UNIFORM LARGE DEVIATIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH MULTIPLICATIVE NOISE

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Abstract. Uniform large deviations for the laws of the paths of the solutions of the stochastic nonlinear Schrödinger equation when the noise converges to zero are presented. The noise is a real multiplicative Gaussian noise. It is white in time and colored in space. The path space considered allows blow-up and is endowed with a topology analogue to a projective limit topology. Thus a large variety of large deviation principle may be deduced by contraction. As a consequence, asymptotics of the tails of the law of the blow-up time when the noise converges to zero are obtained.

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1. Introduction

In the present article, the stochastic nonlinear Schrödinger (NLS) equation with a power law nonlinearity and a multiplicative noise is studied. The deterministic equation occurs as a generic model in many areas of physics: solid state physics, optics, hydrodynamics, plasma physics, molecular biology, field theory. It describes the propagation of slowly varying envelopes of a wave packet in media with both nonlinear and dispersive responses. In some physical applications, see [2, 3, 4, 23], spatial and temporal fluctuations of the parameters of the medium have to be considered and sometimes the only source of phase fluctuation that has significant effect on the dynamics enters linearly as a random potential. This interaction term may for example account for thermal fluctuations or inhomogeneities in the medium. The evolution equation may be written

$$i\frac{\partial u}{\partial t} - (\Delta u + \lambda |u|^{2\sigma} u) = u\xi, \quad \lambda = \pm 1, \tag{1.1}$$

where ξ is a real valued Gaussian noise, it is ideally a space-time white noise with correlation

$$\mathbb{E}\left[\xi(t_1,x_1)\xi(t_2,x_2)\right] = D\delta_{t_1-t_2} \otimes \delta_{x_1-x_2},$$

D is the noise intensity, δ denotes the Dirac mass, u is a complex-valued function of time and space and the space variables x is in $\mathbb{R}^d$. When $\lambda = 1$ the nonlinearity is called focusing or attractive, otherwise it is defocusing or repulsive.

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Since the unbounded operator $-i\Delta$ on $\mathrm{H}^1(\mathbb{R}^d)$ with domain $\mathrm{H}^3(\mathbb{R}^d)$ is skew-adjoint, it generates a unitary group $\left(U(t)=e^{-it\Delta}\right)_{t\in\mathbb{R}}$. Thus $(U(t))_{t\in\mathbb{R}}$ is an isometry on $\mathrm{H}^1(\mathbb{R}^d)$ and there is no smoothing effect in the Sobolev spaces. We are thus unable to treat the space-time white noise and will consider a real valued centered Gaussian noise, white in time and colored in space. Also when $d>1$, which is for example the case in [3], the nonlinearity is never Lipschitz on the bounded sets of $\mathrm{H}^1(\mathbb{R}^d)$. Thankfully, the Strichartz estimates, presented in the next section, show that some integrability property is obtained through "convolution" with the group. This allows us to treat the nonlinearity.

The well posedness of the Cauchy problem associated to the deterministic NLS equation depends on the size of σ. If $\sigma<\frac{2}{d}$, the nonlinearity is subcritical and the Cauchy problem is globally well posed in Ł2 or $\mathrm{H}^1(\mathbb{R}^d)$. If $\sigma=\frac{2}{d}$, critical nonlinearity, or $\frac{2}{d}<\sigma<\frac{2}{d-2}$ when $d\geq 3$ or simply $\sigma>\frac{2}{d}$ otherwise, supercritical nonlinearity, the Cauchy problem is locally well posed in $\mathrm{H}^1(\mathbb{R}^d)$, see [32]. In this latter case, if the nonlinearity is defocusing, the solution is global. In the focusing case some initial data yield global solutions while it is known that other initial data yield solutions which blow up in finite time, see [10, 40]. Two quantities are invariant on every time interval included in the existence time interval of the maximal solution. They are the momentum

$$M(u_d^{u_0}(t))=\|u_d^{u_0}(t)\|_{Ł2}$$

and the Hamiltonian

$$H(u_d^{u_0}(t))=\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u_d^{u_0}(t)|^2dx-\frac{\lambda}{2\sigma+2}\int_{\mathbb{R}^d}|u_d^{u_0}(t)|^{2\sigma+2}dx,$$

where we denote by $u_d^{u_0}$ the solution of the deterministic equation.

In this article, we adopt the formalism of stochastic evolution equations in M-type 2 Banach spaces as presented in [7, 8], see also [16] for the case of an Hilbert space. The real Gaussian noise that we consider hereafter is defined as the time derivative in the sense of distributions of a Wiener process $(W(t))_{t\in[0,+\infty)}$ defined on a real separable Banach space. It is correlated in the space variables and is such that the product make sense in the space considered for the fixed point argument. We write $W=\Phi W_c$, where W_c is a cylindrical Wiener process and Φ is a particular operator. With the Itô notations, the stochastic evolution equation is written

$$i\mathrm{d}u-(\Delta u+\lambda|u|^{2\sigma}u)\mathrm{d}t=u\circ\mathrm{d}W. \tag{1.2}$$

The symbol $\circ$ stands for the Stratanovich product. It follows that the momentum remains an invariant quantity of the stochastic functional flow. We will use the equivalent Itô form

$$i\mathrm{d}u-\left(\Delta u+\lambda|u|^{2\sigma}u-\frac{i}{2}uF_\Phi\right)\mathrm{d}t=u\mathrm{d}W, \tag{1.3}$$

where

$$F_\Phi(x)=\sum_{j\in\mathbb{N}}\left(\Phi e_j(x)\right)^2,\quad x\in\mathbb{R}^d.$$

The initial datum u_0 is a function in $\mathrm{H}^1(\mathbb{R}^d)$. We consider solutions of NLS that are weak solutions in the sense of the analysis of partial differential equations or equivalently mild solutions which satisfy

$$\begin{aligned} u^{u_0}(t) =& U(t)u_0 - \int_0^t U(t-s)\left(i\lambda|u^{u_0}(s)|^{2\sigma}u^{u_0}(s) + \frac{1}{2}u^{u_0}(s)F_\Phi\right)ds \\ & - i\int_0^t U(t-s)(u^{u_0}(s)dW(s)). \end{aligned}$$

In [17, 18], local existence and uniqueness for paths in $\mathrm{H}^1(\mathbb{R}^d)$, respectively in $\mathrm{L}^2(\mathbb{R}^d)$, is proved in the stochastic case when the noise is multiplicative. Global existence when the nonlinearity is defocusing and in the subcritical case when the nonlinearity is focusing are also obtained. Investigations on the blow-up in the stochastic case when the noise is multiplicative appeared in a series of numerical, see [6, 20], and theoretical papers, see [19]. The small noise asymptotics is studied in [30] in the case of an additive noise. There results are applied to the study of the blow-up and to the computation of error probabilities in soliton transmission in fibers. In the second case solitons are carriers for bits, $d = \sigma = \lambda = 1$ and the x and t variables stand for respectively time and space. To make a decision on the received pulse, the momentum is computed over a window. Noise may cause the loss of the signal by essentially shifting the arrival time (timing jitter) of the soliton and degrading its momentum or create from nothing a structure that is large enough and may be mistaken as a soliton. The sample path large deviation principle (LDP) allowed to obtain similar results on the tails of the momentum of the pulse as in [24, 25] where the heuristic method of collective coordinates and the instanton formalism have been used.

In this article, we are again interested in the tails of the laws of the paths of the solution of this random perturbation of a Hamiltonian system when the noise goes to zero. Note that in fiber-optical communications the multiplicative noise may account for Raman noise, it particularly affects the propagation of short pulses, see [23] for more details. However, contrary to the case of an additive noise, here the momentum of a pulse is conserved. Nonetheless we could consider the tails of the law of the timing jitter. We plan to study this application in future works. We are also able to deduce from the sample path large deviations results on the asymptotics of the blow-up time, results are stated in the last section of this article. There also remains many interesting problems which may be studied when a uniform sample path large deviation principle is available. Uniformity with respect to initial data in compact sets is needed when we study the most probable exit points from the boundary of a more geometric domain D than the preceding window. The domain may be a compact neighborhood D of an asymptotically stable equilibrium point. The corresponding asymptotics of the exit time from the domain may also be studied, see [22, 28] in the case of diffusions and [13, 27] in the case of particular SPDEs. Some results on asymptotic stability of solitary waves are available for the translation invariant NLS equation studied herein or with an additional potential accounting for a defect in the homogeneous background, see for example [5, 15, 29, 41]. The solution decomposes then into a solitary wave with temporal fluctuation of the parameters called modulations and a radiative decaying part. Contrary to [13, 27], compactness is a real issue in the NLS equation on an unbounded domain of $\mathbb{R}^d$ when considering random perturbations since the group does not have a smoothing effect. We plan to investigate these types of problems in future works. Uniform LDPs also yield LDPs for the family of invariant measures of Markov transition semi-groups defined by SPDEs, see [12, 39] for the case of some reaction-diffusion equations, when the noise goes to zero and when the measures weakly converge to a Dirac mass on 0, the only stationary solution of the deterministic equation. In the case of the NLS equation there are several invariant measures and little is known about invariant measures in the stochastic case without adding terms accounting for damping or viscosity and considering the space variables in a bounded domain of $\mathbb{R}^d$.

The author is aware of three different types of proof for a LDP when the noise is multiplicative. One uses an extension of Varadhan's contraction principle that may be found in [21] or in [22][Proposition 4.2.3], see also [26]. It requires a sequence of approximation of the measurable Itô map by continuous maps uniformly converging on the level sets of the initial good rate function and that the resulting sequence of family of laws are exponentially good approximations of the laws of the solutions. It is needed that the Itô map takes its values in a metric space and this is not the case when we consider the spaces of exploding paths, see [1, 30]. The second type of proof is strongly based on the estimate given in Proposition 4.1 below, it is the one we adopt in the following. We are indebted to [1] where the proof is written for diffusions that may blow up in finite time. Some aspects of the proof has been simplified in [37]. Also, some assumptions on the coefficient in front of the noise were relaxed. Nonetheless the locally Lipschitz conditions on the drift has been replaced by a globally Lipschitz assumption and boundedness of the drift and coefficient in front of the noise and the framework no longer allows SDEs that blow-up in finite time. The proofs of LDP for stochastic PDEs that we may found for example in [9, 11, 13, 35] follow this second type of proof. Note that in [9, 13] the approach to the SPDE is based on martingale measures and the Brownian sheet field whereas in [11, 35] the approach has an infinite dimensional flavor, like ours. A third type of proof has appeared in [33] for diffusions. It requires C^3 with linear growth diffusion coefficients and is based on the continuity theorem of T. Lyons for rough paths in the $p-$variation topology. Note that the continuity theorem also yields proofs of support theorems for diffusions. Reference [34] presents in this setting the existence of mild solutions of SPDEs when the PDE is linear and the semi-group is analytic, which covers the case of the linear Heat Equation, and when it is driven by a multiplicative γ-hölder time continuous path with value in a space of distributions. The author is not aware of a proof of a LDP in the SPDE setting via the Rough Paths theory. Uniform LDPs are for example proved in [1, 22, 28] for diffusions, and in [13, 38] for particular SPDEs in spaces of Hölder continuous functions in both time and space when the space variables are in a bounded domain.

The paper is organized as follows. Section 2 is devoted to notations and preliminaries and states the uniform LDP in a space of exploding paths where blow-up in finite time is allowed. Since the stronger the topology, the sharper are the estimates, we take advantage of the variety of spaces where the fixed point can be conducted, as it has been done in [30], due to the integrability property and endow the space with a topology analogous to a projective limit topology. The result can be transferred to weaker topologies or more generally by any family of equicontinuous mappings using [38][Proposition 5] which is an extension of Varadhan's contraction principle. The rate function is the infimum of the L^2-norm of the controls, of a deterministic control problem, producing the prescribed path. The mild solution of the control problem is called the skeleton. In section 3, the main tools among which the LDP for the Wiener process, the continuity of the skeleton with respect to the potential on the level sets of the rate function and exponential tail estimates are presented. Section 4 is devoted to the proof of the uniform LDP and Section 5 to an application to the study of asymptotics for the blow-up time.

2. Notations and preliminaries

2.1. **Notations.** Throughout the paper the following notations will be used.

The set of positive integers and positive real numbers are denoted respectively by $\mathbb{N}^*$ and $\mathbb{R}_+^*$, while the set of real numbers different from 0 is denoted by $\mathbb{R}^*$.

For $p \in \mathbb{N}^*$, $\mathrm{L}^p(\mathbb{R}^d)$ is the classical Lebesgue space of complex valued functions. For k in $\mathbb{N}^*$, $\mathrm{W}^{k,p}(\mathbb{R}^d)$ is the associated Sobolev space of $\mathrm{L}^p(\mathbb{R}^d)$ functions with partial derivatives up to level k, in the sense of distributions, in $\mathrm{L}^p(\mathbb{R}^d)$. When $p = 2$, $\mathrm{H}^s(\mathbb{R}^d)$ denotes the fractional Sobolev space of tempered distributions $v \in \mathcal{S}'(\mathbb{R}^d)$ such that the Fourier transform $\hat{v}$ satisfies $(1+|\xi|^2)^{s/2}\hat{v} \in$ Ł2. It is a Hilbert space. We denote by $\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})$ the space of real-valued functions in $\mathrm{H}^s(\mathbb{R}^d)$. The space Ł2 is endowed with the inner product defined by $(u,v)_{Ł2} = \Re \int_{\mathbb{R}^d} u(x)\overline{v}(x)dx$.

