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Correlation, independence, and Bayesian incentives

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Introduction

In a Bayesian framework, when the relevant information is dispersed among several agents, the designer of a decision mechanism must take into account what they know about each other. Many authors have exploited the general properties of such information structures to construct decision procedures or contracts in many different types of allocation problems. The first applications have belonged to public economics and to the study of collective decision making, where the problem is to achieve an optimal allocation of resources, despite market failures due to externalities or to public goods, and despite strategic behavior, free-riding and misreporting of preferences. Starting with Clarke (1971) and Groves (1973), the problem has been framed as the construction of balanced transfers to obtain an optimal allocation compatible with individual incentives. First, it was shown that, in general, there did not exist dominant strategy mechanisms with balanced transfers (see Green and Laffont 1979 and Walker 1980). Followed positive answers in Bayesian frameworks by, among others, d'Aspremont and Gérard-Varet (1975, 1979, 1982), Arrow (1979), Laffont and Maskin (1979), Johnson et al. (1990), d'Aspremont et al. (1990).

Another line of research, starting with Vickrey (1961), is devoted to the design of auctions (Myerson 1981; Milgrom and Weber 1982; Crémer and McLean 1985, 1988 and, for a survey of the early literature, McAfee and

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McMillan 1987). Rather than maximizing the profit of the seller, one can also try to find efficient bilateral or multilateral trading mechanisms (Chatterjee and Samuelson 1983; Wilson 1985; Leininger, et al. 1989; Satterwhaite and Williams 1989). In a more applied vein, Cramton et al. (1987) have studied the related problem of reallocating efficiently the shares of an asset owned by agents who value it differently.

The general problem of mechanism design also has applications in industrial organization. Roberts (1985), Cramton and Palfrey (1990) and others have studied optimal cartel agreements when firms have private information about their own costs. Riordan (1984) has studied agreements between firms and their suppliers, while Bhattacharya et al. (1992) and d'Aspremont et al. (1993) have studied various forms of R&D contracting. The same general framework has also been applied to the internal organization of the firm, see Crémer and Riordan (1987).

The goal of the present paper is to describe the state of the art on Bayesian mechanisms when utility is transferable and only balanced transfers are admissible. New results will be proved along the way, but they will be integrated to the overall picture. This description will be organised around three main questions. The first is:

What is the role of correlation among the types of agents in the resolution of incentive problems with balanced transfers?

Following the first work on Bayesian mechanisms by d'Aspremont and Gérard-Varet (1975) and Arrow (1979), it was generally accepted that independence of types was the central sufficient condition to find Pareto-efficient Bayesian incentive compatible mechanisms (Groves and Ledyard 1980). Even the more general "compatibility condition" (condition C), introduced in d'Aspremont and Gérard-Varet (1979) as a sufficient condition for the existence of efficient Bayesian mechanisms, was often interpreted as a generalisation of independence (see, for instance, Fudenberg et al. 1994). In a two agents framework, where condition C coincides with independence, d'Aspremont et al. (1990) reinforce this intuition by building a counterexample to the existence of efficient Bayesian mechanisms using strong correlation between the types of the agents.

On the other hand, studying optimal auctions, Myerson (1981), Maskin and Riley (1980), and Crémer and McLean (1985, 1988) show that some correlation between types is necessary and sufficient for the seller to do very well indeed, as well as if he had full information about the types of the bidders. Fudenberg and Tirole (1992) survey multi-agents adverse selection problems stressing this feature.

The data is therefore mixed: correlation seems to help in some cases, and gets in the way in others. In this paper, we show that correlation enables us to implement all sorts of decision rules, those taken on behalf of the agents, but also those that go against their best interest. Independence prevents us from implementing decision rules that are not efficient from the viewpoint of the agents, but is useful when we are trying to take decisions in their best interest.

The second main question that we ask is

Can we in general find Bayesian mechanisms which balance the budget?

The early literature on efficient Bayesian mechanisms was motivated by the existence issue. In general, Vickrey-Clarke-Groves mechanisms, which use dominant strategies, cannot balance the budget, and obtaining existence with budget balance was the main reason for the use of the concept of Bayesian equilibrium. In d'Aspremont et al. (1990), we show that generically one can find such mechanisms, as long as there are at least three agents and as long as we try to implement efficient decisions. In the present paper, we reinforce this result by showing that it holds for all decisions, even if they are not efficient. Moreover, the proof will be constructive: we will exhibit a class of balanced transfer rules that, generically, are sufficient to implement any public decision.

The implicit message that transpires from all the attempts to find conditions under which there exist Bayesian mechanisms is that existence is difficult to establish, even with the assumption that utility is transferable. Actually, with this assumption, it is very difficult to find environments where efficient Bayesian mechanisms *do not* exist, and in this paper we provide the first example of an environment with more than two agents for which an efficient mechanism does not exist.

The third question that we ask is:

Is it possible to find Bayesian mechanisms that balance the budget and that have unique equilibria?

Palfrey (1992) surveys the extensive literature on unique Bayesian implementation. We exhibit here new conditions that allow us to go from implementation to unique implementation and show that they hold generically.

The paper is organized as follows. Section 1 presents the general definitions. Section 2 studies a condition on the information structure[1] (called condition B), which is stronger than the compatibility condition, but is necessary and sufficient to guarantee implementation of any decision rule. The main result of this section is that this condition holds generically, and the proof is constructive: in nearly all environments, and for any decision rule, it provides a technique for building a Bayesian mechanism that implements it. In Sect. 3, we return to condition C, providing a simple interpretation, which enables us to link the literature on Bayesian mechanism to the literature on dominant strategy mechanisms. Also, we build a counterexample showing that condition C is not necessary to guarantee implementation of efficient decision rules. Furthermore, we provide the first necessary and sufficient condition in the literature for an information structure to guarantee that any

[1] Information structures, defined formally below, are probability distributions over the types of agents.

efficient decision rule can be implemented, whatever the utility of the agents. Section 4 exhibits what is, to the best of our knowledge, the first proof that there does not always exist efficient Bayesian mechanisms, with three agents. The information structures that we use to construct counterexamples have some interest of their own. Finally, in Sect. 5, we exhibit conditions on the information structures that guarantee unique implementation, and show that they hold generically. We discuss some open questions in the conclusion, Sect. 6.

1 Bayesian incentive compatible mechanisms

1.1 The environment

A planner is designing a mechanism for the set $\mathcal{N} = \{1, \ldots, n\}$ of agents. Except when the opposite is explicitly stated, n will be assumed to be at least equal to 3. These agents may not be the only ones affected by the decision that will be taken, but they are the only ones whose preferences will be taken into account.

A public decision x must be chosen in a set $\mathcal{X}$, which may be finite or infinite. In many problems $\mathcal{X}$ would be an interval of the real line, and would represent the size of some public equipment, or as, in Riordan (1984), the quantity of a good transferred between a supplier and a client. In other problems, it could be a set of real vectors and represent transfers of several goods and services among the agents.

All the private information of agent $i \in \mathcal{N}$ is represented by his type α_i which belongs to a finite set $\mathcal{A}_i$. A vector of possible types is denoted α and is an element of $\mathcal{A} = \prod_{i \in \mathcal{N}} \mathcal{A}_i$.

The payoff of an agent depends on the public decision, his type and, since we assume that utility is transferable, the monetary transfer (which could be negative) that he receives from the planner: if the public decision is x, his type is α_i and he receives a monetary transfer t_i from the planner, the "quasi-linear" payoff of agent i is

$$u_i(x; \alpha_i) + t_i.$$

Most of our results go through if the utility of agent i is a function from $\mathcal{X} \times \mathcal{A}$ into $\Re$, that is if the types of the other agents also influence the value that he attaches to decisions — the so-called "common value" case. In economic terms, this generalization is significant. In many cases, when a public decision is taken agents possess private information that would influence the evaluation of the possible choices by the other agents.

The type of agent i can also affect his beliefs about the types of the other agents. When he is of type α_i these beliefs are represented by a probability distribution $p_i(\alpha_{-i} \mid \alpha_i)$ over $\mathcal{A}_{-i} = \prod_{j \in \mathcal{N}-i} \mathcal{A}_j$, the set of the possible types of the others. We will assume that there exists a probability distribution p on $\mathcal{A}$ such that p_i is obtained from p by taking the conditionalization with respect to α_i, i.e. $p_i(\alpha_{-i} \mid \alpha_i) = p(\alpha_{-i} \mid \alpha_i) = p(\alpha)/p(\alpha_i)$: the beliefs are "consistent".[2] We assume that the marginal distribution $p(\alpha_i)$ is strictly positive for all i and all α_i.

We will need the following notation. For $i \neq j$,

$$\mathscr{A}_{-i-j} = \prod_{k \notin \{i,j\}} \mathscr{A}_k$$

is the set of possible types of agents other than i and j. For every $\alpha_{-i-j} \in \mathscr{A}_{-i-j}$, the distribution $p(\alpha_{-i-j} \mid \alpha_i)$ represents the beliefs of agent i on the types of agents other than himself and j, while $p(\alpha_j \mid \alpha_i)$ are the beliefs of agent i on the type of agent j. (Notice that from this point on, in order to lighten the notation, we are dropping the subscripts on the conditional probabilities.)

We will make repeated use of the notion of *information structure*, that is the 3-tuple $(\mathscr{N}, \mathscr{A}, p)$.

The type of an agent refers to a joint realization of his utility function and probability distribution. When the type varies at least one of these elements changes, and our analysis goes through if only one changes. For instance, there could exist two types α_i and α_i' such that $u_i(x; \alpha_i)$ is equal to $u_i(x; \alpha_i')$ for all $x \in \mathscr{X}$, and that $p(\cdot \mid \alpha_i)$ is not equal to $p(\cdot \mid \alpha_i')$. In this case α_i and α_i' differ only by the information available to agent i.

