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Abstract

We construct a complete space of smooth strictly convex preference relations defined over physical commodities and monetary transfers. This construction extends the classic one by assuming that preferences are monotone in transfers, but not necessarily in all commodities. This provides a natural framework to perform genericity analyses in situations involving inventory costs or decisions under risk.

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1 Introduction

Applications of classical demand theory tend to treat each commodity either as a “good” or as a “bad.” This implies that one can, if necessary, redefine the commodities so as to treat all of them as goods.¹ Thus leisure time is the flip side of hours worked, and clean air or water is the flip side of pollution. The technical translation of this redefinition is that one may focus on preferences that are monotone in each commodity.²

However, such a redefinition is not always warranted. For instance, if a consumer, or a firm, cannot freely dispose of a stock of unwanted commodities, they will typically have to bear inventory costs for holding this stock; for instance, they may need to acquire a new storage facility. In the presence of such costs or, more generally, when the agent’s objective function is an indirect utility function derived under technological constraints, there are situations where more of a commodity actually decreases the agent’s utility, or profit. As a result, one cannot unambiguously classify such a commodity as a good or a bad, as this ultimately depends on how much of it the agent is initially endowed with.

Another example of nonmonotonicity arises in finance, when one considers investors’ preferences over portfolios of assets. Assume for instance that an investor with constant absolute risk-aversion α can invest in l risky assets with payoffs that are jointly normally distributed with mean vector $\bar{\mathbf{a}}$ and covariance matrix Γ , as well as in a risk-free asset. Then his preferences over portfolios of risky and risk-free assets $(\mathbf{q}, t) \in \mathbb{R}^{l+1}$ are represented by

$$u(\mathbf{q}, t) = \mathbf{q}^\top \bar{\mathbf{a}} - \frac{\alpha}{2} \mathbf{q}^\top \Gamma \mathbf{q} + t, \quad (1)$$

and are thus nonmonotone in \mathbf{q} , reflecting that the risk-averse investor does not want to hold an excessively risky asset position.

In such contexts, the standard construction of a complete space of smooth strictly convex preference relations (Mas-Colell (1985, Chapter 2)) has to be amended. We propose such a canonical model of preferences in which nonmonotonicity is allowed for certain commodities, but there is one commodity that is always desirable. In line with (1), this commodity is interpreted as monetary transfers to the consumer. This is in particular relevant for principal-agent models where such transfers are allowed. We extend the standard construction to such

¹See, for instance, Varian (1992, Chapter 7, page 96) or Mas-Colell, Whinston, and Green (1995, Chapter 3, page 42).

²To be fair, much of classical demand theory can be developed by relying on the weaker local-nonsatiation assumption, but monotonicity appears to be the rule in applications. The implications of nonmonotone utility functions for the existence and efficiency of competitive equilibria have been examined in Polemarchakis and Siconolfi (1993), among others. In their approach, however, they take consumers’ utilities as given and do not develop a framework for genericity analysis.

cases, giving rise to a complete space (technically, a G_δ set) \mathbf{P}_{sc} of smooth strictly convex preference relations, enabling genericity analyses for nonmonotone preferences. Thus, one may, for instance, check the robustness of results obtained in portfolio-choice theory under the CARA-normal specification (1), without assuming that investors' primitive preferences over state-contingent consumption have an expected-utility representation.³

2 The Space \mathbf{P}_{sc}

In this section, we formally construct the preference space \mathbf{P}_{sc} . There are $l + 1$ commodities, the last of which represents transfers to the consumer. We shall consider regular preference relations \succeq over an open subset V of \mathbb{R}^{l+1} . We require that V contain the no-trade point $(0, \dots, 0)$, that it be convex with a nonempty interior, and that it be comprehensive with respect to transfers, in the sense that, for all \mathbf{q} , t , and t' , $(\mathbf{q}, t') \in V$ if $(\mathbf{q}, t) \in V$ and $t' > t$. We first impose the following restrictions on \succeq :

- (i) \succeq is closed relative to $V \times V$.
- (ii) \succeq is strictly monotone in transfers: if $(\mathbf{q}, t) \in V$ and $t' > t$, then $(\mathbf{q}, t') \succ (\mathbf{q}, t)$.
- (iii) \succeq is convex: if $(\mathbf{q}, t) \succeq (\mathbf{q}', t')$ and $\lambda \in [0, 1]$, then $\lambda(\mathbf{q}, t) + (1 - \lambda)(\mathbf{q}', t') \succeq (\mathbf{q}', t')$.
- (iv) \succeq has closed upper contour sets relative to \mathbb{R}^{l+1} .
- (v) \succeq has a boundary in $V \times V$ that is a C^2 manifold.

