dot{y}(t) = \alpha \int_{-\tau}^0 k(t+x)\,\phi(x)\,dx + bk(t) + ay(t) \end{cases}$$

with

$$a = \frac{\left(\rho + \frac{\beta(1-s)(\delta+\rho)}{s}\left(1 - \frac{\epsilon_c[\delta(1-s)+\rho]}{\sigma\delta}\right)\right)}{\left(1 - \frac{\beta(1-s)(\delta+\rho)}{s\delta}\right)}$$

$$b = \frac{\frac{\epsilon_c[\delta(1-s)+\rho]}{\sigma}\frac{(1-s)(\delta+\rho)}{s}}{\left(1 - \frac{\beta(1-s)(\delta+\rho)}{s\delta}\right)}(1 - \beta)$$

Initial conditions for this dynamics are such that $k(t) = k_0(t)$ and $y(t) = y_0(t)$, both functions in $C([-\tau, 0])$. Moreover for $t \in [-\tau, 0]$, we need to

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have $y_0(t) = \dot{k}_0(t)$. It follows that $k_0(t)$ has to be given in $C^1([-\tau, 0[) \cap C([-\tau, 0])$, and that it contraints $y_0(t)$ for $t \in [-\tau, 0[$.

The characteristic matrix is

$$\mathcal{J}(\lambda) = \begin{pmatrix} \lambda & -1 \\ -\alpha \int_{-\tau}^{0} e^{\lambda x} \phi(x)\, dx - \beta & \lambda - a \end{pmatrix}$$

Let $n \geq 1$ be the number of eigenvalues with $Re(\lambda) \geq 0$. Assume first that we have a simple eigenvalue $\lambda_0 > 0$ and that this is the only eigenvalue with $Re(\lambda) \geq 0$, i.e. $n = 1$. The criteria for an initial condition $(k_0(t), y_0(t))$ with $t \in [-\tau, 0]$ to be continued to a bounded solution for $t \geq 0$ is then given by

$$Q_{\lambda_0}(k_0(.), y_0(.)) = 0$$

where Q_λ is the spectral projection (The reasonning is the same if we have a set Σ with a finite number $n > 1$ of roots with positive real part, exept that we have to consider the spectral projection on the set $vect\left(e^\theta, \theta \in \Sigma\right)$).

According to the spectral projection formula given by equation (IV.3.3) in [20], then

$$(Q_\lambda(k_0(.), y_0(.)))\,(\theta) = e^{\lambda\theta} H\Bigg(\Bigg((k_0(0), y_0(0))$$
$$+ \left(0, \alpha \int_{-\tau}^{0} \phi(x)\, e^{\lambda_0 x} \int_{x}^{0} e^{-\lambda_0 u} k_0(u)\, du dx\right)\Bigg)\Bigg)$$

where H has to be now determined. We are looking for some H of the form $H = pq^t$, with some vectors $p \in \mathbb{R}^2$ and $q \in \mathbb{R}^2$, such that

$$\begin{cases} \mathcal{J}(\lambda_0)\, p = 0 \\ \mathcal{J}(\lambda_0)^t\, q = 0 \end{cases}$$

where $\mathcal{J}(\lambda_0)^t$ is the transpose matrix of $\mathcal{J}(\lambda_0)$. We thus find that $q = (\lambda_0 - a, 1)$ and $p = (1, \lambda_0)$. Expliciting H, $Q_{\lambda_0}(k_0(.), y_0(.)) = 0$ implies that

$$y_0(0) + (\lambda_0 - a)k_0(0) + \alpha \int_{-\tau}^{0} \phi(x)\, e^{\lambda_0 x} \int_{x}^{0} e^{-\lambda_0 u} k_0(u)\, du dx = 0 \quad (38)$$

Considering now that $k(t) = k_0(t)$ is given for $t \in [-\tau, 0]$ and that $y_0(t)$ must satisfy $y_0(t) = \dot{k}_0(t)$ for $t \in [-\tau, 0]$, when $n = 1$ the only degree of freedom to solve (locally) our problem is to choose $y(t) \in C([-\tau, 0])$ with the condition that $y_0(0)$ is specified as $y_0(0) = -(\lambda_0 - a)k_0(0) - \alpha \int_{-\tau}^{0} \phi(x)\, e^{\lambda_0 x} \int_{x}^{0} e^{-\lambda_0 u} k_0(u)\, du dx$. On the contrary, if $n > 1$ we derive, applying the same argument as above, n constraints similar to (38). But as $y_0(0)$ is the only unknown, there does not exist any solution. $\qquad\square$

5.6 Proof of Theorem 3

For $\tau = 0$ the characteristic equation admits one positive real root and one negative real root. Assuming that $\tau > 0$ implies that an infinite number of roots with negative real part will now exist. Our aim here is to prove that for a non empty set of values of τ, the characteristic equation admits only one positive real root. To prove this, we show that there exists $\bar{\tau} > 0$ such that for $0 < \tau < \bar{\tau}$, there are no pure imaginary roots. According to the proof of Lemma 5.1, under Assumption 2, if there exists a pure imaginary roots $i\omega_0$, with $\omega_0 > 0$, it has to solve

$$\begin{cases} 2\eta\left(\underline{\epsilon}_c\right) + x = 0 \\ \omega_0^2 = x \end{cases}$$

Let us consider the critical value $\underline{\epsilon}_c$ as defined by (18). We get

$$\begin{aligned} \eta\left(\underline{\epsilon}_c\right) &= \underline{\epsilon}_c \frac{(1-s)(\delta+\rho)[\delta(1-s)+\rho]}{\sigma s \delta \tau}\left(\delta\tau - \beta\right) + \frac{\beta(1-s)(\delta+\rho)^2}{s\delta\tau} + \frac{\rho^2}{2} \\ &= \frac{\beta}{s\delta}\left((1-s)(\delta+\rho)^2 - \frac{\delta\beta[\delta(1-s)+\rho]}{\beta - \frac{\delta\tau}{2}}\right)\frac{1}{\tau} + O\left(1\right) \end{aligned}$$

It follows that there exists $\bar{\tau}_1$, such that for $\tau \in (0, \bar{\tau}_1)$, we have

$$(1-s)(\delta+\rho)^2 - \frac{\delta\beta[\delta(1-s)+\rho]}{\beta - \frac{\delta\tau}{2}} > 0$$

if

$$\delta(1-s) + \rho < \frac{(1-s)(\delta+\rho)^2}{\delta} \quad \Leftrightarrow \quad \frac{1}{2+\frac{\rho}{\delta}} < 1 - s$$

If $s \in (0, 1/2)$, the previous condition is always satisfied. We then conclude that when $\tau \in (0, \bar{\tau}_1)$, $\eta\left(\underline{\epsilon}_c\right) > 0$. Let us now consider $\bar{\tau} \equiv \min\{\bar{\tau}_1, 2\beta/\delta\}$. According to Theorem 1, for $\epsilon_c \in [\underline{\epsilon}_c, \epsilon_c^H)$, the characteristic polynomial admits no pure imaginary roots. Thus for $\tau \in (0, \bar{\tau})$ and $\epsilon_c \in [\underline{\epsilon}_c, \epsilon_c^H)$, there exists only one root with positive real part which corresponds to the unique positive real root.

$\square$

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