If I is an interval of $\mathbb{R}$, $(E, \|\cdot\|_E)$ a Banach space and r belongs to $[1,+\infty]$, then $\mathrm{L}^r(I;E)$ is the space of strongly Lebesgue measurable functions f from I into E such that $t \to \|f(t)\|_E$ is in $\mathrm{L}^r(I)$. The space $\mathrm{L}^r(\Omega;E)$, where $(\Omega,\mathcal{F},\mathbb{P})$ is the probability space, is defined similarly.

We recall that a pair $(r(p),p)$ of positive numbers is called an admissible pair if p satisfies $2 \le p < \frac{2d}{d-2}$ when $d > 2$ ($2 \le p < +\infty$ when $d = 2$ and $2 \le p \le +\infty$ when $d = 1$) and $r(p)$ is such that $\frac{2}{r(p)} = d\left(\frac{1}{2} - \frac{1}{p}\right)$. For example $(+\infty, 2)$ is an admissible pair.

Following the denomination of [7, 8], the linear continuous operator Φ is a $\gamma-$radonifying operator from a separable Hilbert space H into a Banach space B if for any complete orthonormal system $(e_j^H)_{j\in\mathbb{N}}$ of H and $(\gamma_j)_{j\in\mathbb{N}}$ a sequence of identically distributed and independent Gaussian random variables on a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, we have

$$\|\Phi\|_{R(H,B)}^2 = \tilde{\mathbb{E}}\left\|\sum_{j\in\mathbb{N}}\gamma_j\Phi e_j^H\right\|_B^2 < \infty.$$

The space of such operators endowed with the norm $\|\cdot\|_{R(H,B)}$ is a Banach space denoted by $R(H,B)$. When B is a Hilbert space $\tilde{H}$, it corresponds to the space of Hilbert-Schmidt operators from H into $\tilde{H}$. We denote by $\mathcal{L}_2(H,\tilde{H})$ the space of Hilbert-Schmidt operators from H into $\tilde{H}$ endowed with the norm

$$\|\Phi\|_{\mathcal{L}_2(H,\tilde{H})} = \mathrm{tr}\,(\Phi\Phi^*) = \sum_{j\in\mathbb{N}}\|\Phi e_j^H\|_{\tilde{H}}^2,$$

where Φ^* denotes the adjoint of Φ and tr the trace. We denote by $\mathcal{L}_2^{s,r}$ the corresponding space for $H = \mathrm{H}^s(\mathbb{R}^d,\mathbb{R})$ and $\tilde{H} = \mathrm{H}^r(\mathbb{R}^d,\mathbb{R})$. The space of linear continuous mappings from a Banach space B into a Banach space $\tilde{B}$ is denoted by $\mathcal{L}_c\left(B,\tilde{B}\right)$.

When A and B are two Banach spaces, $A\cap B$, where the norm of an element is defined as the maximum of the norm in A and in B, is a Banach space. Given an admissible pair $(r(p),p)$ and S and T such that $0\leq S<T$, the space

$$X^{(S,T,p)}=\mathrm{C}\left([S,T];\mathrm{H}^1(\mathbb{R}^d)\right)\cap\mathrm{L}^{r(p)}\left(S,T;\mathrm{W}^{1,p}(\mathbb{R}^d)\right),$$

denoted by $X^{(T,p)}$ when $S=0$ is of particular interest for the NLS equation.

We present hereafter the space of exploding paths, see [30] for the main properties. We add a point Δ to the space $\mathrm{H}^1(\mathbb{R}^d)$ and embed the space with the topology such that its open sets are the open sets of $\mathrm{H}^1(\mathbb{R}^d)$ and the complement in $\mathrm{H}^1(\mathbb{R}^d)\cup\{\Delta\}$ of the closed bounded sets of $\mathrm{H}^1(\mathbb{R}^d)$. This topology induces on $\mathrm{H}^1(\mathbb{R}^d)$ the topology of $\mathrm{H}^1(\mathbb{R}^d)$. The set $\mathrm{C}([0,+\infty);\mathrm{H}^1(\mathbb{R}^d)\cup\{\Delta\})$ is then well defined. If f belongs to $\mathrm{C}([0,+\infty);\mathrm{H}^1(\mathbb{R}^d)\cup\{\Delta\})$ we denote the blow-up time by

$$\mathcal{T}(f)=\inf\{t\in[0,+\infty):f(t)=\Delta\},$$

with the convention that $\inf\emptyset=+\infty$. We may now define the set

$$\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d))=\left\{f\in\mathrm{C}([0,+\infty);\mathrm{H}^1(\mathbb{R}^d)\cup\{\Delta\}):f(t_0)=\Delta\Rightarrow\forall t\geq t_0,\ f(t)=\Delta\right\},$$

it is endowed with the topology defined by the neighborhood basis

$$V_{T,\epsilon}(\varphi_1)=\left\{\varphi\in\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d)):\mathcal{T}(\varphi)>T,\ \|\varphi_1-\varphi\|_{\mathrm{C}([0,T];\mathrm{H}^1(\mathbb{R}^d))}\leq\epsilon\right\},$$

of φ_1 in $\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d))$ given $T<\mathcal{T}(\varphi_1)$ and $\epsilon>0$.

We define $\mathcal{A}(d)$ and $\tilde{\mathcal{A}}(d)$ as $[2,+\infty)$ when $d=1$ or $d=2$ and respectively as $\left[2,\frac{2(3d-1)}{3(d-1)}\right)$ and $\left[2,\frac{2d}{d-1}\right)$ when $d\geq 3$. The space $\mathcal{E}_\infty$ is now defined for any d in $\mathbb{N}^*$ by the set of functions f in $\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d))$ such that for all $p\in\mathcal{A}(d)$ and all $T\in[0,\mathcal{T}(f))$, f belongs to $\mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}(\mathbb{R}^d)\right)$. It is endowed with the topology defined for φ_1 in $\mathcal{E}_\infty$ by the neighborhood basis

$$W_{T,p,\epsilon}(\varphi_1)=\{\varphi\in\mathcal{E}_\infty:\mathcal{T}(\varphi)>T,\ \|\varphi_1-\varphi\|_{X^{(T,p)}}\leq\epsilon\}$$

where $T<\mathcal{T}(\varphi_1)$, $p\in\mathcal{A}(d)$ and $\epsilon>0$. We may show using Hölder's inequality that it is a well defined neighborhood basis. Also the space is a Hausdorff topological space and thus we may consider applying generalizations of Varadhan's contraction principle. If we denote again by $\mathcal{T}:\mathcal{E}_\infty\to[0,+\infty]$ the blow-up time, the mapping is measurable and lower semicontinuous.

We denote by $x\wedge y$ the minimum of the two real numbers x and y and $x\vee y$ their maximum. We finally recall that a rate function I is a lower semicontinuous function and that a good rate function I is a rate function such that for every $c>0$, $\{x:I(x)\leq c\}$ is a compact set.

2.2. **Properties of the group.** We recall in this section the main properties of the group that are used in the article.
The group satisfies the decay estimates: for every $p\geq 2$, $t\neq 0$ and u_0 in $\mathrm{L}^{p'}(\mathbb{R}^d)$, where p' is the conjugate exponent of p, i.e. $\frac{1}{p}+\frac{1}{p'}=1$,

$$\|U(t)u_0\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\leq(4\pi|t|)^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{\mathrm{W}^{1,p'}(\mathbb{R}^d)}$$

and the following Strichartz estimates, see [32, 40]
i/ For every u_0 in $\mathrm{H}^1(\mathbb{R}^d)$ and $(r(p),p)$ an admissible pair, the following linear mapping

$$\begin{aligned}\mathrm{H}^1(\mathbb{R}^d) &\to \mathrm{C}(\mathbb{R};\mathrm{H}^1(\mathbb{R}^d))\cap \mathrm{L}^{r(p)}(\mathbb{R};\mathrm{W}^{1,p}(\mathbb{R}^d))\\ u_0 &\mapsto (t\mapsto U(t)u_0).\end{aligned}$$

is continuous.
ii/ For every T positive and $(r(p),p)$ and $(r(q),q)$ two admissible pairs, if s and ρ are the conjugate exponents of $r(q)$ and q, the following linear mapping

$$\begin{aligned}\mathrm{L}^s(0,T;\mathrm{W}^{1,\rho}(\mathbb{R}^d)) &\to \mathrm{C}([0,T];\mathrm{H}^1(\mathbb{R}^d))\cap \mathrm{L}^{r(p)}(0,T;\mathrm{W}^{1,p}(\mathbb{R}^d))\\ f &\mapsto \Lambda f=\int_0^\cdot U(\cdot-s)f(s)ds\end{aligned}$$

is continuous with a norm that does not depend on T.

2.3. **Statistical properties of the noise.** We consider that W is originated from a cylindrical Wiener process on $\mathrm{L}^2(\mathbb{R}^d,\mathbb{R})$, i.e. $W=\Phi W_c$ where Φ is a $\gamma-$radonifying operator. It is known that for any orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of $\mathrm{L}^2(\mathbb{R}^d,\mathbb{R})$ there exists a sequence of real independent Brownian motions $(\beta_j)_{j\in\mathbb{N}}$ such that the cylindrical Wiener process may be written as the following expansion on a complete orthonormal system $(e_j)_{j\in\mathbb{N}}$ of $\mathrm{L}^2(\mathbb{R}^d,\mathbb{R})$

$$W_c(t,x,\omega)=\sum_{j\in\mathbb{N}}\beta_j(t,\omega)e_j(x).$$

Its direct image in every Banach space B such that the injection of $\mathrm{L}^2(\mathbb{R}^d,\mathbb{R})$ into B is a $\gamma-$radonifying injection is a *bona fide* Wiener process. It is also possible to define the Wiener process as originated from a cylindrical Wiener process on the reproducing kernel Hilbert space (RKHS) of the measure μ, the law of $W(1)$, in such a case the mapping Φ is an injection. The assumption that Φ is $\gamma-$radonifying is exactly the assumption needed to transfer by direct image a cylindrical centered Gaussian measure on a separable Hilbert space to a *bona fide* $\sigma-$additive measure μ on the image Banach space. Remark also that this assumption is too severe to allow the noise to be homogeneous in space. In [18], the assumption on the noise is that $\Phi\in R\left(\mathrm{L}^2(\mathbb{R}^d,\mathbb{R}),\mathrm{W}^{1,\alpha}(\mathbb{R}^d)\right)\cap\mathcal{L}_2^{0,1}$, where $\alpha>2d$. In particular the noise is a real noise. Stronger assumptions on the spacial correlations of the Wiener process than in the additive case, see [18, 30], are imposed in order that the stochastic convolution of the product make sense in the space considered for the fixed point.

In this article we impose the extra assumption (A)

$$\text{for some } s>\tfrac{d}{4}+1,\ \Phi\in\mathcal{L}_2^{0,s}.$$

It is used twice in the proof of the LDP. It is used first to prove the continuity with respect to the potential on the level sets of the rate function of the sample path LDP for the Wiener process. It is needed to produce a process in a real separable Banach space embedded in $\mathrm{W}^{1,\infty}(\mathbb{R}^d)$. Note that in that respect it would be enough to impose that $\Phi\in R(\mathrm{L}2,\mathrm{H}^1(\mathbb{R}^d,\mathbb{R})\cap\mathrm{W}^{1+s,\beta}(\mathbb{R}^d))$ for any $s\beta>\frac{d}{2}$. It is used a second time in the proof of the exponential tail estimates. In that case we also need that $\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})$ is a Hilbert space.

Since the operator Φ belongs to $\mathcal{L}_2^{0,s}$ and thus to $\mathcal{L}_2^{0,0}$ it may be defined through a kernel. This means that for any square integrable function u,

$$\Phi u(x) = \int_{\mathbb{R}^d} \mathcal{K}(x,y)u(y)dy.$$

Now for $(t,s) \in \mathbb{R}^+$, $(x,z) \in (\mathbb{R}^d)^2$, the correlation of our real Gaussian noise $\frac{\partial}{\partial t}W$ is formally given by

$$\mathbb{E}\left[\frac{\partial W}{\partial t}(t+s,x+z)\frac{\partial W}{\partial t}(t,x)\right] = \delta_0(s) \otimes c(x,z),$$

where the distribution $c(x,z)$ is indeed the Ł2 function

$$c(x,z) = \int_{\mathbb{R}^d} \mathcal{K}(x+z,u)\mathcal{K}(x,u)du.$$

In the following we assume that the probability space is endowed with the filtration $\mathcal{F}_t = \mathcal{N} \cup \sigma\{W_s, 0 \le s \le t\}$ where $\mathcal{N}$ denotes the $\mathbb{P}-$null sets.