Properties (i) and (iii) are standard. Property (ii) requires monotonicity of preferences in transfers, but not necessarily in the other commodities. Property (iv) describes the boundary behavior of preferences. Property (v) requires that preferences be sufficiently regular.

Our first task is to characterize the set \mathbf{P} of preferences \succeq that satisfy (i)–(v). The following notation will be useful. Let $U_{(\mathbf{q}, t)}$ and $L_{(\mathbf{q}, t)}$ be the upper and lower contour sets of (\mathbf{q}, t) for \succeq , and let $I_{(\mathbf{q}, t)} \equiv U_{(\mathbf{q}, t)} \cap L_{(\mathbf{q}, t)}$ be the indifference set of (\mathbf{q}, t) for \succeq . Observe by (ii) that $U_{(\mathbf{q}, t)}$ is comprehensive with respect to transfers, just as V . Also denote by cl and ∂ the closure and boundary operators relative to V or $V \times V$, depending on the context. We start with two technical lemmas.

Lemma 1 *If \succeq satisfies (i)–(ii), then, for each $(\mathbf{q}, t) \in V$, $U_{(\mathbf{q}, t)}$ has a nonempty interior relative to \mathbb{R}^{l+1} and $I_{(\mathbf{q}, t)} = \partial U_{(\mathbf{q}, t)}$.*

³Notice in this context that, when the state space is infinite, conducting a genericity analysis in the finite-dimensional space of portfolio choices is mathematically much simpler than doing so in the infinite-dimensional space of state-contingent consumption choices.

Proof. To prove the first claim, observe that $V \setminus L_{(\mathbf{q},t)}$ is open relative to V by (i), and thus relative to \mathbb{R}^{l+1} as V is an open subset of \mathbb{R}^{l+1} . Hence, because $V \setminus L_{(\mathbf{q},t)}$ is nonempty by (ii), $U_{(\mathbf{q},t)} \supset V \setminus L_{(\mathbf{q},t)}$ has a nonempty interior relative to \mathbb{R}^{l+1} . To prove the second claim, observe that, as \succeq is closed relative to $V \times V$ by (i), $U_{(\mathbf{q},t)}$ and $L_{(\mathbf{q},t)}$ are closed relative to V . Therefore, we have

$$\partial U_{(\mathbf{q},t)} \equiv \text{cl}(U_{(\mathbf{q},t)}) \cap \text{cl}(V \setminus U_{(\mathbf{q},t)}) = U_{(\mathbf{q},t)} \cap \text{cl}(V \setminus U_{(\mathbf{q},t)}) \subset U_{(\mathbf{q},t)} \cap L_{(\mathbf{q},t)} = I_{(\mathbf{q},t)}.$$

The reverse inclusion is satisfied if $I_{(\mathbf{q},t)} \subset \text{cl}(V \setminus U_{(\mathbf{q},t)})$, which is obviously true because, for each $(\mathbf{q}', t') \in I_{(\mathbf{q},t)}$, $(\mathbf{q}', t' - \varepsilon) \in V$ for any small enough $\varepsilon > 0$ by openness of V , and $(\mathbf{q}', t') \succ (\mathbf{q}', t' - \varepsilon)$ for any such ε by (ii). The result follows. ■

Lemma 2 *If \succeq satisfies (i)–(iv), then, for each $(\mathbf{q}, t) \in V$, $I_{(\mathbf{q},t)}$ is connected.*

Proof. By Lemma 1, $I_{(\mathbf{q},t)} = \partial U_{(\mathbf{q},t)}$, so that we can focus on the topological properties of $U_{(\mathbf{q},t)}$. Because $U_{(\mathbf{q},t)}$ is a closed convex subset of \mathbb{R}^{l+1} that has a nonempty interior by Lemma 1, two cases may arise (Klee (1953, III.1.6)).

Case 1 Either the asymptotic cone $\mathbf{A}U_{(\mathbf{q},t)}$ of $U_{(\mathbf{q},t)}$ is not a linear subspace. Then $U_{(\mathbf{q},t)}$ is homeomorphic with $\mathbb{R}^l \times [0, 1)$ and $\partial U_{(\mathbf{q},t)}$ with \mathbb{R}^l . In particular, $\partial U_{(\mathbf{q},t)}$ is connected.