All these notions define an *environment*

$$\mathscr{E} = (\{\mathscr{N}, \mathscr{A}, p\}, \mathscr{X}, \{u_i\}_{i \in \mathscr{N}}),$$

which is composed of an information structure, a set of outcomes and utility functions for the agents.

1.2 Implementation

The planner would like the choice of the decision to be sensitive to the state of nature. More precisely, he would like to implement a decision rule s, defined by a function from $\mathscr{A}$ into $\mathscr{X}$: for a vector α of types the public decision $s(\alpha)$ is taken.

A decision rule is efficient, and usually written s^*, if for all $\alpha \in \mathscr{A}$ and all $x \in \mathscr{X}$:

$$\sum_{i \in \mathscr{N}} u_i(s^*(\alpha); \alpha_i) \geq \sum_{i \in \mathscr{N}} u_i(x; \alpha_i).$$

In order to implement a decision rule s, the planner gathers information from the agents. He uses a mechanism in which the strategies of the agents are the messages that they send. Formally, we have the following definitions. A *mechanism* consists of a set of *message spaces*, $\mathscr{M} = \{\mathscr{M}_1, \mathscr{M}_2, \ldots, \mathscr{M}_n\}$,

[2] We make this assumption mainly for ease of presentation. Our results go through if we assume that the informations of the agents are not consistent, i.e., that the $p_i(\cdot \mid \alpha_i)$ are not obtained from a well-defined p.

together with an outcome function $(x(\cdot), t(\cdot))$ that associates to any vector of messages $m \in \mathscr{M}$ a public decision $x(m) \in \mathscr{X}$ and a vector of transfers $t(m) \in \Re^n$. This transfer rule must balance the budget:

$$\sum_{i \in \mathscr{N}} t_i(m) = 0 \quad \text{for all } m \in \mathscr{M}.$$

A mechanism determines a game with incomplete information. A *Bayesian equilibrium* for such a game is a vector of strategies $\tilde{m} = (\tilde{m}_1, \ldots, \tilde{m}_n)$ where each $\tilde{m}_i$ is a function from $\mathscr{A}_i$ to $\mathscr{M}_i$ such that

$$\begin{aligned} &\sum_{\alpha_{-i} \in \mathscr{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)[u_i(x(\tilde{m}(\alpha); \alpha_i) + t_i(\tilde{m}(\alpha))] \\ &\quad \geq \sum_{\alpha_{-i} \in \mathscr{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)[u_i(x(m_i, \tilde{m}_{-i}(\alpha_{-i}); \alpha_i) + t_i(m_i, \tilde{m}_{-i}(\alpha_{-i}))] \\ &\quad \text{for all } i \in \mathscr{N}, \text{and all } \alpha_i \in \mathscr{A}_i \text{ and all } m_i \in \mathscr{M}_i. \end{aligned} \tag{BIC}$$

The mechanism $(\mathscr{M}, x, t)$ *implements* the decision rule s if there exists a Bayesian equilibrium $\tilde{m}$ such that $x(\tilde{m}(\alpha)) = s(\alpha)$ for all[3] $\alpha \in \mathscr{A}$. The mechanism $(\mathscr{M}, x, t)$ *uniquely implements* the decision rule s if it implements s and the Bayesian equilibrium $\tilde{m}$ is unique.

In the first part of the paper, we identify information structures such that all decision rules (or all efficient decision rules) can be implemented. In order to study this issue, we use the revelation principle[4] and restrict ourselves to the class of *direct mechanisms* for which $\mathscr{M}_i = \mathscr{A}_i$ for all $i \in \mathscr{N}$: a message simply consists in announcing a type. Moreover, each agent is induced to announce his true type: $\tilde{m}(\alpha) = \alpha$ for all α. In this case $x(\alpha) = s(\alpha)$: the public decision function x can be identified with the decision rule s which is to be implemented. Finding a direct mechanism (s, t) that implements s is therefore equivalent to finding a (monetary) transfer rule $t : \mathscr{A} \to \Re^n$ which is balanced,

$$\sum_{i \in \mathscr{N}} t_i(\alpha) = 0 \quad \text{for all } \alpha \in \mathscr{A}, \tag{1}$$

and satisfies the Bayesian incentive compatibility constraints

[3] Even if we neglect the issue of unicity of equilibrium, which we tackle in Sect. 5, our use of terms is different from the language in the literature on implementation. In the spirit of that literature, we would have to give ourselves an incentive compatible set of decisions *and* transfers, and find an equivalent mechanism that uniquely implements them (see Palfrey 2002).

[4] See Holmstrom (1977), Dasgupta et al. (1979), Myerson (1979) and Harris and Townsend (1981).

$$\sum_{\alpha_{-i}\in\mathscr{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)[u_i(s(\alpha_i,\alpha_{-i});\alpha_i) + t_i(\alpha_i,\alpha_{-i})]$$
$$\geq \sum_{\alpha_{-i}\in\mathscr{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)[u_i(s(\widetilde{\alpha}_i,\alpha_{-i});\alpha_i) + t_i(\widetilde{\alpha}_i,\alpha_{-i})]$$
$$\text{for all } i \in \mathscr{N}, \text{ and all } (\alpha_i,\widetilde{\alpha}_i) \in \mathscr{A}_i^2, \qquad \text{(B1C}')$$

where

$$\mathscr{A}_i^2 = \{(\alpha_i,\widetilde{\alpha}_i) \mid \alpha_i,\widetilde{\alpha}_i \in \mathscr{A}_i, \alpha_i \neq \widetilde{\alpha}_i\}.$$

We will use the following terminology. An information structure $(\mathscr{N},\mathscr{A},p)$ *guarantees implementation of all public decision rules* [5] if for every outcome set[6] $\mathscr{X}$, every utility functions $u_i : \mathscr{X} \times \mathscr{A}_i \longrightarrow \Re, i = 1,\ldots,n$, and every decision rule s, there exists a balanced transfer rule t such that the associated direct mechanism (s,t) implements s, i.e., satisfies (BIC'). Similar definitions will be used for information structures that guarantee *strict* implementation (when the inequalities (BIC') are strict), *implementation of efficient decision rules* (when s is an efficient decision rule).

All our results hold true if we add an ex-ante individual rationality constraint of the form

$$\sum_{\alpha\in\mathscr{A}} p(\alpha)[u_i(s(\alpha);\alpha_i) + t_i(\alpha)] \geq 0,$$

as long as there is a status quo decision that guarantees each agent a utility of 0. A more general and more precise statement is provided by the following result, which states that if the planner can guarantee an expected aggregate surplus at least equal to $\bar{A}$, then he can allocate this surplus anyway he wishes among the agents.

Theorem 1. *Assume that for some real* $\bar{A}$

$$\sum_{\alpha\in\mathscr{A}} p(\alpha)\left[\sum_{i\in\mathscr{N}} u_i(s(\alpha);\alpha_i)\right] \geq \bar{A},$$

and that there exists a direct mechanism (s,t') *that implements* s *(i.e.,* t' *is balanced and* (s,t') *satisfies* (BIC')*), then for any family of real numbers* $\{B_i\}_{\{i\in\mathscr{N}\}}$ *such that*

[5] Of course, a mechanism designer is only interested in guaranteeing implementation of a specific decision rule. We study implementation of all decision rules because it provides sufficient conditions for the problem of any mechanism designer to be solvable. Our results can be reinterpreted as showing that all mechanism designers can nearly always implement the decision rule they are interested in.

[6] Because we have only a finite set of types, only a finite subset of decisions are really relevant. The fact that the set $\mathscr{X}$ varies does not create any difficulty, and we could keep it fixed without changing the results if its cardinality was at least equal to that of $\mathscr{A}$.

$$\sum_{i\in\mathcal{N}} B_i \leq \bar{A}$$

there exists another direct mechanism (s,t) *that implements* s *and such that*

$$\sum_{\alpha\in\mathcal{A}} p(\alpha)[u_i(s(\alpha);\alpha_i) + t_i(\alpha)] \geq B_i \quad \text{for all } i.$$

Proof. Write $\sum_{\alpha\in\mathcal{A}} p(\alpha)\left[u_i(s(\alpha);\alpha_i) + t_i'(\alpha)\right] = A_i$. Budget balance implies $\sum_{i\in\mathcal{N}} A_i \geq \bar{A}$. For $i \neq 1$ let $t_i(\alpha) = t_i'(\alpha) + B_i - A_i$ and let $t_1(\alpha) = -\sum_{i\neq 1} t_i(\alpha)$. The mechanism (s,t) implements s and is such that the ex ante expected payoff is equal to B_i, for all $i \neq 1$, and larger than or equal to $\bar{A} - \sum_{i\neq 1} B_i \geq B_1$ for $i = 1$. ■

For technical reasons, which will become clear later, and in order to lighten the notation, it is convenient to collapse the utility function of the agents and the decision rule. A decision rule being given, we therefore redefine the functions u_i to be functions from $\mathcal{A} \times \mathcal{A}_i$ into $\Re$, such that $u_i(\alpha';\alpha_i)$ is reinterpreted as the old $u_i(s(\alpha');\alpha_i)$ — this is equivalent to setting $\mathcal{X} = \mathcal{A}$, which we can do without loss of generality.

Incentive compatibility is then written

$$\begin{aligned}
&\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)[u_i(\alpha_i,\alpha_{-i};\alpha_i) + t_i(\alpha_i,\alpha_{-i})] \\
&\quad \geq \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)[u_i(\widetilde{\alpha}_i,\alpha_{-i};\alpha_i) + t_i(\widetilde{\alpha}_i,\alpha_{-i})] \\
&\quad \text{for all } i \in \mathcal{N}, \text{ and all } (\alpha_i,\widetilde{\alpha}_i) \in \mathcal{A}_i^2. \qquad \text{(BIC")}
\end{aligned}$$

When the decision rule is efficient we will use the notation $u_i^*(\alpha',\alpha)$ for $u_i(s^*(\alpha');\alpha)$. Efficiency implies

$$\sum_{i\in\mathcal{N}} u_i^*(\alpha;\alpha_i) \geq \sum_{i\in\mathcal{N}} u_i^*(\alpha';\alpha_i) \text{ for all } \alpha' \neq \alpha. \tag{2}$$

We then say that the u_i^*'s are efficient.

