Optimal Carbon Sequestration Policies in Leaky Reservoirs

Alain Jean-Marie[1,2] Michel Moreaux[3,4] Mabel Tidball[5,2]

May 26, 2014

[1]INRIA and UMR LIRMM, 161 Rue Ada, 34095 Montpellier Cedex 5, France
[2]Part of this work was performed while this author was visiting GERAD, Montreal.
[3]Toulouse School of Economics, IDEI and LERNA, 21 allée de Brienne, 31 000 Toulouse, France
[4]Part of this work was performed while this author was on a one-year academic leave with Inria.
[5]INRA and UMR LAMETA, 2 place P. Viala, 34060 Montpellier Cedex 1, France

Abstract

We study in this report a model of optimal Carbon Capture and Storage in which the reservoir of sequestered carbon is leaky, and pollution eventually is released into the atmosphere. We formulate the social planner problem as an optimal control program and we describe the optimal consumption paths as a function of the initial conditions, the physical constants and the economical parameters. In particular, we show that the presence of leaks may lead to situations which do not occur otherwise, including that of non-monotonous price paths for the energy.

Acknowledgments

The authors wish to acknowledge the fruitful interactions with the attendees of different venues where preliminary versions of this work were presented:

- Journées Économie et Environnement, Montpellier, France, 9 December 2010
- EAERE 2011, Rome, Italy, 1 July 2011
- Seminario ANEODA/Instituto de Matemáticas, Universidad de Valladolid, Valladolid, Spain, 29 February 2012
- Workshop: Energy, Pollution and Environmental Policy GREQAM, Marseille, France, 15 March 2012
- Fondation HEC/Chaire théorie des jeux et gestion/GERAD Montréal, 1 May 2012
- Optimization Days, Montréal, Canada, 8 May 2012
- 12th Vienese Workshop on Optimal Control, Dynamic Games and Non-Linear Dynamics Vienna, Austria, 31 May 2012
- ANR CLEANER Workshop, Annecy, France, 1 February 2013

Chapter 1

Introduction

This report is devoted to the complete solution of an optimal control model with state constraints, arising in the study of economic tradeoffs between energy consumption and pollution management. More precisely, the question is to determine under which circumstances the deployment of Carbon Capture and Storage (CCS) technology is of any help to an economy faced with the potential damages of a high CO_2 concentration in the atmosphere.

The purpose of this document is to serve as technical reference, and provide the mathematical arguments backing up the construction of the solution, as completely as possible. It features a detailed discussion of how theorems from the literature can be applied, why the solution proposed is consistent with their conditions, and also a parametric discussion of the behavior of these solutions.

This introduction follows with a motivation for the problem we study, and a summary of the technical contribution.

1.1 The Economic Relevance of Carbon Storage and Sequestration

The fact that the carbon emissions generated by the use of the fossil fuels could be captured and sequestered is now well documented both empirically and theoretically, and it is now included in the main empirical models of energy uses. Were this option open at a sufficiently low cost for the most potentially polluting primary resource, that is coal, its competitive full cost, including the shadow cost of its pollution power, could be drastically reduced given that coal is abundant at a low extraction cost and can be transformed into energy ready to use for final users at moderately transformation costs. The main problem concerning its future competitiveness is the cost at which its pollution damaging effects can be abated.

Abating the emissions involves two different types of costs. The first one is a monetary cost : capturing, compressing and transporting the captured CO_2 into reservoirs involves money outlays. The second one is a shadow cost because this type of garbage has to be stockpiled somewhere. This problem has been attacked in Lafforgue et al. (2008*a*), Lafforgue et al. (2008*b*). It is not quite clear that sufficient storage capacities would be available for low CO_2 capture and storage costs, in which case the reservoir capacities themselves could have to be seen as scarce resources to which some rents should have to be imputed along an optimal or equilibrium path.

As far as equilibrium paths are concerned there is a very difficult problem about property rights. The reservoirs into which the captured CO_2 is assumed to be confined are in underground places, on which property rights are more or less defined, and differently defined all over the world.

Even if sufficiently large reservoirs are available there exists another problem concerning the security of such reservoirs. Most reservoirs are leaking in the long run, a well-known problem in engineering. The fact that captured CO_2 will eventually return into the atmosphere cannot be ignored when assessing the economic relevance of CCS.

A first investigation of this last problem has been given by Ha-Duong & Keith (2003). Their main conclusion is that "leakage rates on order of magnitude below the discount rate are negligible" (p. 188). Hence leakage is a second order problem as far as the rate of discount is sufficiently high, and probably that other characteristics of the empirical model they use are sufficiently well profiled.

A second batch of investigations has recently been conducted by Gerlagh, Smekens and Van der Zwaan.[1] These papers are mainly empirical papers using and comparing DEMETER and MARKAL models to assess the usefulness of CCS policies. Their results are twofold. First using CCS policies with leaky reservoirs does not permit to escape a big switch to renewable non polluting primary resources if a 450ppmv atmospheric pollution ceiling has to be enforced. But CCS with leaky reservoirs is smoothing the optimal path. A second point concerns the relative competitiveness of coal : "The large scale application of CCS needed for a significantly lower contribution of renewable would be consistent, in terms of climate change control, with the growing expectation that fossil fuels, and in particular coal, will continue to be a dominant form of energy supply during the twenty-first century" Van der Zwaan & Gerlagh (2009, p. 305). As they point out "The economic implications of potential CO_2 leakage associated with the large scale development of CCS have so far been researched in a few studies" (ibidem, p. 306). To our knowledge theoretical studies are even fewer.[2]

The objective of this paper is to try to elucidate some theoretical features of optimal CCS policies with leaky reservoirs and specifically the dynamics of the shadow cost of both carbon stocks and their relation with the mining rent of the nonrenewable resource, determining the long run relative competitiveness of coal and solar energies. The paper has to be seen as mainly exploratory. To conduct the inquiry we adopt the most simple model permitting to isolate the dynamics of captured CO_2, leakage and atmospheric pollution.

Naturally, the presence of leaks, producing an additional flow of pollutant, makes the pressure on the atmospheric stock larger than when there is none, and should favor even more the capture to relax the pressure today. On the other hand, for the same reason, it is not necessarily good to sequestrate too much pollution, since this will make economic conditions worse in the future.

The results presented in this paper show how the optimal consumption paths are modified with respect to the benchmark situation where there are no leaks. In particular, it turn out that over some optimal path, the price of energy is not necessarily monotonous. Non-monotonous price paths in the exploitation of nonrenewable resources have been described before: for a first paper in this direction, see for instance Livernois & Martin (2001). In the present situation, the lack of monotonicity results from a combination of a constraint on the present atmospheric stock of pollution, and a lag effect for the sequestered stock of pollution; such an effect has not been reported in the literature, to the best of our knowledge.

Our analysis reveals other interesting features. First of all, not every possible configuration of atmospheric and sequestered stock is acceptable, thus causing a possible *viability* problem. Other results quantitatively confirm that the presence of leakage does reduce the economic incentive to sequestrate pollution.

1.2 Technical Challenges and Contribution

The model we develop conceals several technical features that are seldom encountered in the literature. First of all, il involves three state variables and three controls, with constraints on the three states and constraints on two of the controls. We are nevertheless able to provide a complete parametric description of solutions when one of the state variables is "saturated". Based on this analysis, the understanding of the case where all three state variables are present appears to be within reach; the details are however not developped in this document.

[1] c.f. Van der Zwaan (2005), Van der Zwaan & Gerlagh (2009) and Van der Zwaan & Smekens (2009).

[2] The contributions of Lontzek & Rickels (2008) et Rickels & Lontzek (2012) are relevant in the context of an underwater sequestration. The study of Augeraud-Veron & Leandri (2013) specifically focuses on the time lag aspect of the sequestration.

In the course of the solution, we identify the presence of a "hidden" viability or controllability constraint, and a "singular" point in the state space. In the vicinity of the viability constraint and of the singular point, optimal trajectories have an unusual behavior, and some adjoint variables (economically interpreted as shadow prices) may be discontinuous.

Related to this unusual behavior is the unusual fact that the so-called *constraint qualification* conditions associated to the optimization problem are not satisfied. Also, classical geometric conditions leading to the regularity of the value function (see *e.g.* Soner (1986)) do not hold. Indeed, the value function turns out not to be differentiable everywhere in the domain of interest.

We contribute to the understanding of the situation by providing a complete description of trajectories, constucted explicitly using the maximum principle, and not *via* a numerical approximation of the value function. This detailed construction allows us to provide as well a complete parametric discussion of the form of optimal trajectories.

The report is organized as follows. We develop the model, its assumptions and notations in Chapter 2. In particular, in Section 2.2 we state the mathematical optimization program representing the social planner problem, and derive the necessary optimality conditions.

In Chapter 3, we prepare the construction of solutions by studying the behavior of optimal trajectories within *phases* characterized by a constant status (free or bound) of the different constraints on states and controls. This allows in particular to eliminate several configurations which cannot be optimal.

In Chapter 4, we construct the solutions of the optimization problem in the situation where the stock of polluting carbon energy is assumed to be infinite (that is, the resource is assumed to be renewable) and the capacity of the reservoir is sufficiently large. While not quite relevant empirically, this analysis provides the essential insights in the behavior of solutions and the complexity of the problem. The first part of the chapter enumerates all possible cases, depending on parameters and the position of the state of the system. The second part (from Section 4.5 onwards) presents the global picture and performs the parametric discussion, including some limiting cases.

Several appendices with the most technical details complete this description. In particular, Appendix E features a numerical illustration in the Linear-Quadratic case.

Chapter 2

The Model

2.1 Model and Assumptions

We consider a global economy in which the energy consumption can be supplied by two primary resources: a nonrenewable polluting source like coal and a clean renewable one as solar plants.

2.1.1 Energy consumption and gross surplus

Let us denote by q the instantaneous energy consumption rate of the final users and by $u(q)$ the instantaneous gross surplus thus generated. The gross surplus function is assumed to satisfy technical assumptions that will be specified as Assumption 1 on p. 8.

The function $u'(q)$, is the inverse demand function and its inverse, the direct demand function, is denoted by $q^d(p)$. Under Assumption 1, the function q^d is strictly positive and strictly decreasing.

2.1.2 The nonrenewable polluting resource

Let $X(t)$ be the stock of coal available at time t, $X^0 = X(0)$ be its initial endowment, and $x(t)$ be the instantaneous extraction rate: $\dot{X}(t) = -x(t)$. The current average transformation cost of coal into useful energy is assumed to be constant and is denoted by c_x. We denote by $\widetilde{x}$ the nonrenewable energy consumption when its marked price is equal to c_x and coal is the only energy supplier: $u'(\widetilde{x}) = c_x$.

Burning coal for producing useful energy implies a flow of pollution emissions proportional to the coal thus burned. Let ζ be the unitary pollution contents of coal so that the gross emission flow amounts to $\zeta x(t)$. This gross emission flow can be either freely relaxed into the atmosphere or captured to be stockpiled into underground reservoirs however at some cost.

Let c_s be the average capturing and sequestrating cost of the potential pollution generated by the exploitation of coal. Let us denote by $s(t)$ this part of the potential flow $\zeta x(t)$ which is captured and sequestered. Then the sequestration cost amounts to $c_s s(t)$. The remaining flow of carbon $\zeta x(t) - s(t) \geq 0$ goes directly into the atmosphere.

2.1.3 Pollution stocks and leakage effects

We take two pollution stocks explicitly into account, the atmospheric stock denoted by $Z(t)$ and the sequestered stock denoted by $S(t)$. As previously stated, the atmospheric stock Z is first fed by the non-captured pollution emissions, resulting from the use of coal, that is $\zeta x(t) - s(t)$. This atmospheric stock is self-regenerating at some constant proportional rate α.[1] However, Z is also fed by the leaks of the sequestered pollution stock S. We assume that leaks are proportional to

[1] This self-regeneration effect may be seen as some kind of leakage of the atmosphere reservoir towards some other natural reservoirs not explicitly modeled in the present setting. For models taking explicitly into account such questions, see for example Lontzek and Rickels (2008) or Rickels and Lontzek (2008).

the stock and denote by β the leakage rate. Taking into account both this leakage effect and the above self-regeneration effect, we get the dynamics of the atmospheric stock:

$$\dot{Z}(t) \ = \ \zeta x(t) - s(t) + \beta S(t) - \alpha Z(t) \ .$$

Since the sequestered stock is just fed by the sequestered pollution, we have:

$$\dot{S}(t) \ = \ s(t) - \beta S(t) \ .$$

The flows and stocks of energy and pollution are illustrated in Figure 2.1.

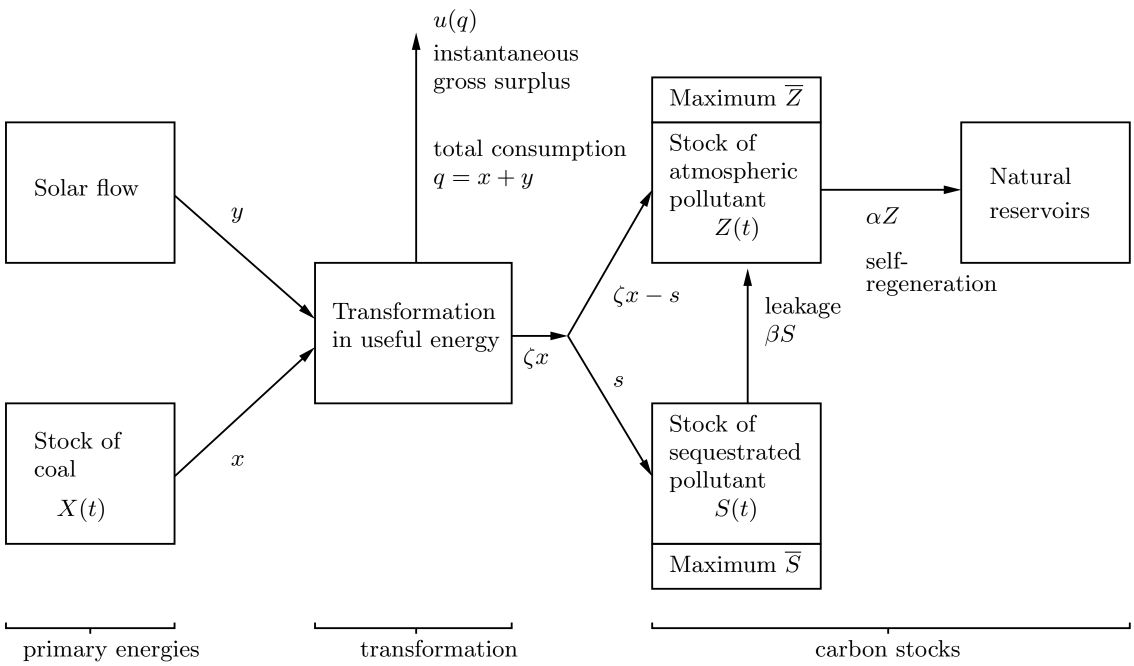

Figure 2.1: Flows and stocks of energy and pollution

We assume that the sequestered stock is limited by a known constant capacity $\overline{S}$. It is acceptable that $\overline{S}$ be set sufficiently large to be never saturated. In every case, it is assumed that no cost has to be incurred for maintaining the captured stock S into reservoirs. The only costs are the above capture costs $c_s s(t)$.

2.1.4 Atmospheric pollution damages

There are two main ways for modeling the atmospheric pollution damages. A most favored way by some economists is to postulate some damage function, the higher is the atmospheric pollution stock $Z(t)$, the larger are the current damages at the same time t. Generally, this function is assumed to be convex. The other way is to assume that, as far as the atmospheric pollution stock is kept under some critical level $\overline{Z}$, the damages are not so large. However, around the critical level $\overline{Z}$, the damages are strikingly increasing, so that, whatever what could have been gained by following a path generating an overrun at $\overline{Z}$, the damages would counterbalance the gains.[2] We adopt the second way of modeling damages pioneered by Chakravorty et al. (2006), and therefore assume that the loss generated by Z are negligible provided that Z be maintained under some level $\overline{Z}$, but is infinitely costly once $Z(t)$ overruns $\overline{Z}$.[3]

[2]Some authors use simultaneously both approaches.

[3]As pointed out by Amigues et al. (2011), taking into account both *small* and *catastrophic* damages does not change the main qualitative characteristics of optimal paths.

We denote by $\overline{x}$ the maximum coal consumption when the atmospheric pollution stock is at its ceiling $\overline{Z}$, no part of the gross pollution flow ζx is captured ($s = 0$) and the stock of sequestered pollution is nil:

$$\dot{Z} \;=\; 0 \;=\; \zeta\overline{x} - \alpha\overline{Z} \qquad \Longrightarrow \qquad \overline{x} \;=\; \frac{\alpha}{\zeta}\overline{Z}\ .$$

We denote by $\overline{p}$ the corresponding energy price assuming that coal is the only energy supplier: $\overline{p} = u'(\overline{x})$.

Clearly there exists an effective constraint on coal consumption if and only if $\overline{p} > c_x$ or equivalently $\overline{x} < \widetilde{x}$ and simultaneously the coal initial endowment X^0 is sufficiently large.

2.1.5 The renewable clean energy

The other primary resource is a renewable clean energy. Let $y(t)$ be its instantaneous consumption rate. We assume that its average cost, denoted by c_y, is constant. We denote by $\widetilde{y}$ the renewable energy consumption when the renewable one is the only energy supplier: $u'(\widetilde{y}) = c_y$. The consumption of renewable energy is assumed to be limited by a known constant $\overline{y}$. It is acceptable that $\overline{y}$ be set larger than $\widetilde{y}$.

Both c_x and c_y include all that has to be supported to supply ready to use energy to the final users. Hence, once these costs are supported the two types of energy are perfect substitutes for the final user and we may define the total energy consumption as $q = x + y$.

2.2 The Social Planner problem

The social planner problem is to maximize the social welfare. The social welfare W is the sum of the discounted net current surplus, taking into account the gross surplus $u(q)$ and the production or capture costs. We assume that the social rate of discount ρ, $\rho > 0$, is constant throughout time.

Accordingly, the social planner faces the following optimization problem:

$$\max_{s(\cdot),x(\cdot),y(\cdot)} \int_0^{\infty} \left[u(x(t) + y(t)) - c_s s(t) - c_x x(t) - c_y y(t)\right] e^{-\rho t}\mathrm{d}t \tag{2.2.1}$$

given the controlled dynamics:[4]

$$\left\{\begin{array}{rcl} \dot{X} &=& -x \\ \dot{Z} &=& -\alpha Z + \beta S + \zeta x - s \\ \dot{S} &=& -\beta S + s\ , \end{array}\right. \tag{2.2.2}$$

the initial conditions $(X(0), Z(0), S(0)) = (X^0, Z^0, S^0)$, and the constraints on state variables and controls:

$$Z(t) \;\leq\; \overline{Z} \tag{2.2.3}$$
$$S(t) \;\leq\; \overline{S} \tag{2.2.4}$$
$$X(t) \;\geq\; 0 \tag{2.2.5}$$
$$y(t) \;\geq\; 0 \tag{2.2.6}$$
$$y(t) \;\leq\; \overline{y} \tag{2.2.7}$$
$$s(t) \;\geq\; 0 \tag{2.2.8}$$
$$s(t) \;\leq\; \zeta x(t) \tag{2.2.9}$$

for all t. Other physically relevant constraints ($S \geq 0$, $Z \geq 0$) are automatically satisfied by the dynamics and are not explicitly taken into account. This follows from the fact that $Z = 0$ implies

[4]An alternate parametrization of the control is in terms of "cleaned carbon" consumption $x_c = s/\zeta$ and "dirty carbon" consumption $x_d = x - s/\zeta$. With these controls instead of x and s, the dynamics become: $\dot{Z} = -\alpha Z + \beta S + \zeta x_d$ and $\dot{S} = -\beta S + \zeta x_c$. The constraints on control are then $x_c \geq 0$ and $x_d \geq 0$.