2.4. **The law of the solution in the space of exploding paths.** In the following we restrict ourselves to the case where

$$\begin{cases} \frac{1}{2} \le \sigma & \text{if } d = 1,2, \\ \frac{1}{2} \le \sigma < \frac{2}{d-2} & \text{if } d \ge 3. \end{cases}$$

Proposition 2.1. *The solution u^{u_0} defines a random variable with values in $\mathcal{E}_\infty$.*

Proof. The proof of existence and uniqueness of a maximal solution in [18], in the case where $\sigma \ge \frac{1}{2}$, is obtained as follows. Take $p \in \mathcal{A}(d)$ and $T > 0$. The mapping

$$\begin{aligned}\mathcal{F}_{R,u_0}(u)(t) =& U(t)u_0 - i\lambda \int_0^t U(t-s)\left(\theta\left(\frac{\|u\|_{X^{(s,p)}}}{R}\right)|u(s)|^{2\sigma}u(s)\right)ds \\ &- i\int_0^t U(t-s)\left(u(s)dW(s)\right) - \frac{1}{2}\int_0^t U(t-s)\left(u(s)F_\Phi\right)ds\end{aligned}$$

is a contraction in $\mathrm{L}^{r(p)}\left(\Omega, X^{(T^*,p)}\right)$ provided T^*, depending on p, $\|u_0\|_{\mathrm{H}^1(\mathbb{R}^d)}$ and R, is small enough. The fixed point is denoted by $u_R^{u_0}$. The solution can be extended to the whole interval $[0,T]$ and is a measurable mapping from $(\Omega, \mathcal{F})$ to $X^{(T,p)}$ with its Borel $\sigma-$field. The blow-up time is defined as the increasing limit of the approximate blow-up time, see also [30],

$$\tau_R = \inf\{t \in \mathbb{R}^+ : \|u_R^{u_0}\|_{X^{(t,p)}} \ge R\}.$$

The solution u^{u_0} is then such that $u^{u_0} = u_R^{u_0}$ on $[0,\tau_R]$. Finally it is proved that the limit of the approximate blow-up time corresponds indeed to the blow-up of the $\mathrm{H}^1(\mathbb{R}^d)$ norm. Consequently, the solution is an element of $\mathcal{E}_\infty$. The topology of $\mathcal{E}_\infty$ is defined by a countable basis of neighborhoods, see [30], thus it suffices to show that for $\varphi_1 \in \mathcal{E}_\infty$, $T < \mathcal{T}(\varphi_1)$, $p \in \mathcal{A}(d)$ and $\epsilon > 0$, $\{u^{u_0} \in W_{T,p,\epsilon}(\varphi_1)\}$ is an element of $\mathcal{F}$. Since $T < \mathcal{T}(\varphi_1)$ there exists $R > 0$ such that $\|\varphi_1\|_{X^{(T,p)}} \le R$ and every φ in $W_{T,p,\epsilon}(\varphi_1)$ satisfies $\|\varphi\|_{X^{(T,p)}} \le R + \epsilon$. The conclusion follows from the fact that

$$\{u^{u_0} \in W_{T,p,\epsilon}(\varphi_1)\} = \left\{u_{R+\epsilon}^{u_0} \in W_{T,p,\epsilon}(\varphi_1)\right\} = \left\{\|\varphi_1 - u_{R+\epsilon}^{u_0}\|_{X^{(T,p)}} \le \epsilon\right\}.$$

□

We denote by $\mu^{u^{u_0}}$ its law and, for $\epsilon>0$, by $\mu^{u^{\epsilon,u_0}}$ the law in $\mathcal{E}_\infty$ of the mild solution of

$$\left\{\begin{array}{l} i\mathrm{d}u^{\epsilon,u_0}=\left(\Delta u^{\epsilon,u_0}+\lambda|u^{\epsilon,u_0}|^{2\sigma}u^{\epsilon,u_0}-\frac{i\epsilon}{2}F_\Phi u^{\epsilon,u_0}\right)\mathrm{d}t+\sqrt{\epsilon}u^{\epsilon,u_0}\mathrm{d}W.\\ u^{\epsilon,u_0}(0)=u_0\end{array}\right.$$

2.5. **The uniform large deviation principle.** The rate function of the LDP involves the deterministic control problem

$$\left\{\begin{array}{l} i\frac{\mathrm{d}}{\mathrm{d}t}u=\Delta u+\lambda|u|^{2\sigma}u+u\Phi h,\\ u(0)=u_0\ \in \mathrm{H}^1(\mathbb{R}^d),\\ h\in \mathrm{L}^2\left(0,+\infty;\mathrm{L}^2(\mathbb{R}^d)\right).\end{array}\right.$$

We may write the mild solution, also called skeleton, as

$$S^c(u_0,h)=U(t)u_0-i\int_0^t U(t-s)\left[S^c(u_0,h)(s)\left(\lambda|S^c(u_0,h)(s)|^{2\sigma}+\Phi h(s)\right)\right]ds.$$

If we replace Φh by $\frac{\partial f}{\partial t}$ where f belongs to $\mathrm{H}^1_0\left([0,+\infty);\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})\right)$ which is the subspace of $\mathrm{C}\left([0,+\infty);\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})\right)$ of functions null at time 0, square integrable in time and with square integrable in time time derivative, we write

$$S(u_0,f)=U(t)u_0-i\int_0^t U(t-s)\left[S(u_0,f)(s)\left(\lambda|S(u_0,f)(s)|^{2\sigma}+\frac{\partial f}{\partial s}\right)\right]ds.$$

In the following we denote by $K\subset\subset \mathrm{H}^1(\mathbb{R}^d)$ the fact that K is a compact set of $\mathrm{H}^1(\mathbb{R}^d)$ and by $Int(A)$ the interior of A.

Theorem 2.2. *The family of measures $\left(\mu^{u^{\epsilon,u_0}}\right)_{\epsilon>0}$ satisfies a uniform LDP on $\mathcal{E}_\infty$ of speed ϵ and good rate function*

$$\begin{aligned} I^{u_0}(w)&=\inf_{f\in\mathrm{C}([0,+\infty);\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})):w=S(u_0,f)} I^W(f)\\ &=\frac{1}{2}\inf_{h\in\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d)):w=S^c(u_0,h)}\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))},\end{aligned}$$

where I^W is the rate function of the sample path LDP for the Wiener process, i.e. $\forall K\subset\subset \mathrm{H}^1(\mathbb{R}^d),\ \forall A\in\mathcal{B}\left(\mathcal{E}_\infty\right),$

$$\begin{aligned} &-\sup_{u_0\in K}\inf_{w\in Int(A)} I^{u_0}(w)\le \underline{\lim}_{\epsilon\to 0}\epsilon\log\inf_{u_0\in K}\mathbb{P}\left(u^{\epsilon,u_0}\in A\right)\\ &\le \overline{\lim}_{\epsilon\to 0}\epsilon\log\sup_{u_0\in K}\mathbb{P}\left(u^{\epsilon,u_0}\in A\right)\le -\inf_{w\in\overline{A},u_0\in K} I^{u_0}(w).\end{aligned}$$

Remark 2.3. Writing $\frac{\partial h}{\partial t}$ instead of h in the optimal control problem leads to a rate function consisting in the minimisation of $\frac{1}{2}\|h\|^2_{H^1_0(0,+\infty;\mathrm{L}2)}$. Specifying only the law μ of $W(1)$ and dropping Φ in the control problem would lead to a rate function consisting in the minimisation of $\frac{1}{2}\|h\|^2_{H^1_0(0,+\infty;H_\mu)}$, where H_μ stands for the RKHS of μ.

3. The main tools

In the two next sections C is a constant which may have a different value from line to line and also within the same line.

3.1. **Large deviations for the Wiener process.**

Proposition 3.1. *The family of laws of* $(\sqrt{\epsilon}W)_{\epsilon>0}$ *satisfies in* $\mathrm{C}([0,+\infty);\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))$ *a LDP of speed* ϵ *and good rate function*

$$I^W(f)=\frac{1}{2}\inf_{h\in\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d)):f=\int_0^\cdot\Phi h(s)ds}\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))}.$$

Proof. The result follows from the general LDP for Gaussian measures on a real separable Banach space, see [21], the fact that the laws of the restrictions of the paths $\sqrt{\epsilon}W$ on $\mathrm{C}\left([0,T];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})\right)$ and $\mathrm{L}^2\left(0,T;\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})\right)$ have same RKHS and Dawson-Gärtner's theorem which allows to deduce the LDP on $\mathrm{C}\left([0,+\infty);\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})\right)$ with the topology of the uniform convergence on the compact sets of $[0,+\infty)$. □

In the following we denote by C_a the set

$$\begin{aligned} C_a &= \left\{f\in\mathrm{C}([0,+\infty);\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})):I^W(f)\le a\right\},\\ &=\Bigg\{f\in\mathrm{C}([0,+\infty);\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})):f(0)=0,\ \tfrac{\partial f}{\partial t}\in\mathrm{im}\Phi\\ &\quad\text{and }\left\|\Phi^{-1}_{|(\ker\Phi)^\perp}\tfrac{\partial f}{\partial t}\right\|_{\mathrm{L}^2(0,+\infty;\mathrm{L}2)}\le\sqrt{2a}\Bigg\}.\end{aligned}$$

3.2. **Continuity of the skeleton with respect to the initial data and control on the level sets of the rate function of the Wiener process.** The continuity with respect to the control on the level sets of the rate function of the Wiener process is used herein to prove that the rate function is a good rate function and to prove the lower bound of the LDP, see 4.1. In that respect our proof is closer to the proof of [9, 13]. The authors of [11, 35] use some slightly different arguments and do not prove this continuity but prove separately the compactness of the level sets of the rate function, the lower and upper bounds of the LDP using the usual characterization in metric spaces. Remark also that in the applications of the LDP, see for example Section 5, we need to compute infima of the rate function, or of the rate function of a LDP deduced by contraction, on particular sets and the continuity proves to be very useful. We prove the stronger continuity with respect to the control and initial datum as suggested in [13] but we will not need the continuity with respect to the initial datum in the proof of the uniform LDP.

Proposition 3.2. *For every* $u_0\in\mathrm{H}^1(\mathbb{R}^d)$*,* $a>0$ *and* $f\in C_a$*,* $S(u_0,f)$ *exists and is uniquely defined. Moreover, it is a continuous mapping from* $\mathrm{H}^1(\mathbb{R}^d)\times C_a$ *into* $\mathcal{E}_\infty$*, where* C_a *has the topology induced by that of* $\mathrm{C}([0,+\infty);\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))$.

Proof. Let $\mathfrak{F}$ denote the mapping such that

$$\mathfrak{F}(u,u_0,f)=U(t)u_0-i\int_0^t U(t-s)\left[u(s)\left(\lambda|u(s)|^{2\sigma}+\frac{\partial f}{\partial s}\right)\right]ds.$$

Let a and r be positive, f be in C_a and u_0 be such that $\|u_0\|_{\mathrm{H}^1(\mathbb{R}^d)}\le r$, set $R=2cr$ where c is the norm of the linear continuous mapping of the $i/$ of the Strichartz estimates. From $i/$ and $ii/$ of the Strichartz estimates along with Hölder's inequality, the Sobolev injections and the continuity of Φ, for any T positive, p in $\mathcal{A}(d)$ and u and v in $X^{(T,p)}$ and any ν in $\left(0,1-\frac{\sigma(d-2)}{2}\right)$, which is a well defined interval since $\sigma<\frac{2}{d-2}$, there exists C positive such that

$$\begin{aligned}
&\|\mathfrak{F}(u,u_0,f)\|_{X^{(T,p)}}\\
&\leq c\|u_0\|_{\mathrm{H}^1(\mathbb{R}^d)} + CT^{\nu}\|u\|^{2\sigma+1}_{X^{(T,p)}} + CT^{\frac{1}{2}-\frac{1}{r(p)}}\|u\|_{\mathrm{C}([0,T];\mathrm{H}^1(\mathbb{R}^d))}\left\|\frac{\partial f}{\partial s}\right\|_{\mathrm{L}^2\left(0,T;\mathrm{W}^{1,\frac{r(p)d}{2}}(\mathbb{R}^d)\right)}\\
&\leq c\|u_0\|_{\mathrm{H}^1(\mathbb{R}^d)} + CT^{\nu}\|u\|^{2\sigma+1}_{X^{(T,p)}} + C\sqrt{a}T^{\frac{1}{2}-\frac{1}{r(p)}}\|u\|_{X^{(T,p)}}.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
&\|\mathfrak{F}(u,u_0,f)-\mathfrak{F}(v,u_0,f)\|_{X^{(T,p)}}\\
&\leq C\left[T^{\nu}\left(\|u\|^{2\sigma}_{X^{(T,p)}}+\|v\|^{2\sigma}_{X^{(T,p)}}\right)+T^{\frac{1}{2}-\frac{1}{r(p)}}\sqrt{a}\right]\|u-v\|_{X^{(T,p)}}.
\end{aligned}$$

Thus, for $T=T^*_{r,a,p}$ small enough depending on r, a and p, the ball centered at 0 of radius R is invariant and the mapping $\mathfrak{F}(\cdot,u_0,f)$ is a $\frac{3}{4}-$contraction. We denote by $S^0(u_0,f)$ the unique fixed point of $\mathfrak{F}(\cdot,u_0,f)$ in $X^{(T^*_{r,a,p},p)}$.

Also when T is positive, we can solve the fixed point problem on any interval $\left[kT^*_{r,a,p},(k+1)T^*_{r,a,p}\right]$ with $1\leq k\leq\left\lfloor\frac{T}{T^*_{r,a,p}}\right\rfloor$. The fixed point is denoted by $S^k(u_k,f)$ where $u_k=S^{k-1}(u_{k-1},f)(kT^*_{r,a,p})$, as long as $\|u_k\|_{\mathrm{H}^1(\mathbb{R}^d)}\leq r$. Existence and uniqueness of a maximal solution $S(u_0,f)$ follows. It coincides with $S^k(u_k,f)$ on the above intervals when it is defined. We may also show that the blow-up time corresponds to the blow-up of the $\mathrm{H}^1(\mathbb{R}^d)-$norm, thus $S(u_0,f)$ is an element of $\mathcal{E}_\infty$.