2 Implementing all decision rules

In this section, we characterize information structures that guarantee implementation of all decision rules and we show in Subsect. 2.2 that, as long as there are at least three agents, nearly all information structures satisfy this property. We make no assumption about the objectives of the planner, and assume that the net aggregate transfers between him and the agents are equal to 0. For instance, one could imagine the following types of circumstances: The planner already knows the utility of consumers and must gather only information from firms about their costs. Then, the agents of our framework are the firms, and it is not the sum of their utilities that must be maximized.

2.1 Condition B

We will need the following condition introduced by d'Aspremont and Gérard-Varet (1982):

Condition 1 (Condition B). *An information structure satisfies condition B if and only if there exists a balanced tranfer rule t^B such that for all $i \in \mathcal{N}$ and all $(\alpha_i, \widetilde{\alpha}_i) \in \mathcal{A}_i^2$ we have*

$$\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i^B(\alpha_{-i}, \alpha_i) p(\alpha_{-i} \mid \alpha_i) > \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i^B(\alpha_{-i}, \widetilde{\alpha}_i) p(\alpha_{-i} \mid \alpha_i). \tag{3}$$

The following result was first proved in d'Aspremont et al. (1987) and Johnson et al. (1990). We present a much more transparent proof.

Theorem 2. *Condition B guarantees strict implementation of all decision rules. If the information structure $(\mathcal{N}, \mathcal{A}, p)$ guarantees implementation of all decision rules, then condition B holds.*

Proof. It is easy to prove that if condition B holds any decision rule can be implemented: use the transfers satisfying (3) multiplied by a positive number large enough that, from the point of view of the agents, the monetary incentives to reveal their true types dominate the incentives they would have to lie in order to modify the decision.

It is more difficult to show that condition B holds for every information structure $(\mathcal{N}, \mathcal{A}, p)$ that guarantees implementation of all decision rules. Choose any $(i, \alpha_i^0) \in \mathcal{N} \times \mathcal{A}_i$ and define an environment, $\mathcal{E}^{(i,\alpha_i^0)}$, for which the information structure is $(\mathcal{N}, \mathcal{A}, p)$ and whose characterization is completed as follows:

$$u_i(\alpha_{-i}, \alpha_i^0; \alpha_i^0) = -1 \text{ for all } \alpha_{-i},$$
$$u_j(\alpha; \widetilde{\alpha}_j) = 0 \text{ if } (j, \alpha_j, \widetilde{\alpha}_j) \neq (i, \alpha_i^0, \alpha_i^0).$$

Notice that the decision rule that we are implementing minimizes the sum of the utilities of the agents. Because it can be implemented, there exists a balanced transfer function $t^{(i,\alpha_i^0)}$ that satisfies the following equations,

$$\begin{aligned}\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} & p(\alpha_{-i} \mid \alpha_i^0)\left[t_i^{(i,\alpha_i^0)}(\alpha_{-i}, \alpha_i^0) - t_i^{(i,\alpha_i^0)}(\alpha_{-i}, \widetilde{\alpha}_i)\right] \\ &\geq \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} p(\alpha_{-i} \mid \alpha_i^0)\left[u_i(\alpha_{-i}, \widetilde{\alpha}_i; \alpha_i^0) - u_i(\alpha_{-i}, \alpha_i^0; \alpha_i^0)\right] = 1 \quad \text{for all } \widetilde{\alpha}_i \neq \alpha_i^0,\end{aligned} \tag{4}$$

and

$$\begin{aligned}\sum_{\alpha_{-j} \in \mathcal{A}_{-j}} & p(\alpha_{-j} \mid \alpha_j)\left[t_j^{(i,\alpha_i^0)}(\alpha_{-j}, \alpha_j) - t_j^{(i,\alpha_i^0)}(\alpha_{-j}, \widetilde{\alpha}_j)\right] \\ &\geq \sum_{\alpha_{-j} \in \mathcal{A}_{-j}} p(\alpha_{-j} \mid \alpha_j)\left[u_j(\alpha_{-j}, \widetilde{\alpha}_j; \alpha_j) - u_j(\alpha_{-j}, \alpha_j; \alpha_j)\right] = 0 \\ &\qquad \text{for all } (j, \alpha_j) \neq (i, \alpha_i^0) \text{ and } \widetilde{\alpha}_j \neq \alpha_j.\end{aligned} \tag{5}$$

Repeating this construction for every $i \in \mathcal{N}$ and every $\alpha_i^0 \in \mathcal{A}_i$, and using Eqs. (4) and (5) the transfer rule obtained by summing up the $t^{(i,\alpha_i^0)}$'s over all $i \in \mathcal{N}$ and all $\alpha_i^0 \in \mathcal{A}_i$ satisfies Eq. (3), and the result is proved. ■

If condition B holds, in order to implement a decision rule, the planner needs to know very little about the utility functions of the agents: he only needs to know an upper bound on the maximum willingness to pay to change the decision,

$$\max_{i,\alpha_i,\widetilde{\alpha},\alpha'} |u_i(\alpha';\alpha_i) - u_i(\widetilde{\alpha};\alpha_i)|.$$

As is well-known, guaranteeing implementation of any *efficient* decision rule can be obtained by assuming a condition of independence of types.[7] We now show that independence of types precludes implementation of *all*[8] decision rules. Formally, for any agent i, we say that the types α_i and $\widetilde{\alpha}_i$ are *independent* if

$$p(\alpha_{-i} \mid \alpha_i) = p(\alpha_{-i} \mid \widetilde{\alpha}_i) \text{ for all } \alpha_{-i} \in \mathcal{A}_{-i}.$$

Agent i has *free beliefs* if all pairs of types α_i and $\widetilde{\alpha}_i$ are independent. *Independence of types* holds when all agents have free beliefs.

Corollary 1. *If condition B holds no two types of any agent can be independent, and therefore in any information structure that guarantees implementation of all decision rules no two types of any agent can be independent.*

Proof. For condition B to hold Eq. (3) must hold, and it must hold with the roles of α_i and $\widetilde{\alpha}_i$ inverted, which is clearly impossible if they are independent. ■

As we will see in Sect. 4, the reverse implication does not hold. There exist information structures such that no two types of any agent are independent, and such that condition B does not hold.

The conclusions of Theorem 2 can be extended to the common value case where the utilities of the agents depend not only on their types but also on the types of the other agents. The only if part holds, since it is stronger than needed, and the if part can easily be adapted.

This enables us to weaken the budget balance constraint:

Corollary 2. *If the budget balance condition (1) is replaced by*

[7] See d'Aspremont and Gérard-Varet (1975) and Arrow (1979).

[8] One could weaken the requirement that all decision rules be implemented. For instance, with independence of types, Laffont and Maskin (1979) show that it is impossible to implement a decision rule that maximizes a strictly concave and increasing function of the u_i's, even if one limits oneself to quadratic valuation functions.

$$\sum_{i \in \mathcal{N}} t_i(\alpha) = R(\alpha) \quad \text{for all } \alpha \in \mathcal{A}$$

for some function $R : \mathcal{A} \to \Re$, condition B is still necessary for the implementation of all decision rules, and sufficient for the strict implementation of all decision rules.

Proof. By the discussion of the common value case that precedes the statement of the theorem, we can implement the decision rule $u_i'(\widetilde{\alpha}; \alpha) = u_i(\widetilde{\alpha}, \alpha_i) + R(\alpha)/n$, and this implies the corollary. ∎

If in the proof of Corollary 2 one assumes that all u_i's are uniformly equal to zero, we obtain the following corollary:

Corollary 3. *An information structure satisfies condition B if and only if for all functions $R : \mathcal{A} \to \Re$, there exists a transfer rule t^B such that for all $\alpha \in \mathcal{A}$*

$$\sum_{i \in \mathcal{N}} t_i^B(\alpha) = R(\alpha),$$

and such that for all $i \in \mathcal{N}$ and all $(\alpha_i, \widetilde{\alpha}_i) \in \mathcal{A}_i^2$ we have:

$$\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i^B(\alpha_{-i}, \alpha_i) p(\alpha_{-i} \mid \alpha_i) > \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i^B(\alpha_{-i}, \widetilde{\alpha}_i) p(\alpha_{-i} \mid \alpha_i).$$

2.2. Genericity

It is rather remarkable that, as we will show in this subsection, condition B holds for nearly all information structures as long as there are at least three agents, each with at least two types, and therefore that nearly all information structures guarantee strict implementation of all decision rules. The proof is constructive: for nearly all environments, it provides a technique of construction of transfers that allow implementation of all decision rules.

For a given set $\mathcal{A}$ of states of nature, the set of vectors of probabilities $p(\alpha)$ is the simplex of $\Re^{\mathcal{A}}$. A property holds for nearly all information structures if it holds in an open and dense (according to the standard topologies) subset of all probability distributions p in the simplex corresponding to any $\mathcal{N}$.

Theorem 3. *If $\mathcal{N}$ contains at least three agents, each with at least two types, condition B holds for nearly all information structures.*

The technique of proof uses "scoring rules" introduced by Good (1952), discussed by Savage (1974), and applied to Bayesian implementation by Johnson et al. (1990): An agent who maximizes expected value will state his true probability distribution over events, when in each state of nature he is paid proportionally to the logarithm of the probability that he announced ex-ante for that state.