$\dot{Z} = \beta S + \zeta x - s \geq 0$ and likewise, $S = 0$ implies $\dot{S} \geq 0$. A natural constraint on the control is $x(t) \geq 0$: this constraint is implied by (2.2.8) and (2.2.9), and we do not refer to it explicitly in the remainder.

The maximization in (2.2.1) involves *admissible* control functions $s(\cdot)$, $x(\cdot)$, $y(\cdot)$, that is, piecewise continuous functions. Pairs of control vectors and state trajectories such that controls are piecewise continuous, trajectories solve the state equation (2.2.2) and both satisfy all constraints (2.2.3)–(2.2.9), will be called admissible pairs.

2.2.1 Assumptions on costs and parameters

The results we obtain are valid under the following composite assumption.[5]

We assume not only that the cost of the renewable energy is higher than the cost of the nonrenewable one, but furthermore that c_y is higher than $\overline{p}$. We assume also that $c_x \leq \overline{p}$ as discussed in Section 2.1.4. The function $u(\cdot)$ obeys standard assumptions, with the possibility (but not the requirement) that $u'(0) = +\infty$. In summary:

Assumption 1. The function $u : [0, \infty) \to \mathbb{R}$ is a function of class C^2, strictly increasing and strictly concave. It is assumed that $c_s > 0$, and

$$\lim_{x\to\infty} u'(x) \ < \ c_x \ < \ \overline{p} \ < \ c_y \ < \ u'(0) \ , \tag{2.2.10}$$

or equivalently, $0 < \widetilde{y} < \overline{x} < \widetilde{x}$. Other parameters are such that: $\alpha > 0$, $\beta > 0$, $\rho > 0$ and $\zeta > 0$.

These assumptions on the cost parameters of the model are summarized in Figure 2.2, which also recapitulates the notation

$$\overline{x} \ = \ \frac{\alpha \overline{Z}}{\zeta} \qquad \widetilde{y} \ = \ q^d(c_y) \qquad \widetilde{x} \ = \ q^d(c_x) \qquad \overline{p} \ = \ u'(\overline{x}) \ .$$

The following unit system proves useful in calculations and interpretations (see Section 2.3.1 for the missing notation λ_X etc. (adjoint variables or shadow prices) and ν_S, ν_Z or γ_s etc. (Lagrange multipliers)). The unit T_c refers to "tons of CO_2" whereas the unit T_p refers to "tons of pollutant".

α, β, ρ	in	s^{-1}	ζ	in	T_p/T_c
X	in	T_c	Z, S	in	T_p
λ_X	in	$\$/T_c$	λ_Z, λ_S	in	$\$/T_p$
$x(\cdot), y(\cdot)$	in	T_c/s	$s(\cdot)$	in	T_p/s
$u(\cdot)$	in	$\$/s$	$u'(\cdot), c_x, c_y$	in	$\$/T_c$
$q^d(\cdot)$	in	T_c/s	c_s	in	$\$/T_p$
$\underline{\gamma}_y, \overline{\gamma}_y$	in	$\$/T_c$	γ_s, γ_{sx}	in	$\$/T_p$
			ν_S, ν_Z	in	$\$/T_p/s$

2.2.2 Literature and particular cases

The model generalizes several previous models of the literature, which can be recovered using particular values of the parameters.

No reservoirs, no capture The model where capture is not possible has been studied in Chakravorty et al. (2006).

When $\beta \to \infty$ in the present model, then whatever is captured in the stock is immediately leaked into the atmosphere. The model therefore reduces to the case without reservoir and without capture (since capturing is more costly than not capturing).

[5]The standard of the literature is to place assumptions separately on $u(\cdot)$ and on other parameters. This results in unnecessarily strong assumptions like Inada's $u'(0) = \infty$.

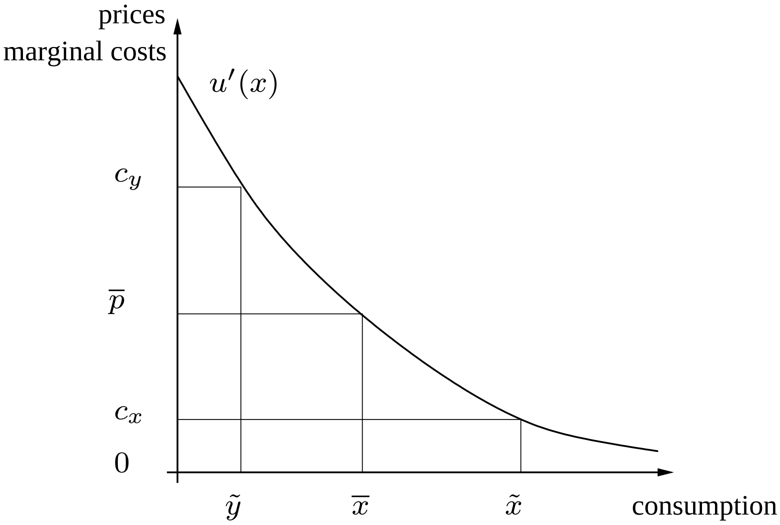

Figure 2.2: Assumptions on marginal costs

The model without capture also shows up when the capture cost c_s is very large so as to make sequestration economically suboptimal (see Section 4.5.4). Equivalently, the control s can be forced to be 0. The difference with $\beta = \infty$ is however that the standing stock of sequestered carbon will empty only progressively. If the initial condition is an empty stock, then there is no difference.

No leakage The case $\beta = 0$ models the situation where reservoirs do not leak.

This model is studied in Lafforgue et al. (2008*a*), which actually considers the case of multiple reservoirs with different sequestration costs. Each reservoir has a finite capacity. The flow of clean energy $\overline{y}$ is never binding, which is equivalent to assuming that $\overline{y} \geq \tilde{y}$.

In Lafforgue et al. (2008*b*), only one reservoir is considered, it has a finite capacity $\overline{S}$, and in addition the maximally available flow of clean energy $\overline{y}$ is possibly binding.

In both papers, an additional assumption is made: $c_s < (c_x - \overline{p})/\xi$. In the forthcoming analysis, this situation will be called "c_s small", see Section 4.5.1.

2.3 Main elements for finding the solution of the social planner problem

We shall use the maximum principle in order to identify the solutions to this optimization problem. In this paragraph, we first state the first-order conditions for the problem, next review the theorems on which we base the solution method.

2.3.1 First order conditions

Let us denote by L the current-value Lagrangian of the problem. Introducing λ_X, λ_Z and λ_S as adjoint variables, ν_Z, ν_S and ν_X as Lagrange multipliers for state constraints, γ_s, γ_{sx}, $\underline{\gamma}_y$ and $\overline{\gamma}_y$ as Lagrange multipliers for control constraints, the Lagrangian writes as:

$$\begin{aligned} L(y, x, s, X, Z, S) \quad = \quad & u(x+y) - c_s s - c_x x - c_y y \\ & + \lambda_X[-x] + \lambda_Z[-\alpha Z + \beta S + \zeta x - s] + \lambda_S[-\beta S + s] \\ & + \nu_Z[\overline{Z} - Z] + \nu_S[\overline{S} - S] + \nu_X X \end{aligned} \qquad (2.3.1)$$

$$+\gamma_s s + \gamma_{sx}(\zeta x - s) + \underline{\gamma}_y y + \overline{\gamma}_y(\overline{y} - y)\ .$$

The "classical" first order conditions are then the following. First, optimality of the control yields:

$$\frac{\partial L}{\partial s} = 0 \quad \Longleftrightarrow \quad c_s + \lambda_Z = \lambda_S + \gamma_s - \gamma_{sx} \tag{2.3.2}$$

$$\frac{\partial L}{\partial x} = 0 \quad \Longleftrightarrow \quad u'(x+y) = c_x + \lambda_X - \zeta\lambda_Z - \zeta\gamma_{sx} \tag{2.3.3}$$

$$\frac{\partial L}{\partial y} = 0 \quad \Longleftrightarrow \quad u'(x+y) = c_y - \underline{\gamma}_y + \overline{\gamma}_y\ , \tag{2.3.4}$$

together with the constraints and slackness conditions:

$$\gamma_{sx} \geq 0, \qquad \zeta x - s \geq 0 \quad \text{and} \quad \gamma_{sx}[\zeta x - s] = 0 \tag{2.3.5}$$

$$\gamma_s \geq 0, \qquad s \geq 0 \quad \text{and} \quad \gamma_s s = 0 \tag{2.3.6}$$

$$\underline{\gamma}_y \geq 0, \qquad y \geq 0 \quad \text{and} \quad \underline{\gamma}_y y = 0 \tag{2.3.7}$$

$$\overline{\gamma}_y \geq 0, \qquad \overline{y} - y \geq 0 \quad \text{and} \quad \overline{\gamma}_y[\overline{y} - y] = 0\ . \tag{2.3.8}$$

Next, the dynamics of the adjoint variables are

$$\dot{\lambda}_X = \rho\lambda_X - \frac{\partial L}{\partial X} \quad \Longleftrightarrow \quad \dot{\lambda}_X \ = \ \rho\lambda_X - \nu_X \tag{2.3.9}$$

$$\dot{\lambda}_Z = \rho\lambda_Z - \frac{\partial L}{\partial Z} \quad \Longleftrightarrow \quad \dot{\lambda}_Z \ = \ (\rho+\alpha)\lambda_Z + \nu_Z \tag{2.3.10}$$

$$\dot{\lambda}_S = \rho\lambda_S - \frac{\partial L}{\partial S} \quad \Longleftrightarrow \quad \dot{\lambda}_S \ = \ (\rho+\beta)\lambda_S - \beta\lambda_Z + \nu_S\ , \tag{2.3.11}$$

with the constraints:

$$\nu_X \geq 0, \qquad X \geq 0 \quad \text{and} \quad \nu_X\ X = 0 \tag{2.3.12}$$

$$\nu_Z \geq 0, \qquad \overline{Z} - Z \geq 0 \quad \text{and} \quad \nu_Z[\overline{Z} - Z] = 0 \tag{2.3.13}$$

$$\nu_S \geq 0, \qquad \overline{S} - S \geq 0 \quad \text{and} \quad \nu_S[\overline{S} - S] = 0\ . \tag{2.3.14}$$

Finally, we have the transversality conditions:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_X X \ = \ 0 \tag{2.3.15}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda_Z Z \ = \ 0 \tag{2.3.16}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda_S S \ = \ 0\ . \tag{2.3.17}$$

2.3.2 Elimination of suboptimal controls

For technical reasons related to the possibility that $u'(0)$ be infinite, it is convenient to add more constraints to the control problem, knowing that these will be satisfied by any optimal control.

Lemma 2.1. *Under Assumption 1, any solution to the control problem (2.2.1) with constraints (2.2.2)–(2.2.9) is such that* $x(t) + y(t) \geq \widetilde{y}$ *for virtually all* t*.*

Proof. Assume that (y, x, s) is a control such that $x(t) + y(t) < \widetilde{y}$ for $t \in I$, some nonempty interval. Modify this strategy into: $x^\dagger(t) = x(t)$, $y^\dagger(t) = \widetilde{y} - x(t)$ for $t \in I$, while not changing $s(t)$ nor the strategy outside of interval I. Since the solution to the differential system (2.2.2) is not changed, this is also an admissible strategy. We show that it yields a larger profit. Indeed, the difference in profits can be written as:

$$J - J^\dagger \ = \ \int_I \left[(u(x(t)+y(t)) - c_y(x(t)+y(t))) - (u(\widetilde{y}) - c_y\widetilde{y})\right] e^{-\rho t}\mathrm{d}t\ .$$

The function $y \mapsto v(q) = u(q) - c_y q$ has derivative $v'(q) = u'(q) - c_y$. By Assumption 1 and the definition of $\widetilde{y}$, this is positive for $0 < q < \widetilde{y}$. As a consequence, $v(\cdot)$ is strictly increasing on the interval $[0, \widetilde{y}]$ and for every t, $v(x(t)+y(t)) < v(\widetilde{y})$. Therefore, $J - J^\dagger < 0$ and the strategy (y, x, s) cannot be optimal. □

2.3.3 Sufficient optimality conditions

We will base our solution on the two following results, which provide *sufficient* conditions for optimality. The difference between these theorems lies in the set of assumptions and the type of optimal trajectories they allow for. While the first one (Theorem 2.1) allows for jumps in the adjoint variables, it needs stronger C^2 assumptions than the second one (Theorem 2.2), which concerns continuous adjoint variables, but needs only quasi-concave assumptions on the constraints. In order to use these theorems to solve our problem, we will need to introduce an extra constraint, which turns out not to be C^2. Hence the need for both results.

The first statement is that of Seierstad & Sydsæter (1987, Theorem 11, p. 385).

Theorem 2.1 (Seierstad & Sydsæter (1987), Theorem 11). *Consider the infinite-horizon optimal control problem:*

$$\max_{\mathbf{u}(\cdot)} \int_0^\infty f_0(\mathbf{x}(t), \mathbf{u}(t), t)\, dt$$

where the state vector $\mathbf{x}(\cdot)$ *belongs to* $\mathbb{R}^n$, *the control vector* $\mathbf{u}(\cdot)$ *belongs to some fixed convex set* $U \subset \mathbb{R}^r$, *and* $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$ *with initial conditions* $\mathbf{x}(0) = \mathbf{x}^0$. *Assume that admissible trajectories must satisfy the vector of* s *constraints:*

$$g_j(\mathbf{x}(t), \mathbf{u}(t), t) \geq 0, j = 1, \ldots, s', \qquad g_j(\mathbf{x}(t), \mathbf{u}(t), t) = \bar{g}_j(\mathbf{x}(t), t) \geq 0, j = s'+1, \ldots, s,$$

as well as the terminal conditions

$$\liminf_{t \to \infty} x_i(t) = x_i^1, i = 1, \ldots, \ell, \quad \liminf_{t \to \infty} x_i(t) \geq x_i^1, i = \ell + 1, \ldots, m,$$

and no condition for $i = m+1, \ldots, n$.

Assume that:

a) f_0, f *and* g_j *for* $j = 1, \ldots, s'$ *have derivatives w.r.t.* $\mathbf{x}$ *and* $\mathbf{u}$, *and that these derivatives are continuous.*

b) $\bar{g}_j$ *is* C^2 *for* $j = s'+1, \ldots, s$,

c) g_j *is a quasi-concave function of* $(\mathbf{x}, \mathbf{u})$, *for all* t *and* $j = 1, \ldots, s$.

If there exists an admissible pair $(\mathbf{x}^*(t), \mathbf{u}^*(t))$, *together with a piecewise continuous and piecewise continuously differentiable vector function* $\mathbf{p}(t)$ *with jump points* $0 < \tau_1 < \ldots < \tau_N$, *a piecewise-continuous function* $\mathbf{q}(t)$ *and* $2N$ *vectors* β_k^-, β_k^+, $k = 1, \ldots, N$ *in* $\mathbb{R}^s$ *such that, defining*

$$\begin{aligned} H(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) &:= f_0(\mathbf{x}, \mathbf{u}, t) + \mathbf{p} \cdot f(\mathbf{x}, \mathbf{u}, t) \\ L(\mathbf{x}, \mathbf{u}, \mathbf{p}, \mathbf{q}, t) &:= H(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) + \mathbf{q} \cdot g(\mathbf{x}, \mathbf{u}, t)\ , \end{aligned}$$

d) for virtually all t, *and all* $\mathbf{u} \in U$, $\dfrac{\partial L}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t), \mathbf{q}(t), t) \cdot (\mathbf{u} - \mathbf{u}^*) \leq 0$,

e) for virtually all t, $\dot{\mathbf{p}}(t) = -\dfrac{\partial L}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t), \mathbf{q}(t), t)$,

f) the Hamiltonian is a concave function of $(\mathbf{x}, \mathbf{u})$, *for all* t,

g) for all t *and* $j = 1, \ldots, s$, $q_j(t) \geq 0$ *and* $= 0$ *if* $g_j(\mathbf{x}^*(t), \mathbf{u}^*(t), t) > 0$, *h) for each* $i = 1, \dots, n$ *and* $k = 1, \dots, N$,

$$p_i(\tau_k^-) - p_i(\tau_k^+) \;=\; \sum_{j=1}^{s} \beta_{kj}^{+} \frac{\partial g_j}{\partial x_i}(\mathbf{x}^*(\tau_k), \mathbf{u}^*(\tau_k^+), \tau_k) + \beta_{kj}^{-} \frac{\partial g_j}{\partial x_i}(\mathbf{x}^*(\tau_k), \mathbf{u}^*(\tau_k^-), \tau_k) \;, \quad (2.3.18)$$

i) for each $k = 1, \dots, N$ *and* $\mathbf{u} \in U$, $\beta_k^{\pm} \cdot \dfrac{\partial g}{\partial \mathbf{u}}(\mathbf{x}^*(\tau_k), \mathbf{u}^*(\tau_k^{\pm}), \tau_k) \cdot (\mathbf{u} - \mathbf{u}^*(\tau_k^{\pm})) \leq 0$,

j) for each $j = 1, \dots, s$ *and* $k = 1, \dots, N$, $\beta_{kj}^{\pm} \geq 0$, *and* $= 0$ *if* $g_j(\mathbf{x}^*(\tau_k), \mathbf{u}^*(\tau_k^{\pm}), \tau_k) > 0$,

k) and for all admissible $\mathbf{x}(t)$, $\liminf_{t \to \infty} \mathbf{p}(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t)) \geq 0$,

then the pair $(\mathbf{x}^*(t), \mathbf{u}(t))$ *is catching-up-optimal.*

The second statement is that of Seierstad & Sydsæter (1977, Theorems 6 and 10), where the notation "g_j" replaces the original notation "h_j".

Theorem 2.2 (Seierstad & Sydsæter (1977), Theorems 6 and 10)**.** *Consider the infinite-horizon optimal control problem:*

$$\max_{\mathbf{u}(\cdot)} \int_0^\infty f_0(\mathbf{x}(t), \mathbf{u}(t), t)\, dt$$

where the state vector $\mathbf{x}(\cdot)$ *belongs to* $\mathbb{R}^n$, *the control vector* $\mathbf{u}(\cdot)$ *belongs to* $\mathbb{R}^r$, *and* $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$ *with initial conditions* $\mathbf{x}(0) = \mathbf{x}^0$. *Assume that admissible trajectories must satisfy the vector of* s *constraints:*

$$g_j(\mathbf{x}(t), \mathbf{u}(t), t) \geq 0, j = 1, \dots, s$$

as well as the terminal conditions $\lim_{t \to \infty} x_i(t) = x_i^1$, $i = 1, \dots, n$.

Assume that:

a) f_0, f *are continuous on the set* $\{(\mathbf{x}, \mathbf{u}, t) \mid ((\mathbf{x}, \mathbf{u}) \in A(t)\}$, *where* $A(t) = \{(\mathbf{x}, \mathbf{u}) : g_j(\mathbf{x}, \mathbf{u}, t) \geq 0, j = 1, \dots, s\}$.