We shall now prove the continuity. Take u_0 in $\mathrm{H}^1(\mathbb{R}^d)$, a positive and f in C_a. From the definition of the neighborhood basis of the topology of $\mathcal{E}_\infty$, it is enough to see that for ϵ positive, $T<\mathcal{T}\left(S(u_0,f)\right)$, and $p\in\mathcal{A}(d)$, there exists η positive such that for every $\tilde{u}_0$ in $\mathrm{H}^1(\mathbb{R}^d)$ and g in C_a satisfying $\|u_0-\tilde{u}_0\|_{\mathrm{H}^1(\mathbb{R}^d)}+\|f-g\|_{\mathrm{C}([0,T];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))}\leq\eta$ then $\|S(u_0,f)-S(\tilde{u}_0,g)\|_{X^{(T,p)}}\leq\epsilon$. We set

$$r=\|S(u_0,f)\|_{X^{(T,p)}}+1,\ N=\left\lfloor\frac{T}{T^*_{r,a,p}}\right\rfloor,\ \delta_{N+1}=\frac{\epsilon}{N+1}\wedge 1,$$

and define for $k\in\{0,...,N\}$, δ_k and η_k such that $0<\delta_k<\delta_{k+1}$, $0<\eta_k<\eta_{k+1}<1$ and

$$\left\|S^{k+1}(u_k,f)-S^{k+1}(\tilde{u}_k,g)\right\|_{X^{(kT^*_{r,a,p},(k+1)T^*_{r,a,p},p)}}\leq\delta_{k+1},$$

if

$$\|u_k-\tilde{u}_k\|_{\mathrm{H}^1(\mathbb{R}^d)}+\|f-g\|_{\mathrm{C}([0,\infty);\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))}\leq\eta_{k+1}.$$

It is possible to choose $(\delta_k)_{k=0,\dots,N}$ as long as we prove the continuity for $k=0$. The maximal solution $S(\tilde{u}_0,g)$ then necessarily satisfies $\mathcal{T}\left(S(\tilde{u}_0,g)\right)>T$. We conclude setting $\eta=\eta_1$ and using the triangle inequality.

We now prove the continuity for $k=0$. First note that $\|u_0\|_{\mathrm{H}^1(\mathbb{R}^d)}\leq r$ and $\|\tilde{u}_0\|_{\mathrm{H}^1(\mathbb{R}^d)}\leq r$ since $\eta<1$, as a consequence if Υ denotes $\left\|S^0(u_0,f)-S^0(\tilde{u}_0,g)\right\|_{X^{\left(T^*_{r,a,p},p\right)}}$ we obtain that

$$\begin{aligned}
\Upsilon&=\left\|\mathfrak{F}(S^0(u_0,f),u_0,f)-\mathfrak{F}(S^0(\tilde{u}_0,g),\tilde{u}_0,g)\right\|_{X^{\left(T^*_{r,a,p},p\right)}}\\
&\leq\left\|\mathfrak{F}(S^0(u_0,f),u_0,f)-\mathfrak{F}(S^0(u_0,f),\tilde{u}_0,g)\right\|_{X^{\left(T^*_{r,a,p},p\right)}}\\
&\quad+\left\|\mathfrak{F}(S^0(u_0,f),\tilde{u}_0,g)-\mathfrak{F}(S^0(\tilde{u}_0,g),\tilde{u}_0,g)\right\|_{X^{\left(T^*_{r,a,p},p\right)}},
\end{aligned}$$

thus since $\mathfrak{F}(\cdot,\tilde{u}_0,g)$ is a $\frac{3}{4}-$contraction,

$$\begin{aligned}\Upsilon &\leq 4\left\|\mathfrak{F}(S^0(u_0,f),u_0,f)-\mathfrak{F}(S^0(u_0,f),\tilde{u}_0,g)\right\|_{X\left(T^*_{r,a,p},p\right)}\\ &\leq 4\left(c\|u_0-\tilde{u}_0\|_{\mathrm{H}^1(\mathbb{R}^d)}+\left\|\int_0^{\cdot}U(\cdot-s)S^0(u_0,f)\frac{\partial(f-g)}{\partial s}(s)ds\right\|_{X\left(T^*_{r,a,p},p\right)}\right).\end{aligned}$$

Using Hölder's inequality and taking $p<\tilde{p}$ we can bound the second term, denoted by T_2, by

$$\begin{aligned}T_2\leq&\left(\left\|\int_0^{\cdot}U(\cdot-s)S^0(u_0,f)\frac{\partial(f-g)}{\partial s}(s)ds\right\|^{\theta}_{\mathrm{C}\left(\left[0,T^*_{r,a,p}\right];\mathrm{H}^1(\mathbb{R}^d)\right)}\right.\\ &\left.\times\left\|\int_0^{\cdot}U(\cdot-s)S^0(u_0,f)\frac{\partial(f-g)}{\partial s}(s)ds\right\|^{1-\theta}_{\mathrm{L}^{r(\tilde{p})}\left(0,T^*_{r,a,p};\mathrm{W}^{1,\tilde{p}}(\mathbb{R}^d)\right)}\right)\\ &\vee\left\|\int_0^{\cdot}U(\cdot-s)S^0(u_0,f)\frac{\partial(f-g)}{\partial s}(s)ds\right\|_{\mathrm{C}\left(\left[0,T^*_{r,a,p}\right];\mathrm{H}^1(\mathbb{R}^d)\right)},\end{aligned}$$

where $\theta=\frac{\tilde{p}-p}{\tilde{p}-2}$. From the $ii/$ of the Strichartz estimates we can bound the second term of the product by $C\left(\sqrt{a}\left(T^*_{r,a,p}\right)^{\frac{1}{2}-\frac{1}{r(\tilde{p})}}R\right)^{1-\theta}$. It is now enough to show that when g is close enough to f, the first term in the above can be made arbitrarily small.

Take $n\in\mathbb{N}$ and set for $i\in\{0,...,n\}$, $t_i=\frac{iT^*_{r,a,p}}{n}$ and $S^{0,n}(u_0,f)=U(t-t_i)\left(S(u_0,f)(t_i)\right)$ when $t_i\leq t<t_{i+1}$. As it has been done previously we can compute the following bound,

$$\begin{aligned}&\left\|\int_0^{\cdot}U(\cdot-s)\left(S^0(u_0,f)-S^{0,n}(u_0,f)\right)\tfrac{\partial(f-g)}{\partial s}(s)ds\right\|_{\mathrm{C}\left(\left[0,T^*_{r,a,p}\right];\mathrm{H}^1(\mathbb{R}^d)\right)}\\ &\leq C\sqrt{a}\left(T^*_{r,a,p}\right)^{\frac{1}{2}-\frac{1}{r(p)}}\left\|S^0(u_0,f)-S^{0,n}(u_0,f)\right\|_{\mathrm{C}\left(\left[0,T^*_{r,a,p}\right];\mathrm{H}^1(\mathbb{R}^d)\right)}\\ &\leq C\sqrt{a}\left(T^*_{r,a,p}\right)^{\frac{1}{2}-\frac{1}{r(p)}}\\ &\sup_{i\in\{0,\dots,n-1\}}\left\|\int_{t_i}^{\cdot}U(\cdot-s)\left[S^0(u_0,f)\left(\lambda\left|S^0(u_0,f)\right|^{2\sigma}+\tfrac{\partial f}{\partial s}\right)\right](s)ds\right\|_{\mathrm{C}([t_i,t_{i+1}];\mathrm{H}^1(\mathbb{R}^d))}\\ &\leq C\sqrt{a}\left(T^*_{r,a,p}\right)^{\frac{1}{2}-\frac{1}{r(p)}}\left[R^{2\sigma+1}\left(\tfrac{T^*_{r,a,p}}{n}\right)^{\nu}+\left(\tfrac{T^*_{r,a,p}}{n}\right)^{\frac{1}{2}-\frac{1}{r(p)}}R\sqrt{a}\right],\end{aligned}$$

which can be made arbitrarily small for large n. Finally it remains to bound the $\mathrm{C}\left(\left[0,T^*_{r,a,p}\right];\mathrm{H}^1(\mathbb{R}^d)\right)-$norm of

$$\begin{aligned}&\int_0^t U(t-s)S^{0,n}(u_0,f)\tfrac{\partial(f-g)}{\partial s}(s)ds\\ &=\sum_{i=0}^{n-1}\int_{t_i\wedge t}^{t_{i+1}\wedge t}U(t-t_i\wedge t)\left(S^0(u_0,f)(t_i\wedge t)\tfrac{\partial(f-g)}{\partial s}(s)\right)ds\\ &=\sum_{i=0}^{n-1}U(t-t_i\wedge t)\left[S^0(u_0,f)(t_i\wedge t)\left((f-g)(t_{i+1}\wedge t)-(f-g)(t_i\wedge t)\right)\right]\end{aligned}$$

since, from the Sobolev injections and the fact that $U(t-t_i\wedge t)$ is a group on $\mathrm{H}^1(\mathbb{R}^d)$, for any $i\in\{0,...,n-1\}$ and v in $\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})$,

$$\left\|U(t-t_i\wedge t)\left(S^0(u_0,f)(t_i\wedge t)v\right)\right\|_{\mathrm{H}^1(\mathbb{R}^d)}\leq C\left\|S^0(u_0,f)(t_i\wedge t)\right\|_{\mathrm{H}^1(\mathbb{R}^d)}\|v\|_{\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})},$$

and from the fact that $\frac{\partial(f-g)}{\partial s}$ is Bochner integrable. We finally obtain that an upper bound can be written as $rCn\|f-g\|_{\mathrm{C}([0,T];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))}$, which for fixed n can be made arbitrarily small taking g sufficiently close to f. □

3.3. **Exponential tail estimates.** In the following we denote by $c\left(\frac{r(p)d}{2}\right)$ and $c(\infty)$ the norms of the linear continuous embeddings $\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})\subset \mathrm{W}^{1,\frac{r(p)d}{2}}(\mathbb{R}^d,\mathbb{R})$ and $\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})\subset \mathrm{W}^{1,\infty}(\mathbb{R}^d,\mathbb{R})$. Exponential tail estimates for the $\mathrm{L}^\infty-$norm of stochastic integrals in $\mathrm{L}^p(0,1)$ is proved in [9] but here we need the following

Lemma 3.3. *Assume that ξ is a point-predictable process, that p is such that $p\in\tilde{\mathcal{A}}(d)$, that $T>0$ and that there exists $\eta>0$ such that $\|\xi\|^2_{\mathrm{C}([0,T];\mathrm{H}^1(\mathbb{R}^d))}\le\eta$ a.s., then for every $t\in[0,T]$, and $\delta>0$,*

$$\mathbb{P}\left(\sup_{t_0\in[0,T]}\left\|\int_0^{t_0}U(t-s)\xi(s)dW(s)\right\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\ge\delta\right)\le\exp\left(1-\frac{\delta^2}{\kappa(\eta)}\right),$$

where

$$\kappa(\eta)=\frac{4c\left(\frac{r(p)d}{2}\right)^2T^{1-\frac{4}{r(p)}}(d+1)(d+p)\|\Phi\|^2_{\mathcal{L}_2^{0,s}}}{1-\frac{4}{r(p)}}\eta.$$

Proof. Let us denote by $g_a(f)=\left(1+a\|f\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{\frac{1}{p}}$ the real-valued function parametrized by $a>0$ and by M the martingale defined by $M(t_0)=\int_0^{t_0}U(t-s)\xi(s)dW(s)$. The function g_a is twice Fréchet differentiable, the first and second derivatives at point $M(t)$ in the direction h are denoted by $Dg_a(M(t)).h$ and $D^2g_a(M(t),M(t)).(h,h)$, they are continuous. Also, the second derivative is uniformly continuous on the bounded sets. By the Itô formula the following decomposition holds

$$g_a(M(t_0))=1+E_a(t_0)+R_a(t_0),$$

where $E_a(t_0)$ is equal to

$$\int_0^{t_0}Dg_a(M(s)).U(t-s)\xi(s)dW(s)-\frac{1}{2}\int_0^t\|Dg_a(M(s)).U(t-s)\xi(s)\|^2_{\mathcal{L}_2(\text{Ł}2,\mathbb{R})}ds$$

and $R_a(t_0)$ to

$$\frac{1}{2}\left(\int_0^{t_0}\|Dg_a(M(s)).U(t-s)\xi(s)\|^2_{\mathcal{L}_2(\text{Ł}2,\mathbb{R})}ds\right.$$
$$\left.+\int_0^{t_0}\sum_{j\in\mathbb{N}}D^2g_a(M(s),M(s)).(U(t-s)\xi(s)\Phi e_j,U(t-s)\xi(s)\Phi e_j)ds\right),$$

where $(e_j)_{j\in\mathbb{N}}$ is a complete orthonormal system of Ł2. We denote by

$$q(u,h)=\Re\left[\int_{\mathbb{R}^d}\overline{u}|u|^{p-2}hdx+\sum_{k=1}^d\int_{\mathbb{R}^d}\overline{\partial_{x_j}u}|\partial_{x_j}u|^{p-2}\partial_{x_j}hdx\right],$$

then

$$Dg_a(u).h=a\left(1+a\|u\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{\frac{1}{p}-1}q(u,h)$$

and

$$D^2 g_a(u,u).(h,g) = a^2(1-p)\left(1+a\|u\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{\frac{1}{p}-2} q(u,h)q(u,g)$$
$$+\, a\left(1+a\|u\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{\frac{1}{p}-1}\left[\frac{p}{2}\Re\int_{\mathbb{R}^d}\left(|u|^{p-2}\overline{g}h + \sum_{k=1}^d |\partial_{x_j}u|^{p-2}\overline{\partial_{x_j}g}\partial_{x_j}h\right)dx\right.$$
$$\left.+\frac{p-2}{2}\int_{\mathbb{R}^d}\left(\overline{u}^2|u|^{p-4}gh + \overline{\partial_{x_j}u}^2|\partial_{x_j}u|^{p-4}\partial_{x_j}g\partial_{x_j}h\right)dx\right].$$