Proof. Addition and subtraction on the indices of agents are defined modulo n so that $n+1 \equiv 1$ and $1-1 \equiv n$. For all i and all α_i we assume (and this holds generically) that $p_i(\alpha_{-i-(i-1)} \mid \alpha_i) > 0$ for all $\alpha_{-i-(i-1)}$, and that $p_i(\alpha_{-i-(i+1)} \mid \alpha_i) > 0$ for all $\alpha_{-i-(i+1)}$. Define $t^B(\alpha)$ by

$$\begin{aligned} t_i^B(\alpha) = {} & [\log p_i(\alpha_{-i-(i-1)} \mid \alpha_i) - \log p_i(\alpha_{-i-(i-1)} \mid \alpha_{i-1})] \\ & + [\log p_i(\alpha_{-i-(i+1)} \mid \alpha_i) - \log p_i(\alpha_{-i-(i+1)} \mid \alpha_{i+1})]. \end{aligned}$$

The negative terms are constant in α_i and do not influence the incentives of agent i, but they ensure that the transfer rule is balanced. This implies, for all i, all α_i and all $\widetilde{\alpha}_i$,

$$\begin{aligned} & \sum_{\alpha_{-i} \in \mathscr{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)[t_i(\alpha_{-i}, \widetilde{\alpha}_i) - t_i(\alpha_{-i}, \alpha_i)] \\ & \quad = \sum_{\alpha_{-i} \in \mathscr{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)[\log \frac{p_i(\alpha_{-i-(i-1)} \mid \tilde{\alpha}_i)}{p_i(\alpha_{-i-(i-1)} \mid \alpha_i)} + \log \frac{p_i(\alpha_{-i-(i+1)} \mid \tilde{\alpha}_i)}{p_i(\alpha_{-i-(i+1)} \mid \alpha_i)}] \\ & \quad = \sum_{\alpha_{-i-(i-1)} \in \mathscr{A}_{-i-(i-1)}} p(\alpha_{-i-(i-1)} \mid \alpha_i) \log \frac{p_i(\alpha_{-i-(i-1)} \mid \tilde{\alpha}_i)}{p_i(\alpha_{-i-(i-1)} \mid \alpha_i)} \\ & \qquad + \sum_{\alpha_{-i-(i+1)} \in \mathscr{A}_{-i-(i+1)}} p(\alpha_{-i-(i+1)} \mid \alpha_i) \log \frac{p_i(\alpha_{-i-(i+1)} \mid \tilde{\alpha}_i)}{p_i(\alpha_{-i-(i+1)} \mid \alpha_i)} \\ & \quad \leq \log \sum_{\alpha_{-i-(i-1)} \in \mathscr{A}_{-i-(i-1)}} p(\alpha_{-i-(i-1)} \mid \alpha_i) \frac{p_i(\alpha_{-i-(i-1)} \mid \tilde{\alpha}_i)}{p_i(\alpha_{-i-(i-1)} \mid \alpha_i)} \\ & \qquad + \log \sum_{\alpha_{-i-(i+1)} \in \mathscr{A}_{-i-(i+1)}} p(\alpha_{-i-(i+1)} \mid \alpha_i) \frac{p_i(\alpha_{-i-(i+1)} \mid \tilde{\alpha}_i)}{p_i(\alpha_{-i-(i+1)} \mid \alpha_i)} \\ & \quad = 0 \end{aligned}$$

where the inequality is a consequence of the concavity of the function log.

Since, each i has at least two types, generically for all i, all α_i and all $\widetilde{\alpha}_i$, either $p_i(\alpha_{-i-(i-1)} \mid \alpha_i) \neq p_i(\alpha_{-i-(i-1)} \mid \tilde{\alpha}_i)$ for some $\alpha_{-i-(i-1)}$, or $p_i(\alpha_{-i-(i+1)} \mid \alpha_i) \neq p_i(\alpha_{-i-(i+1)} \mid \tilde{\alpha}_i)$ for some $\alpha_{-i-(i+1)}$, and because the function log is strictly concave, the above inequality is generically strict: it holds in an open and dense subset of all probability distributions p. Since the intersection of open and dense subsets is open and dense[9] condition B holds on an open and dense subset of the set of probability distributions, which proves the result. ■

In contrast, condition B never holds with only two agents. Indeed, d'Aspremont and Gérard-Varet (1982) show that in the two-agent case condition C, which is defined and shown to be weaker than condition B in the

[9] See, for instance, Hirsch and Smale (1974), Sect. 7.3.

next section, is equivalent to independence of types. By corollary 1, this implies that condition B never holds for two agents. Also, with more than two agents but some without private information (i.e., of a single type), genericity fails (see Forges et al. 2002).

3 Implementing efficient decision rules

In many cases, the planner acts on behalf of the agents, and tries to implement an efficient decision rule. In this section, we study the environments in which he can do so.

When condition B holds, the planner has tools to provide strict incentives to the agents to reveal their true types, the transfers t_i^B. Multiplied by a large enough constant they overwhelm any incentive to "lie" that would be created by the desire to influence the choice of the public decision. We introduce a condition C which is more modest – it simply states that we can collect from the agents any aggregate transfer, dependent on the state of nature, without inciting them to lie – and we will show in turn the following properties:

1. Condition C guarantees implementation of efficient decision rules[10];
2. It is equivalent[11] to, but simpler to interpret than, another condition introduced by d'Aspremont and Gérard-Varet (1982) (also called C, but called here C^*, as it is the dual form of C) and to LINK, a complex condition introduced by Johnson et al. (1990);
3. It is obtained, in a precise sense, by allowing some independence in condition B;
4. It is not necessary for the implementability of all efficient decision rules. However, we present a necessary and sufficient, but quite complicated, condition in Theorem 9.

In d'Aspremont et al. (2003), we show that condition C is more general (i.e., *strictly* less restrictive) than *all* the other sufficient conditions[12] for the existence of balanced-transfer Bayesian mechanisms that have been proposed in the literature, except for LINK, to which it is equivalent. Here we prove other properties[13] of condition C.

[10] The direct proof given here (see theorem 4 below) is also given in d'Aspremont et al. (2003), where we also show that condition C is equivalent to "guaranteeing budget balance": condition C holds if and only if whenever a BIC-mechanism exists, then there exists another one with a balanced transfer rule.

[11] This is shown in Theorem 8 below. This theorem, together with Theorem 2 in d'Aspremont and Gérard-Varet (1982), provides an alternative proof of Theorem 4.

[12] Matsushima's (1991) *regularity condition*, Chung's (1999) *weak regularity* condition (which is equivalent to Assumption I(i) in Aoyagi (1998)) and Fudenberg et al. (1994, 1995) *pairwise identifiability* condition.

[13] Theorems 5, 6 and 7 below.

3.1 Condition C

We weaken condition B, in the equivalent form presented in Corollary 3, by replacing the strict inequalities by weak inequalities:

Condition 2 (Condition C). *An information structure satisfies condition C if and only if for every function $R : \mathscr{A} \to \Re$, there exists a transfer rule t^C such that for all $\alpha \in \mathscr{A}$*

$$\sum_{i \in \mathscr{N}} t_i^C(\alpha) = R(\alpha) \tag{6}$$

and such that for all $i \in \mathscr{N}$ and all $(\alpha_i, \widetilde{\alpha}_i) \in \mathscr{A}_i^2$ we have:

$$\sum_{\alpha_{-i} \in \mathscr{A}_{-i}} t_i^C(\alpha_{-i}, \alpha_i) p(\alpha_{-i} \mid \alpha_i) \geq \sum_{\alpha_{-i} \in \mathscr{A}_{-i}} t_i^C(\alpha_{-i}, \widetilde{\alpha}_i) p(\alpha_{-i} \mid \alpha_i). \tag{7}$$

It is clear that condition C holds whenever condition B holds. Combined with Theorem 3 this provides an alternate, and much simpler, proof of the result of d'Aspremont et al. (1990) that condition C holds generically.

To see why condition C guarantees implementation of efficient decision rules, we can imagine that the planner has set up two bureaus and sent to the first, the "preference bureau", a description of the functions u_i^*. The bureau sends back a Vickrey-Clark-Groves transfer rule t^G that implements this efficient decision rule in dominant strategies, but without, in general, balancing the budget:

$$u_i^*(\alpha; \alpha_i) + t_i^G(\alpha) \geq u_i^*(\alpha_{-i}, \widetilde{\alpha}_i; \alpha_i) + t_i^G(\alpha_{-i}, \widetilde{\alpha}_i) \quad \text{for all } i \in \mathscr{N}, \text{ all } \alpha \in \mathscr{A} \text{ and all } \widetilde{\alpha}_i \in \mathscr{A}_i.$$

Let $R = -\sum_{i \in \mathscr{N}} t_i^G$ be the deficit. The preference bureau sends a message to a second bureau called the "beliefs bureau", stating "we have been able to provide incentives to the agents, but we are left with an aggregate deficit R".

If condition C holds, the beliefs bureau is able to find transfers t^C that satisfy Eqs. (6) and (7). It is immediate that the transfer rule $t = t^G + t^C$ balances the budget and provides the correct incentives. We have therefore proved the following theorem.

Theorem 4. *Any information structure that satisfies condition C guarantees implementation of efficient decision rules.*

Verifying that condition C holds looks like a formidable task: we have to check that the system composed of Eqs. (6) and (7) has a solution for any function R. The following result shows that the task, although computationally heavy, is still manageable.

Theorem 5. *Condition C holds if and only if for any $\alpha \in \mathscr{A}$ there exist transfer rules $t^{(\alpha,+)}$ and $t^{(\alpha,-)}$ such that*

$$\sum_{i\in\mathcal{N}} t_i^{(\alpha,+)}(\alpha) = 1, \quad \sum_{i\in\mathcal{N}} t_i^{(\alpha,-)}(\alpha) = -1$$
$$\sum_{i\in\mathcal{N}} t_i^{(\alpha,+)}(\alpha') = \sum_{i\in\mathcal{N}} t_i^{(\alpha,-)}(\alpha') = 0 \quad \text{for all } \alpha' \in \mathcal{A},\ \ \alpha' \neq \alpha,$$

and such that for all $i \in \mathcal{N}$ and all $(\alpha_i, \widetilde{\alpha}_i) \in \mathcal{A}_i^2$ condition (7) holds, when t^C is replaced by $t^{(\alpha,+)}$ or $t^{(\alpha,-)}$.