If there exists an admissible pair $(\mathbf{x}^*(t), \mathbf{u}^*(t))$, *together with a continuous and piecewise continuously differentiable vector function* $\mathbf{p}(t)$, *and a piecewise-continuous function* $\mathbf{q}(t)$ *such that, defining*

$$\begin{aligned} H(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) &:= f_0(\mathbf{x}, \mathbf{u}, t) + \mathbf{p} \cdot f(\mathbf{x}, \mathbf{u}, t) \\ L(\mathbf{x}, \mathbf{u}, \mathbf{p}, \mathbf{q}, t) &:= H(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) + \mathbf{q} \cdot g(\mathbf{x}, \mathbf{u}, t) \;, \end{aligned}$$

the following conditions hold for all t *where* $\mathbf{q}(t)$ *and* $\mathbf{u}(t)$ *are continuous:*

b) $\dfrac{\partial L}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t), \mathbf{q}(t), t) = 0$,

c) $\dot{\mathbf{p}}(t) = -\dfrac{\partial L}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}(t), \mathbf{q}(t), t)$,

d) the Hamiltonian is concave in $(\mathbf{x}, \mathbf{u})$, *and differentiable at* $(\mathbf{x}^*(t), \mathbf{u}^*(t))$,

e) $q_j(t) \geq 0$ *and* $= 0$ *if* $g_j(\mathbf{x}^*(t), \mathbf{u}^*(t), t) > 0$, *for all* $j = 1, \dots, s$,

f) g_j *is a quasi-concave function of* $(\mathbf{x}, \mathbf{u})$, *and differentiable at* $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ *for all* $j = 1, \dots, s$,

g) and for all admissible $\mathbf{x}(t)$, $\liminf_{t \to \infty} \mathbf{p}(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t)) \geq 0$,

then the pair $(\mathbf{x}^*(t), \mathbf{u}(t))$ *is catching-up-optimal.*

Applied to our problem, these theorems provide respectively Corollary 2.1 below and Corollary 2.2 in Section 2.3.4. In order to state them, we first give the detail of the correspondence between the notations of the theorem and that of our problem.

We have a state $\mathbf{x} = (X, Z, S)$ $(n = 3)$ and a control $\mathbf{u} = (y, x, s)$ $(r = 3)$. The cost function is $f_0 = e^{-\rho t}(u(x+y) - c_s s - c_x x - c_y y)$ and the dynamics f are specified by (2.2.2). The constraints are enumerated as (omitting the argument (X, Z, S, y, x, s, t)):

$$g_1 = y, \quad g_2 = \overline{y} - y, \quad g_3 = s, \quad g_4 = \zeta x - s,$$

$$\bar{g}_5 = X, \quad \bar{g}_6 = \overline{Z} - Z, \quad \bar{g}_7 = \overline{S} - S.$$

These correspond, respectively, to constraints (2.2.6) and (2.2.7) (g_1 and g_2), (2.2.8) and (2.2.9) (g_3 and g_4), (2.2.5), (2.2.3), and (2.2.4). We have $s' = 4$ and $s = 7$. There are no constraints *a priori* on the behavior of the state trajectory as $t \to \infty$. In other words, we take $\ell = m = 0$.

The constraints have some specific features: they are all linear, and they depend either on control variables, or state variables, but not both. As a consequence, partial derivatives are constant, some being null. Also, the constraints expressed in (2.3.18) and requirement i) of Theorem 2.1 involve disjoint sets of parameters $\beta^{\pm}_{kj}$: those can therefore be chosen independently.

Concretely, evaluating (2.3.18) we obtain the simpler requirement: for $i = 1, 2, 3$ (that is, for $x_i = X, Z, S$),

$$p_i(\tau_k^-) - p_i(\tau_k^+) \;=\; \sum_{j=5}^{7} (\beta^+_{kj} + \beta^-_{kj}) \frac{\partial g_j}{\partial x_i} \,. \tag{2.3.19}$$

Each state variable appears in exactly one of the constraints g_5, g_6 and g_7, which leads to:

$$p_1(\tau_k^-) - p_1(\tau_k^+) \;=\; (\beta^+_{k5} + \beta^-_{k5}), \quad p_2(\tau_k^-) - p_2(\tau_k^+) \;=\; -(\beta^+_{k6} + \beta^-_{k6}), \quad p_3(\tau_k^-) - p_3(\tau_k^+) \;=\; -(\beta^+_{k7} + \beta^-_{k7}).$$

Equivalently, since $\beta^{\pm}_{kj} \geq 0$ according to requirement j),

$$p_1(\tau_k^-) - p_1(\tau_k^+) \;\geq\; 0, \quad p_2(\tau_k^-) - p_2(\tau_k^+) \;\leq\; 0, \quad p_3(\tau_k^-) - p_3(\tau_k^+) \;\leq\; 0. \tag{2.3.20}$$

On the other hand, requirement i) boils down to:

$$\beta^{\pm}_k \cdot \frac{\partial g}{\partial \mathbf{u}} \cdot (\mathbf{u} - \mathbf{u}^*(\tau_k^{\pm})) \;=\; \sum_{j=1}^{4} \beta^{\pm}_{kj} \, \frac{\partial g_j}{\partial \mathbf{u}} \cdot (\mathbf{u} - \mathbf{u}^*(\tau_k^{\pm})) \;\leq\; 0 \;, \tag{2.3.21}$$

and this is satisfied with equality, choosing $\beta^{\pm}_{kj} = 0$, $j = 1, \ldots, 4$.

Corollary 2.1. *Assume there exist:*

- *a vector of continuous functions $(X, Z, S)(t)$, a bounded vector function $(y, x, s)(t)$, satisfying equations (2.2.2)–(2.2.9),*
- *a vector function $\lambda(t) = (\lambda_X, \lambda_Z, \lambda_S)(t)$ such that λ_X and λ_S are continuous and continuously differentiable, and λ_Z piecewise continuously differentiable, a piecewise-continuous vector function $\gamma(t) = (\underline{\gamma}_y, \overline{\gamma}_y, \gamma_s, \gamma_{sx}, \nu_Z, \nu_S, \nu_X)(t)$, satisfying equations (2.3.2)–(2.3.14) for all t, (2.3.9)–(2.3.11) for virtually every t, and conditions (2.3.15)–(2.3.17),*
- *a sequence of time instants $0 < \tau_1 < \ldots < \tau_N$, where $Z(\tau_k^-) < \overline{Z}$ and $Z(\tau_k^+) = \overline{Z}$, such that λ_Z is continuous except at the τ_k, and*

$$\lambda_Z(\tau_k^-) - \lambda_Z(\tau_k^+) \;\leq\; 0. \tag{2.3.22}$$

Then the pair $(\mathbf{x}^*(t), \mathbf{u}(t))$ is catching-up-optimal for the criterion (2.2.1).

Proof. We shall check the conditions of Theorem 2.1, using the correspondence of notation detailed above. Using Lemma 2.1, it is possible to choose the set U of Theorem 2.1 as $U = \{(y, x, s) \in \mathbb{R}^3_+ \mid x + y \geq \widetilde{y}\}$. It is a convex set.

The pair $((X, Z, S), (y, x, s))$ is admissible, by assumption. In addition, we define the vector functions $\mathbf{p}(t) = e^{-\rho t}\lambda(t)$ and $\mathbf{q} = e^{-\rho t}\gamma(t)$. By assumptions on λ and γ, $\mathbf{p}$ is piecewise continuous and piecewise continuously differentiable, and $\mathbf{q}$ is piecewise-continuous. We now check *a*) to *k*).

a): given the definition of f_0, continuity and differentiability are satisfied from Assumption 1. Then we have

$$\frac{\partial f_0}{\partial y} = e^{-\rho t}(u'(x+y) - c_y), \quad \frac{\partial f_0}{\partial x} = e^{-\rho t}(u'(x+y) - c_x), \quad \frac{\partial f_0}{\partial s} = e^{-\rho t}(-c_s).$$

By Assumption 1, and thanks to the fact that $x + y > 0$ on the set U, these derivatives exist and are continuous; f is linear hence C^∞; this is the case also for constraints g_j, $j = 1, \ldots, 4$;

b): the constraints $\bar{g}_j$, $j = 5, 6, 7$ are also linear, hence C^∞;

c): the constraints are all linear, hence concave, hence quasi-concave;

d): the inequality of this requirement is satisfied with equality, since Equations (2.3.2)–(2.3.4) are equivalent to the assumption $\partial L/\partial \mathbf{u} = 0$;

e): is also satisfied by assumption, since Equations (2.3.9)–(2.3.11) are equivalent to the assumption $\dot{\mathbf{p}} = -\partial L/\partial \mathbf{x}$;

f): the Hamiltonian of the problem is given by the two first lines in the Lagrangian (2.3.1). It is a linear, hence concave, function of the state (X, Z, S) (although not strictly concave), and a concave function of the control (y, x, s), thanks to the concavity of the function $u(\cdot)$ in Assumption 1. The Hamiltonian is therefore a concave function of $(\mathbf{x}, \mathbf{u})$;

g): is satisfied, consequence of conditions (2.3.5)–(2.3.14);

h): by assumption, λ_X and λ_S are continuous, hence (2.3.18) (or the equivalent (2.3.19)) holds for $i = 1, 3$ by choosing $\beta^\pm_{kj} = 0$. By assumption (2.3.22) on the jumps of λ_Z, it is sufficient to choose $\beta^+_{k6} = -e^{-\rho\tau_k}(\lambda_Z(\tau_k^-) - \lambda_Z(\tau_k^+)), \beta^-_{k6} = 0$ in order to have $\beta^\pm_{k6} \geq 0$ and comply with (2.3.18);

i): is satisfied with equality by setting $\beta^\pm_{kj} = 0$, $j = 1, \ldots, 4$ (see the preliminary discussion);

j): is satisfied trivially for $j = 1, \ldots, 4$ by the choice made in *i*). Likewise for $j = 5, 7$ by picking $\beta^\pm_{kj} = 0$. Given the choices of $\beta^\pm_{k6}$ in *h*), and the assumption on jump instants τ_k which specifies that the constraint is always bound after the jump, we indeed have $\beta^\pm_{k6} \geq 0$ and $\beta^-_{k6} = 0$ since $Z(\tau_k^-) < \overline{Z}$;

k): since the state variables X, Z and S are bounded by the system of constraints,[6] Conditions (2.3.15)–(2.3.17) imply respectively

$$\lim_{t\to\infty} e^{-\rho t}\lambda_X(t) = \lim_{t\to\infty} p_1(t) = 0, \quad \lim_{t\to\infty} p_2(t) = 0, \quad \lim_{t\to\infty} p_3(t) = 0.$$

This in turn implies that $\lim_{t\to\infty} \mathbf{p}(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*(t)) = 0$ for every admissible trajectory $\mathbf{x}$, since by the boundedness assumption on controls, the difference $\mathbf{x}(t) - \mathbf{x}^*(t)$ is also bounded.

□

Our task is therefore to exhibit solutions to the first-order conditions, with bounded controls, which are continuous, or if not continuous, which satisfy the jump condition (2.3.22).

[6] This argument holds *sensu stricto* when $\overline{S}$ and $\overline{Z}$ are finite. However, it holds also when $\overline{S} = +\infty$ and $\beta > 0$, because there is a finite admissible domain, see Section 2.3.4.

2.3.4 The admissible domain of S and Z

Since $\beta > 0$, the model exhibits a *viability* or *controllability* problem that we study in this section.

Assume that for some reason, $x(t) = s(t) = 0$ over some interval of time. Then the dynamics of S and Z are given by:

$$\dot{S}(t) \; = \; -\beta S(t) \quad \text{and} \quad \dot{Z}(t) \; = \; \beta S(t) - \alpha Z(t) \; .$$

Let t^0 be some time instant in this interval and let us denote by S^0 and Z^0 the stocks of S and Z at this time: $S^0 \equiv S(t^0)$ and $Z^0 \equiv Z(t^0)$. Integrating the above system, we obtain for all t (in the case $\alpha \neq \beta$; see Footnote 7 for the case $\alpha = \beta$):

$$S(t) \; = \; S^0 e^{-\beta(t-t^0)} \tag{2.3.23}$$

$$Z(t) \; = \; Z^0 e^{-\alpha(t-t^0)} - S^0 \frac{\beta}{\alpha-\beta}\left(e^{-\alpha(t-t^0)} - e^{-\beta(t-t^0)}\right) \; . \tag{2.3.24}$$

Eliminating t with (2.3.23), we get the family of trajectories in the (S, Z) space:

$$Z(S; S^0, Z^0) \; = \; \left(\frac{S}{S^0}\right)^{\alpha/\beta}\left(Z^0 - \frac{\beta}{\alpha-\beta}S^0\right) \; + \; \frac{\beta}{\alpha-\beta}S \; .$$

These curves depend upon α and β and, structurally, only upon α/β. As a function of S, Z is first increasing and next decreasing whatever $\alpha > 0$ and $\beta > 0$ may be. The maximum is attained when $\dot{Z} = 0$, that is, $Z = \beta S/\alpha$. The family of these curves is illustrated in Figure 2.3. The movement is going from the right to the left though time. Under the line $Z = \beta S/\alpha$, the leaks flow βS is higher than the self-regeneration flow αZ so that the atmospheric stock of pollutant increases, whereas above the line the reverse holds and the atmospheric stock decreases.

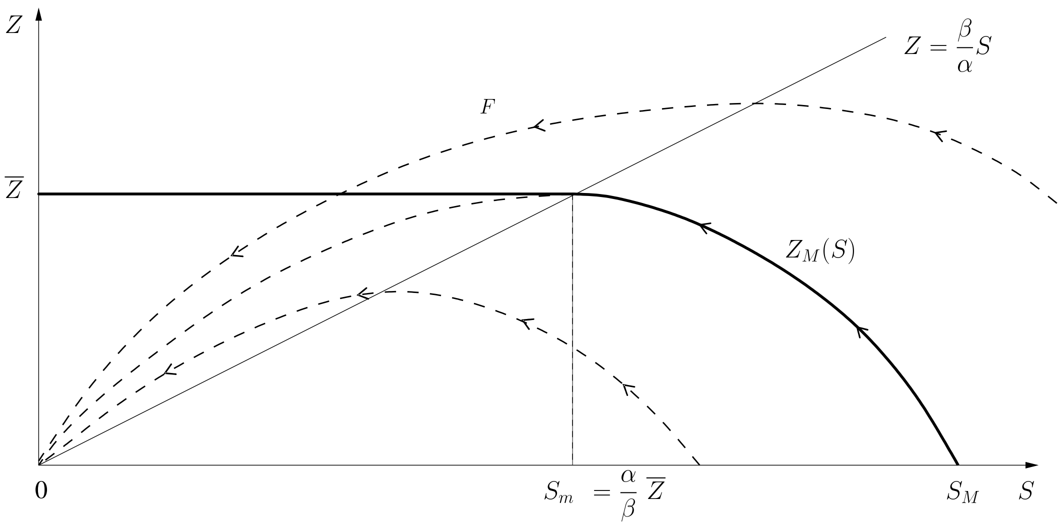

Figure 2.3: Admissible (S, Z) pairs

Among these trajectories, let $Z_M(S)$ be the one, the maximum of which is equal to $\overline{Z}$, S_m the value of S for which this maximum is attained, and S_M the (strictly) positive value of S for which $Z_M(S) = 0$. Clearly, $S_M > S_m$. Given that the maxima of $Z(\cdot)$ are located along the line $Z = (\alpha/\beta)S$, we get for $Z = \overline{Z}$:

$$S_m \; = \; \frac{\alpha}{\beta}\,\overline{Z} \; . \tag{2.3.25}$$

Then

$$Z_M(S) \; = \; Z(S; S_m, \overline{Z}) \; = \; \frac{\beta}{\alpha-\beta}\left(S - \overline{Z}\left(\frac{S}{S_m}\right)^{\alpha/\beta}\right) \; .$$

It follows that $S_M = \overline{Z}(\alpha/\beta)^{\alpha/(\alpha-\beta)}$, and it can be verified that $S_M > S_m$ for all values of α and β.[7]

For any $S \in (S_m, S_M]$, the control vector $(s, \zeta x - s)$ points *outwards*, and it is easy to see that for any initial position located above the curve $Z = Z_M(S)$, and for any control, the trajectory will necessarily exit the domain $\{Z \leq \overline{Z}\}$. Such a trajectory is not viable. Likewise, if a non-zero control is applied at any point of the curve $(S, Z_M(S))$, then the trajectory will necessarily exit the domain $\{Z \leq \overline{Z}\}$, whatever control is applied later on.[8]

Therefore, the set of *viable* initial states (S^0, Z^0) is delimited by the constraint

$$Z \;\leq\; \widetilde{Z}(S) \tag{2.3.26}$$

where the function $\widetilde{Z}$ is defined on $[0, S_M]$ as:

$$\widetilde{Z}(S) \;=\; \begin{cases} \overline{Z}, & 0 \leq S \leq S_m \\ Z_M(S), & S_m \leq S \leq S_M. \end{cases} \tag{2.3.27}$$

This function is continuous since $Z_M(S_m) = \overline{Z}$, decreasing and concave. It is differentiable because $Z'_M(S_m) = 0$. However, the derivative $\widetilde{Z}'(S)$ is not differentiable at $S = S_m$.

Since this viability constraint holds for every admissible trajectory, it is possible to add it to the optimization problem (2.2.1)–(2.2.9) without changing its solution. Doing so, we shall be able to handle the situation where the optimal trajectory lies on the boundary of the domain. This situation cannot be handled by Theorem 2.2 because, as it turns out, the evolution of adjoint variables is not defined by (2.3.9)–(2.3.11).

Replacing the constraint (2.2.3) by the more general (2.3.26), rewritten as

$$\widetilde{Z}(S) - Z \;\geq\; 0 \tag{2.3.28}$$

entails the following modifications. In the Lagrangian (2.3.1), the term "$\nu_Z[\overline{Z} - Z]$" must be replaced with "$\nu_Z[\widetilde{Z}(S) - Z]$". Condition (2.3.13) then becomes

$$\nu_Z \geq 0, \qquad \widetilde{Z}(S) - Z \geq 0 \text{ and } \nu_Z[\widetilde{Z}(S) - Z] = 0\ , \tag{2.3.29}$$

and Equation (2.3.11) must be replaced with

$$\dot{\lambda}_S \;=\; \rho\lambda_S - \frac{\partial L}{\partial S} \quad\Longleftrightarrow\quad \dot{\lambda}_S \;=\; (\rho+\beta)\lambda_S - \beta\lambda_Z + \nu_S - \nu_Z\widetilde{Z}'(S)\ . \tag{2.3.30}$$

In the correspondence established with the notation of Theorem 2.2, we have to set

$$g_6 \;=\; \widetilde{Z}(S) - Z\ .$$

This constraint is not linear anymore. It is continuous, differentiable, but not C^2 because the derivative is not continuous at $S = S_m$. It is not possible to apply Theorem 2.1 to this variant of the problem. When applied to state variable $x_3 = S$, Condition (2.3.19) yields now

$$p_3(\tau_k^-) - p_3(\tau_k^+) \;=\; \widetilde{Z}'(S(\tau_k))(\beta_{k6}^+ + \beta_{k6}^-) - (\beta_{k7}^+ + \beta_{k7}^-)$$

but since $\widetilde{Z}'(\cdot) \leq 0$, this still means $p_3(\tau_k^-) - p_3(\tau_k^+) \leq 0$. We aim at solutions where p_3 is continuous (equivalently, λ_S continuous) anyway.