From the series expansion of the Hilbert-Schmidt norm along with Hölder's inequality, we obtain that $R_a(t_0)$ is bounded above by

$$\frac{a(d+1)}{2}\int_0^{t_0}\left[a(d+1)\sum_{j\in\mathbb{N}}\|U(t-s)\xi(s)\Phi e_j\|^2_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\left(\frac{\|M(s)\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}}{\left(1+a\|M(s)\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{\frac{1}{p}}}\right)^{2(p-1)}\right.$$
$$-(p-1)a(d+1)\left(1+a\|M(s)\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{\frac{1}{p}-2}\sum_{j\in\mathbb{N}} q\left(M(s),U(t-s)\xi(s)\Phi e_j\right)^2$$
$$\left.(p-1)\|M(s)\|^{p-2}_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\left(1+a\|M(s)\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{\frac{1}{p}-1}\sum_{j\in\mathbb{N}}\|U(t-s)\xi(s)\Phi e_j\|^2_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right]ds.$$

Since the term in parenthesis in the first part is a decreasing function of $\|M(s)\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}$, the second term is non positive and the following calculation holds

$$\|M(s)\|^{p-2}_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\left(1+a\|M(s)\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{\frac{1}{p}-1}$$
$$= a^{\frac{2}{p}-1}\left(a\|M(s)\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{1-\frac{2}{p}}\left(1+a\|M(s)\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{\frac{1}{p}-1}$$
$$\le a^{\frac{2}{p}-1}\left(1+a\|M(s)\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{1-\frac{2}{p}}\left(1+a\|M(s)\|^p_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{\frac{1}{p}-1} \le a^{\frac{2}{p}-1},$$

we obtain that

$$R_a(t_0) \le \frac{(d+1)(d+p)a^{\frac{2}{p}}}{2}\int_0^{t_0}\sum_{j\in\mathbb{N}}\|U(t-s)\xi(s)\Phi e_j\|^2_{\mathrm{W}^{1,p}(\mathbb{R}^d)}ds.$$

Finally from the decay estimates, see Section 2.2, Hölder's inequality and the Sobolev injections we obtain that for any $t_0 \in [0,T]$,

$$R_a(t_0) \le \frac{2(d+1)(d+p)a^{\frac{2}{p}}}{4}c\left(\frac{r(p)d}{2}\right)^2\|\Phi\|^2_{\mathcal{L}_2^{0,s}}\eta\int_0^T |t-s|^{-\frac{4}{r(p)}}ds,$$

the integral is finite since we have made the assumption that $p < \frac{2d}{d-1}$ and thus

$$R_a(t_0) \le \frac{\kappa(\eta)a^{\frac{2}{p}}}{4}.$$

Also calculation using the fact that $(\exp(E_a(t_0)))_{t_0\in[0,T]}$ is a martingale, indeed the Novikov condition is satisfied from the above and Doob's inequality leads

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t_0\in[0,T]}\left\|\int_0^{t_0}U(t-s)\xi(s)dW(s)\right\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\geq\delta\right)\\
&=\mathbb{P}\left(\sup_{t_0\in[0,T]}\exp\left(g_a(M(t_0))\right)\geq\exp\left((1+a\delta^p)^{\frac{1}{p}}\right)\right)\\
&\leq\mathbb{P}\left(\sup_{t_0\in[0,T]}\exp\left(E_a(t_0)\right)\geq\exp\left((1+a\delta^p)^{\frac{1}{p}}-1-\frac{\kappa(\eta)a^{\frac{2}{p}}}{4}\right)\right)\\
&\leq\exp\left(-(1+a\delta^p)^{\frac{1}{p}}+1+\frac{\kappa(\eta)a^{\frac{2}{p}}}{4}\right)\\
&\leq e\exp\left(-a^{\frac{1}{p}}\delta+\frac{\kappa(\eta)a^{\frac{2}{p}}}{4}\right).
\end{aligned}$$

The last inequality holds for arbitrary a positive. Minimizing on a one finally obtains the desired estimate. □

Proposition 3.4 (Exponential tail estimates). *If Z is defined by $Z(t)=\int_0^t U(t-s)\xi(s)dW(s)$ such that there exists η positive such that $\|\xi\|^2_{\mathrm{C}([0,T];\mathrm{H}^1(\mathbb{R}^d))}\leq\eta$ a.s., then for any p in $\tilde{\mathcal{A}}(d)$, T and δ positive,*

$$\mathbb{P}\left(\|Z\|_{\mathrm{C}([0,T];\mathrm{H}^1(\mathbb{R}^d))}\geq\delta\right)\leq 3\exp\left(-\frac{\delta^2}{\kappa_1(\eta)}\right)$$
$$\mathbb{P}\left(\|Z\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p}(\mathbb{R}^d))}\geq\delta\right)\leq c\exp\left(-\frac{\delta^2}{\kappa_2(\eta)}\right)$$

where $c=2e+\exp\left((2ek_0!)^{\frac{1}{k_0}}\right)$, $k_0=2\vee\min\{k\in\mathbb{N}:2k\geq r(p)\}$

$$\kappa_1(\eta)=T4c\left(\infty\right)^2\|\Phi\|^2_{\mathcal{L}_2^{0,s}}\eta,$$

$$\kappa_2(\eta)=\frac{8c\left(\frac{r(p)d}{2}\right)^2T^{1-\frac{2}{r(p)}}(d+1)(d+p)\|\Phi\|^2_{\mathcal{L}_2^{0,s}}}{1-\frac{4}{r(p)}}\eta.$$

Proof. **The first estimate.** We recall that in $\mathrm{H}^1(\mathbb{R}^d)$ we may write, using the series expansion of the Wiener process and the fact that $(U_t)_{t\in\mathbb{R}}$ is a unitary group, see [30], $Z(t)=U(t)\int_0^t U(-s)\xi(s)dW(s)$. Since $U(-s)$ is an isometry, one obtains that for every s in $[0,T]$,

$$\begin{aligned}
\|U(-s)\xi(s)\Phi\|_{\mathcal{L}_2(\mathrm{L}^2,\mathrm{H}^1(\mathbb{R}^d))}&\leq\|L_s\|_{\mathcal{L}_c(\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}),\mathrm{H}^1(\mathbb{R}^d))}\|\Phi\|_{\mathcal{L}_2^{0,s}}\\
&\leq c\left(\infty\right)\|L_s\|_{\mathcal{L}_c(\mathrm{W}^{1,\infty}(\mathbb{R}^d,\mathbb{R}),\mathrm{H}^1(\mathbb{R}^d))}\|\Phi\|_{\mathcal{L}_2^{0,s}}\\
&\leq c\left(\infty\right)\|\xi(s)\|_{\mathrm{H}^1(\mathbb{R}^d)}\|\Phi\|_{\mathcal{L}_2^{0,s}}
\end{aligned}$$

where L is such that $L_su=\xi(s)u$. Consequently, we obtain that

$$\int_0^T\|U(-s)\xi(s)\Phi\|^2_{\mathcal{L}_2^{0,1}}ds\leq c\left(\infty\right)^2\|\Phi\|^2_{\mathcal{L}_2^{0,s}}T\|\xi(s)\|^2_{\mathrm{C}([0,T],\mathrm{H}^1(\mathbb{R}^d))}ds.$$

We conclude using Theorem 2.1 of [36]. The result is indeed stated for operator integrands from a Hilbert space H into H and it still holds when the operator takes its value in another Hilbert space.

The second estimate. From Markov's inequality it is enough to show that

$$\mathbb{E}\left[\exp\left(\frac{1}{\kappa_2(\eta)}\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|^2_{\mathrm{L}^{r(p)}(0,T;\mathrm{W}^{1,p}(\mathbb{R}^d))}\right)\right]\le c.$$

For $k\ge k_0$, Jensen's inequality along with Fubini's theorem, Lemma 3.3, the change of variables and the integration by parts formulae give that

$$\begin{aligned}
&\mathbb{E}\left[\left(\tfrac{1}{\sqrt{\kappa_2(\eta)}}\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p}(\mathbb{R}^d))}\right)^{2k}\right]\\
&\le \tfrac{1}{T}\int_0^T \mathbb{E}\left[\left(\tfrac{T}{(\kappa_2(\eta))^{\frac{r(p)}{2}}}\left\|\int_0^t U(t-s)\xi(s)dW(s)\right\|^{r(p)}_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\right)^{\frac{2k}{r(p)}}\right]dt\\
&\le \tfrac{1}{T}\int_0^T\int_0^\infty \mathbb{P}\left(\left\|\int_0^t U(t-s)\xi(s)dW(s)\right\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)}\ge\left(\tfrac{(\kappa_2(\eta))^{\frac{r(p)}{2}}}{T}\right)^{\frac{1}{r(p)}}u^{\frac{1}{2k}}\right)dudt\\
&\le \tfrac{1}{T}\int_0^T\int_0^\infty e\exp\left(-\tfrac{\kappa_2(\eta)}{T^{\frac{2}{r(p)}}}\tfrac{u^{\frac{1}{k}}}{\kappa(\eta)}\right)dudt\\
&\le e\int_0^\infty \exp\left(-2u^{\frac{1}{k}}\right)du = e\int_0^\infty kv^{k-1}\exp(-2v)dv = 2e\int_0^\infty v^k\exp(-2v)dv.
\end{aligned}$$

Thus, using Fubini's theorem one obtains that

$$\begin{aligned}
&\mathbb{E}\left[\left(\tfrac{1}{\sqrt{\kappa_2(\eta)}}\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p}(\mathbb{R}^d))}\right)^{2k_0}\right]\\
&\le k_0!\sum_{k\ge k_0}\tfrac{1}{k!}\mathbb{E}\left[\left(\tfrac{1}{\sqrt{\kappa_2(\eta)}}\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p}(\mathbb{R}^d))}\right)^{2k}\right]\\
&\le k_0!\sum_{k\ge k_0}\tfrac{1}{k!}2e\int_0^\infty v^k\exp(-2v)dv\\
&\le k_0!\sum_{k\in\mathbb{N}}\tfrac{1}{k!}2e\int_0^\infty v^k\exp(-2v)dv = 2ek_0!,
\end{aligned}$$

hence using Hölder's inequality one obtains

$$\begin{aligned}
&\sum_{k=0}^{k_0-1}\tfrac{1}{k!}\mathbb{E}\left[\left(\tfrac{1}{\sqrt{\kappa_2(\eta)}}\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p}(\mathbb{R}^d))}\right)^{2k}\right]\\
&=\sum_{k=0}^{k_0-1}\tfrac{1}{\kappa_2(\eta)^k k!}\mathbb{E}\left[\left\{\left(\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p}(\mathbb{R}^d))}\right)^{2k_0}\right\}^{\frac{k}{k_0}}\right]\\
&\le\sum_{k=0}^{k_0-1}\tfrac{1}{\kappa_2(\eta)^k k!}\mathbb{E}\left[\left(\left\|\int_0^{\cdot} U(\cdot-s)\xi(s)dW(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p}(\mathbb{R}^d))}\right)^{2k_0}\right]^{\frac{k}{k_0}}\\
&\le\sum_{k=0}^{k_0-1}\frac{\left[(2ek_0!)^{\frac{1}{k_0}}\right]^k}{k!}\le\exp\left((2ek_0!)^{\frac{1}{k_0}}\right).
\end{aligned}$$

The end of the proof is now straightforward. □

4. Proof of the uniform large deviation principle

4.1. **Almost continuity of the Itô map.** The proof of the uniform large deviation principle now relies on

Proposition 4.1. *For every positive a, R and ρ, u_0 in $\mathrm{H}^1(\mathbb{R}^d)$, f in C_a, $T < \mathcal{T}(S(u_0,f))$, p in $\mathcal{A}(d)$, there exists positive ϵ_0, γ and r such that for every ϵ in $(0,\epsilon_0]$ and $\tilde{u}_0$ in $B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0,r)$,*

$$\epsilon \log \mathbb{P}\left(\left\| u^{\epsilon,\tilde{u}_0} - S(u_0,f) \right\|_{X^{(T,p)}} \geq \rho; \left\| \sqrt{\epsilon} W - f \right\|_{\mathrm{C}([0,T];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))} < \gamma \right) \leq -R.$$

Proof. Take u_0 in $\mathrm{H}^1(\mathbb{R}^d)$, f in C_a, a, R and ρ positive, $T < \mathcal{T}(S(u_0,f))$ and p in $\mathcal{A}(d)$.