Case 2 Or the asymptotic cone $\mathbf{A}U_{(\mathbf{q},t)}$ of $U_{(\mathbf{q},t)}$ is an $(l + 1 - k)$ -dimensional linear subspace, for some integer $k \leq l + 1$. Because $U_{(\mathbf{q},t)}$ is comprehensive with respect to transfers, we must have $k \leq l$ and $(\mathbf{0}, 1) \in \mathbf{A}U_{(\mathbf{q},t)}$. As $\mathbf{A}U_{(\mathbf{q},t)}$ is a linear subspace, it follows that $(\mathbf{0}, -1) \in \mathbf{A}U_{(\mathbf{q},t)}$. This implies that $(\mathbf{q}, t') \succeq (\mathbf{q}, t)$ for all $t' < t$, which is ruled out by (ii). This case is thus impossible. The result follows. ■

Let \mathbf{U} be the set of quasiconcave C^2 functions $u : V \rightarrow \mathbb{R}$ such that $\partial u / \partial t > 0$ and such that $u^{-1}([v, \infty))$ is closed in \mathbb{R}^{l+1} for all $v \in \mathbb{R}$. Lemmas 1–2 then imply the following representation result.

Lemma 3 *\succeq satisfies (i)–(v) if and only if it admits a utility function $u \in \mathbf{U}$.*

Proof. (Direct part.) Suppose that \succeq is representable by u . Then \succeq trivially satisfies (i)–(iii). Next, as $u^{-1}([v, \infty))$ is closed in \mathbb{R}^{l+1} for all $v \in \mathbb{R}$, \succeq satisfies (iv). Finally, because u clearly has no critical point, that is, $\partial u \neq 0$ over V , it follows as in Mas-Colell (1985, Proposition 2.3.5) that \succeq satisfies (v).

(Indirect part.) By (ii), \succeq is locally nonsatiated, and by (v), $\partial \succeq$ is a C^2 manifold in $V \times V$. Hence \succeq is of class C^2 (Mas-Colell (1985, Definition 2.3.4)). Moreover, by Lemmas

1–2, \succeq has connected indifference sets $I_{(\mathbf{q},t)}$. Hence it admits a C^2 utility function u over V with no critical point (Mas-Colell (1985, Proposition 2.3.9)). That u is quasiconcave follows from (iii). To show that $\partial u/\partial t > 0$, observe first from (ii) that $\partial u/\partial t \geq 0$. Now, suppose, by way of contradiction, that $(\partial u/\partial t)(\mathbf{q}, t) = 0$. Then $(\partial u/\partial \mathbf{q})(\mathbf{q}, t) \neq \mathbf{0}$ as u has no critical point. Thus the hyperplane through (\mathbf{q}, t) orthogonal to $\partial u(\mathbf{q}, t)$ that supports the convex set $U_{(\mathbf{q},t)}$ is vertical. It follows that the strict upper contour set of (\mathbf{q}, t) for \succeq , $U_{(\mathbf{q},t)} \setminus L_{(\mathbf{q},t)}$, strictly lies on one side or the other of this hyperplane, which is ruled out by (ii). Hence $\partial u/\partial t > 0$, as claimed. Finally, that $u^{-1}([v, \infty))$ is closed in \mathbb{R}^{l+1} for all $v \in \mathbb{R}$ is a direct consequence of (iv). The result follows. \blacksquare

We know from Lemma 3 that any preference relation in \mathbf{P} can be represented by some function in \mathbf{U} and, conversely, that any function in \mathbf{U} represents a preference relation in \mathbf{P} . For each $u \in \mathbf{U}$, let $P(u) \in \mathbf{U} \times \mathbf{U}$ be the preference relation represented by u . In line with Mas-Colell (1985, Chapter 2, Section 4), a topology on \mathbf{P} can be constructed as follows. Note that \mathbf{U} is a subspace of $C^2(V)$, the Polish space of real-valued C^2 functions over V endowed with the topology of uniform convergence over compact subsets of V of functions and of their derivatives up to the order 2 (Mas-Colell (1985, Chapter 1, K.1.2)). Then endow \mathbf{P} with the identification topology from P , that is, let O be open in \mathbf{P} if $P^{-1}(O)$ is open in \mathbf{U} . It will be convenient to work with a normalized space of utility functions, $\mathbf{U}_d \equiv \{u \in \mathbf{U} : u(\mathbf{0}, t) = t \text{ for all } t \text{ such that } (\mathbf{0}, t) \in V\}$. Notice that this normalization differs from the usual radial one (Wold and Juréen (1953), Kannai (1970)), reflecting that preferences are monotone in transfers, but not necessarily in the other commodities. We are now ready to complete the characterization of \mathbf{P} .