Proof. To prove necessity, it suffices, under condition C, to choose the proper R to get $t^C = t^{(\alpha,+)}$ (resp. $t^C = t^{(\alpha,-)}$): by putting $R(\alpha') = 0$ for all $\alpha' \neq \alpha$ and $R(\alpha) = 1$ (resp. $R(\alpha) = -1$). To show that the conditions of the theorem are sufficient, assume that they hold. The transfers required for condition C can then be built by the formula

$$t_i^C(\alpha) = \sum_{\{\alpha':R(\alpha')\geq 0\}} R(\alpha')t_i^{(\alpha',+)}(\alpha) + \sum_{\{\alpha':R(\alpha')\leq 0\}} (-R(\alpha'))t_i^{(\alpha',-)}(\alpha). \qquad \blacksquare$$

It is well known that independence of the types of agents implies that condition C^* holds (as already said, we will define C^* below, and prove that it is equivalent to C). This can be weakened.[14] For instance, it is sufficient to assume that one agent has free beliefs. The following theorem generalizes these known results, and uses a much simpler proof, thanks to the use of the primal version of condition C and Theorem 5. We say that agents i and j have their types *independent of each other* if : $p(\alpha_j \mid \alpha_i)p(\alpha_i) = p(\alpha_j)p(\alpha_i) = p(\alpha_i \mid \alpha_j)p(\alpha_j)$. Then:

Theorem 6. *If there exists two agents whose types are independent of each other condition C holds.*

Proof. Assume that the types of agents 1 and 2 are independent. Pick any $\alpha' \in \mathcal{A}$, and construct $t^{(\alpha',+)}$ as follows:

$$t_i^{(\alpha',+)}(\alpha) = 0 \text{ for all } i \geq 3 \text{ and all } \alpha \in \mathcal{A},$$
$$t_1^{(\alpha',+)}(\alpha) = t_2^{(\alpha',+)}(\alpha) = 0 \text{ except:}$$
$$t_1^{(\alpha',+)}(\alpha') = 1 - p(\alpha'_2),\ \ t_2^{(\alpha',+)}(\alpha') = p(\alpha'_2),$$
$$t_1^{(\alpha',+)}(\alpha'_{-2}, \alpha_2) = -p(\alpha'_2),\ \ t_2^{(\alpha',+)}(\alpha'_{-2}, \alpha_2) = p(\alpha'_2) \ \text{ if } \alpha_2 \neq \alpha'_2$$

It is easy to check that these transfers satisfy the conditions of Theorem 5. In particular, to see that they satisfy condition (7), note that if agent 1 announces α'_1, his expected payment is

[14] We show that very little independence is required to guarantee implementation of efficient decision rules. Another extension is to keep independence unchanged and to strengthen the results. For instance, Crémer and Riordan (1985) show that if the types of all agents are independent, there exists a balanced *BIC* mechanism in which all but one agent have dominant strategies.

$$[1 - p(\alpha_2')]p(\alpha_2') - p(\alpha_2')[1 - p(\alpha_2')] = 0,$$

which is equal to his expected payment if he announces any other α_1. (Remember that, because the types of agents 1 and 2 are independent, $p(\alpha_2 \mid \alpha_1) = p(\alpha_2)$ for all α_1 and α_2.) Transfers $t^{(\alpha',-)}$ can simply be taken equal to $-t^{(\alpha',+)}$. ■

Finally, it is immediate to prove the following theorem.

Theorem 7. *Condition C holds if an agent has only one possible type ($\mathscr{A}_i$ contains only one element).*

Proof. If agent 1 has only one type use $t_1 = R$, with the other t_i's identically equal to 0. ■

3.2 Dual form: Condition C^*

In this subsection we define a dual form of condition C, condition C^*, first defined[15] in d'Aspremont and Gérard-Varet (1982), and which will be useful later. We need:

Definition 1 (Fundamental dual system). *The system of equations*

$$p(\alpha_{-i} \mid \alpha_i)\left[\sum_{\widetilde{\alpha}_i \neq \alpha_i} \lambda_i(\widetilde{\alpha}_i, \alpha_i)\right] - \left[\sum_{\widetilde{\alpha}_i \neq \alpha_i} p(\alpha_{-i} \mid \widetilde{\alpha}_i)\lambda_i(\alpha_i, \widetilde{\alpha}_i)\right] = \mu(\alpha)$$
$$\text{for all } i \in \mathscr{N} \text{ and all } \alpha \in \mathscr{A},$$

whose unknowns are the functions $\lambda_i : \mathscr{A}_i^2 \to \Re^+$ and *$\mu : \mathscr{A} \to \Re$ is called the fundamental dual system of equations associated with the information structure $(\mathscr{N}, \mathscr{A}, p)$.*

Then the condition can be simply stated:

Condition 3 (condition C*). *An information structure satisfies condition C^* if all solutions of the associated fundamental dual system of equations satisfy $\mu(\alpha) = 0$* **for all** *$\alpha \in \mathscr{A}$.*

Condition C^* is simply the dual of condition C, as the following theorem shows.

Theorem 8. *Condition C^* holds if and only if condition C holds.*

[15] This condition is close but slightly weaker than the "compatibility condition" introduced by d'Aspremont and Gérard-Varet (1979). See Johnson et al. (1990).

Proof. Condition C^* can be rewritten: Whatever the function $\mu : \mathscr{A} \to \Re$, not uniformly equal to zero, there exists no family of functions $\lambda_i : \mathscr{A}_i^2 \to \Re^+$ that satisfy

$$p(\alpha_{-i} \mid \alpha_i) \sum_{\tilde{\alpha}_i \neq \alpha_i} \lambda_i(\tilde{\alpha}_i, \alpha_i) - \sum_{\tilde{\alpha}_i \neq \alpha_i} p(\alpha_{-i} \mid \tilde{\alpha}_i) \lambda_i(\alpha_i, \tilde{\alpha}_i) = \mu(\alpha)$$

$$\text{for all } i \in \mathscr{N} \text{ and all } \alpha \in \mathscr{A}.$$

This can in turn be rewritten: Whatever the function $R : \mathscr{A} \to \Re$, not uniformly equal to zero, there exists no family of functions $\lambda_i : \mathscr{A}_i^2 \to \Re^+$ and no function $w : \mathscr{A} \to \Re$ that satisfy

$$p(\alpha_{-i} \mid \alpha_i) \sum_{\widetilde{\alpha_i \neq \alpha_i}} \lambda_i(\tilde{\alpha}_i, \alpha_i) - \sum_{\widetilde{\alpha_i \neq \alpha_i}} p(\alpha_{-i} \mid \tilde{\alpha}_i) \lambda_i(\alpha_i, \tilde{\alpha}_i) = w(\alpha)$$

$$\text{for all } i \in \mathscr{N} \text{ and all } \alpha \in \mathscr{A}.$$

$$\sum_{\alpha \in \mathscr{A}} w(\alpha) R(\alpha) > 0$$

Using (e.g., as in Lemma 1 of d'Aspremont et al. (1990)) the duality of linear inequalities,[16] it follows straightforwardly that this condition is equivalent to the existence of a t^C that satisfies the required conditions. ■

In d'Aspremont et al. (2003), we show that an information structure $(\mathscr{N}, \mathscr{A}, p)$ satisfies condition B if and only if it satisfies condition C and for all $i \in \mathscr{N}$ and all $(\alpha_i, \tilde{\alpha}_i) \in \mathscr{A}_i^2$ there exists $\alpha_{-i} \in \mathscr{A}_{-i}$ such that

$$p(\alpha_{-i} \mid \alpha_i) \neq p(\alpha_{-i} \mid \tilde{\alpha}_i).$$

This shows that condition B is equivalent to condition C plus a no independence of types condition. This answers in part the question that we asked in the introduction about the role of correlation and independence in implementation: Correlation (as in condition B) enables us to implement any decision rule; independence allows us also to implement efficient decision rules.

3.3 There exists information structures that do not satisfy condition C but guarantee implementation of efficient decision rules

As soon as there are more than two agents, it becomes very difficult to find examples of information structures that do not satisfy condition C.[17] In this subsection we present examples of such information structures that still

[16] That is, using some theorem of the alternative: see for instance Fan (1956), Gale (1960), or Mangasarian (1969).

[17] As we have already mentioned, d'Aspremont and Gérard-Varet (1982) show that with two agents Condition C holds only if types are independent. This leaves open the question when there are at least three agents, as we know that in this case condition C holds for a much wider class of information structure.

guarantee implementation of efficient decision rules. Hence, condition C is not necessary to guarantee efficient implementation.

We will show that the information structures represented on Fig. 1 guarantees implementation of efficient decision rules and does not satisfy condition C.

Efficiency turns out to be a tricky concept to handle in this type of frameworks, and the proof is rather involved. The reader can skip it without loss of continuity.

We begin by the following lemma. To the best of our knowledge, this is the first *necessary* and sufficient condition presented in the literature for an information structure to guarantee implementation of *efficient* decision rules.