Step 1: Change of measure to center the Wiener process around f. The function f in C_a is such that there exists h in $\mathrm{L}^2\left(0,T;\mathrm{L}^2(\mathbb{R}^d)\right)$ such that $f(\cdot) = \int_0^{\cdot} \Phi h(s)ds$ and $\frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))} \leq a$. We denote by W^ϵ the process defined by

$$\begin{aligned} W^\epsilon(t) = W(t) - \frac{1}{\sqrt{\epsilon}} \int_0^t \frac{\partial f}{\partial s} ds &= W(t) - \frac{1}{\sqrt{\epsilon}} \int_0^t \Phi h(s) ds \\ &= \Phi\left(W_c(t) - \frac{1}{\sqrt{\epsilon}} \int_0^t h(s) ds \right). \end{aligned}$$

Since $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \|h(s)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \right) ds \right] < +\infty$, the Novikov condition is satisfied and the Girsanov theorem gives that W^ϵ is a $\mu-$Wiener process on $\mathrm{C}\left([0,T];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})\right)$ under the probability $\mathbb{P}^\epsilon$ defined by

$$\left.\frac{\mathrm{d}\mathbb{P}^\epsilon}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\frac{1}{\sqrt{\epsilon}} \int_0^t (h, dW_c(s))_{\mathrm{L}^2(\mathbb{R}^d)} - \frac{1}{2\epsilon} \int_0^T \|h(s)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} ds \right).$$

Set $U_\epsilon(t) = \exp\left(-\frac{1}{\sqrt{\epsilon}} \int_0^t (h, dW_c(s))_{\mathrm{L}^2(\mathbb{R}^d)} \right)$ and λ such that $a - \lambda < -R$ and denote by A the event

$$\left\{ \left\| u^{\epsilon,\tilde{u}_0} - S(u_0,f) \right\|_{X^{(T,p)}} \geq \rho; \left\| \sqrt{\epsilon} W - f \right\|_{\mathrm{C}([0,T];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))} < \gamma \right\},$$

then

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{E}_{\mathbb{P}_\epsilon}\left\{ \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_\epsilon} \mathbb{1}_{A \cap \left\{ U_\epsilon(T) \leq \exp\left(\frac{\lambda}{\epsilon}\right) \right\}} \right\} + \mathbb{P}\left(U_\epsilon(T) > \exp\left(\frac{\lambda}{\epsilon} \right) \right) \\ &\leq \mathbb{E}_{\mathbb{P}_\epsilon}\left\{ \mathbb{1}_A \exp\left(\frac{\lambda}{\epsilon} + \frac{1}{2\epsilon} \int_0^T \|h(s)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} ds \right) \right\} + \exp\left(-\frac{\lambda}{\epsilon} \right) \mathbb{E}\left(U_\epsilon(T) \right) \\ &\leq \exp\left(\frac{\lambda + a}{\epsilon} \right) \mathbb{P}_\epsilon(A) + \exp\left(\frac{a - \lambda}{\epsilon} \right). \end{aligned}$$

The last inequality follows from the fact that

$$\left(\exp\left(-\frac{1}{\sqrt{\epsilon}} \int_0^t (h, dW_c(s))_{\mathrm{L}^2(\mathbb{R}^d)} - \frac{1}{2\epsilon} \int_0^t \|h(s)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} ds \right) \right)_{t \in [0,T]}$$

is a uniformly integrable martingale. Finally we see that it is sufficient to prove that there exists positive ϵ_0, γ and r such that for every ϵ in $(0,\epsilon_0]$ and $\tilde{u}_0$ in $B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0,r)$,

$$\epsilon \log \mathbb{P}_\epsilon(A) \leq -R - \lambda - a,$$

or equivalently that

$$\epsilon \log \mathbb{P}_\epsilon\left(\left\| v^{\epsilon,\tilde{u}_0} - S(u_0,f) \right\|_{X^{(T,p)}} \geq \rho; \left\| \sqrt{\epsilon} W_\epsilon \right\|_{\mathrm{C}([0,T];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))} < \gamma \right) \leq -R - \lambda - a,$$

where $v^{\epsilon,\tilde{u}_0}$ satisfies $v^{\epsilon,\tilde{u}_0}(0)=\tilde{u}_0$ and

$$idv^{\epsilon,\tilde{u}_0}=\left(\Delta v^{\epsilon,\tilde{u}_0}+\lambda|v^{\epsilon,\tilde{u}_0}|^{2\sigma}v^{\epsilon,\tilde{u}_0}+\frac{\partial f}{\partial t}v^{\epsilon,\tilde{u}_0}-\frac{i\epsilon}{2}F_\phi v^{\epsilon,\tilde{u}_0}\right)dt+\sqrt{\epsilon}v^{\epsilon,\tilde{u}_0}dW_\epsilon.$$

Step 2: Reduction to estimates of the stochastic convolution.
Remark, this is standard fact, that the unboundedness of the drift and coefficient of the Wiener process is not a limitation since the result of Proposition 4.1 is local. A localisation argument will therefore be used to overcome the apparent difficulty. We replace T by

$$\tau_\rho=\inf\left\{t:\ \|v^{\epsilon,\tilde{u}_0}-S(u_0,f)\|_{X^{(t,p)}}\geq\rho\right\}\wedge T.$$

Since $T<\mathcal{T}(S(u_0,f))$, $v^{\epsilon,\tilde{u}_0}$ necessarily satisfies

$$\|v^{\epsilon,\tilde{u}_0}\|_{X^{(\tau_\rho,p)}}\leq\rho+\|S(u_0,f)\|_{X^{(\tau_\rho,p)}}=D.$$

Using the estimates used in the proofs of Proposition 3.1 herein and of Theorem 4.1 of [18], with a truncation in front of the nonlinearity of the form $\theta\left(\frac{\|S(u_0,f)\|_{X^{(s,p)}}}{D}\right)$ and $\theta\left(\frac{\|v^{\epsilon,\tilde{u}_0}\|_{X^{(s,p)}}}{D}\right)$, one may obtain for some $t>0$ and $\nu\in\left(0,1-\frac{\sigma(d-2)}{2}\right)$

$$\begin{aligned}\left\|v^{\epsilon,\tilde{u}_0}-S(u_0,f)\right\|_{X^{(t\wedge\tau_\rho,p)}}&\leq C\|\tilde{u}_0-u_0\|_{\mathrm{H}^1(\mathbb{R}^d)}+\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot-s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(t\wedge\tau_\rho,p)}}\\
&+C\left[(t\wedge\tau_\rho)^\nu(D^{2\sigma})(1+D)+(t\wedge\tau_\rho)^{\frac12-\frac{1}{r(p)}}\sqrt{a}+\epsilon(t\wedge\tau_\rho)^{1-\frac{2}{r(p)}}\right]\left\|v^{\epsilon,\tilde{u}_0}-S(u_0,f)\right\|_{X^{(t\wedge\tau_\rho,p)}}\\
&+C\epsilon(t\wedge\tau_\rho)^{1-\frac{2}{r(p)}}\|S(u_0,f)\|_{X^{(t\wedge\tau_\rho,p)}}.\end{aligned}$$

Set $\epsilon\leq1$, then for $t=t^*$ small enough we obtain

$$\begin{aligned}\left\|v^{\epsilon,\tilde{u}_0}-S(u_0,f)\right\|_{X^{(t^*\wedge\tau_\rho,p)}}\leq&2\left(C\|\tilde{u}_0-u_0\|_{\mathrm{H}^1(\mathbb{R}^d)}+C\epsilon D\right.\\
&\left.+\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot-s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}}\right).\end{aligned}$$

Set $N=\left\lfloor\frac{\tau_\rho}{t^*\wedge\tau_\rho}\right\rfloor$, and for i in $\{0,...,N\}$, $T_i=iT^*$ and $T_{N+1}=T$. The previous bound also holds for $\left\|v^{\epsilon,\tilde{u}_0}-S(u_0,f)\right\|_{X^{(T_i,T_{i+1},p)}}$ for every i in $\{0,...,N\}$, replacing $\|y-x\|_{\mathrm{H}^1(\mathbb{R}^d)}$ by $\left\|v^{\epsilon,\tilde{u}_0}(T_i)-S(u_0,f)(T_i)\right\|_{\mathrm{H}^1(\mathbb{R}^d)}$.
As for i in $\{1,...,N\}$, $\left\|v^{\epsilon,\tilde{u}_0}(T_i)-S(u_0,f)(T_i)\right\|_{\mathrm{H}^1(\mathbb{R}^d)}\leq\left\|v^{\epsilon,\tilde{u}_0}-S(u_0,f)\right\|_{X^{(T_{i-1},T_i,p)}}$, we obtain using the triangle inequality that

$$\begin{aligned}\left\|v^{\epsilon,\tilde{u}_0}-S(u_0,f)\right\|_{X^{(\tau_\rho,p)}}\leq&2(N+1)\left(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot-s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}}+C\epsilon D\right)\\
&+2C\sum_{i=1}^{N-1}\left\|v^{\epsilon,\tilde{u}_0}-S(u_0,f)\right\|_{X^{(T_{i-1},T_i,p)}}+2C\|u_0-\tilde{u}_0\|_{\mathrm{H}^1(\mathbb{R}^d)}\\
\leq&2(N+1)\left(\sum_{i=0}^{N-1}(2C)^i\right)\left(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot-s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}}\right.\\
&\left.+C\epsilon D\right)+(2C)^N\|u_0-\tilde{u}_0\|_{\mathrm{H}^1(\mathbb{R}^d)}.\end{aligned}$$

We may suppose that $2C>1$ and thus it is enough to show that there exists positive ϵ_0, γ and r such that $(2C)^N r<\rho$ and for every ϵ in $(0,\epsilon_0]$ and $\tilde{u}_0$ in $B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0,r)$,

$$\epsilon\log\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot-s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}}+C\epsilon D\geq\frac{(2C-1)(\rho-(2C)^N r)}{2(N+1)((2C)^N-1)};\right.$$
$$\left.\|\sqrt{\epsilon}W_\epsilon\|_{\mathrm{C}([0,T];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))}<\gamma\right)\leq -R-\lambda-a.$$

Step 3: The case of the stochastic convolution. We now need to see that for fixed u_0 in $\mathrm{H}^1(\mathbb{R}^d)$, f in C_a, a, R and ρ positive, $T<\mathcal{T}(S(u_0,f))$ and p in $\mathcal{A}(d)$, there exists ϵ_0, γ and r positive such that for all ϵ in $(0,\epsilon_0]$ and $\tilde{u}_0$ in $B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0,r)$,

$$\epsilon\log\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot-s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}}\geq\rho;\left\|\sqrt{\epsilon}W_\epsilon\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))}<\gamma\right)\leq -R.$$

We introduce an approximation of $v^{\epsilon,\tilde{u}_0}$ similar to that of the proof of Proposition 3.1. That is for n in $\mathbb{N}$ and i in $\{0,...,n\}$, we set $t_i=\frac{i\tau_\rho}{n}$ and define $v^{\epsilon,\tilde{u}_0,n}$ when $t_i\leq t<t_{i+1}$ by

$$v^{\epsilon,\tilde{u}_0,n}(t)=U(t-t_i)\left(v^{\epsilon,\tilde{u}_0}(t_i)\right).$$

For any δ positive we may write

$$\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot-s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}}\geq\rho;\left\|\sqrt{\epsilon}W_\epsilon\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})}<\gamma\right)$$
$$\leq\ \mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot-s)\left(v^{\epsilon,\tilde{u}_0}(s)-v^{\epsilon,\tilde{u}_0,n}(s)\right)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}}\geq\frac{\rho}{2};\right.$$
$$\left.\left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))}<\delta\right)$$
$$+\mathbb{P}_\epsilon\left(\left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))}\geq\delta\right)$$
$$+\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}}\geq\frac{\rho}{2};\left\|\sqrt{\epsilon}W_\epsilon\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))}<\gamma;\right.$$
$$\left.\left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))}<\delta\right)$$

Bound for the first term. From Proposition 3.4, $C\exp\left(-\frac{\rho^2}{4\epsilon(\kappa_1(\delta^2)\vee\kappa_2(\delta^2))}\right)$ is an upper bound for the first term. Thus for any ϵ positive and δ small enough

$$\epsilon\log\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^\cdot U(\cdot-s)\left(v^{\epsilon,\tilde{u}_0}(s)-v^{\epsilon,\tilde{u}_0,n}(s)\right)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}}\geq\frac{\rho}{2};\right.$$
$$\left.\left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))}<\delta\right)$$

is less than $-R-1$.