Theorem 2.2 and Lemma 2.1 yield then the following corollary.

Corollary 2.2. *Assume there exist:*

- *a vector of continuous functions $(X, Z, S)(t)$, a bounded vector function $\mathbf{u}(t) = (y, x, s)(t)$, satisfying equations (2.2.2), (2.3.28), (2.2.4)–(2.2.9), and $x(t) + y(t) \geq \widetilde{y}$ for all t,*

---

[7] These formulas must be modified in the limit case $\alpha = \beta$. In that case, we have $\overline{Z} = S_m$, then

$$Z(t) \;=\; Z^0 e^{-\alpha(t-t^0)} + S^0\alpha(t-t^0)e^{-\alpha(t-t^0)} \qquad S(t) \;=\; S^0 e^{-\alpha(t-t^0)} \quad \text{and} \quad Z_M(S) \;=\; S \;-\; S\,\log\frac{S}{S_m}\ .$$

The value where this function vanishes is $S_M = eS_m$.

[8] This problem obviously occurs only if $\overline{S} > S_m$.

- *a vector function $\lambda(t) = (\lambda_X, \lambda_Z, \lambda_S)(t)$ which is continuous and continuously differentiable, a piecewise-continuous vector function $\gamma(t) = (\underline{\gamma}_y, \overline{\gamma}_y, \gamma_s, \gamma_{sx}, \nu_Z, \nu_S, \nu_X)(t)$, satisfying equations (2.3.2)–(2.3.12), (2.3.29), (2.3.14)–(2.3.10) and (2.3.30) for all t where $\mathbf{u}(t)$ and $\gamma(t)$ are continuous, and conditions (2.3.15)–(2.3.17).*

Then the pair $(\mathbf{x}^*(t), \mathbf{u}(t))$ is catching-up-optimal for the criterion (2.2.1).

Proof. We check the conditions of Theorem 2.2 using again the correspondence of notations established in Section 2.3.3. In addition, Theorem 2.2 introduces the set

$$\begin{aligned} A(t) &= \{(\mathbf{x}, \mathbf{u}) : g_j(\mathbf{x}, \mathbf{u}, t) \geq 0, j = 1, \ldots, s\} \\ &= \{(X, Z, S, y, x, s) : Z \leq \overline{Z}, S \leq \overline{S}, X \geq 0, 0 \leq y \leq \overline{y}, 0 \leq s \leq \zeta x, x + y \geq \widetilde{y}\} \,. \end{aligned}$$

Adding the constraint $x + y \geq \widetilde{y}$ to the problem is possible by virtue of Lemma 2.1, and it excludes the possibility that $x + y = 0$ from the set $A(t)$. Then f_0 is not only continuous, but also differentiable on $A(t)$ even if $u'(0)$ is allowed to be infinite. Since f is clearly diferentiable, we see that the Hamiltonian is differentiable for every admissible control, so *a fortiori* at any candidate optimal control $(\mathbf{x}^*(t), \mathbf{u}^*(t))$ as in Condition d) of Theorem 2.2. This condition and Condition a) are therefore satisfied.

Let us now check the remaining conditions of Theorem 2.2. We have already remarked the continuity of f_0 and f. Condition d) holds as in Theorem 2.1 f) because the Hamiltonian is not affected by this change in the constraints. It is therefore still is concave and, as observed above, differentiable as a function of (X, Z, S, y, x, s).

Conditions b), c) and e) holds by construction. Condition f) holds due to the concavity of $\widetilde{Z}(S)$. Finally, Condition g) holds as for Condition k) of Theorem 2.1 in the proof of Corollary 2.1. □

2.3.5 On the lack of necessary conditions

The usual practice is to consider that the first-order conditions listed in Section 2.3.1 are in fact necessary, with the adjoint variables in some precise class of functions (continuous, piecewise continuous, ...). For models with state constraints, this actually requires that these constraints be "qualified".

Consider for instance Theorem 9, Chapter 6, page 381 of Seierstad & Sydsæter (1987), with the same notation as in Section 2.3.3. The problem has here seven constraints, but if $(X(t), S(t), Z(t)) = (X(t), S_m, \overline{Z})$ (with $X(t) > 0$) and $(y(t), x(t), s(t)) = (0, \overline{x}, \zeta\overline{x})$ is a candidate state/control pair, only constraints

$$g_1 = y, \qquad g_4 = \zeta x - s, \quad \text{and} \quad \bar{g}_6 = \overline{Z} - Z$$

are active. According to the theorem, the third one must be converted into

$$h_6(y, x, s, X, S, Z) \;=\; \frac{\partial \bar{g}_6}{\partial X}(-x) + \frac{\partial \bar{g}_6}{\partial S}(-\beta S + s) + \frac{\partial \bar{g}_6}{\partial Z}(-\alpha Z + \beta S + \zeta x - s) \;=\; \alpha Z - \beta S - \zeta x + s \,.$$

The following matrix $M(t)$ should then have rank 3:

$$M(t) \;=\; \begin{bmatrix} \partial g_1/\partial y & \partial g_1/\partial x & \partial g_1/\partial s \\ \partial g_4/\partial y & \partial g_4/\partial x & \partial g_4/\partial s \\ \partial h_6/\partial y & \partial h_6/\partial x & \partial h_6/\partial s \end{bmatrix} \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & -1 \\ 0 & -\zeta & 1 \end{bmatrix} .$$

However, the rank of $M(t)$ is clearly 2. This constraint qualification condition does not hold. The condition of Theorem 8, Chapter 6, page 378 of Seierstad & Sydsæter (1987) fails as well.

Chapter 3

Preliminary results

3.1 Introduction to the solution

The central object of our analysis is the "phase", which we define as a piece of optimal path for which the set of active constraints on states or controls is constant. A complete optimal trajectory is necessarily decomposed into a succession of such phases. The method consists then in "gluing" together pieces of trajectory, each one being in some phase.

This chapter is devoted to the individual analysis of the different possible phases. The assembly of pieces of trajectories will be done in Chapter 4 for a simplification of the model. The complete solution for the model presented in Chapter 2 is left for future research.

The combinatorics of the exploration of phases is quite large *a priori*. Constraints (2.2.3)–(2.2.5) provide 2 situations each, constraints (2.2.6) and (2.2.7) provide 3, and the set of constraints (2.2.8)–(2.2.9) provide 4 distinct situations, for a potential total of 96 phases.

We choose to disregard the limit $\overline{y}$ on the flow of renewable resource y, as well as capacity constraints $\overline{S}$ on the reservoir S. This simplification will allow us to concentrate on the importance of the self-regeneration rate α, the leakage rate β and the capture cost c_s on the shape of optimal extraction paths. Indeed, in Chapter 4 we will provide a complete classification of optimal trajectories, according to the position of c_s with respect to various thresholds defined with the other parameters.

Ignoring the constraints $\overline{y}$ and $\overline{S}$ reduces the number of possible phases to 32. We will see however than only 9 phases are actually useful in the construction of optimal trajectories.

For this restricted problem, Corollary 2.1 and Corollary 2.2 take the following form. The proof for these variants is easily adapted from the original proofs with the aid of Footnote 6 on page 14.

Corollary 3.1. *Assume there exist:*

- *a vector of continuous functions* $(X, Z, S)(t)$*, a vector function* $(y, x, s)(t)$*, satisfying equations (2.2.2), (2.2.3), (2.2.5), (2.2.6), (2.2.8) and (2.2.9),*
- *a vector function* $\lambda(t) = (\lambda_X, \lambda_Z, \lambda_S)(t)$ *such that* λ_X *and* λ_S *are continuous and continuously differentiable, and* λ_Z *piecewise continuously differentiable, a piecewise-continuous vector function* $\gamma(t) = (\gamma_y, \gamma_s, \gamma_{sx}, \nu_Z, \nu_X)(t)$*, satisfying equations (2.3.2)–(2.3.13) for all t (with* $\gamma_Y = 0$*), (2.3.9)–(2.3.11) for virtually every t (with* $\nu_S = 0$*), and conditions (2.3.15)–(2.3.17),*
- *a sequence of time instants* $0 < \tau_1 < \ldots < \tau_N$*, where* $Z(\tau_k^-) < \overline{Z}$ *and* $Z(\tau_k^+) = \overline{Z}$*, such that* λ_Z *is continuous except at the* τ_k*, and*

$$\lambda_Z(\tau_k^-) - \lambda_Z(\tau_k^+) \ \leq \ 0. \tag{3.1.1}$$

Then the pair $(\mathbf{x}^*(t), \mathbf{u}(t))$ *is catching-up-optimal for the criterion (2.2.1).*

Corollary 3.2. *Assume there exist:*

- *a vector of continuous functions* $(X, Z, S)(t)$*, a vector function* $\mathbf{u}(t) = (y, x, s)(t)$*, satisfying equations (2.2.2), (2.3.28), (2.2.5), (2.2.6), (2.2.8) and (2.2.9),*
- *a vector function* $\lambda(t) = (\lambda_X, \lambda_Z, \lambda_S)(t)$ *which is continuous and continuously differentiable, a piecewise-continuous vector function* $\gamma(t) = (\gamma_y, \gamma_s, \gamma_{sx}, \nu_Z, \nu_X)(t)$*, satisfying equations (2.3.2)–(2.3.12) (with* $\gamma_Y = 0$*), (2.3.29), (2.3.9)–(2.3.11) (with* $\nu_S = 0$*) and (2.3.30) for all* t *where* $\mathbf{u}(t)$ *and* $\gamma(t)$ *are continuous, and conditions (2.3.15)–(2.3.17).*

Then the pair $(\mathbf{x}^*(t), \mathbf{u}(t))$ *is catching-up-optimal for the criterion (2.2.1).*

Guided by these theoretical results, we look for trajectories which are continuous inside each phase: the only discontinuities which we will consider are related with the change in the status of the constraint $Z = \overline{Z}$: in some situations, λ_Z will be allowed to jump when this constraint becomes active.

In the different sections of this chapter, we analyze separately the dynamics of each phase. We adopt the following common notation: t^0 denotes an arbitrary time instant at which the trajectory is within the phase under study. The corresponding values of the state, adjoint variables and multipliers are denoted with the same superscript as in X^0, S^0, Z^0, λ_Z^0 etc. We express the value of the different relevant trajectories as a function of t and these "initial" values. They hold whether t is smaller or larger than t^0, as long as both time instants lie in an interval where the system stays in the phase without interruption.

We begin with general observations about the phases which are "interior" with respect to state constraints. In Section 3.2, we characterize the evolution of adjoint variables in such phases. Next, in Section 3.3, we simplify the problem by ruling out certain configurations for the optimal control. Then, we give the details of state and adjoint variable trajectories in the remaining phases. We start with phases located in the interior of the domain, in Section 3.4. Finally, we turn to the boundary, and describe phases such that the atmospheric stock has reached its ceiling (Section 3.5).

3.2 The system in the interior

When no state constraint is active, the dynamics of the adjoint variables take a particularly simple form, which yields closed-form expressions.

The interior of the domain, which we will denote by $\mathcal{D}$, is defined by the set of strict inequalities:

$$\mathcal{D} \;=\; \left\{(X, S, Z) \in \mathbb{R}^3 \mid 0 \;<\; X(t)\;, \quad 0 \;<\; S(t) \quad 0 \;<\; Z(t) \;<\; \widetilde{Z}(S(t))\right\}\;, \tag{3.2.1}$$

where the function $\widetilde{Z}$ has been defined in (2.3.27). At time instants where the state lies in $\mathcal{D}$, the Lagrange multipliers ν_X, ν_S and ν_Z vanish because of (2.3.12)–(2.3.14), and the dynamics of adjoint variables (2.3.9)–(2.3.11) reduce to

$$\left\{\begin{array}{rcl} \dot{\lambda}_X &=& \rho\lambda_X \\ \dot{\lambda}_Z &=& (\rho+\alpha)\lambda_Z \\ \dot{\lambda}_S &=& (\rho+\beta)\lambda_S - \beta\lambda_Z\;. \end{array}\right. \tag{3.2.2}$$

It follows that for every t, t^0 in any period where $X > 0$,

$$\lambda_X(t) \;=\; \lambda_X^0\; e^{\rho(t-t^0)}\;. \tag{3.2.3}$$

3.2.1 Dynamics of the adjoint variables

We concentrate now on λ_S and λ_Z. Integrating the dynamical system:

$$\left\{\begin{array}{rcl} \dot{\lambda}_Z &=& (\rho+\alpha)\lambda_Z \\ \dot{\lambda}_S &=& (\rho+\beta)\lambda_S - \beta\lambda_Z \end{array}\right.$$

under initial conditions at t^0 yields:

$$\lambda_Z(t) = \lambda_Z^0 e^{(\rho+\alpha)(t-t^0)} \tag{3.2.4}$$

$$\lambda_S(t) = \lambda_S^0 e^{(\rho+\beta)(t-t^0)} - \frac{\beta}{\alpha-\beta}\lambda_Z^0 \left(e^{(\rho+\alpha)(t-t^0)} - e^{(\rho+\beta)(t-t^0)}\right) . \tag{3.2.5}$$

The pair $(\lambda_S(t), \lambda_Z(t))$ therefore lies on the curve:

$$\lambda_S = \lambda_S^0 \left(\frac{\lambda_Z}{\lambda_Z^0}\right)^{\frac{\rho+\beta}{\rho+\alpha}} - \frac{\beta}{\alpha-\beta}\left(\lambda_Z - \lambda_Z^0 \left(\frac{\lambda_Z}{\lambda_Z^0}\right)^{\frac{\rho+\beta}{\rho+\alpha}}\right) .$$

When $\alpha = \beta$, these formulas must be modified as follows:

$$\lambda_S(t) = \left(\lambda_S^0 + \alpha(t-t^0)\lambda_Z^0\right) e^{(\rho+\alpha)(t-t^0)}$$

$$\lambda_S = \frac{\lambda_Z}{\lambda_Z^0}\left(\lambda_S^0 - \lambda_Z^0 \frac{\alpha}{\rho+\alpha}\log\left(\frac{\lambda_Z}{\lambda_Z^0}\right)\right) .$$

3.2.2 Dynamics of ratios

Define the ratio variables:

$$r(t) := \frac{Z(t)}{S(t)} \qquad r_\lambda(t) := \frac{\lambda_S(t)}{\lambda_Z(t)} .$$

As above, assume for adjoint variables that the system is in the interior (phases that will be named "A", "B" and "L" later on). For the state variables, assume that no control is applied to the system (phases that will be named "L", "U" or "T" later on). It is straightforward to check that the ratios thus defined satisfy the autonomous, first-order differential equations:

$$\dot{r} = (\beta-\alpha)r + \beta \qquad \dot{r}_\lambda = (\beta-\alpha)r_\lambda - \beta ,$$

which do not depend on ρ. Integrating leads to the solutions:

$$r(t) = \left(r(t_0) + \frac{\beta}{\beta-\alpha}\right) e^{(\beta-\alpha)(t-t_0)} - \frac{\beta}{\beta-\alpha}$$

$$r_\lambda(t) = \left(r_\lambda(t_0) - \frac{\beta}{\beta-\alpha}\right) e^{(\beta-\alpha)(t-t_0)} + \frac{\beta}{\beta-\alpha} .$$

When $\alpha = \beta$, these formulas take the form:

$$r(t) = r(t^0) + \alpha(t-t^0) \qquad r_\lambda(t) = r_\lambda(t^0) - \alpha(t-t^0) .$$

As an application of these formulas, observe that the time necessary for the system to go from a position (S^0, Z^0) to (S^1, Z^1) depends only on the ratios $r^0 = Z^0/S^0$ and $r^1 = Z^1/S^1$. The value of this duration is given by:

$$t_1 - t_0 = \frac{1}{\beta-\alpha}\log\left(\frac{r^1 + \frac{\beta}{\beta-\alpha}}{r^0 + \frac{\beta}{\beta-\alpha}}\right) = \frac{1}{\beta-\alpha}\log\left(\frac{(\beta-\alpha)r^1+\beta}{(\beta-\alpha)r^0+\beta}\right) ,$$

when $\alpha \neq \beta$, and $t_1 - t_0 = (r^1 - r^0)/\alpha$ when $\alpha = \beta$. In particular, when $Z(t_0) = 0$, we have $r^0 = 0$ and:

$$t_1 - t_0 = \frac{1}{\beta-\alpha}\log\left(\frac{\beta-\alpha}{\beta}r^1 + 1\right) .$$

Likewise for adjoint variables: the time necessary for the system to go from a position where the ratio is $r_\lambda^0 = \lambda_S^0/\lambda_Z^0$ to one where the ratio is $r_\lambda^1 = \lambda_S^1/\lambda_Z^1$ is given by:

$$t_1 - t_0 = \frac{1}{\beta-\alpha}\log\left(\frac{(\beta-\alpha)r_\lambda^1 - \beta}{(\beta-\alpha)r_\lambda^0 - \beta}\right) ,$$

when $\alpha \neq \beta$, and $t_1 - t_0 = -(r_\lambda^1 - r_\lambda^0)/\alpha$ when $\alpha = \beta$.

3.2.3 Invariants

Another formulation of the previous results is that the following quantities are invariant over time:

$$\left(\frac{Z(t)}{S(t)} + \frac{\beta}{\beta-\alpha}\right) e^{-(\beta-\alpha)t}, \qquad \left(\frac{\lambda_S(t)}{\lambda_Z(t)} - \frac{\beta}{\beta-\alpha}\right) e^{-(\beta-\alpha)t},$$

as long as the state remains in Phase L, T or U for the first quantity, or in Phase A, B or L for the second one. As a consequence, the line $\{r_\lambda = \beta/(\beta-\alpha)\} = \{(\beta-\alpha)\lambda_S = \beta\lambda_Z\}$ is invariant, and so is the sign of $r_\lambda - \beta/(\beta-\alpha)$. If $\beta > \alpha$, trajectories starting with $r_\lambda(t^0) > \beta/(\beta-\alpha)$ go to $+\infty$, and trajectories with $r_\lambda(t^0) < \beta/(\beta-\alpha)$ go to $-\infty$, as $t \to +\infty$. All trajectories tend to $\beta/(\beta-\alpha) > 0$ when $t \to -\infty$. If $\beta < \alpha$, the converse situation occurs: all trajectories tend to $\beta/(\beta-\alpha) < 0$ when $t \to +\infty$, and the limit when $t \to -\infty$ is $\pm\infty$ with the sign of $r_\lambda(t^0) - \beta/(\beta-\alpha)$.

The following quantities are also constant on trajectories in the interior of the domain $\mathcal{D}$ when it is optimal to apply no control (Phase L):

$$(S(t)\lambda_S(t) + Z(t)\lambda_Z(t))\, e^{-\rho t}$$

$$(-\beta S(t)\lambda_S(t) + (-\alpha Z(t) + \beta S(t))\lambda_Z(t))\, e^{-\rho t}\ .$$

Some of these results will be useful for proving that certain trajectories satisfy certain constraints, for instance in Section 4.2.1, or when applying transversality conditions, see Section 3.5.5.

3.3 Elimination of impossible phases

When the state of the system is not bound by a constraint, the structure of the cost function allows to eliminate controls that are necessarily suboptimal. This allows to eliminate certain phases from the construction of a solution.

Our first result is a sort of "bang-bang" principle for the capture control s in the interior of the domain.