Theorem 9. *An information structure* $(\mathcal{N}, \mathcal{A}, p)$ *guarantees implementation of efficient decision rules if and only if for all* λ_i's *and* μ's *solution of the fundamental dual system the system of equations in the unknowns* $v(\alpha;\widetilde{\alpha})$, $\alpha \neq \widetilde{\alpha}$:

$$\sum_{\widetilde{\alpha}\neq\alpha} v(\alpha;\widetilde{\alpha}) - \sum_{\widetilde{\alpha}_{-i}\neq\alpha_{-i}} v(\widetilde{\alpha}_{-i},\alpha_i;\alpha_{-i},\alpha_i) = \sum_{\widetilde{\alpha}_i\neq\alpha_i} \lambda_i(\widetilde{\alpha}_i,\alpha_i)p(\alpha_{-i} \mid \alpha_i)$$

$$\text{for all } \alpha \text{ and all } i, \tag{8}$$

$$\sum_{\widetilde{\alpha}_{-i}} v(\widetilde{\alpha}_{-i},\widetilde{\alpha}_i;\alpha) = \sum_{\widetilde{\alpha}_i\neq\alpha_i} \lambda(\alpha_i,\widetilde{\alpha}_i)p(\alpha_{-i} \mid \widetilde{\alpha}_i)$$

$$\text{for all } \alpha_{-i}, \alpha_i, \widetilde{\alpha}_i, \widetilde{\alpha}_i \neq \alpha_i \text{ and all } i, \tag{9}$$

$$v(\alpha;\widetilde{\alpha}) \geq 0 \text{ for all } \alpha \neq \widetilde{\alpha}, \tag{10}$$

has a solution where the v*'s are not all equal to 0.*

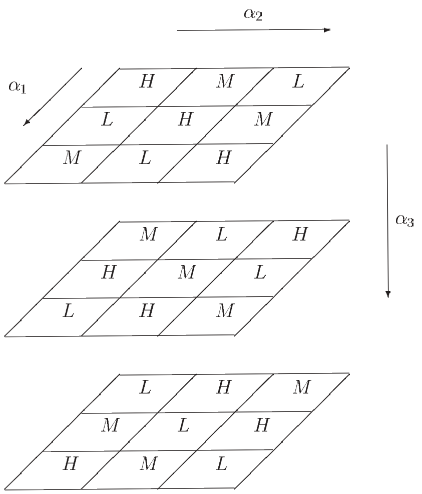

Fig. 1. This figure represents a probability structure that does not satisfy condition C but guarantees implementation of efficient decision rules. For instance, the probability that $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 1$ is equal to M, and the probability that $\alpha_1 = 3$, $\alpha_2 = 1$, $\alpha_3 = 3$ is equal to H. We assume $H \geq M \geq L$, with at least one of these inequalities strict, and, of course, $H + M + L = 1/9$

Proof. Using again the theory of duality of linear inequalities, Lemma 1 of d'Aspremont et al. (1990) shows that, given utility functions u_i for each agent $i \in \mathcal{N}$ and an information structure $(\mathcal{N}, \mathcal{A}, p)$, there exists a balanced Bayesian mechanism if and only if the system of equations consisting of the fundamental dual system and

$$\sum_{i\in\mathcal{N}}\sum_{\mathcal{A}_i^2}\lambda_i(\widetilde{\alpha}_i,\alpha_i)\left[\sum_{\alpha_{-i}\in\mathcal{N}_{-i}}p(\alpha_{-i}\mid\alpha_i)[u_i(\widetilde{\alpha}_i,\alpha_{-i};\alpha_i)-u_i(\alpha_i,\alpha_{-i};\alpha_i)]\right]>0 \tag{11}$$

does not have a solution.

Hence an information structure guarantees implementation of efficient decision rules if and only if for all u_i^*'s satisfying Eq. (2) the system consisting of the fundamental dual system and Eq. (11) does not have a solution. It follows that an information structure guarantees implementation of efficient decision rules if and only if for all λ_i's and all μ's, solution of the fundamental dual system – the system of equations composed of Eq. (2) and Eq. (11) – does not have a solution. This proves the lemma as the system consisting of Eqs. (8) to (10) is the dual of the system consisting of Eqs. (2) and (11), with the unknowns being the utilities. ■

In order to prove that the information structure represented by Fig. 1 guarantees efficient implementation we prove that it satisfies the conditions of Theorem 9.

From the fundamental dual system we obtain

$$\begin{aligned}\mu(1,1,1)&=H[\lambda_1(2,1)+\lambda_1(3,1)]-L\lambda_1(1,2)-M\lambda_1(1,3)\\&=H[\lambda_2(2,1)+\lambda_2(3,1)]-M\lambda_2(1,2)-L\lambda_3(1,3);\\\mu(1,1,2)&=M[\lambda_1(2,1)+\lambda_1(3,1)]-H\lambda_1(1,2)-L\lambda_1(1,3))\\&=M[\lambda_2(2,1)+\lambda_2(3,1)]-L\lambda_2(1,2)-H\lambda_3(1,3);\\\mu(1,1,3)&=L[\lambda_1(2,1)+\lambda_1(3,1)]-M\lambda_1(1,2)-H\lambda_1(1,3)\\&=L[\lambda_2(2,1)+\lambda_2(3,1)]-H\lambda_2(1,2)-M\lambda_3(1,3).\end{aligned} \tag{12}$$

We define $x_2=\lambda_1(1,2)-\lambda_2(1,3)$, $x_3=\lambda_1(1,3)-\lambda_2(1,2)$ and $x_1=x_2+x_3$, and we substitute in (12). Remembering that we cannot have $H=M=L$, we then obtain $x_1=x_2=x_3=0$. Hence, $\lambda_1(1,2)=\lambda_2(1,3)$, and $\lambda_1(1,3)=\lambda_2(1,2)$.

Using repeatedly similar arguments, we can prove that for any solution of the fundamental dual system there exist two non-negative real numbers, ρ and σ such that

$$\begin{aligned}\rho&=\lambda_1(1,2)=\lambda_2(1,3)=\lambda_3(1,3)=\lambda_1(2,3)\\&=\lambda_2(2,1)=\lambda_3(2,1)=\lambda_1(3,1)=\lambda_2(3,2)=\lambda_3(3,2)\end{aligned} \tag{13}$$

and that the other λ_i's are equal to σ. One can check that any system of λ_i's satisfying these properties is, associated with the relevant μ's, a solution to the fundamental dual system.

We can finally verify that the system of equations (8) to (10) has a solution where the values of $v(\alpha;\widetilde{\alpha})$ are represented on the following table:

		H	$p(\widetilde{\alpha})M$	L
$p(\alpha)$	H	$2(\rho L+\sigma M)/3$	$\rho/3$	$\sigma/3$
	M	$\sigma/3$	$2(\rho H+\sigma L)/3$	$\rho/3$
	L	$\rho/3$	$\sigma/3$	$2(\rho M+\sigma H)/3$

(For instance, if $p(\alpha)=H$ and $p(\widetilde{\alpha})=L$, then $v(\alpha;\widetilde{\alpha})=\sigma/3$.)

From (12) and (13), $\mu(1,1,1)=2H\rho-L\sigma-M\sigma$, which can be made different from 0. Hence, the information structure does not satisfy condition C.

This proves the result.

4. There exist environments in which efficient decision rules cannot be implemented

We have proved that condition C is sufficient but not necessary to guarantee implementation of efficient decision rules; we know that nearly all information structures satisfy condition C; a natural conjecture is that all information structures guarantee implementation of efficient decision rules. In this section, we show that this conjecture is false by displaying an example of an environment where efficient mechanisms cannot be implemented.

Before beginning, we would like to point out that the literature is implicitly misleading on this point. Many papers produce examples of sufficient conditions for existence of BIC mechanisms, leaving the impression that it is difficult to prove existence. Actually, finding an information structure that does not guarantee existence is the difficult part. In d'Aspremont et al. (1990), we have shown that with two agents there do not always exist Bayesian mechanisms that implement efficient decision rules. The restriction to two agents can create very special results, as witnessed, for instance, by the fact that in this case condition C is equivalent to independence. In this section, we show that the fact that efficient decision rules cannot be implemented holds more generally.

We consider an environment with three agents, each having four types. The states of nature are allocated in four subsets, which are described on Fig. 2.

We assume that the information structure satisfies have

$$p(\alpha)=\begin{cases}\frac{1}{16}-3\varepsilon & \text{if } \alpha\in H,\\ \varepsilon & \text{if } \alpha\notin H,\end{cases}$$

for some $\varepsilon<1/49$.

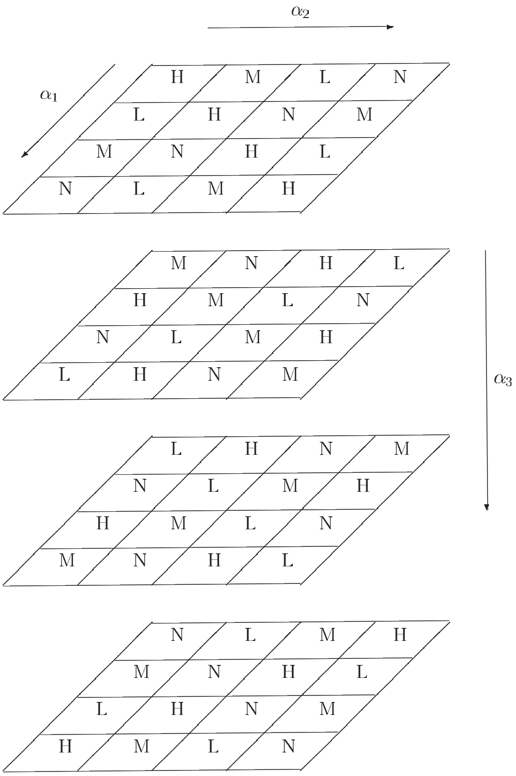

Fig. 2. The allocation of the states of nature in 4 subsets for the example of Sect. 4

The utility functions of the agents are[18]

$$u_i(\alpha; \alpha_i) = 0 \text{ for all } \alpha \text{ and all } i;$$
$$u_i(\widetilde{\alpha}_i, \alpha_{-i}; \alpha_i) = 1 \text{ if } \alpha \in H \text{ and } (\widetilde{\alpha}_i, \alpha_{-i}) \in M;$$
$$u_i(\widetilde{\alpha}; \alpha_i) = -1 \text{ in all other cases.}$$

(Small variations in these utility functions would not change the result). If the transfers are equal to 0, for any α_i there is exactly one type $\widetilde{\alpha}_i$ such that agent i would have unambiguous incentives to announce $\widetilde{\alpha}_i$ rather than α_i. For instance, for

$$(\alpha_1, \widetilde{\alpha}_1) \in \{(1,3), (2,1), (3,4), (4,2)\} \tag{14}$$

agent 1 of type α_1 has an unambiguous incentive to announce $\widetilde{\alpha}_1$ as, depending on the announcement of the other agents, this would never decrease his utility and would sometimes increase it.