Bound for the second term. The second term $\mathbb{P}_\epsilon\left(\left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))}\geq\delta\right)$ is bounded by the sum of

$$\mathbb{P}_\epsilon\left(\sup_{i\in\{0,\dots,n-1\}}\sqrt{\epsilon}\left\|\int_{t_i}^\cdot U(t-s)v^{\epsilon,\tilde{u}_0}(s)dW_\epsilon(s)\right\|_{\mathrm{C}([t_i,t_{i+1}];\mathrm{H}^1(\mathbb{R}^d))}\geq\frac{\delta}{2}\right)$$

which from Proposition 3.4 is less than $3n\exp\left(-\frac{C\delta^2 n}{\tau_\rho D^2}\right)$ and

$$\mathbb{P}_\epsilon\left(\sup_{i\in\{0,\dots,n-1\}}\left\|\int_{t_i}^\cdot U(t-s)\left[\lambda\left|v^{\epsilon,\tilde{u}_0}(s)\right|^{2\sigma}+\frac{\partial f}{\partial s}(s)-\frac{i\epsilon}{2}F_\Phi\right]v^{\epsilon,\tilde{u}_0}(s)ds\right\|_{\mathrm{C}([t_i,t_{i+1}];\mathrm{H}^1(\mathbb{R}^d))}\geq\frac{\delta}{2}\right).$$

The second term is equal to zero for n large enough. Indeed, with calculations similar to that of the proof of Theorem 4.1 of [18] and of Proposition 3.1, we obtain that for $\epsilon < 1$,

$$\sup_{i\in\{0,\dots,n-1\}} \left\| \int_{t_i}^{\cdot} U(t-s)\left[\lambda \left|v^{\epsilon,\tilde{u}_0}(s)\right|^{2\sigma} + \frac{\partial f}{\partial s}(s) - \frac{i\epsilon}{2}F_\Phi\right] v^{\epsilon,\tilde{u}_0}(s)ds \right\|_{\mathrm{C}([t_i,t_{i+1}];\mathrm{H}^1(\mathbb{R}^d))}$$

$$\leq C\left[\left(\frac{\tau_\rho}{n}\right)^{\nu} D^{2\sigma+1} + \left(\frac{\tau_\rho}{n}\right)^{\frac{1}{2}-\frac{1}{r(p)}} D\sqrt{a} + \frac{1}{2}\left(\frac{\tau_\rho}{n}\right)^{1-\frac{2}{r(p)}} D\right].$$

Finally, for every δ positive and $0<\epsilon<1$, for n large enough

$$\epsilon \log \mathbb{P}_\epsilon\left(\left\|v^{\epsilon,\tilde{u}_0} - v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} \geq \delta\right) < -R-1.$$

Thus choosing first δ small enough and then n large enough we obtain that for any $0<\epsilon<\frac{1}{2\log(2)}$,

$$\begin{aligned}\epsilon \log \Big\{ \mathbb{P}_\epsilon \Big(\sqrt{\epsilon}\left\|\int_0^{\cdot} U(\cdot-s)\left(v^{\epsilon,\tilde{u}_0}(s)-v^{\epsilon,\tilde{u}_0,n}(s)\right)dW_\epsilon(s)\right\|_{X^{(\tau_\rho,p)}} \geq \tfrac{\rho}{2};\\ \left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} < \delta\Big)\\ +\mathbb{P}_\epsilon\left(\left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} \geq \delta\right)\Big\} \leq -R-\tfrac{1}{2}\end{aligned}$$

Bound for the third term. Fix δ as above. Remark first that the last condition

$$\left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} < \delta$$

implies that $\left\|v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} < D+\delta$. Denote by

$$\underline{t} = \max\{t_i : t_i \leq t,\ i\in\{0,...,n\}\}$$

and by E the event

$$E = \left\{\left\|\sqrt{\epsilon}W_\epsilon\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))} < \gamma\ ;\ \left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} < \delta\right\}.$$

As $p<\frac{2(3d-1)}{3(d-1)}$ there exists $p<\tilde{p}<\frac{2d}{d-1}$ and η positive such that $1-\frac{p-2}{\tilde{p}-2}\left(1+\frac{2}{r(\tilde{p})}+\eta\right)$ is positive. Thus from Hölder's inequality, setting $\theta=\frac{\tilde{p}-p}{\tilde{p}-2}$, the third term is bounded above by

$$\begin{aligned} &\mathbb{P}_\epsilon\Big(\sqrt{\epsilon}\left\|\int_{\underline{\cdot}}^{\cdot} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(\tilde{p})}(0,\tau_\rho;\mathrm{W}^{1,\tilde{p}}(\mathbb{R}^d))} \geq n^{\left(1+\frac{2}{r(\tilde{p})}+\eta\right)\frac{1}{2}};\\ &\qquad\qquad \left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} < \delta\Big)\\ +\ &\mathbb{P}_\epsilon\Big(\sqrt{\epsilon}\left\|\int_{\underline{\cdot}}^{\cdot} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))}^{\theta} n^{\left(1+\frac{2}{r(\tilde{p})}+\eta\right)\frac{1-\theta}{2}} \geq \tfrac{\rho}{8};\\ &\qquad\qquad \left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} < \delta\Big)\\ +\ &\mathbb{P}_\epsilon\Big(\sqrt{\epsilon}\left\|\int_{\underline{\cdot}}^{\cdot} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} \geq \tfrac{\rho}{8};\\ &\qquad\qquad \left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} < \delta\Big)\\ +\ &\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^{\underline{\cdot}} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(p)}(0,\tau_\rho;W^{1,p}(\mathbb{R}^d))} \geq \tfrac{\rho}{8};E\right)\\ +\ &\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^{\underline{\cdot}} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} \geq \tfrac{\rho}{8};E\right).\\ +\ &\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_0^{\underline{\cdot}} U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} \geq \tfrac{\rho}{8};E\right).\end{aligned}$$

The first probability is bounded from above by

$$\sum_{i=0}^{n-1}\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_{t_i}^{t_{i+1}}U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(\tilde{p})}(t_i,t_{i+1};W^{1,\tilde{p}}(\mathbb{R}^d))}\geq n^{\left(-1+\frac{2}{r(\tilde{p})}+\eta\right)\frac{1}{2}};\ \left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))}<\delta\right)$$

which from Proposition 3.4 is less than

$$nC\exp\left(-\frac{n^{-1+\frac{2}{r(\tilde{p})}+\eta}}{C\left(\frac{\tau_\rho}{n}\right)^{1-\frac{2}{r(\tilde{p})}}\epsilon(\delta+D)^2}\right).$$

Thus for n large enough, for any ϵ and δ positive,

$$\epsilon\log\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_{-}^{\cdot}U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(\tilde{p})}(0,\tau_\rho;\mathrm{W}^{1,\tilde{p}}(\mathbb{R}^d))}\geq n^{\left(1+\frac{2}{r(\tilde{p})}+\eta\right)\frac{1}{2}};\ \left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))}<\delta\right)<-R-1.$$

The first exponential tail estimate of Proposition 3.4 gives that the second probability is less than

$$nC\exp\left(-\frac{\rho^2}{Cn^{\left(1+\frac{2}{r(\tilde{p})}+\eta\right)(1-\theta)}\left(\frac{\tau_\rho}{n}\right)\epsilon(\delta+D)^2}\right),$$

and, from the choice of $\tilde{p}$, for n large enough, for any ϵ and δ positive,

$$\epsilon\log\mathbb{P}_\epsilon\left(\sqrt{\epsilon}\left\|\int_{-}^{\cdot}U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|^{\theta}_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))}n^{\left(1+\frac{2}{r(\tilde{p})}+\eta\right)\frac{1-\theta}{2}}\geq\frac{\rho}{3};\ \left\|v^{\epsilon,\tilde{u}_0}-v^{\epsilon,\tilde{u}_0,n}\right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))}<\delta\right)<-R-1.$$

The same holds for the third probability more clearly.
The decay estimates of Section 2.2 along with Hölder's inequality give that the mapping

$$w\mapsto U(t-t_j)v^{\epsilon,\tilde{u}_0}(t_j)w$$

from $\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})$ to $\mathrm{W}^{1,p}(\mathbb{R}^d)$ is continuous. Thus, we may write

$$\begin{aligned}&\left\|\int_0^{\cdot}U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p}(\mathbb{R}^d))}\\&=\left\|\sum_{i=1}^{n-1}\mathbb{1}_{t_i\leq t<t_{i+1}}\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}U(t-t_j)v^{\epsilon,\tilde{u}_0}(t_j)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(p)}(0,T;W^{1,p}(\mathbb{R}^d))}\\&\leq\sum_{i=1}^{n-1}\sum_{j=0}^{i-1}\left\|\int_{t_j}^{t_{j+1}}U(t-t_j)v^{\epsilon,\tilde{u}_0}(t_j)dW_\epsilon(s)\right\|_{\mathrm{L}^{r(p)}(t_i,t_{i+1};W^{1,p}(\mathbb{R}^d))}\\&\leq C\left(\frac{(n-1)(n-2)}{2}\right)\left(\frac{\tau_\rho}{n}\right)^{-\frac{2}{r(p)}}D\gamma,\end{aligned}$$

and obtain that, for any n in $\mathbb{N}$, for γ small enough the fourth probability is equal to zero.
Similarly we write, using the continuity of the group and Hölder's inequality,

$$\begin{aligned}&\left\|\int_0^{\cdot}U(\cdot-s)v^{\epsilon,\tilde{u}_0,n}(s)dW_\epsilon(s)\right\|_{\mathrm{C}(0,T;\mathrm{H}^1(\mathbb{R}^d))}\\&=\max_{i=1,\dots,n-1}\left\|\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}U(t_i-t_j)v^{\epsilon,\tilde{u}_0}(t_j)dW_\epsilon(s)\right\|_{\mathrm{H}^1(\mathbb{R}^d)}\\&\leq\sum_{j=0}^{n-1}\|v^{\epsilon,\tilde{u}_0}(t_j)\|_{\mathrm{H}^1(\mathbb{R}^d)}\|W_\epsilon(t_{j+1})-W_\epsilon(t_j)\|_{\mathrm{H}^s(\mathbb{R}^d,\mathbb{R})}\\&\leq 2nD\gamma.\end{aligned}$$

Thus, for any n in $\mathbb{N}$, for γ small enough the fifth probability is equal to zero. Finally, when δ is fixed, for n large enough and a particular choice of γ depending on n and δ, we obtain that for any $0<\epsilon<\frac{1}{2\log(2)}$,

$$\epsilon \log \mathbb{P}_\epsilon \Big(\sqrt{\epsilon} \left\| \int_0^\cdot U(\cdot - s) v^{\epsilon,\tilde{u}_0,n}(s) dW_\epsilon(s) \right\|_{X^{(\tau_\rho,p)}} \geq \frac{\rho}{2}; \| \sqrt{\epsilon} W_\epsilon \|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))} < \gamma; \\ \left\| v^{\epsilon,\tilde{u}_0} - v^{\epsilon,\tilde{u}_0,n} \right\|_{\mathrm{C}([0,\tau_\rho];\mathrm{H}^1(\mathbb{R}^d))} < \delta \Big) \leq -R - \tfrac{1}{2}.$$

We have now proved Step 3 and thus Proposition 4.1. $\square$

Remark 4.2. In Step 2 we did not use the Gronwall inequality, which is often used in that case. Instead we split the norm in many parts because in the case of the Schrödinger group it is needed, in order to use the integrability property governed by $ii/$ of the Strichartz estimates, to keep the convolution with the group.

Remark 4.3. Revisiting Step 2 we may see that a uniform LDP holds with the same good rate function and same speed for the laws of the paths of the solutions for an equation with an additional term of the form $f\left(u^{\epsilon,u_0},\epsilon,t,x\right)$ in the drift. It is needed that there exists (s,ρ) conjugate exponents of an admissible pair $(r(q),q)$ such that for every positive T such that $\|\psi\|_{X^{(T,p)}}<+\infty$, $\|f\left(\psi,\epsilon,.,*\right)\|_{\mathrm{L}^s(0,T;\mathrm{W}^{1,\rho}(\mathbb{R}^d))}$ is bounded and goes to zero as ϵ goes to zero. This extra term may be an external potential accounting for damping or amplification going to zero along with the noise intensity. In [14, 31] for example, extra terms are added to the equation to account for nonlinear, respectively linear, damping. In [24] a small amplification added to compensate for loss due to a small amplitude modulation is considered. The amplitude modulation makes sense considering initial data in spaces of spatially localized functions used when studying the blow-up of deterministic and stochastic NLS equations.

Remark 4.4. The proof of local existence and uniqueness of the solutions of the stochastic NLS equation in $\mathrm{H}^1(\mathbb{R}^d)$ holds with more general nonlinearities and when the noise enters nonlinearly under some Lipschitz assumptions. We may adapt our proof of the uniform LDP to cover those NLS equations.

4.2. **End of the proof.** We prove hereafter how the almost continuity along with Proposition 2.1 and 3.1 allow to prove the uniform LDP for the laws of the solutions in the space $\mathcal{E}_\infty$ when the noise goes to zero.
Suppose that I^{u_0} is the rate function of the LDP then from Proposition 3.1, since its level sets are the direct image by $S(u_0,\cdot)$ of the level sets C_a which are compact, it is a good rate function.
The set A is a Borel set of $\mathcal{E}_\infty$ and u_0 is some initial datum in $\mathrm{H}^1(\mathbb{R}^d)$.
An upper bound. In the case where $\inf_{w\in\overline{A}} I^{u_0}(w)=0$ there is nothing to prove. Otherwise, take $0<a<\inf_{w\in\overline{A}} I^{u_0}(w)$ and $R>a$. Suppose that f is such that $I^W(f)\leq a$, then

$$I^{u_0}(S(u_0,f)) \leq a < \inf_{w\in\overline{A}} I^{u_0}(w),$$

thus $S(u_0,f)\notin\overline{A}$ and there exists an elementary neighborhood of $S(u_0,f)$ of the form

$$V_{u_0,f}=\{v\in\mathcal{E}_\infty:\ \mathcal{T}(v)>T \text{ and } \|v-S(u_0,f)\|_{X^{(T,p)}}<\rho_{u_0,f}\}$$

for some (p,T) such that $V_{u_0,f}\subset \overline{A}^c$. Also, from Proposition 3.4, there exists $\epsilon_{u_0,f}$, $\gamma_{u_0,f}$ and $r_{u_0,f}$ positive such that for every $\epsilon\le\epsilon_{u_0,f}$ and $\tilde{u}_0$ in $B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0,r_{u_0,f})$,

$$\epsilon\log\mathbb{P}\left(\left\|u^{\epsilon,\tilde{u}_0}-S(u_0,f)\right\|_{X^{(T,p)}}\ge\rho_{u_0,f};\left\|\sqrt{\epsilon}W-f\right\|_{\mathrm{C}([0,T];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))}<\gamma_{u_0,f}\right)\le -R.$$

Let denote by $O_{u_0,f}$ the set

$$O_{u_0,f}=B_{\mathrm{C}([0,T];\mathrm{H}^s(\mathbb{R}^d,\mathbb{R}))}(f,\gamma_{u_0,f}).$$