Lemma 3.1. *Assume that $c_s \neq 0$. Consider a piece of optimal trajectory located in the interior of the domain $\mathcal{D}$, such that $x(t) > 0$. Then for every time instant t, either $s(t) = 0$, or $s(t) = \zeta x(t)$.*

Proof. Assume by contradiction that $0 < s(t) < \zeta x(t)$. Then by (2.3.5) and (2.3.6), we have $\gamma_s(t) = \gamma_{sx}(t) = 0$. Then, (2.3.2) reduces to:

$$-\,c_s - \lambda_Z(t) + \lambda_S(t) \;=\; 0\ . \tag{3.3.1}$$

Differentiating, we must have, over some time interval, $\dot\lambda_Z(t) = \dot\lambda_S(t)$. Using (2.3.10) and (2.3.11), this implies in turn that

$$(\rho+\alpha)\lambda_Z \;=\; (\rho+\beta)\lambda_S - \beta\lambda_Z \tag{3.3.2}$$

because $\nu_Z = 0$. Replacing in (3.3.1), we find that necessarily, $(\rho+\beta)c_s = \alpha\lambda_Z$. If $\alpha = 0$, this is not possible since $c_s \neq 0$. If $\alpha > 0$, the adjoint variables are necessarily constant and equal to:

$$\lambda_Z \;=\; \frac{\rho+\beta}{\alpha}\, c_s \qquad \lambda_S \;=\; \frac{\rho+\beta+\alpha}{\alpha}\, c_s\ .$$

However, these functions do not solve the differential system (3.2.2), unless $c_s = 0$. This is excluded by assumption, hence the contradiction. □

We observe that in the case $c_s = 0$, the reasoning above leads to the conclusion that $\lambda_S = \lambda_Z$ if $\alpha = 0$ and $\lambda_S = \lambda_Z = 0$ if $\alpha > 0$. We discuss further this situation in Section 4.6.4.

Next, we rule out the possibility that both the renewable resource and the nonrenewable resource be used at the same time.

Lemma 3.2. *Assume that $c_s > 0$. Consider a piece of optimal trajectory located in the interior of the domain $\mathcal{D}$. Then either $x(t) > 0$ or $y(t) > 0$ but not both.*

Proof. Assume by contradiction that $x(t) > 0$ and $y(t) > 0$. Then $\gamma_y(t) = 0$ and the first-order conditions (2.3.2)–(2.3.4) reduce to: $x + y = \widetilde{y}$ and

$$0 = -c_s - \lambda_Z + \lambda_S + \gamma_s - \gamma_{sx} \qquad (3.3.3)$$

$$0 = c_y - c_x - \lambda_X + \zeta\lambda_Z + \zeta\gamma_{sx} \, . \qquad (3.3.4)$$

According to Lemma 3.1 (which is applicable since $x > 0$ and $c_s > 0$), either $s = 0$ and $\gamma_{sx} = 0$, or $s = \zeta x$ and $\gamma_s = 0$. In the first case, differentiating Equation (3.3.4) gives $\dot{\lambda}_X = \zeta\dot{\lambda}_Z$ or equivalently with (3.2.2): $\rho\lambda_X = \zeta(\rho + \alpha)\lambda_Z$. Then the adjoint variables are necessarily constant and equal to

$$\lambda_Z = \frac{c_y - c_x}{\alpha\zeta} \qquad \lambda_X = \frac{\rho + \alpha}{\rho}\,\frac{c_y - c_x}{\alpha} \, .$$

However, these functions do not solve the differential system (3.2.2): a contradiction.

In the second case, Equation (3.3.3) provides the identity $\lambda_Z + \gamma_{sx} = \lambda_S - c_s$, and replacing this into (3.3.4) yields:

$$0 = c_y - c_x - \zeta c_s - \lambda_X + \zeta\lambda_S \, .$$

Then $\dot{\lambda}_X = \zeta\dot{\lambda}_S = \zeta(\rho+\beta)\lambda_S - \zeta\beta\lambda_Z = \rho\lambda_X$. Eliminating λ_X between these equations, we arrive successively at: $C := c_y - c_x - \zeta c_s = \zeta(\lambda_S - \lambda_Z)$, $\dot{\lambda}_S = \dot{\lambda}_Z$, $(\rho + \alpha + \beta)\lambda_Z = (\rho + \beta)\lambda_S$. We have three linear algebraic equations linking λ_X, λ_S and λ_Z. If $\alpha \neq 0$, this linear system has a unique solution providing three constant functions, all proportional to C. But the unique constant solution to (3.2.4)–(3.2.5) is null. This entails $\gamma_{sx} = -c_s < 0$, which is not consistent. If $\alpha = 0$, it follows that $\lambda_Z = \lambda_S$. But this also implies $\gamma_{sx} = -c_s < 0$. We reach a contradiction in every case. □

3.4 Dynamics in interior phases

Given Lemmas 3.1 and 3.2, the optimal control on an interior piece of trajectory reduces to one of the three alternatives: either $y = 0$, $s = 0$, $x > 0$, or $y = 0$, $s = \zeta x$, $x > 0$, or $y = \widetilde{y}$, $x = s = 0$.

We name the first situation Phase "A": it is characterized by the absence of constraints on the state, zero capture and exclusive consumption of nonrenewable energy.

We name the second situation Phase "B": it is characterized by the absence of constraints on the state, total capture of the emissions due to nonrenewable energy.

The third situation is called Phase "L".

We analyze the dynamics of the system in these three phases.

3.4.1 Dynamics when capture is nil (Phase A)

Phase A corresponds to the situation where the resource is not exhausted ($X(t) > 0$), the ceiling is not reached ($Z(t) < \overline{Z}$), and no sequestration occurs ($s(t) = 0$). See Appendix A.1 on page 76.

Consumption is directly given by the first order equation (2.3.3):

$$x = q^d(c_x + \lambda_X - \zeta\lambda_Z) \qquad (3.4.1)$$

and the value of the adjoint variable $\lambda_X(t)$ is known from (3.2.3), and that of $\lambda_Z(t)$ from (3.2.4):

$$\lambda_X(t) = \lambda_X^0 e^{\rho(t-t^0)} \qquad \lambda_Z(t) = \lambda_Z^0 e^{(\rho+\alpha)(t-t^0)} \, .$$

The integration of the dynamical system for the state variables gives:

$$X(t) = X^0 - \int_t^{t^0} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta\lambda_Z(u))\mathrm{d}u \qquad (3.4.2)$$

$$Z(t) = Z^0 e^{-\alpha(t-t^0)} + S^0 \frac{\beta}{\alpha - \beta}\left(e^{-\beta(t-t^0)} - e^{-\alpha(t-t^0)}\right)$$

$$
+ \ \zeta \int_{t^0}^{t} e^{-\alpha(t-u)} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta \lambda_Z(u)) \mathrm{d}u \tag{3.4.3}
$$

$$
S(t) \ = \ S^0 \ e^{-\beta(t-t^0)} \ . \tag{3.4.4}
$$

3.4.2 Dynamics when capture is maximal (Phase B)

Phase B corresponds to the situation where the resource is not exhausted ($X(t) > 0$), the ceiling is not reached ($Z(t) < \overline{Z}$), and maximal sequestration occurs ($s(t) = \zeta x(t)$). See Appendix A.2 on page 77.

Consumption is directly given by the first order equations:

$$
x \ = \ q^d(c_x + \lambda_X - \zeta \lambda_S + \zeta c_s) \tag{3.4.5}
$$

and the value of λ_X is given by (3.2.3) and that of $\lambda_S(t)$ is given by (3.2.5), that is:

$$
\lambda_S(t) \ = \ \lambda_S^0 e^{(\rho+\beta)(t-t^0)} \ - \ \frac{\beta}{\alpha-\beta} \lambda_Z^0 \left(e^{(\rho+\alpha)(t-t^0)} - e^{(\rho+\beta)(t-t^0)} \right) \ .
$$

The integration of the dynamical system for the state variables gives:

$$
X(t) \ = \ X^0 \ - \ \int_{t}^{t^0} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta \lambda_S(u) + \zeta c_s) \mathrm{d}u \tag{3.4.6}
$$

$$
Z(t) \ = \ Z^0 e^{-\alpha(t-t^0)} \ + \ \beta \int_{t^0}^{t} e^{\alpha(u-t)} S(u) \mathrm{d}u \tag{3.4.7}
$$

$$
S(t) \ = \ S^0 \ e^{-\beta(t-t^0)} \ + \ \zeta \int_{t^0}^{t} e^{\beta(u-t)} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta \lambda_S(u) + \zeta c_s) \mathrm{d}u \ . \tag{3.4.8}
$$

3.4.3 Dynamics with only renewable energy consumption (Phase L)

Phase L corresponds to the situation where $x = s = 0$, and $y = \tilde{y}$, while the state is inside the domain: $X > 0$ and $Z < \widetilde{Z}(S)$. It is summarized in Appendix A.3 on page 78.[1]

The trajectories of both the state and the adjoint variables follow the "free" forms (2.3.24)–(2.3.23) and (3.2.4)–(3.2.5), that is:

$$
\begin{aligned}
Z(t) \ &= \ Z^0 e^{-\alpha(t-t^0)} - S^0 \frac{\beta}{\alpha-\beta} \left(e^{-\alpha(t-t^0)} - e^{-\beta(t-t^0)} \right) \\
S(t) \ &= \ S^0 e^{-\beta(t-t^0)} \\
\lambda_Z(t) \ &= \ \lambda_Z^0 e^{(\rho+\alpha)(t-t^0)} \\
\lambda_S(t) \ &= \ \lambda_S^0 e^{(\rho+\beta)(t-t^0)} \ - \ \frac{\beta}{\alpha-\beta} \lambda_Z^0 \left(e^{(\rho+\alpha)(t-t^0)} - e^{(\rho+\beta)(t-t^0)} \right) \ ,
\end{aligned}
$$

together with $X(t) = X^0$.

3.5 Boundary Phases

The boundary of the admissible domain is the frontier of the domain $\mathcal{D}$ defined in (3.2.1). The part of most interest in the analysis is the curve $\{(S, \widetilde{Z}(S)), 0 \le S \le S_M\}$, itself decomposed into the "ceiling" phase $Z = \overline{Z}$ and $0 \le S \le S_m$, and the curve $Z = Z_M(S)$ for $S_m \le S \le S_M$ (see (2.3.27).

The rest of the boundary is made of parts of the lines $S = 0$ and $Z = 0$. On the former, the dynamics is as in Phase A (Section 3.4.1). On the latter, no optimal trajectory can stay.

[1] These assumptions on control are also in the definition of Phase T and Phase U to be described in Sections 3.5.5 and 3.5.6, in which the state is on the boundary.

When $Z(t) = \overline{Z}$ over some interval of time, the dynamics (2.2.2) imply that the control is constrained by

$$\zeta x - s \;=\; \alpha \overline{Z} - \beta S \;=\; \beta(S_m - S) \;=\; \zeta \overline{x} - \beta S \; . \tag{3.5.1}$$

We analyze the consequences in this section, depending on whether s is further constrained to be 0, interior ($0 < s < \zeta x$) or constrained at its maximum ($s = \zeta x$).

3.5.1 Dynamics in Phase P (constrained atmospheric stock, no capture)

If capture is further constrained to be 0, this actually determines the consumption

$$x(t) \;=\; \frac{\beta}{\zeta}(S_m - S(t)) \; . \tag{3.5.2}$$

We call this situation Phase "P", see Appendix A.4 on page 79.

In such a phase, the values of the adjoint variables can be directly deduced from the first order conditions (2.3.2)–(2.3.4) and the dynamical system (2.3.10)–(2.3.11)

$$\lambda_Z(t) \;=\; \frac{1}{\zeta}\left(c_x + \lambda_X^0 e^{\rho(t-t^0)} - u'\left(\frac{\beta}{\zeta}(S_m - S^0 e^{-\beta(t-t^0)})\right)\right) \tag{3.5.3}$$

$$\lambda_S(t) \;=\; \lambda_S^0 e^{(\rho+\beta)(t-t^0)} - \beta \int_{t^0}^{t} e^{(\rho+\beta)(t-u)} \lambda_Z(u) \mathrm{d}u \tag{3.5.4}$$

$$\begin{aligned}\nu_Z(t) \;=\; & \frac{\rho}{\zeta}\lambda_X(t) - \frac{\beta^2}{\zeta^2} S u''\left(\frac{\beta}{\zeta}(S_m - S^0 e^{-\beta(t-t^0)})\right) \\ & -(\rho+\alpha)\left(c_x + \lambda_X(t) - u'\left(\frac{\beta}{\zeta}(S_m - S^0 e^{-\beta(t-t^0)})\right)\right) \; .\end{aligned} \tag{3.5.5}$$

The state variables are:

$$X(t) \;=\; X^0 \;+\; \frac{S^0}{\zeta}(1 - e^{-\beta(t-t^0)}) \;-\; \overline{x}(t - t^0) \tag{3.5.6}$$

$$S(t) \;=\; S^0 \; e^{-\beta(t-t^0)} \; . \tag{3.5.7}$$

Along every optimal path in this phase, the fact that $s(t) = 0$ must imply by (2.3.6) that $\gamma_s(t) = c_s + \lambda_Z(t) - \lambda_S(t) \geq 0$. It is also necessary that $\nu_Z \geq 0$.

3.5.2 Dynamics in Phase Q (constrained atmospheric stock, free capture)

Phase Q corresponds to the situation where the resource is not exhausted ($X(t) > 0$), the ceiling is reached ($Z(t) = \overline{Z}$), and sequestration occurs, but not all emissions are sequestered ($0 < s(t) < \zeta x(t)$). It is described in Appendix A.5 on page 80.

The use of the first order conditions and the dynamical system leads to the following derivation. First, the first-order condition for s provides the identity:

$$\lambda_S(t) = \lambda_Z(t) + c_s \; . \tag{3.5.8}$$

Then, differentiating and using the dynamics on λ_S, we obtain:

$$\dot{\lambda}_S \;=\; \dot{\lambda}_Z \;=\; \rho \lambda_Z + (\rho + \beta) c_s \; .$$

The adjoint variable for S is obtained by integrating Equation (2.3.11). The value of λ_Z is then deduced from (3.5.8). These are:

$$\lambda_Z(t) \;=\; e^{\rho(t-t^0)}\left(\lambda_Z^0 + c_s \frac{\rho+\beta}{\rho}\right) \;-\; c_s \frac{\rho+\beta}{\rho} \tag{3.5.9}$$

$$\lambda_S(t) \;=\; e^{\rho(t-t^0)}\left(\lambda_Z^0 + c_s\frac{\rho+\beta}{\rho}\right) \;-\; c_s\frac{\beta}{\rho}\;. \tag{3.5.10}$$

Finally, we also have the following expressions for ν_Z:

$$\begin{aligned}\nu_Z(t) &= (\rho+\beta)\lambda_S(t) - (\rho+\alpha+\beta)\lambda_Z(t)\\ &= (\rho+\beta)c_s - \alpha\lambda_Z(t) \;=\; (\rho+\alpha+\beta)c_s - \alpha\lambda_S(t)\;.\end{aligned}$$

Let us focus on the trajectory of the adjoint variable vector $(\lambda_Z(t), \lambda_S(t))$. If it happens that

$$0 \;=\; \lambda_Z^0 + c_s\frac{\rho+\beta}{\rho}\;, \tag{3.5.11}$$

then both quantities are constant and the system (3.5.9)–(3.5.10) is stationary at point

$$\Omega \;=\; \left(\;-\;c_s\frac{\rho+\beta}{\rho},\;-\;c_s\frac{\beta}{\rho}\;\right)\;. \tag{3.5.12}$$

If Condition (3.5.11) is not satisfied, then the vector $(\lambda_Z(t), \lambda_S(t))$ moves away from Ω on the line $\lambda_S = \lambda_Z + c_s$. In that case, whatever the value of λ_Z^0, we have: $\lim_{t\to-\infty}(\lambda_Z(t), \lambda_S(t)) = \Omega$. We shall make use of this property in our analysis in Chapter 4.

The dynamics for X and S are given by:

$$\dot{X} = -x \qquad \dot{S} = \zeta(x - \overline{x})\;.$$

Since the values of consumption and capture are respectively given by:

$$\begin{aligned}x(t) &= q^d(c_x + \lambda_X^0 e^{\rho(t-t^0)} - \zeta\lambda_Z(t)) &(3.5.13)\\ s(t) &= \zeta x(t) - \beta(S_m - S(t)) \;=\; \zeta(x(t) - \overline{x}) + \beta S(t)\;, &(3.5.14)\end{aligned}$$

they are integrated as:

$$\begin{aligned}X(t) &= X^0 \;-\; \int_{t^0}^{t} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta\lambda_Z(u))\mathrm{d}u &(3.5.15)\\ S(t) &= S^0 \;+\; \zeta\int_{t^0}^{t} q^d(c_x + \lambda_X^0 e^{\rho u} - \zeta\lambda_Z(u))\mathrm{d}u \;-\; \zeta\overline{x}(t-t^0)\;, &(3.5.16)\end{aligned}$$

with $\lambda_Z(t)$ given by (3.5.9).

3.5.3 Dynamics in Phase R (constrained atmospheric stock and renewable energy consumption)

Phase R corresponds to the situation where the resource is not exhausted ($X(t) > 0$), the ceiling is reached ($Z(t) = \overline{Z}$), no sequestration occurs, but there is mixed consumption of the renewable *and* nonrenewable resource ($x(t) > 0$ and $y(t) > 0$). It is described in Appendix A.6 on page 81.

Given the first order conditions and the ceiling constraint, the consumptions are given by:

$$\begin{aligned}x &= \frac{\beta}{\zeta}(S_m - S) &(3.5.17)\\ y &= \frac{\beta}{\zeta}(S - S_{\widetilde{y}})\;, &(3.5.18)\end{aligned}$$

where we have introduced the quantity:

$$S_{\widetilde{y}} \;=\; \frac{\zeta}{\beta}(\overline{x} - \widetilde{y})\;. \tag{3.5.19}$$

Since $x > 0$ and $y > 0$, it is necessary that $S_{\widetilde{y}} < S < S_m$.

The dynamics of adjoint variables are integrated explicitly as:

$$\begin{aligned}
\lambda_Z &= \frac{1}{\zeta}\lambda_X^0 e^{\rho(t-t^0)} - \frac{c_y - c_x}{\zeta} \\
\lambda_S &= \lambda_S^0 e^{(\rho+\beta)(t-t^0)} - \beta \int_{t^0}^{t} e^{(\rho+\beta)(t-u)} \lambda_Z(u) \mathrm{d}u \\
&= \lambda_S^0 e^{(\rho+\beta)(t-t^0)} + \frac{1}{\zeta}\lambda_X^0 e^{\rho(t-t^0)}(1 - e^{\beta(t-t^0)}) - \frac{c_y - c_x}{\zeta}\frac{\beta}{\rho+\beta}(1 - e^{(\rho+\beta)(t-t^0)}) \ .
\end{aligned}$$

It follows that:

$$\nu_Z = -\alpha\lambda_Z + \rho\frac{c_y - c_x}{\zeta} \ .$$

Consider an initial condition $(X^0, \overline{Z}, S^0)$ at time t^0, such that $S^0 \in (S_{\widetilde{y}}, S_m)$. The dynamics of Phase R imply that:

$$\begin{aligned}
S(t) &= S^0 e^{-\beta(t-t^0)} \\
X(t) &= X^0 - \int_{t^0}^{t} \left(\overline{x} - \frac{\beta}{\zeta}S(u)\right) \mathrm{d}u \\
&= X^0 - \overline{x}(t - t^0) + \frac{1}{\zeta}(S^0 - S(t)) \ .
\end{aligned}$$

Eliminating the variable t as: $\beta(t - t^0) = \log(S^0/S(t))$, we see that the trajectory is the curve:

$$X = X^0 + \frac{\overline{x}}{\beta}\log\frac{S}{S^0} + \frac{1}{\zeta}(S^0 - S) \ . \tag{3.5.20}$$

Observe that these curves are increasing and concave in the interval $S \in [S_{\widetilde{y}}, S_m]$, and their derivative is 0 when $S = S_m$.