Proposition 1 \mathbf{U}_d is homeomorphic with \mathbf{P} under the natural map P .

Proof. We must prove that P restricted to \mathbf{U}_d is one-to-one, onto, continuous, and open.

(One-to-one.) Let u and u' in \mathbf{U}_d such that $P(u) = P(u')$. Then $u = \xi \circ u'$, where $\xi : u'(V) \rightarrow \mathbb{R}$ is C^2 , increasing, and regular (Mas-Colell (1985, Proposition 2.3.11)). But for each $v \in u'(V)$, $\xi(v) = \xi(u'(\mathbf{0}, v)) = u(\mathbf{0}, v) = v$, so that $u = u'$.

(Onto.) Let $\succeq \in \mathbf{P}$. From Lemma 3, there exists $u \in \mathbf{U}$ such that $\succeq = P(u)$. In addition, from (ii) one can renormalize u such that $\text{range}(u(\mathbf{0}, \cdot)) = \mathbb{R}$. Define $u' : V \rightarrow \mathbb{R}$ implicitly by $u(\mathbf{q}, t) = u(\mathbf{0}, u'(\mathbf{q}, t))$. Clearly $P(u') = \succeq$. We now check that $u' \in \mathbf{U}_d$. As $\partial u/\partial t > 0$, $u'(\mathbf{0}, t) = t$ for all t such that $(\mathbf{0}, t) \in V$. That u' is quasiconcave follows from the observation that $\{(\mathbf{q}, t) \in V : u'(\mathbf{q}, t) \geq v\} = \{(\mathbf{q}, t) \in V : u(\mathbf{q}, t) \geq u(\mathbf{0}, v)\}$ for all $v \in \mathbb{R}$; this also implies that $(u')^{-1}([v, \infty))$ is closed in \mathbb{R}^{l+1} for any such v . That u' is C^2 follows from the

implicit function theorem along with the fact that $\partial u/\partial t > 0$. That $\partial u'/\partial t > 0$ follows from $(\partial u/\partial t)(\mathbf{q}, t) = (\partial u/\partial t)(0, u'(\mathbf{q}, t))(\partial u'/\partial t)(\mathbf{q}, t)$, using again the fact that $\partial u/\partial t > 0$. Thus $u' \in \mathbf{U}_d$, as claimed.

(Continuous.) This follows from the definition of the topology of \mathbf{P} .

(Open.) Mimic the proof of Mas-Colell (1985, Proposition 2.4.2)). Hence the result. ■

Preferences in \mathbf{P} are not necessarily strictly convex. We add this as a further restriction:

- (vi) \succeq is strictly convex: if $(\mathbf{q}, t) \succeq (\mathbf{q}', t')$, $(\mathbf{q}, t) \neq (\mathbf{q}', t')$, and $\lambda \in (0, 1)$, then $\lambda(\mathbf{q}, t) + (1 - \lambda)(\mathbf{q}', t') \succ (\mathbf{q}', t')$.

Finally, to obtain a topologically complete space of preferences, we require preferences to be nonlinear, even in a local sense. To do so, observe that because a utility function $u \in \mathbf{U}_d$ representing a preference $\succeq \in \mathbf{P}$ has no critical point, the Gaussian curvature $\kappa(\mathbf{q}, t)$ of the indifference set $I_{(\mathbf{q}, t)}$ at (\mathbf{q}, t) is well defined and given by

$$\kappa(\mathbf{q}, t) \equiv \frac{1}{\|\partial u\|^3} \begin{vmatrix} -\partial^2 u & \partial u \\ -\partial u^\top & 0 \end{vmatrix} (\mathbf{q}, t),$$

see, for instance, Debreu (1972). The last restriction we impose on preferences is that this curvature nowhere vanishes.

- (vii) Any point of V is regular for \succeq , that is, $\kappa \neq 0$ over V .

Preferences that satisfy (vi)–(vii) are said to be *differentiably strictly convex* (Mas-Colell (1985, Definition 2.6.1)). We can now define our fundamental space of preferences as the space \mathbf{P}_{sc} of preferences over V that satisfy (i)–(vii). According to Lemma 1, \mathbf{P}_{sc} can be seen as a subset of \mathbf{U}_d and, hence, of $C^2(V)$. Our final result is that \mathbf{P}_{sc} is topologically complete, as desired, and that it is contractible.