The family $(O_{u_0,f})_{f\in C_a}$ is a covering by open sets of the compact set C_a, thus there exists a finite sub-covering of the form $\bigcup_{i=1}^N O_{u_0,f_i}$. We can now write

$$\begin{aligned}\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\le&\mathbb{P}\left(\left\{u^{\epsilon,\tilde{u}_0}\in A\right\}\cap\left\{\sqrt{\epsilon}W\in\bigcup_{i=1}^N O_{u_0,f_i}\right\}\right)+\mathbb{P}\left(\sqrt{\epsilon}W\notin\bigcup_{i=1}^N O_{u_0,f_i}\right)\\ \le&\sum_{i=1}^N\mathbb{P}\left(\left\{u^{\epsilon,\tilde{u}_0}\in A\right\}\cap\left\{\sqrt{\epsilon}W\in O_{u_0,f_i}\right\}\right)+\mathbb{P}\left(\sqrt{\epsilon}W\notin C_a\right)\\ \le&\sum_{i=1}^N\mathbb{P}\left(\left\{u^{\epsilon,\tilde{u}_0}\notin V_{u_0,f_i}\right\}\cap\left\{\sqrt{\epsilon}W\in O_{u_0,f_i}\right\}\right)+\exp\left(-\frac{a}{\epsilon}\right),\end{aligned}$$

for $\epsilon\le\epsilon_0$ for some ϵ_0 positive. Thus for $\epsilon\le\epsilon_0\wedge\left(\bigwedge_{i=1}^m\epsilon_{u_0,f_i}\right)$ we obtain for $\tilde{u}_0$ in $B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0,r_{u_0})$ where $r_{u_0}=\bigwedge_{i=1}^m r_{u_0,f_i}$,

$$\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\le\ N\exp\left(-\frac{R}{\epsilon}\right)+\exp\left(-\frac{a}{\epsilon}\right),$$

and

$$\epsilon\log\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\le\epsilon\log 2+(\epsilon\log N-R)\vee(-a).$$

Finally, there exists r_{u_0} such that for any $\tilde{u}_0$ in $B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0,r_{u_0})$,

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\le -a.$$

Since a is arbitrary, we obtain,

$$\overline{\lim}_{\epsilon\to 0,\tilde{u}_0\to u_0}\epsilon\log\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\le -\inf_{w\in\overline{A}}I^{u_0}(w).$$

A lower bound. Suppose that $\inf_{w\in Int(A)}I^{u_0}(w)<+\infty$, otherwise there is nothing to prove, and take w in $Int(A)$ such that $I^{u_0}(w)<+\infty$.
The continuity of $S(u_0,\cdot)$ along with the compactness the level set $C_{I^{u_0}(w)+1}$ give that there exists f such that $w=S(u_0,f)$ and $I^{u_0}(w)=I^W(f)$. Take $V_{u_0,f}$ an elementary neighborhood of $S(u_0,f)$ included in A and $O_{u_0,f}$ defined as previously, η positive and $R>I^{u_0}(w)+\eta$. We obtain

$$\begin{aligned}\exp\left(-\frac{R-\eta}{\epsilon}\right)&\le\exp\left(-\frac{I^W(f)}{\epsilon}\right)\\ &\le\mathbb{P}\left(\sqrt{\epsilon}W\in O_{u_0,f}\right)\\ &\le\mathbb{P}\left(\left\{u^{\epsilon,\tilde{u}_0}\notin V_{u_0,f}\right\}\cap\left\{\sqrt{\epsilon}W\in O_{u_0,f}\right\}\right)+\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right).\end{aligned}$$

Thus there exists r_{u_0} and ϵ_0 positive such that for all $\tilde{u}_0$ in $B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0,r_{u_0})$ and $\epsilon\le\epsilon_0$,

$$-R+\eta\le\epsilon\log 2+\left(\epsilon\log\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right)\right)\vee(-R)$$

and there exists $\epsilon_1\le\epsilon_0$ such that for all $\epsilon\le\epsilon_1$,

$$-I^{u_0}(w)\le\epsilon\log 2+\epsilon\log\mathbb{P}\left(u^{\epsilon,\tilde{u}_0}\in A\right).$$

As a consequence, we obtain that for every u_0 in $\mathrm{H}^1(\mathbb{R}^d)$, there exists r_{u_0} positive such that for every $\tilde{u}_0$ in $B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0, r_{u_0})$,

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(u^{\epsilon,\tilde{u}_0} \in A\right) \geq -I^{u_0}(w)$$

and

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(u^{\epsilon,\tilde{u}_0} \in A\right) \geq -\inf_{w\in Int(A)} I^{u_0}(w)$$

since w in $Int(A)$ is arbitrary.

Uniformity with respect to initial data in compact sets. The uniform LDP follows from the previous bounds along with Corollary 5.6.15 of [22]. We now give the proof of the lower bound which is not written in the previous reference. Let K be a compact set in $\mathrm{H}^1(\mathbb{R}^d)$. Suppose that $\sup_{\tilde{u}_0\in K}\inf_{w\in Int(A)} I^{\tilde{u}_0}(w) < +\infty$, otherwise there is nothing to prove and take δ positive. For any u_0 in K, there exists r_{u_0} positive such that for every $\epsilon \leq \epsilon_{u_0}$,

$$\epsilon \log \inf_{\tilde{u}_0\in B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0,r_{u_0})} \mathbb{P}\left(u^{\epsilon,\tilde{u}_0} \in A\right) \geq -\sup_{\tilde{u}_0\in K}\inf_{w\in Int(A)} I^{\tilde{u}_0}(w) - \delta.$$

The set of balls $B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0, r_{u_0})$ is a covering of K by open sets thus there exists a sub-covering of K of the form $\bigcup_{i=1}^m B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0^i, r_{u_0^i})$ and for $\epsilon \leq \bigwedge_{i=1}^m \epsilon_{u_0^i}$,

$$\begin{aligned}\epsilon \log \inf_{\tilde{u}_0\in K} \mathbb{P}\left(u^{\epsilon,\tilde{u}_0} \in A\right) &\geq \epsilon \log \inf_{\tilde{u}_0\in \bigcup_{i=1}^m B_{\mathrm{H}^1(\mathbb{R}^d)}(u_0^i,r_{u_0^i})} \mathbb{P}\left(u^{\epsilon,\tilde{u}_0} \in A\right)\\ &\geq -\sup_{\tilde{u}_0\in K}\inf_{w\in Int(A)} I^{\tilde{u}_0}(w) - \delta,\end{aligned}$$

conclusion follows since δ is arbitrary.

5. Applications to the blow-up time

In this section the equation with a focusing nonlinearity is considered. In this case, it is known that some solutions of the deterministic equation blow up in finite time for critical or supercritical nonlinearities. If B is a Borel set of $[0, +\infty]$,

$$\mathbb{P}\left(\mathcal{T}\left(u^{\epsilon,u_0}\right) \in B\right) = \mu^{u^{\epsilon,u_0}}\left(\mathcal{T}^{-1}(B)\right).$$

Thus the uniform LDP for the family $\left(\mu^{u^{\epsilon,u_0}}\right)_{\epsilon>0}$ gives that for $K \subset\subset \mathrm{H}^1(\mathbb{R}^d)$,

$$-\sup_{u_0\in K}\inf_{u\in Int(\mathcal{T}^{-1}(B))} I^{u_0}(u) \leq \underline{\lim}_{\epsilon\to 0}\epsilon \log \inf_{u_0\in K} \mathbb{P}\left(\mathcal{T}(u^{\epsilon,u_0}) \in B\right)$$

and that

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \sup_{u_0\in K} \mathbb{P}\left(\mathcal{T}(u^{\epsilon,u_0}) \in B\right) \leq -\inf_{u\in \overline{\mathcal{T}^{-1}(B)}, u_0\in K} I^{u_0}(u).$$

Since $\mathcal{T}$ is lower semicontinuous the sets $(T, +\infty]$ and $[0, T]$ are particularly interesting. We recall, see [30] for more details, that for every $T > 0$, $\overline{\mathcal{T}^{-1}((T, +\infty])} = \mathcal{E}_\infty$ and $Int\left(\mathcal{T}^{-1}([0, T])\right) = \emptyset$. Thus for the two types of sets, at least one bound is trivial. Considering the approximate blow-up time allows us to obtain two interesting bounds and to treat intervals of the form $(S, T]$ where $0 \leq S < T$. We do not consider this latter question in the article. We finally recall that when $T < \mathcal{T}(u_d^{u_0})$ the LDP gives that

$$\lim_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}(u^{\epsilon,u_0}) > T\right) = 0,$$

indeed this is not a large deviation event, we obtain similarly that when $T > \mathcal{T}(u_d^{u_0})$ the LDP gives that

$$\lim_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\mathcal{T}(u^{\epsilon,u_0}) \le T\right) = 0.$$

5.1. **Probability that blow-up occur before time T.**

Proposition 5.1. *If $T < \mathcal{T}_K^i = \inf_{u_0 \in K} \mathcal{T}\left(u_d^{u_0}\right)$, where $K \subset\subset \mathrm{H}^1(\mathbb{R}^d)$, then there exists c positive such that*

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \sup_{u_0 \in K} \mathbb{P}\left(\mathcal{T}(u^{\epsilon,u_0}) \le T\right) \le -c.$$

Proof. Since $\mathcal{T}$ is lower semicontinuous, $\mathcal{T}^{-1}([0,T])$ is a closed set. Suppose now that there exists a sequence (u_n, h_n) in $K \times \mathrm{L}^2(0,+\infty; \mathrm{L}2)$ such that

$$\mathcal{T}\left(S^c(u_n, h_n)\right) \le T$$

and $\lim_{n \to \infty} h_n = 0$. Since K is a compact set we may extract a subsequence $u_{\varphi(n)}$ such that $u_{\varphi(n)}$ converges to some $\tilde{u}$. Also, if we denote by $f_n(\cdot) = \int_0^{\cdot} \Phi h_n(s) ds$, f_n converges to zero in $\mathrm{C}\left([0,+\infty); \mathrm{H}^s(\mathbb{R}^d, \mathbb{R})\right)$ and satisfies $\mathcal{T}\left(S(u_n, f_n)\right) \le T$. Also there exists a positive such that for every n in $\mathbb{N}$, f_n in C_a. The semicontinuity of $\mathcal{T}$ along with Proposition 3.2 give that

$$T \ge \underline{\lim}_{n \to \infty} \mathcal{T}\left(S(u_{\varphi(n)}, f_{\varphi(n)})\right) \ge \mathcal{T}\left(S(\tilde{u}, 0)\right) \ge \mathcal{T}_K^i > T,$$

which is contradictory. □

When $K = \{u_0\}$ we obtain the same result as in Proposition 5.5 of [30].

5.2. **Probability that blow-up occur after time T.** In the following we consider the case $d = 2$ or $d = 3$ and a cubic nonlinearity, i.e. $\sigma = 1$. In that case blow-up may occur.

Proposition 5.2. *Let U^{u_0} be the solution of the free Schrödinger equation with initial data u_0 in $\mathrm{H}^r(\mathbb{R}^d)$ where $r > \frac{d}{2} \vee s$, assume that $T > \mathcal{T}\left(u_d^{u_0}\right)$ and that $\operatorname{span}\left\{|U^{u_0}(t)|^2, t \in [0, 2T]\right\}$ belongs to the range of Φ. There exists c positive such that*

$$\underline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\mathcal{T}(u^{\epsilon,u_0}) > T\right) \ge -c.$$

Remark 5.3. It is known for example that for some Gaussian initial data u_0, see for example [6], the solutions of NLS blows up in finite time. Also, the solutions of the free equation are smooth and strongly decreasing at infinity, thus it may easily be checked that it is possible to define an Hilbert-Schmidt operator Φ such that the last assumption holds.

Proof. Define F^{u_0} by $F^{u_0}(t) = -\int_0^{t \wedge 2T} |U^{u_0}(s)|^2 ds$. The control is such that $S(u_0, F^{u_0}) = U^{u_0}$ on $[0, 2T]$, which does not blow up, thus $\mathcal{T}\left(S(u_0, F^{u_0})\right) \ge 2T$. Also, F^{u_0} belongs to $\mathrm{C}\left([0,+\infty), \mathrm{H}^s(\mathbb{R}^d, \mathbb{R})\right)$ since for $r > \frac{d}{2}$ $\mathrm{H}^r(\mathbb{R}^d)$ is an algebra and U^{u_0} belongs to $\mathrm{C}([0,+\infty), \mathrm{H}^r(\mathbb{R}^d))$. Finally, from the assumption on Φ, there exists h in $\mathrm{L}^2(0,+\infty; \mathrm{L}^2(\mathbb{R}^d))$ setting $h = 0$ after $2T$ such that $\Phi h(s) = |U^{u_0}(s)|^2 \mathbb{1}_{s \le 2T}$ and F^{u_0} belongs to C_a for some a positive. We thus obtain that F^{u_0} belongs to

$$\left\{f \in \mathrm{C}\left([0,+\infty), \mathrm{H}^s(\mathbb{R}^d, \mathbb{R})\right) : \mathcal{T}\left(S(u_0, f)\right) > T\right\}$$

and that $I^W(F^{u_0}) \le a < +\infty$. □

Remark 5.4. We obtain a result on compact sets K in $\mathrm{H}^r(\mathbb{R}^d)$ for $T > \sup_{u_0 \in K} \mathcal{T}\left(u_d^{u_0}\right)$ provided that span $\left\{|U^{u_0}(t)|^2, t \in [0,2T], u_0 \in K\right\}$ belongs to the range of Φ restricted to a ball of $\mathrm{L}^2\left(0,2T;\mathrm{L}^2(\mathbb{R}^d)\right)$.

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