Let us now consider the multiplier:

$$\begin{aligned}
\gamma_s(t) &= c_s - \frac{\rho}{\rho+\beta}\frac{c_y - c_x}{\zeta} + e^{(\rho+\beta)(t-t^0)}\left(-\lambda_S^0 + \frac{1}{\zeta}\lambda_X^0 - \frac{\beta}{\beta+\rho}\frac{c_y - c_x}{\zeta}\right) \\
&= c_s - \bar{c}_s + e^{(\rho+\beta)(t-t^0)}\left(-\lambda_S^0 + \frac{1}{\zeta}\lambda_X^0 - \frac{\beta}{\beta+\rho}\frac{c_y - c_x}{\zeta}\right) \ .
\end{aligned} \tag{3.5.21}$$

The constant $\bar{c}_s$ is defined as

$$\bar{c}_s = \frac{\rho}{\rho+\beta}\frac{c_y - c_x}{\zeta} \ . \tag{3.5.22}$$

Consequently, assuming that the term between the last parentheses is positive, there exists a finite value $\bar{t}_s$ at which $\gamma_s(\bar{t}_s) = 0$ if, and only if, $c_s < \bar{c}_s$.

3.5.4 Dynamics in Phase S (constrained atmospheric stock and maximal capture)

Phase S corresponds to the situation where the resource is not exhausted ($X(t) > 0$), the ceiling is reached ($Z(t) = \overline{Z}$), maximal sequestration occurs ($s(t) = \zeta x(t)$). This phase is described in Appendix A.7 on page 82.

Since Z is constant, $\dot{Z} = 0$ and therefore from (2.2.2), it is necessary that $\beta S = \alpha\overline{Z}$, that is, $S = S_m$. As a consequence, the trajectory is stationary at the point $(S_m, \overline{Z})$. This implies in turn that $\dot{S} = 0$ and then $x = \overline{x}$.

The integration of the dynamics of the adjoint variables yields the following expressions:

$$\lambda_Z(t) = \frac{\rho+\beta}{\beta}(c_s - \frac{\overline{p} - c_x}{\zeta}) + \frac{1}{\zeta}\lambda_X^0 e^{\rho(t-t^0)} \tag{3.5.23}$$

$$\lambda_S(t) \;=\; c_s \;-\; \frac{\overline{p}-c_x}{\zeta} \;+\; \frac{1}{\zeta}\lambda_X^0 e^{\rho(t-t^0)} \;. \tag{3.5.24}$$

This in turn provides the value of the multiplier: from the first-order condition

$$\lambda_S \;=\; c_s + \lambda_Z + \gamma_{sx} \;, \tag{3.5.25}$$

we obtain

$$\gamma_{sx} \;=\; \frac{\rho}{\beta\zeta}(\overline{p}-c_x) \;-\; \frac{\rho+\beta}{\beta}c_s \;=\; \frac{\rho+\beta}{\beta}(\hat{c}_s - c_s) \;, \tag{3.5.26}$$

where we have defined the particular value for c_s:

$$\hat{c}_s \;=\; \frac{\rho}{\rho+\beta}\frac{\overline{p}-c_x}{\zeta} \;. \tag{3.5.27}$$

The value of γ_{sx} is constant over time. It is positive if and only if $c_s \leq \hat{c}_s$.

The state trajectory is simply given by:

$$X(t) = X^0 - \overline{x}(t-t^0) \qquad S(t) = S_m \;. \tag{3.5.28}$$

3.5.5 Dynamics in Phase T (exhausted nonrenewable resource)

Phase T is like Phase L (Section 3.4.3), but the state is supposed to be $X = 0$ and is therefore on one boundary of $\mathcal{D}$. It is described in Appendix A.8 on page 83. In that case, the set of feasible controls is reduced to $\{(y,x,s), y \geq 0, x = 0, s = 0\}$, because of the constraint $X \geq 0$.

Assuming that the phase is terminal, the transversality conditions (2.3.15)–(2.3.17) must hold. Since $X = 0$, (2.3.15) is clearly satisfied. On the other hand, we have seen in Section 3.2.3 that $(Z\lambda_Z + S\lambda_Z)e^{-\rho t}$ is constant. But according to (2.3.16) and (2.3.17), $Z\lambda_Z e^{-\rho t} \to 0$ and $S\lambda_S e^{-\rho t} \to 0$. This constant must therefore be 0. Then, $Z\lambda_Z + S\lambda_Z$ is also 0 for all t, which is possible only if:

$$\lambda_Z(t) \;=\; \lambda_S(t) \;=\; 0 \;. \tag{3.5.29}$$

Then the first-order condition (2.3.3) gives the value of λ_X:

$$\lambda_X(t) \;=\; c_y - c_x \;. \tag{3.5.30}$$

Since $y > 0$, we have $\gamma_y = 0$. From the first-order equations (2.3.2) and (2.3.3), the other multipliers satisfy the following constraints:

$$0 \;=\; c_s - \gamma_s + \gamma_{sx}, \qquad 0 \;=\; \zeta\gamma_{sx} \;.$$

The second one implies $\gamma_{sx} = 0$. Replacing in the first one, we have $\gamma_s = c_s$.

3.5.6 Dynamics in Phase U (no consumption of the nonrenewable resource)

A singular situation is encountered in the case where the state of the system is located on the curve $Z = Z_M(S)$, which forms a boundary of the admissible domain when $S_m \leq S \leq S_M$, while at the same time $X > 0$.

In that case, the set of feasible controls is reduced to $\{(y,x,s), y \geq 0, x = 0, s = 0\}$, because of the viability constraint. The difference with Phase L, where $X > 0$ and $Z < \widetilde{Z}(S)$, has no impact on the dynamics. Whatever the value of y ($y = \widetilde{y}$ is the optimal one), the trajectory is forced to follow the boundary, according to the state equations of Section 3.4.3, until $S(t) = S_m$.

The analysis of the dynamics of this phase will take place in Section 4.4.2.4 on page 46.

Chapter 4

Unexhaustible resources

We study in this chapter the model introduced in Chapter 2.1, in the case where the resource stock X is assumed to be infinite, and there are no constraints on the stock of sequestered carbon: $\overline{S} = +\infty$, nor on the rate of consumption of clean energy: $\overline{y} = +\infty$.

Formally, the problem is the same as exposed in Section 2.2, except that there is no dynamics of the stock X. The system is described by the two variables $Z(t)$ and $S(t)$.

The first-order conditions associated with this new problem are easily obtained from that of the general problem by setting formally $\lambda_X = 0$.

As mentioned in Chapter 3, we consider that optimal trajectories are decomposed in a succession of phases, characterized by the set of constraints that are active. We shall use the same phase names as in that chapter, and ignore the variable X.

Optimal trajectories will be constructed *backwards*. We shall first identify which phases are possibly terminal, that is, contain the infinite part of the trajectory. Then we shall find which phases can possibly be "glued" to these terminal phases, and so on until an optimal trajectory starting from all possible initial states in the feasible domain has been identified.

Optimal trajectories will be identified with the help of Corollaries 3.1 and 3.2. Several lemmas will successively identify optimal trajectories starting from initial states in locations of the state space. Occasionally, we will identify pieces of trajectories satisfying the first-order conditions: these will be confirmed as optimal trajectories when glued together with another

Several requirements of Corollaries 3.1 and 3.2 will be satisfied by construction and will not be checked explicitly on each candidate optimal trajectory. For instance, continuity of the state trajectory and of the adjoint variable λ_S is implicit. Likewise, it turns out that control trajectories are always bounded, as required. The bulk of proofs will therefore be devoted to checking that the state evolves in the correct domain, and that conditions on Lagrange multipliers are satisfied.

As it turns out, the only possible terminal phases are located on the boundary of the domain. The backwards construction will then involve first phases on the boundary (Phases P, Q, R, S and U in the terminology of Chapter 3), then phases of the interior (Phases A, B and L). Phase T identified in Chapter 3 is not relevant here since it is characterized by $X = 0$.

In the course of the analysis, several qualitatively different behaviors will emerge, depending on the value of the parameters of the model. We choose to classify this cases according to the value of c_s. Several critical values for this parameter will be identified along the way, as functions of the other parameters. One of them has already been defined in (3.5.27):

$$\hat{c}_s = \frac{\rho}{\rho + \beta} \frac{\overline{p} - c_x}{\zeta} .$$

For future reference in this chapter, we also recall some critical values on the variable S already encountered in (2.3.25) and (3.5.19):

$$S_m = \frac{\alpha \overline{Z}}{\beta} \quad \text{and} \quad S_{\widetilde{y}} = \frac{\zeta}{\beta}(\overline{x} - \widetilde{y}) ,$$

with the equivalent:

$$S_m \;=\; \frac{\zeta\overline{x}}{\beta}, \qquad \overline{x} \;=\; \frac{\beta S_m}{\zeta}, \qquad S_m - S_{\widetilde{y}} \;=\; \frac{\zeta}{\beta}\widetilde{y}\ .$$

This chapter is organized as follows. In Sections 4.1 and 4.2, we look successively at the phases defined in Chapter 3 and we identify which ones are possibly terminal. Since they turn out to be located on the boundary of the valid state space, we study in Section 4.3 the cases where an optimal trajectory may follow this boundary. In Section 4.4, we look at the way optimal trajectories in the interior connect to the boundary. Finally, in Section 4.5, we review the findings by presenting the different solutions to the problem, classified according to the value of the parameter c_s. Some concluding remarks are presented in Section 4.6.

4.1 Terminal phases

The first question we address is that of the behavior of the trajectory when $t \to \infty$. As a consequence of the first-order conditions and the transversality conditions (2.3.16)–(2.3.17), only a few phases are consistent with the infinite part of the trajectory.

In this report, we call "terminal phase" a phase for which there exists an optimal trajectory and some t^0 for which the trajectory is within the phase for all $t \geq t^0$. Among terminal phases, some are possibly stationary, in the sense that the trajectories of all variables (state, costate, control etc.) remain constant.

We stick to the convention of Chapter 3 that t^0 denotes the arbitrary time instant inside the phase currently under study.

4.1.1 Terminal P phase

In Phase P (see Appendix A.4 on page 79, and Section 3.5.1), $Z = \overline{Z}$, $s = 0$, $y = 0$ and $x = \overline{x} - \beta S/\zeta$. The evolution of S is "free", and $S(t) = S^0 e^{-\beta(t-t^0)}$.

The first-order equations provide the value of λ_Z, see (3.5.3):

$$\lambda_Z(t) \;=\; \frac{1}{\zeta}\left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S(t))\right) \;=\; \frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^0 e^{-\beta(t-t^0)})\right)\ . \tag{4.1.1}$$

In this last expression, both terms are negative. The second one tends to 0 as $t \to +\infty$. Accordingly,

$$\lim_{t\to+\infty} \lambda_Z(t) \;=\; \frac{c_x - \overline{p}}{\zeta} < 0.$$

Next, the expression found for λ_S in (3.5.4) is:

$$\begin{aligned}
\lambda_S(t) \;&=\; \lambda_S^0 e^{(\rho+\beta)(t-t^0)} - \beta\int_{t^0}^{t} e^{(\rho+\beta)(t-v)}\lambda_Z(v)\mathrm{d}v \\
&=\; \lambda_S^0 e^{(\rho+\beta)(t-t^0)} - \frac{\beta}{\zeta}\int_{t^0}^{t} e^{(\rho+\beta)(t-v)}\left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^0 e^{-\beta(v-t^0)})\right)\mathrm{d}v \\
&=\; e^{(\rho+\beta)(t-t^0)}\left[\lambda_S^0 \;-\; \frac{\beta}{\zeta}\int_0^{t-t^0} e^{-(\rho+\beta)v}\left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^0 e^{-\beta v})\right)\mathrm{d}v\right]\ .
\end{aligned} \tag{4.1.2}$$

Invoking the transversality condition (2.3.17), that is:

$$\lim_{t\to\infty} e^{-\rho t}\lambda_S S \;=\; 0\ ,$$

with $S(t) = S^0 e^{-\beta(t-t^0)}$, we get *for* $S^0 \neq 0$,

$$\lambda_S^0 \;=\; \frac{\beta}{\zeta}\int_0^{\infty} e^{-(\rho+\beta)v}\left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^0 e^{-\beta v})\right)\mathrm{d}v\ . \tag{4.1.3}$$

Finally, replacing in (4.1.2), we obtain the value for the λ_S:

$$\begin{aligned}
\lambda_S(t) &= \frac{\beta}{\zeta} e^{(\rho+\beta)(t-t^0)} \int_{t-t^0}^{\infty} e^{-(\rho+\beta)v} \left(c_x - u'(\overline{x} - \frac{\beta}{\zeta} S^0 e^{-\beta v}) \right) \mathrm{d}v \\
&= \frac{1}{\zeta} \, L(S^0 e^{-\beta(t-t^0)}) \\
&= \frac{\beta}{\rho+\beta} \frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta} \, M(S^0 e^{-\beta(t-t^0)}) \,,
\end{aligned}$$

where we have defined the functions

$$\begin{aligned}
L(S) &= \beta \int_0^{\infty} e^{-(\rho+\beta)v} \left(c_x - u'(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) \right) \mathrm{d}v && (4.1.4) \\
M(S) &= \beta \int_0^{\infty} e^{-(\rho+\beta)v} \left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) \right) \mathrm{d}v \,. && (4.1.5)
\end{aligned}$$

The properties of these functions are studied in Appendix B. In particular, $M(S) \leq 0$, so that the formula for λ_S above gives a negative value because both terms in its right-hand side are negative.

The value of γ_s can be written as, introducing the constant $\hat{c}_s$ defined in (3.5.27), as:

$$\begin{aligned}
\gamma_s(t) &= \lambda_Z(t) - \lambda_S(t) + c_s \\
&= c_s \;+\; \frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta} \left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta} S^0 e^{-\beta(t-t^0)}) \right) \;-\; \frac{\beta}{\rho+\beta} \frac{c_x - \overline{p}}{\zeta} \;-\; \frac{1}{\zeta} M(S^0 e^{-\beta(t-t^0)}) \\
&= c_s - \hat{c}_s \;+\; \frac{1}{\zeta} \left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta} S^0 e^{-\beta(t-t^0)}) \right) \;-\; \frac{1}{\zeta} M(S^0 e^{-\beta(t-t^0)}) \,. && (4.1.6)
\end{aligned}$$

The previous reasoning applies only to $S^0 \neq 0$, when the value of λ_S is computed. Assume now that $S^0 = 0$, so that $S(t) = 0$ for all t in the phase. This is the case without capture, which has been studied in Chakravorty et al. (2006). The transversality condition (2.3.17) is automatically satisfied. In that case, from the solutions obtained in Section 3.5.1, and given that $\beta S_m/\zeta = \overline{x}$, we obtain:

$$\begin{aligned}
\lambda_Z(t) &= \frac{c_x - \overline{p}}{\zeta} && (4.1.7) \\
\lambda_S(t) &= \lambda_S^0 e^{(\rho+\beta)(t-t^0)} \;+\; \frac{\beta}{\rho+\beta} \frac{c_x - \overline{p}}{\zeta} (1 - e^{(\rho+\beta)(t-t^0)}) \\
&= e^{(\rho+\beta)(t-t^0)} \left(\lambda_S^0 \;-\; \frac{\beta}{\rho+\beta} \frac{c_x - \overline{p}}{\zeta} \right) \;+\; \frac{\beta}{\rho+\beta} \frac{c_x - \overline{p}}{\zeta} \,. && (4.1.8)
\end{aligned}$$

In that case, the function γ_s is:

$$\begin{aligned}
\gamma_s(t) &= c_s + \frac{\rho}{\rho+\beta} \frac{c_x - \overline{p}}{\zeta} \;-\; e^{(\rho+\beta)(t-t^0)} \left(\lambda_S^0 \;-\; \frac{\beta}{\rho+\beta} \frac{c_x - \overline{p}}{\zeta} \right) \\
&= c_s - \hat{c}_s \;-\; e^{(\rho+\beta)(t-t^0)} \left(\lambda_S^0 \;-\; \frac{\beta}{\rho+\beta} \frac{c_x - \overline{p}}{\zeta} \right) \,.
\end{aligned}$$

Since the system is motionless, it is expected that the function $\gamma_s(\cdot)$ will be positive, whatever the value of t and t^0, since t^0 has been arbitrarily chosen within the phase. The only way this can happen is to chose

$$\lambda_S^0 \;=\; \frac{\beta}{\rho+\beta} \frac{c_x - \overline{p}}{\zeta} \,,$$

which implies, for all t:

$$\lambda_S(t) \;=\; \frac{\beta}{\rho+\beta} \frac{c_x - \overline{p}}{\zeta} \qquad \gamma_s(t) \;=\; c_s - \hat{c}_s \,.$$

Finally, the formulas established for λ_S, λ_Z and γ_s hold for all $S^0 \geq 0$. We have identified in passing the point

$$P_\infty \ := \ \left(\frac{c_x - \overline{p}}{\zeta}, \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta}\right) \tag{4.1.9}$$

which represents the values of adjoint variables in the space (λ_Z, λ_S) at the stationary state $(0, \overline{Z})$ as well as the limit of these variables, when $t \to \infty$ when the system is in the terminal Phase P.

We can now prove the following result.

Lemma 4.1. *Phase P can be terminal only if $c_s > \hat{c}_s$. In that case, the entry point in Phase P is such that $S(t^0) \leq S_{\widetilde{y}}$. Under this condition, the following trajectory is optimal: $x(t) = \overline{x} - \frac{\beta}{\zeta}S(t)$, $s(t) = y(t) = 0$, and*

$$\begin{aligned}
S &= S(t^0)e^{-\beta(t-t^0)} && (4.1.10)\\
Z &= \overline{Z} \\
\lambda_Z &= \frac{1}{\zeta}\left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S)\right) \ = \ \frac{c_x - \overline{p}}{\zeta} + \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S)\right) && (4.1.11)\\
\lambda_S &= \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} + \frac{1}{\zeta}M(S) && (4.1.12)\\
\gamma_s &= c_s - \hat{c}_s + \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S)\right) - \frac{1}{\zeta}M(S) && (4.1.13)\\
\gamma_{sx} &= 0 \\
\underline{\gamma}_y &= c_y - u'(\overline{x} - \frac{\beta}{\zeta}S) && (4.1.14)
\end{aligned}$$

on the interval $t \in [t^0, \infty)$.

Proof. We check the conditions of Corollary 3.1. If the phase is permanent, then the conditions $\nu_Z(t) \geq 0$, $\gamma_s(t) \geq 0$ and $\underline{\gamma}_y(t) \geq 0$ must hold for all value of t.