[18] It is easy to check that this utility function satisfies condition (2).

Assume that there existed an efficient mechanism. Let $(\alpha_1, \widetilde{\alpha}_1)$ satisfy (14). For agent 1 of type α_1 not to have any incentive to announce that he is of type $\widetilde{\alpha}_1$ (i.e., for condition (BIC) to hold), we must have

$$\begin{aligned}
&\left(\frac{1}{4} - 12\varepsilon\right)[\sum_{(\alpha_1,\alpha_{-1})\in H} t_1(\alpha_1, \alpha_{-1})] + 4\varepsilon \sum_{(\alpha_1,\alpha_{-1})\notin H} [t_1(\alpha_1, \alpha_{-1})] \\
&\quad \geq \left(\frac{1}{4} - 12\varepsilon\right)[4 + \sum_{(\alpha_1,\alpha_{-1})\in H} t_1(\widetilde{\alpha}_1, \alpha_{-1})] + 4\varepsilon \sum_{(\alpha_1,\alpha_{-1})\notin H} [-1 + t_1(\widetilde{\alpha}_1, \alpha_{-1})] \\
&\quad = \left(\frac{1}{4} - 12\varepsilon\right)[4 + \sum_{(\widetilde{\alpha}_1,\alpha_{-1})\in M} t_1(\widetilde{\alpha}_1, \alpha_{-1})] + 4\varepsilon \sum_{(\widetilde{\alpha}_1,\alpha_{-1})\notin M} [-1 + t_1(\widetilde{\alpha}_1, \alpha_{-1})].
\end{aligned}$$

Add the four inequalities corresponding to the four pairs $(\alpha_1, \widetilde{\alpha}_1)$, we obtain

$$\begin{aligned}
&\left(\frac{1}{4} - 12\varepsilon\right) \sum_{\alpha\in H} t_1(\alpha) + 4\varepsilon \sum_{\alpha\notin H} t_1(\alpha) \\
&\quad \geq \left(\frac{1}{4} - 12\varepsilon\right)[16 + \sum_{\alpha\in M} t_1(\alpha)] + 4\varepsilon \sum_{\alpha\notin M} [-1 + t_1(\alpha)].
\end{aligned}$$

We can use the same reasoning[19] for the two other agents to obtain, for all i,

$$\begin{aligned}
&\left(\frac{1}{4} - 12\varepsilon\right) \sum_{\alpha\in H} t_i(\alpha) + 4\varepsilon \sum_{\alpha\notin H} t_i(\alpha) \\
&\quad \geq \left(\frac{1}{4} - 12\varepsilon\right)[16 + \sum_{\alpha\in M} t_i(\alpha)] + 4\varepsilon \sum_{\alpha\notin M} [-1 + t_i(\alpha)],
\end{aligned}$$

Adding the three inequalities corresponding to the three agents yields

$$\begin{aligned}
&\left(\frac{1}{4} - 12\varepsilon\right) \sum_{\alpha\in H} (t_1(\alpha) + t_2(\alpha) + t_3(\alpha)) + 4\varepsilon \sum_{\alpha\notin H} (t_1(\alpha) + t_2(\alpha) + t_3(\alpha)) \\
&\quad \geq \left(\frac{1}{4} - 12\varepsilon\right)[48 + \sum_{\alpha\in M} (t_1(\alpha) + t_2(\alpha) + t_3(\alpha))] \\
&\quad + 4\varepsilon[-3 + \sum_{\alpha\notin M} (t_1(\alpha) + t_2(\alpha) + t_3(\alpha))],
\end{aligned}$$

which is by the budget balance equation (1) is equivalent to

$$0 \geq 12 - 588\varepsilon \geq 12 - 588 \times \frac{1}{49} > 0,$$

and establishes the contradiction.

[19] For instance, agent 2 of type 1 must not announce that he is of type 2, agent 2 of type 2 not announce that he is of type 4, agent 2 of type 3 not announce that he is of type 1, agent 2 of type 4 not announce that he is of type 3.

5 Unique implementation

The results that we have presented up to now are subject to the traditional criticism: they show that there exists one equilibrium with the requisite properties, but there might exist other equilibria that do not have these characteristics. Nothing guarantees that the "good" equilibrium will obtain. Following the seminal contribution of Maskin (1977, 1985) on Nash implementation under complete information, a number of authors (e.g., Postlewaite and Schmeidler 1986; Palfrey and Srivastava 1991; Jackson 1991) have addressed the problem of unique implementation under incomplete information. They mainly concentrate on necessary conditions, such as the extension of Maskin's monotonicity condition, called "Bayesian monotonicity", and incentive compatibility. The Bayesian monotonocity condition restricts jointly the utilities and the probabilities. But, as shown in Palfrey (1992), the possibility of ensuring unique implementation (under Bayesian incentive compatibility and transferable utilities) may result from conditions imposed *on the belief structure only*. As in Maskin's original work, unique implementation is obtained by enlarging the message space of the direct mechanism to non-type messages, and these are of the kind introduced in Ma et al. (1988) and used by Mookherjee and Reichelstein (1990). Applying a similar technique of enlargement of the message space, we introduce a much weaker condition on the probabilities alone to guarantee unique implementation of decision rules. We will prove the following theorem:[20]

Theorem 10. *Consider any direct mechanism (s, t) which is Bayesian incentive compatible and budget-balanced. Assume that*

$$p(\alpha_{-i} \mid \alpha_i) \neq p(\widetilde{\alpha}_{-i} \mid \widetilde{\alpha}_i) \text{ whenever } \alpha_{-i} \neq \widetilde{\alpha}_{-i}. \tag{15}$$

Then, there exists another mechanism $(\mathscr{M}, x, \tau)$ that uniquely implements s.

Condition (15) holds for nearly all information structures[21] Indeed, because the conditional probabilities are continuous functions of the $p(\alpha)$'s it is straightforward that the set of information structures that satisfy it contains an open neighbourhood of any of its elements. The remaining task is to show that if some information structure does not satisfy it, one can find another probability structure arbitrarily close that satisfies this property. For that task, the argument used to derive the genericity of condition C in d'Aspremont et al.

[20] This is a generalization of Proposition 3 in d'Aspremont et al. (1999).

[21] This condition plays a role similar to that of condition NCD (No Consistent Deceptions) introduced by Matsushima (1990). Such conditions are indispensable if we are to find conditions on information structures alone that guarantee unique implementation. To see this, consider the case where the same utility function is attached to two different types. We can only guarantee unique implementation if the types generate different probability distributions over the types of the other agents.

(1990) can be adapted. It consists in proving that any information structure that does not satisfy the condition can be slightly modified in order to reduce the number of equalities between conditional probabilities. A sequence of such modifications will lead to an information structure that satisfies the condition. In summary, the conclusion of the theorem holds for nearly all information structures. By the genericity of condition B, which guarantees implementation of all decision rules, this theorem has thus an important corollary:

Corollary 4. *For nearly all information structures, all decision rules can be uniquely implemented.*

Notice that condition (15) is compatible with independence. Indeed, the reasoning which leads to Corollary 4 can be repeated if we constrain the types to be independent. Now, we know that with independence all decision rules cannot be implemented, but we know that efficient decision rules can be implemented. This yields the following corollary.

Corollary 5. *For nearly all information structures satisfying independence, all efficient decision rules can be uniquely implemented.*

The proof of the theorem can be divided in two steps, each given by a lemma which is of some interest *per se*. Before presenting the first lemma, we notice that condition (15) implies the following condition:

$$\begin{aligned}&\text{For all } i \neq j \text{ and all } (\alpha_i, \alpha_j) \in \mathscr{A}_i \times \mathscr{A}_j,\\ &\quad p(\alpha_{-i-j}, \alpha_j \mid \alpha_i) > 0, \text{ for some } \alpha_{-i-j} \in \mathscr{A}_{-i-j}.\end{aligned} \tag{16}$$

Indeed, if (16) did not hold, we would have $p(\alpha_{-i-j}, \alpha_j \mid \alpha_i) = 0$ for all α_{-i-j}, which contradicts (15).

Lemma 1. *Consider any direct mechanism (s, t) which is Bayesian incentive compatible and budget-balanced. Assume that for every $i \neq j$, condition (16) holds and that for all i and all non trivial bijections*[22] *$\gamma : \mathscr{A}_{-i} \to \mathscr{A}_{-i}$, there exist $\widetilde{\alpha}_i \in \mathscr{A}_i$ and $\widetilde{t}_i : \mathscr{A}_{-i} \to \Re$ such that*

$$\begin{cases} \sum\limits_{\alpha_{-i} \in \mathscr{A}_{-i}} \widetilde{t}_i(\alpha_{-i}) p(\gamma(\alpha_{-i}) \mid \widetilde{\alpha}_i) > 0, \\ \sum\limits_{\alpha_{-i} \in \mathscr{A}_{-i}} \widetilde{t}_i(\alpha_{-i}) p(\alpha_{-i} \mid \alpha_i) < 0, \quad \text{for all } \alpha_i \neq \widetilde{\alpha}_i. \end{cases} \tag{17}$$

Then, there exists another mechanism $(\mathscr{M}, x, \tau)$ which has a unique equilibrium $\widetilde{m}$ and implements s.