Proposition 2 \mathbf{P}_{sc} is a contractible Polish space.

Proof. We first prove that \mathbf{P}_{sc} is a Polish space. Let $\{t_n\}$ be a sequence in \mathbb{R} decreasing to $\inf\{t \in \mathbb{R} : (\mathbf{0}, t) \in V\}$, and let $\{K_n\}$ be a countable collection of compact convex sets covering V . Then \mathbf{P}_{sc} is the intersection of the following countable families of open sets:

$$\left\{ u \in C^2(V) : \frac{\partial u}{\partial t}(\mathbf{q}, t) > 0 \text{ for all } (\mathbf{q}, t) \in K_n \right\},$$

$$\left\{ u \in C^2(V) : \text{there exists } \varepsilon > 0 \text{ such that } u(\mathbf{q}, t) < u(\mathbf{0}, t_n) \right.$$

$$\left. \text{if } (\mathbf{q}, t) \in K_n \text{ and } \inf\{\|(\mathbf{q}', t') - (\mathbf{q}, t)\| : (\mathbf{q}', t') \in \mathbb{R}^{l+1} \setminus V\} \leq \varepsilon \right\},$$

$$\left\{ \begin{array}{l} u \in C^2(V) : \max \{ |u(\mathbf{0}, t) - t| : t \in [-n, n] \text{ and } (\mathbf{0}, t) \in V \} < \frac{1}{n} \\ u \in C^2(V) : \text{there exists } \xi : u(V) \rightarrow \mathbb{R} \text{ such that } \partial\xi > 0 \text{ over } u(V) \\ \text{and } \partial^2(\xi \circ u) \text{ is negative definite over } K_n \end{array} \right\}.$$

The first family deals with the monotonicity in transfers (property (ii)), the second family with the boundary behavior (property (iv)), the third family with the normalization, and the fourth family with the differential strict convexity of preferences (properties (vi)–(vii)), bearing in mind that differentiably strictly convex preferences can be represented over any compact convex subset K of V by a C^2 utility function u with no critical point such that $\partial^2 u$ is negative definite over K (Mas-Colell (1985, Proposition 2.6.4)). Hence \mathbf{P}_{sc} is a G_δ in the Polish space $C^2(V)$ and thus, by Alexandrov's lemma (Mas-Colell (1985, Chapter 1, A.3.4)), a Polish space itself in the relative topology.

To prove that \mathbf{P}_{sc} is contractible, we exhibit a contraction $h : \mathbf{P}_{sc} \times [0, 1] \rightarrow \mathbf{P}_{sc}$, that is, we show that the identity function on \mathbf{P}_{sc} is homotopic to a constant function. The proof follows Mas-Colell (1985, Proposition 2.6.7), with some adjustments. Pick an arbitrary $\bar{\succ} \in \mathbf{P}_{sc}$ with corresponding utility function $\bar{u} \in \mathbf{U}_d$. To each $(u, \xi) \in \mathbf{U}_d \times [0, 1]$ we associate a utility function $u_\xi \in \mathbf{U}_d$ as follows. We first let $u_0 \equiv u$ and $u_1 \equiv \bar{u}$. For all $\xi \in (0, 1)$ and $(\mathbf{q}, t) \in V$, we then let $\mu_\xi(\mathbf{q}, t) \in (0, 1/\xi)$ be the unique solution to

$$\bar{u}(\mathbf{q}, \mu_\xi(\mathbf{q}, t)t) = u\left(\mathbf{q}, \left[\frac{1 - \xi\mu_\xi(\mathbf{q}, t)}{1 - \xi} \right] t\right)$$

and we let $u_\xi(\mathbf{q}, t) \equiv \bar{u}(\mathbf{q}, \mu_\xi(\mathbf{q}, t)t)$. What this transformation does is that, to each t' such that $(\mathbf{0}, t') \in V$, it assigns an indifference curve $u_\xi^{-1}(t')$ that is the vertical convex combination of $\bar{u}^{-1}(t')$ and $u^{-1}(t')$ with weights ξ and $1 - \xi$. (Bear in mind that, by normalization, $\bar{u}(\mathbf{0}, t') = u(\mathbf{0}, t') = t'$.) One can then verify that the mapping $h : (u, \xi) \mapsto u_\xi$ is the desired contraction. Hence the result. ■

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