From (2.3.10), we have $\nu_Z = \dot{\lambda}_Z - (\rho+\alpha)\lambda_Z$. From (4.1.1), we obtain

$$\dot{\lambda}_Z(t) \ = \ -\frac{\beta^2}{\zeta^2}S^0 u''(\overline{x} - \frac{\beta}{\zeta}S(t)) \ ,$$

and since $u''(\cdot) \leq 0$, $\dot{\lambda}_Z \geq 0$. Therefore λ_Z is increasing and since its limit as $t \to +\infty$ is negative, it is always negative. As a consequence, $\nu_Z \geq 0$.

Given that $\underline{\gamma}_y(t) = u'(x(t)) - c_y$ and since $u'(\cdot)$ is decreasing, we have: $\underline{\gamma}_y \geq 0 \iff x \geq \widetilde{y} \iff \overline{x} - \beta S/\zeta \geq \widetilde{y} \iff S \leq S_{\widetilde{y}}$.

Turning now to $\gamma_s(t)$, we see that the two last terms in (4.1.6) both tend to 0 as $t \to \infty$, since $u'(\overline{x}) = \overline{p}$ and $M(0) = 0$ (see Appendix B). Therefore, $\lim_{t\to+\infty}\gamma_s(t) = c_s - \hat{c}_s$ and a *necessary condition* for $\gamma_s(s)$ to be positive for all $t \geq t^0$ is:

$$c_s \ \geq \ \hat{c}_s \ .$$

On the other hand, the condition $c_s < \hat{c}_s$ is sufficient for the existence of a t^0 such that $\gamma_s(t) > 0$ for all $t \geq t^0$, since in that case $\lim_{t\to\infty}\gamma_s(t) > 0$. □

Remark: Actually, we can show that $\gamma_s(\cdot)$ is increasing under the additional condition that u' is convex. Indeed, according to (4.1.13), we have:

$$\dot{\gamma}_s(t) \ = \ \dot{S}(t)\frac{1}{\zeta}\ \left(\frac{\beta}{\zeta}u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) - M'(S)\right) \ .$$

We know that $\dot{S} < 0$. On the other hand, it is shown in the proof of Lemma B.2 (page 86) that if $u'(\cdot)$ is convex, then

$$M'(S) \ \geq \ \frac{\beta}{\rho+2\beta}\ \frac{\beta}{\zeta}\ u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) \ .$$

Therefore,

$$\frac{\beta}{\zeta}u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) - M'(S) \;\leq\; -\frac{\beta}{\zeta}\,\frac{\rho+\beta}{\rho+2\beta}\,u''\left(\overline{x} - \frac{\beta}{\zeta}S\right)$$

is negative. As a consequence, $\dot{\gamma}_s$ is positive.

4.1.2 Terminal S phase

The assumptions made in Phase S are that: $Z = \overline{Z}$, $s = \zeta x$ and $y = 0$. According to the results of Section 3.5.4, the value of S is constant as well, $S = S_m$, and $x = \overline{x}$. Then, $s = \zeta\overline{x}$. This implies $\gamma_s = 0$. Since here X is assumed to be infinite, the values of the adjoint variables in (3.5.23) and (3.5.24) have to be replaced with:

$$\lambda_Z = \frac{\rho+\beta}{\beta}\left(c_s + \frac{c_x - \overline{p}}{\zeta}\right) \qquad\qquad \lambda_S = c_s + \frac{c_x - \overline{p}}{\zeta}\ .$$

The value of γ_{sx} is still as in (3.5.26):

$$\gamma_{sx} = \frac{\rho+\beta}{\beta}(\hat{c}_s - c_s)\ .$$

Clearly, $\gamma_{sx} \geq 0$ if and only if $c_s \leq \hat{c}_s$.

Finally, from (2.3.10), we get $\nu_Z = -(\rho+\alpha)\lambda_Z$. If $c_s \leq \hat{c}_s$, then $c_s + (c_x - \overline{p})/\zeta = c_s - (1 + \beta/\rho)\hat{c}_s < 0$. We then have $\lambda_Z < 0$ and $\nu_Z > 0$.

All conditions of Corollary 3.1 being satisfied, we have proved the following result:

Lemma 4.2. *Phase S can be terminal if and only if $c_s \leq \hat{c}_s$. In that case, the following trajectory is optimal: $S(t) = S_m$, $Z(t) = \overline{Z}$, $x(t) = \overline{x}$, $s(t) = \zeta\overline{x}$, $y(t) = 0$ and*

$$\lambda_Z = \frac{\rho+\beta}{\beta}\left(c_s + \frac{c_x - \overline{p}}{\zeta}\right) \qquad (4.1.15)$$

$$\lambda_S = c_s + \frac{c_x - \overline{p}}{\zeta} \qquad (4.1.16)$$

$$\gamma_s = 0$$

$$\gamma_{sx} = \frac{\rho+\beta}{\beta}(\hat{c}_s - c_s) \qquad (4.1.17)$$

$$\underline{\gamma}_y = c_y - \overline{p} \qquad (4.1.18)$$

on any time interval.

4.1.3 Terminal Q phase

Phase Q may be terminal in the very specific case $c_s = \hat{c}_s$, see Lemma 4.4 in Section 4.2.2.

4.1.4 Conclusion on terminal phases

Summing up the results on terminal phases, we have the following dichotomy:

if $c_s < \hat{c}_s$: the point $(S_m, \overline{Z})$ is terminal and stationary,

if $c_s > \hat{c}_s$: Phase P is terminal and the point $(0, \overline{Z})$ is stationary.

In the next section, we review the other possible phases and show that the cases identified above are actually the only terminal phases, except in the limit case $c_s = \hat{c}_s$. In the process, we establish properties that will be used to construct complex trajectories.

The case $c_s < \hat{c}_s$ is the most interesting from the point of view of Economics, since it is the one where capture of CO_2 is optimal in the long run. This is the case studied in (Lafforgue et al. 2008*a*),(Lafforgue et al. 2008*b*) in the case $\beta = 0$: their assumption is that $c_s < (\overline{p} - c_x)/\zeta$, and this last quantity is precisely $\hat{c}_s$ when $\beta = 0$.

4.2 Non-terminal Phases

We now show that phases A, B, L, Q, R and U cannot be terminal. Doing so, we obtain some insight on the way these phases may begin or end.

4.2.1 Phases A, B and L

The common feature of these three phases is that the adjoint variables evolve "freely" according to the equations (3.2.2) analyzed in Section 3.2.

It can be verified, for instance using the results of Section 3.2.3, that

$$\lim_{t\to\infty} \lambda_Z(t) = -\infty \qquad \lim_{t\to\infty} \lambda_S(t) = +\infty \qquad \lim_{t\to\infty} \lambda_S(t) - \lambda_Z(t) = +\infty$$

under the following conditions: $\lambda_Z^0 < 0$, $\lambda_S^0 < 0$ and either (a) $\beta \leq \alpha$ or (b) $\beta > \alpha$, and $\lambda_S^0 > \beta/(\beta-\alpha)\lambda_Z^0$.

According to first-order condition (2.3.2), we have

$$\gamma_{sx}(t) - \gamma_s(t) \;=\; \lambda_S(t) - \lambda_Z(t) - c_s \quad \to \quad +\infty$$

as $t \to \infty$. If $x(t) > 0$ (Phase A or B), then only one of γ_s and γ_{sx} can be different from 0. Since both are positive, it means that eventually $\gamma_{sx}(t) > 0$ and $\gamma_s = 0$. In other words, the trajectory cannot stay in Phase A forever, and must necessarily enter Phase B, unless the state variable hits the boundary first.

When the trajectory is in Phase B, the consumption is given (see (3.4.5)) by:

$$x \;=\; q^d(c_x + \zeta c_s - \zeta\lambda_S)\;.$$

Then, when $t \to \infty$, $x(t)$ becomes necessarily strictly larger than $\widetilde{x}$, according to Assumption 1. It is actually possible that $x(t)$ tends to infinity if $\lim_{x\to\infty} u'(x)$ is finite. In every situation, we have (see Appendix A.2 or Section 3.4.2):

$$\dot{Z} + \dot{S} \;=\; \zeta x - \alpha Z \;=\; \zeta(x - \overline{x}) + \alpha(\overline{Z} - Z) \;>\; \zeta(\widetilde{x} - \overline{x}) \;>\; 0\;.$$

As a consequence, we have $\lim_{t\to\infty}(Z(t)+S(t)) = +\infty$, but this is not possible because the domain of Phase B is bounded. So Phase B must end in finite time, when the trajectory hits the boundary or, as we shall see, if $\underline{\gamma}_y(t) = 0$.

Finally, consider a trajectory perpetually in Phase L. According to Conditions (2.3.2)–(2.3.4) (see also Appendix A.3 or Section 3.4.3), given that $y = \widetilde{y}$, we must have:

$$\begin{aligned} \gamma_s - \gamma_{sx} &= \lambda_Z - \lambda_S + c_s \\ \zeta\gamma_{sx} &= c_x - c_y - \zeta\lambda_Z\;, \end{aligned}$$

and both γ_s and γ_{sx} must be positive (Conditions (2.3.5)–(2.3.6)). But by a linear combination of these two equations, we obtain:

$$\zeta\gamma_s \;=\; -\,\zeta\lambda_S + \zeta c_s + c_x - c_y \quad \to \quad -\infty$$

as $t \to \infty$. This is a contradiction. Phase L cannot be terminal. It is necessary that the consumption x becomes nonnegative at some point in time.

We have therefore proved:

Lemma 4.3. *None of the three "interior" phases (Phase A, Phase B and Phase L) can be terminal.*

4.2.2 Phase Q

In Phase Q, characterized by $Z = \overline{Z}$, $y = 0$ and $0 < s < \zeta x$, the dynamics of the state are $\dot{Z} = 0$ and $\dot{S} = \zeta(x - \overline{x})$. The first-order equations imply the relationship

$$\lambda_Z - \lambda_S + c_s = 0. \tag{4.2.1}$$

The values of consumption and capture, specialized from Section 3.5.2, are respectively given by:

$$\begin{aligned} x(t) &= q^d(c_x - \zeta\lambda_Z(t)) &(4.2.2)\\ s(t) &= \zeta x(t) - \beta(S_m - S(t)) \;=\; \zeta(x(t) - \overline{x}) + \beta S(t)\ , &(4.2.3)\end{aligned}$$

and the constraints $s > 0$ and $s < \zeta x$ are satisfied as long as, respectively, $x > \overline{x} - \beta S/\zeta$ and $S < S_m$.

The adjoint variables are given by Equations (3.5.9) and (3.5.10) which we recall here:

$$\begin{aligned} \lambda_Z(t) &= e^{\rho(t-t^0)}\left(\lambda_Z^0 + c_s\frac{\rho+\beta}{\rho}\right) - c_s\frac{\rho+\beta}{\rho} &(4.2.4)\\ \lambda_S(t) &= e^{\rho(t-t^0)}\left(\lambda_Z^0 + c_s\frac{\rho+\beta}{\rho}\right) - c_s\frac{\beta}{\rho}\ . &(4.2.5)\end{aligned}$$

As observed in Section 3.5.2 (on page 24), if

$$0 \;=\; \lambda_Z^0 + c_s\frac{\rho+\beta}{\rho}$$

then $(\lambda_Z(t), \lambda_S(t))$ is stationary at point Ω defined by (3.5.12). In that case, consumption is $x = q^d(c_x + \zeta c_s(\rho+\beta)/\rho) = q^d(\overline{p} + \zeta(\rho+\beta)(c_s - \hat{c}_s)/\rho)$ (is constant as well) and $\dot{S} = \zeta(x - \overline{x})$. The value of $\underline{\gamma}_y$ is:

$$\underline{\gamma}_y = c_y - c_x - \zeta c_s\frac{\rho+\beta}{\rho} \;=\; \zeta\frac{\rho+\beta}{\rho}(\bar{c}_s - c_s)\ ,$$

where $\bar{c}_s$ has been defined in (3.5.22) on page 26. Therefore, $\underline{\gamma}_y$ is is positive as long as $c_s \le \bar{c}_s$. Clearly, under Assumption 1, $\hat{c}_s < \bar{c}_s$. In the special case $c_s = \hat{c}_s$, then $x = \overline{x}$ and $S(t)$ is constant. The phase can therefore *a priori* be terminal.

In other cases, $(\lambda_Z(t), \lambda_S(t))$ moves away from Ω and tends to infinity. If $\lambda_Z^0 < -c_s(\rho+\beta)/\rho$, then $\lambda_Z(t)$ tends to $-\infty$ when $t \to \infty$, so that the first-order condition on y (2.3.4):

$$0 \le \underline{\gamma}_y = c_y - u'(x) = c_y - c_x + \zeta\lambda_Z$$

is eventually violated. If $\lambda_Z^0 > -c_s(\rho+\beta)/\rho$, then $\lambda_Z(t)$ tends to $+\infty$ when $t \to \infty$, so $c_x - \zeta\lambda_Z(t)$ tends to $-\infty$. According to Assumption 1, the value of $x(t) = q^d(c_x - \zeta\lambda_Z(t))$ tends to infinity, possibly in finite time. Since $\dot{S}(t) = \zeta(x(t) - \overline{x})$, this implies that $S(t)$ tends to infinity, which is clearly not possible.

The results can be summarized as:

Lemma 4.4. *Phase Q is terminal if, and only if $c_s = \hat{c}_s$. In that case, the following constant trajectory is optimal: $S(t) = S^0$, $Z(t) = \overline{Z}$, $x(t) = \overline{x}$, $s(t) = \beta S^0$*

$$\begin{aligned} \lambda_Z &= -\frac{\rho+\beta}{\rho}\hat{c}_s \;=\; \frac{c_x - \overline{p}}{\zeta}\\ \lambda_S &= -\frac{\beta}{\rho}\hat{c}_s \;=\; \frac{\beta}{\rho+\beta}\,\frac{c_x - \overline{p}}{\zeta}\\ \underline{\gamma}_y &= c_y - \overline{p}\end{aligned}$$

and $\gamma_s = \gamma_{sx} = 0$ on any time interval and for any $0 \le S^0 \le S_m$.

4.2.3 Phase R

In Phase R, $Z = \overline{Z}$, $y > 0$ and $s = 0$. The dynamics of this phase can be specialized from the equations of Section 3.5.3.

In particular, we have $S(t) = S^0 e^{-\beta(t-t^0)}$ but also, according to (3.5.18), $y(t) = (\beta/\zeta)(S(t) - S_{\widetilde{y}})$. Therefore, as $t \to \infty$, the value of y cannot remain positive. Another possibility is that γ_s may become negative. In any case, Phase R cannot be terminal.

We can state the following result. The piece of trajectory we identify is not termed as optimal because it is not described for values of t after the trajectory has exited Phase R.

Lemma 4.5. *Under Assumption 1, Phase R is never terminal. The following configuration is a solution to the first order equations and the system of constraints:*

$$
\begin{aligned}
S &= S^0 e^{-\beta(t-t^0)} \\
Z &= \overline{Z} \\
\lambda_Z &= -\frac{c_y - c_x}{\zeta} && (4.2.6)\\
\lambda_S &= \left(\lambda_S^0 + \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta}\right) e^{(\rho+\beta)(t-t^0)} - \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta} && (4.2.7)\\
x &= \frac{\beta}{\zeta}(S_m - S) \\
y &= \tilde{y} - x = \frac{\beta}{\zeta}(S - S_{\widetilde{y}}) \\
\gamma_s &= c_s - \bar{c}_s - \left(\lambda_S^0 + \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta}\right) e^{(\rho+\beta)(t-t^0)}
\end{aligned}
$$

together with $\gamma_{sx} = \underline{\gamma}_y = 0$, as long as $S(t) \geq S_{\widetilde{y}}$ and $\gamma_s(t) \geq 0$.

Proof. In order to apply Corollary 3.1, we must check the constraint $\nu_Z \geq 0$. From (2.3.10), we have $\nu_Z = -(\rho + \alpha)\lambda_Z$. Since $\lambda_Z < 0$, we have $\nu_Z > 0$. □

4.2.4 Phase U

When in Phase U, $S(t) = S^0 e^{-\beta(t-t^0)}$ therefore the state $(S(t), Z(t))$ eventually reaches $(S_m, \overline{Z})$. This phase cannot be terminal.

4.3 Optimal trajectories on the boundary

In this section, we take the first steps at constructing optimal trajectories by connecting individual phases together. As a result of the analysis of Section 4.1, we know that *whatever the value of c_s*, all optimal trajectories eventually end up on one boundary of the domain $\mathcal{D}$, namely, the curve defined in (2.3.27) as:

$$
\widetilde{Z}(S) = \begin{cases} \overline{Z} & \text{if } S \leq S_m \\ Z_M(S) & \text{if } S_m \leq S \leq S_M. \end{cases}
$$

For convenience, we refer to it as "the" boundary in the following.

It is therefore reasonable to suppose that some optimal trajectories will follow this boundary until the final state. This solution strategy turns out to work for $c_s \geq \hat{c}_s$ and we make this assumption in this section. The case $c_s < \hat{c}_s$ will be addressed in Section 4.4.3.

The computation of optimal trajectories can be decomposed in two sub-problems: A) computing the optimal trajectory on the curve $Z = \widetilde{Z}(S)$, and B) computing the optimal way to join the curve. We address the first sub-problem in this section: in Section 4.3.1, we address the first problem by showing how boundary phases P, Q, R and U can be glued together; we synthetize the findings in Section 4.3.2. The second sub-problem will be addressed in Section 4.4, where we show how trajectories coming from the inside of the domain can connect to the boundary.

The following convention is adopted throughout: when a function $f(\cdot)$ of time (state, adjoint variable, Lagrange multiplier) refers to a generic trajectory in Phase ϕ, it will be denoted as $f^{(\phi)}$.

4.3.1 Junction between phases on the boundary

The possible phases for states on the boundary $Z = \widetilde{Z}(S)$ are phases P, Q and R for $S \leq S_m$ and Phase U for $S \in [S_m, S_M]$. Connection between Phase U and other phases occurs when $S(t) = S_m$.

4.3.1.1 Phases Q/P

Assume that a trajectory begins at time t^0 in state $(S^0, \overline{Z})$ and in phase Q, then enters phase P at time t^{QP}, then stays in that phase forever. Denote $S^{QP} = S(t^{QP})$.

In Phase Q, the equations of the state and adjoint variables are given in Section 4.2.2. In Phase P, they are given in Section 4.1.1. Continuity for the state $S(t)$ writes as:

$$S^{QP} := S^{(Q)}(t^{QP}) \;=\; S^0 \;+\; \zeta \int_{t^0}^{t^{QP}} q^d(c_x - \zeta\lambda_Z^{(Q)}(u))\mathrm{d}u \;-\; \zeta\overline{x}(t^{QP} - t^0)\;. \tag{4.3.1}$$

We try to construct a trajectory such that the adjoint variables $\lambda_Z(\cdot)$ and $\lambda_S(\cdot)$ are continuous at $t = t^{QP}$. For $t < t^{QP}$, these functions are given by formulas for Phase Q, and for $t > t^{QP}$, they are given by formulas for terminal Phase P. Therefore: equating (4.2.4) and (4.1.11) on the one hand, and (4.2.5) and (4.1.12) on the other hand (after the appropriate change of variable in the formulas for the Phase P), we obtain the continuity equations (using the functions $L(\cdot)$ and $M(\cdot)$ which have been defined in equations (4.1.4) and (4.1.5) on p. 30):

$$\begin{aligned}\lambda_Z^{(Q)}(t^{QP}) \;&=\; e^{\rho(t^{QP}-t^0)}\left(\lambda_Z^0 + \frac{\rho+\beta}{\rho}c_s\right) - \frac{\rho+\beta}{\rho}c_s \\ &=\; \lambda_Z^{(P)}(t^{QP}) \;=\; \frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right)\end{aligned} \tag{4.3.2}$$

$$\begin{aligned}\lambda_S^{(Q)}(t^{QP}) \;&=\; c_s \;+\; e^{\rho(t^{QP}-t^0)}\left(\lambda_Z^0 + c_s\frac{\rho+\beta}{\rho}\right) \\ &=\; \lambda_S^{(P)}(t^{QP}) \;=\; \frac{1}{\zeta}\;L(S^{QP})\;.\end{aligned} \tag{4.3.3}$$

The unknown quantities in these equations are: $t^{QP} - t^0$, S^{QP} and λ_Z^0. We have to discuss under which conditions there exists a solution to this system.