Proof. We will actually show something stronger: we will not only show that $x(\widetilde{m}(\alpha)) = s(\alpha)$ but also that $\tau(\widetilde{m}(\alpha)) = t(\alpha)$ for all α. To build the new

[22] That is, bijections that are not the identity mapping.

mechanism let us define first the set Θ^i of functions $\theta^i : \mathscr{A}_{-i} \to \Re$ such that either $\theta^i(\alpha_{-i}) = 0$ for all $\alpha_{-i} \in \mathscr{A}_{-i}$, or

$$\sum_{\alpha_{-i} \in \mathscr{A}_{-i}} \theta^i(\alpha_{-i}) p(\alpha_{-i} \mid \alpha_i) < 0 \quad \text{for all } \alpha_i \in \mathscr{A}_i. \tag{18}$$

For every i we let $\mathscr{M}_i = \mathscr{A}_i \times \Theta^i$. A message of agent i consists in announcing a type α_i in $\mathscr{A}_i$ and a function θ^i in Θ^i. We define

$$x((\alpha_1, \theta^1), \ldots, (\alpha_n, \theta^n)) = s(\alpha),$$
$$\tau_i((\alpha_1, \theta^1), \ldots, (\alpha_n, \theta^n)) = t_i(\alpha) + \theta^i(\alpha_{-i}) - \sum_{j \in \mathscr{N}_{-i}} \frac{1}{n-1} \theta^j(\alpha_{-j}) \text{ for all } i.$$

It is clear that this transfer rule is balanced. It has a simple interpretation: each agent i participates in the original mechanism, but is also allowed to ask for contributions from the other agents, as long as these contributions are guaranteed to give him a negative payoff in the truthtelling equilibrium.

In the augmented mechanism, for every agent i, let the equilibrium strategy of i be written $(\tilde{m}_i^\alpha, \tilde{m}_i^\theta)$ with $\tilde{m}_i^\alpha : \mathscr{A}_i \to \mathscr{A}_i$ and $\tilde{m}_i^\theta : \mathscr{A}_i \to \Theta^i$. It is an equilibrium for every agent i of type α_i to announce the message $(\alpha_i, 0)$, because truthtelling is an equilibrium of the original mechanism, and because the "extra" transfers can only yield negative expected payoffs when the other agents tell the truth. For the same reason there exist no equilibrium in which all agents announce their true types ($\tilde{m}_j^\alpha(\alpha_j) = \alpha_j$ for all j and all α_j) and in which we have $\tilde{m}_i^\theta \neq 0$ for some agent i. Therefore in any other candidate equilibrium, at least one agent, say agent 1, must lie about his type: $\tilde{m}_1^\alpha(\alpha_1) \neq \alpha_1$ for some α_1. However, whatever j (lying or not), $\tilde{m}_j^\alpha$ has to be a bijection. Otherwise, some $\alpha_j'' \in \mathscr{A}_j$ would never be announced by agent j, and by (16), for any $i \neq j$ and for M large enough, any positive multiple of the function defined by

$$\theta^i(\alpha_{-i}) = \begin{cases} 1 & \text{if } \alpha_j \neq \alpha_j'', \\ -M & \text{if } \alpha_j = \alpha_j'', \end{cases}$$

would belong to Θ^i, and agent i would have no best response (a large multiple of θ^i would always be better than any chosen announcement). So, choose $i \neq 1$ and let γ be the inverse function of $\tilde{m}_{-i}^\alpha$. Then γ is a bijection from $\mathscr{A}_{-i}$ to $\mathscr{A}_{-i}$ which is different from the identity, so that, by hypothesis, there are some $\tilde{\alpha}_i \in \mathscr{A}_i$ and some $\tilde{t}_i : \mathscr{A}_i \to \Re$ satisfying (17). Any θ^i equal to $\lambda \tilde{t}_i$, with $\lambda > 0$, belongs to Θ^i and, because the greater the λ the better the response for agent i of type $\tilde{\alpha}_i$, we cannot get any equilibrium other than the truthtelling one. The lemma is proved. ■

Theorem 10 is now a simple consequence of the second lemma:

Lemma 2. *If condition (15) holds, then for all i and all non trivial bijections* $\gamma : \mathscr{A}_{-i} \to \mathscr{A}_{-i}$, *there exist* $\tilde{\alpha}_i \in \mathscr{A}_i$ *and* $\tilde{t}_i : \mathscr{A}_{-i} \to \Re$ *satisfying (17).*

Proof. Choose any agent i and any bijection $\gamma : \mathscr{A}_{-i} \to \mathscr{A}_{-i}$, not equal to the identity. There exists a state of nature $\tilde{\alpha}$ such that $\tilde{\alpha}_{-i} \neq \gamma^{-1}(\tilde{\alpha}_{-i})$ (or $\gamma(\tilde{\alpha}_{-i}) \neq \tilde{\alpha}_{-i}$) and such that

$$p(\tilde{\alpha}_{-i} \mid \tilde{\alpha}_i) > p(\alpha_{-i} \mid \alpha_i), \quad \text{for all } \alpha \in \mathscr{A} \text{ with } \alpha_{-i} \neq \tilde{\alpha}_{-i}. \tag{19}$$

We obtain

$$p(\tilde{\alpha}_{-i} \mid \tilde{\alpha}_i) > p(\gamma^{-1}(\tilde{\alpha}_{-i}) \mid \alpha_i) \text{ for all } \alpha_i. \tag{20}$$

Now, for any $\eta > 0$, let the transfer function $\tilde{t}_i$ be defined by

$$\begin{aligned} &\tilde{t}_i(\gamma^{-1}(\tilde{\alpha}_{-i})) = 1 \\ &\tilde{t}_i(\alpha_{-i}) = -\eta \quad \text{for all } \alpha_{-i} \neq \gamma^{-1}(\tilde{\alpha}_{-i}). \end{aligned}$$

Note that

$$\begin{aligned} &\sum_{\alpha_{-i} \in \mathscr{A}_{-i}} \tilde{t}_i(\alpha_{-i}) p(\gamma(\alpha_{-i}) \mid \tilde{\alpha}_i) \\ &\qquad = \tilde{t}_i(\gamma^{-1}(\tilde{\alpha}_{-i})) p(\tilde{\alpha}_{-i} \mid \tilde{\alpha}_i) - \eta(1 - p(\tilde{\alpha}_{-i} \mid \tilde{\alpha}_{-i})) \\ &\qquad = (1+\eta) p(\tilde{\alpha}_{-i} \mid \tilde{\alpha}_i) - \eta \end{aligned}$$

and that for $\alpha_i \neq \tilde{\alpha}_i$

$$\begin{aligned} &\sum_{\alpha_{-i} \in \mathscr{A}_{-i}} \tilde{t}_i(\alpha_{-i}) p(\alpha_{-i} \mid \alpha_i) \\ &\qquad = \tilde{t}_i(\gamma^{-1}(\tilde{\alpha}_{-i})) p(\gamma^{-1}(\tilde{\alpha}_{-i}) \mid \alpha_i) - \eta(1 - p(\gamma^{-1}(\tilde{\alpha}_{-i}) \mid \alpha_i)) \\ &\qquad = (1+\eta) p(\gamma^{-1}(\tilde{\alpha}_{-i}) \mid \alpha_i) - \eta. \end{aligned}$$

Then, equation (20) implies

$$\sum_{\alpha_{-i} \in \mathscr{A}_{-i}} \tilde{t}_i(\alpha_i) p(\gamma(\alpha_{-i}) \mid \tilde{\alpha}_i) > \sum_{\alpha_{-i} \in \mathscr{A}_{-i}} \tilde{t}_i(\alpha_i) p(\alpha_{-i} \mid \alpha_i) \quad \text{for all } \alpha_i \neq \tilde{\alpha}_i.$$

Choose η just large enough so that the left hand side of this inequality is positive while the right hand side is negative for all α_i. Then the transfer function $\tilde{t}_i$ satisfies (17). The result follows. ■

6 Conclusion

In this paper we have solved the problem of implementation of all decision rules with transferable utility: condition B is necessary and sufficient for a solution to exist. We have also shown that, for nearly all information structures, any decision rule can be implemented by constructing a balanced transfer rule of the "scoring rule" type. This constructive result, under condition B, is an important adjunction to the other (but disjoint) case where a balanced transfer rule can be explicitly given to implement any efficient decision rule, namely the case of independence of types.

This implies that we can solve constructively the existence problem for a large part of the information structures that satisfy condition C, which is weaker than both condition B and independence, and, as shown in d'Aspremont et al. (2003) weaker than all other conditions that have been proposed in the literature. Furthermore, condition C has an immediate interpretation. However, the paper has shown that it is not necessary to guarantee implementation of efficient decision rules. To prove that, we have provided a necessary and sufficient condition for the implementation of efficient decision rules, but the interpretation of this new condition remains to be clarified.

Finally, we have produced conditions that guarantee unique implementation. It seems very difficult to weaken them significantly, without making joint assumptions on the utility functions and on the probability distributions. Similar techniques are applicable to auctions, as shown in d'Aspremont et al. (1999).

Many questions which we do not tackle here are still open, even if we concentrate on efficient mechanisms in the first best sense. More work should be devoted to issues such as risk aversion and limited transferability,[23] ex-post and interim individual rationality constraints[24] (ex-ante individual rationality creates no difficulty as theorem 1 shows), and the comparisons of various concepts of efficiency, durability, renegotiation proofness or robustness to coalitions.[25]

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[23] See d'Aspremont and Gérard-Varet (1992).

[24] See Makowski and Mezzetti (1991).

[25] See, for example, Holmstrom and Myerson (1983), Tirole (1986), Rovesti (1992), Crémer (1996), and Forges et al. (2002).

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