We first determine S^{QP}. Eliminating the factor of $e^{\rho(t^{QP}-t^0)}$ between Equations (4.3.2), (4.3.3), we obtain the equality:

$$\frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right) \;+\; c_s \;=\; \frac{1}{\zeta}L(S^{QP}) \;=\; \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}M(S^{QP})\;.$$

This is actually equivalent to require that the function $\gamma_s^{(P)}$ given by Equation (4.1.13) is equal to 0 at $t = t^{QP}$, which gives directly this formula. Rewriting this equation gives the form:

$$\zeta(c_s - \hat{c}_s) \;+\; \overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP}) \;=\; M(S^{QP})\;. \tag{4.3.4}$$

The unique unknown quantity in this equation is S^{QP}. The existence of solutions to this equation is the topic of Lemma B.2 in the Appendix (p. 86). It states, among other properties, that the solution S^{QP} of (4.3.4) exists and is unique when

$$\hat{c}_s \leq c_s \leq c_{sm} \tag{4.3.5}$$

and $u'(\cdot)$ is convex. We have introduced the critical cost:

$$c_{sm} \;:=\; \frac{c_y - c_x}{\zeta} + \frac{1}{\zeta}L(S_{\widetilde{y}}) \;=\; \frac{c_y - c_x}{\zeta} + \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} + \frac{1}{\zeta}M(S_{\widetilde{y}}) \;=\; \hat{c}_s + \frac{c_y - \overline{p}}{\zeta} + \frac{1}{\zeta}M(S_{\widetilde{y}}) \,. \tag{4.3.6}$$

This constant c_{sm} is such that $c_{sm} > \bar{c}_s$, as proved in (B.0.4), p. 86. Observe also that $L(S_{\widetilde{y}}) < 0$, and therefore $c_{sm} < (c_y - c_x)/\zeta$.

Once S^{QP} has been determined, the remaining unknowns can be computed as well. First, from (4.3.2):

$$e^{\rho(t^{QP}-t^0)}\left(\lambda_Z^0 + \frac{\rho+\beta}{\rho}c_s\right) \;=\; \frac{\rho+\beta}{\rho}(c_s - \hat{c}_s) \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right) \,.$$

Then, in Phase Q (that is, for $t < t^{QP}$), the function $\lambda_Z^{(Q)}$ can be written as:

$$\lambda_Z^{(Q)}(t) \;=\; e^{\rho(t-t^{QP})}\left[\frac{\rho+\beta}{\rho}(c_s - \hat{c}_s) \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right)\right] \;-\; \frac{\rho+\beta}{\rho}c_s \,. \tag{4.3.7}$$

Under the condition (4.3.5), the term inside brackets is positive (Lemma B.3). Then the function $\lambda_Z^{(Q)}(t)$ is negative and increasing, and it is bounded on the interval $(-\infty, t^{QP}]$: its limit when $t \to -\infty$ is $-(\rho+\beta)c_s/\rho$. This limit is the point Ω introduced in Section 3.5.2, Equation (3.5.12).

The condition $\underline{\gamma}_y(t) \geq 0$, or equivalently, $\lambda_Z(t) \geq (c_x - c_y)/\zeta$ is required for Phase Q. Given the value of $\lambda_Z(t)$ in (4.3.7), this condition is equivalent to:

$$\begin{aligned}
\frac{c_x - c_y}{\zeta} \;&\leq\; e^{\rho(t-t^{QP})}\left[\frac{\rho+\beta}{\rho}(c_s - \hat{c}_s) \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right)\right] \;-\; \frac{\rho+\beta}{\rho}c_s \\
\frac{c_x - c_y}{\zeta} \;+\; \frac{\rho+\beta}{\rho}c_s \;&\leq\; e^{\rho(t-t^{QP})}\left[\frac{\rho+\beta}{\rho}(c_s - \hat{c}_s) \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right)\right] \,.
\end{aligned}$$

The left-hand side of this inequality is $(\rho+\beta)(c_s - \bar{c}_s)/\rho$. The inequality is therefore automatically satisfied if $c_s \leq \bar{c}_s$. If $c_s > \bar{c}_s$, it is equivalent to:

$$t - t_{QP} \;\;\geq\;\; \frac{1}{\rho}\,\log\left(\frac{\frac{c_x - c_y}{\zeta} + c_s\frac{\rho+\beta}{\rho}}{\frac{1}{\zeta}\left(c_x - u'\left(\overline{x} - \frac{\beta}{\zeta}S^{QP}\right)\right) + c_s\frac{\rho+\beta}{\rho}}\right) \,. \tag{4.3.8}$$

We conclude that $\underline{\gamma}_y(t) \geq 0$ for all t if $c_s \leq \bar{c}_s$, and for all t satisfying (4.3.8) if $c_s > \bar{c}_s$.

Since the function $\lambda_Z^{(Q)}$ is increasing, then $\lambda_Z^{(Q)}(t) \leq \lambda_Z^{(Q)}(t^{QP})$, then it follows that

$$x^{(Q)}(t) \;=\; q^d(c_x - \zeta\lambda_Z^{(Q)}(t)) \;<\; \overline{x} - \frac{\beta}{\zeta}S^{QP} \;<\; \overline{x} \,.$$

As a consequence, $\dot{S}^{(Q)}(t) = \zeta(x(t) - \overline{x}) < -\beta S^{QP}/\zeta < 0$. This property implies that equation (4.3.1) can be solved for every value of $S^0 \in [S^{QP}, S_m]$: the solution gives the value of $t^{QP} - t^0$.

We summarize the solution just constructed in the following result.

Lemma 4.6. *Assume that Assumption 1 holds, that $u'(\cdot)$ is convex, and that $\hat{c}_s \leq c_s \leq c_{sm}$. Let t^0 be an arbitrary time instant. Denote with S^{QP} the unique solution to Equation (4.3.4), and with t^{QP} the unique solution to equation (4.3.1). Then the following trajectory is optimal:*

for $S^0 \leq S^{QP}$: *there is no Phase Q; the trajectory is in Phase P, starting from $S(t^0) = S^0$, as described in Lemma 4.1;*

for $S^0 > S^{QP}$**:** *for* $t \in [t^0, t^{QP}]$*, the trajectory is in Phase Q, starting from* $S(t^0) = S^0$*, and described by Equations* (4.2.2)–(4.2.5) *(equivalently, Equations* (4.3.7) *and* (4.2.1) *for adjoint variables); for* $t \in [t^{QP}, \infty)$*, the trajectory is in Phase P, starting from* $S(t^{QP}) = S^{QP}$*, as described in Lemma 4.1, for every value of* t^0*, restricted to satisfy Condition* (4.3.8) *in case* $\bar{c}_s < c_s \leq c_{sm}$*.*

Proof. The only constraint not checked yet is $\nu_Z \geq 0$. From (2.3.10), $\nu_Z = \dot{\lambda}_Z - (\rho + \alpha)\lambda_Z$. We have observed that λ_Z is negative and increasing. This difference is therefore always positive. □

The result is not explicit on the exact range of values or S^0 for which the trajectory starts in Phase Q. We come back to this point in Section 4.3.1.3.

4.3.1.2 Phases R/P

Assume the system is in Phase R at time t^0, with initial position $(S^0, \overline{Z})$, and that it passes from Phase R to Phase P at time t^{RP} and location $S^{RP} := S(t^{RP})$ which will be determined soon.

When in Phase R, the evolution of state and adjoint variables is given by (see (4.2.6) and (4.2.7)):

$$\begin{aligned}
S^{(R)}(t) &= S^{(R)}(t^0)e^{-\beta(t-t^0)} \\
\lambda_Z^{(R)}(t) &= -\frac{c_y - c_x}{\zeta} \\
\lambda_S^{(R)}(t) &= \left(\lambda_S^0 + \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta}\right) e^{(\rho+\beta)(t-t^0)} \;-\; \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta}\;. \qquad (4.3.9)
\end{aligned}$$

On the other hand, the equations for a terminal Phase P, starting in $S(t^{RP}) = S^{RP}$ are (see (4.1.11) and (4.1.12) on page 31):

$$\begin{aligned}
\lambda_Z^{(P)} &= \frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{RP}e^{(t-t^{RP})})\right) \\
\lambda_S^{(P)} &= \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}M(S^{RP}e^{-\beta(t-t^{RP})})\;.
\end{aligned}$$

The continuity of the adjoint variables imposes that $\lambda_Z^{(P)}(t^{RP}) = \lambda_Z^{(R)}(t^{RP})$ and $\lambda_S^{(P)}(t^{RP}) = \lambda_S^{(R)}(t^{RP})$. The first condition implies:

$$\begin{aligned}
\frac{c_x - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{RP})\right) &= -\frac{c_y - c_x}{\zeta} \\
\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{RP})) &= \overline{p} - c_y \\
S^{RP} &= \frac{\zeta}{\beta}(\overline{x} - \tilde{y}) \;=\; S_{\tilde{y}}\;, \qquad (4.3.10)
\end{aligned}$$

according to the definition of $S_{\tilde{y}}$ in (3.5.19). The second condition implies:

$$\left(\lambda_S^0 + \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta}\right) \;=\; e^{-(\rho+\beta)(t^{RP}-t^0)}\left[\frac{\beta}{\rho+\beta}\,\frac{c_y - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}M(S^{RP})\right]\;.$$

Using the value of S^{RP} determined in (4.3.10) and replacing in (4.3.9), we obtain the value of $\lambda_S^{(R)}(t)$ for $t \leq t^{RP}$:

$$\lambda_S^{(R)}(t) \;=\; e^{(\rho+\beta)(t-t^{RP})}\left[\frac{\beta}{\rho+\beta}\,\frac{c_y - \overline{p}}{\zeta} \;+\; \frac{1}{\zeta}M(S_{\tilde{y}})\right] \;-\; \frac{\beta}{\rho+\beta}\,\frac{c_y - c_x}{\zeta}\;.$$

Introducing the number c_{sm} defined in (4.3.6), the term inside brackets is actually:

$$\frac{\beta}{\rho+\beta}\frac{c_y-\overline{p}}{\zeta} + \frac{1}{\zeta}M(S_{\widetilde{y}}) = \frac{\beta}{\rho+\beta}\frac{c_y-\overline{p}}{\zeta} - \frac{c_y-c_x}{\zeta} - \frac{\beta}{\rho+\beta}\frac{c_x-\overline{p}}{\zeta} = c_{sm} - \bar{c}_s \ ,$$

where $\bar{c}_s$ is defined in (3.5.22). In other terms, the function $\lambda_S^{(R)}(t)$ is:

$$\lambda_S^{(R)}(t) = (c_{sm}-\bar{c}_s)e^{(\rho+\beta)(t-t^{RP})} - \frac{\beta}{\rho+\beta}\frac{c_y-c_x}{\zeta} \ . \tag{4.3.11}$$

Finally, the value of $\gamma_s^{(R)}$ is computed as:

$$\begin{aligned} \gamma_s^{(R)}(t) &= \lambda_Z^{(R)}(t) - \lambda_S^{(R)}(t) + c_s \\ &= c_s - \bar{c}_s - (c_{sm}-\bar{c}_s)e^{(\rho+\beta)(t-t^{RP})} \ . \end{aligned} \tag{4.3.12}$$

We have seen above that $c_{sm} - \bar{c}_s > 0$. The function $\lambda_S^{(R)}$ is therefore increasing, and the function $\gamma_s^{(R)}$ is decreasing on the interval $(-\infty, t^{RP}]$. Its limit when $t \to -\infty$ is $c_s - \bar{c}_s$ and its value at $t = t^{RP}$ is $\gamma_s^{(R)}(t^{RP}) = c_s - c_{sm}$. The function $\gamma_s^{(R)}$ is positive on the interval $(-\infty, t^{RP}]$ if and only if this value is positive, and this is equivalent to: $c_s \geq c_{sm}$. As argued in Section 4.2.3 (Lemma 4.5), $\nu_Z \geq 0$ in Phase R.

The results are summarized as follows.

Lemma 4.7. *Assume that Assumption 1 holds and $c_s \geq c_{sm}$. Then for every $S^0 \in [S_{\widetilde{y}}, S_m]$ the following trajectory is optimal. Let $t^{RP} = t^0 - \beta^{-1}\log(S_{\widetilde{y}}/S^0)$. For $t \in [t^0, t^{RP}]$: the trajectory is in Phase R, as described in Lemma 4.5; for $t \in [t^{RP}, \infty)$: the trajectory is in Phase P, as described in Lemma 4.1.*

4.3.1.3 Phases R/Q

In Section 4.3.1.1, we have left open the issue of whether Phase Q can start from any initial $S^0 \in [S^{QP}, S_m]$, where S^{QP} solves Equation (4.3.4). We resolve this issue by considering the possibility that a Phase R precedes Phase Q.

Assume the system is in phase R at time t^0, with initial position $(S^0, \overline{Z})$, and that it passes from phase R to phase Q at time t^{RQ} and location $S^{RQ} = S(t^{RQ})$.

When in Phase R, the evolution of the state is $S(t) = S^0 e^{-\beta(t-t^0)}$ and that of the adjoint variables is given by (4.2.6) and (4.2.7), see also Section 4.3.1.2. The condition $S(t^{RQ}) = S^{RQ}$ provides the value of $t^{RQ} - t^0 = -\beta^{-1}\log(S^{RQ}/S^0)$.

On the other hand, assuming that Phase Q is followed by Phase P, we have the form (4.3.7) for $\lambda_Z^{(Q)}$, and we have the relation $\lambda_S^{(Q)} = \lambda_Z^{(Q)} + c_s$ which characterizes Phase Q. The continuity of the adjoint variables imposes that $\lambda_Z^{(Q)}(t^{RQ}) = \lambda_Z^{(R)}(t^{RQ})$ and $\lambda_S^{(Q)}(t^{RQ}) = \lambda_S^{(R)}(t^{RQ})$. The first condition writes just as:

$$\lambda_Z^{(Q)}(t^{RQ}) = -\frac{c_y-c_x}{\zeta} \ .$$

According to (4.3.8) on page 37 and the reasoning preceding it, this equation can be solved only for $c_s > \bar{c}_s$ and we get:

$$t_{RQ} - t_{QP} = \frac{1}{\rho}\log\left(\frac{\frac{c_x-c_y}{\zeta} + c_s\frac{\rho+\beta}{\rho}}{\frac{1}{\zeta}\left(c_x - u'\left(\overline{x} - \frac{\beta}{\zeta}S^{QP}\right)\right) + c_s\frac{\rho+\beta}{\rho}}\right) \ . \tag{4.3.13}$$

Remember that S^{QP} itself depends on c_s since it is defined as the solution of (4.3.4). The continuity of λ_S at $t = t^{RQ}$ provides the value of λ_S^0, and then:

$$\lambda_S^{(R)}(t) = (c_s - \bar{c}_s)\, e^{(\rho+\beta)(t-t^{RQ})} - \frac{\beta}{\rho+\beta}\frac{c_y-c_x}{\zeta} \ . \tag{4.3.14}$$

As in Section 4.3.1.2, we conclude that if $c_s \geq \bar{c}_s$, then $\lambda_S^{(R)}$ is increasing, and $\gamma_s^{(R)}$ is decreasing. It is 0 at $t = t^{RQ}$, therefore it is positive for $t \leq t^{RQ}$.

Finally, dynamics of the state in Phase Q are given by (3.5.16), which yields

$$\begin{aligned}
S(t) &= S^{QP} + \zeta \int_{t^{QP}}^{t} (x^{(Q)}(t) - \overline{x}) \mathrm{d}t \;=\; S^{QP} + \zeta \int_{t^{QP}}^{t} (q^d(c_x - \lambda_Z^{(Q)}(t)) - \overline{x}) \; \mathrm{d}t \\
&= S^{QP} - \zeta \overline{x}(t - t^{QP}) \\
&\quad + \zeta \int_{t^{QP}}^{t} q^d \left(c_x - \zeta \left(e^{\rho(t - t^{QP})} \left[\frac{\rho + \beta}{\rho} c_s + \frac{1}{\zeta} \left(c_x - u'(\overline{x} - \frac{\beta}{\zeta} S^{QP}) \right) \right] \; - \; \frac{\rho + \beta}{\rho} c_s \right) \right) \mathrm{d}t \\
&= S^{QP} - \zeta \overline{x}(t - t^{QP}) \\
&\quad + \zeta \int_{0}^{t - t^{QP}} q^d \left(c_x + \frac{\rho + \beta}{\rho} \zeta c_s - e^{\rho v} \left[\frac{\rho + \beta}{\rho} \zeta c_s + c_x - u'(\overline{x} - \frac{\beta}{\zeta} S^{QP}) \right] \right) \mathrm{d}v \; .
\end{aligned}$$

In particular, the value of the stock S at the time the system passes from Phase R to Phase Q is given by:

$$\begin{aligned}
S^{RQ} &= S^{QP} - \zeta \overline{x}(t^{RQ} - t^{QP}) \\
&\quad + \zeta \int_{0}^{t^{RQ} - t^{QP}} q^d \left(c_x + \frac{\rho + \beta}{\rho} \zeta c_s - e^{\rho v} \left[\frac{\rho + \beta}{\rho} \zeta c_s + c_x - u'(\overline{x} - \frac{\beta}{\zeta} S^{QP}) \right] \right) \mathrm{d}v \; .
\end{aligned} \tag{4.3.15}$$

Depending on the value of c_s, this value S^{RQ} is smaller than S_m or not.

This leads us to introduce a new threshold for c_s: this value c_{sQ} is such that Phase R "just disappears" at the stock value $S = S_m$. More precisely, we have simultaneously:

$$S^{(Q)}(t^{RQ}) \;=\; S_m, \qquad \lambda_Z^{(Q)}(t^{RQ}) \;=\; \frac{c_x - c_y}{\zeta} \quad \text{or, equivalently:} \quad x^{(Q)}(t^{RQ}) \;=\; \widetilde{y} \; .$$

Given the formula above for S^{RQ}, we have the equivalent form:

$$\begin{aligned}
S_m &= S^{QP} - \zeta \overline{x}(t^{RQ} - t^{QP}) \\
&\quad + \zeta \int_{0}^{t^{RQ} - t^{QP}} q^d \left(c_x + \frac{\rho + \beta}{\rho} \zeta c_s - e^{\rho v} \left[\frac{\rho + \beta}{\rho} \zeta c_s + c_x - u'(\overline{x} - \frac{\beta}{\zeta} S^{QP}) \right] \right) \mathrm{d}v \; .
\end{aligned} \tag{4.3.16}$$

The values of S^{QP} and $t^{RQ} - t^{QP}$ are given respectively by (4.3.4) and (4.3.13). They are themselves functions of c_s. The number c_{sQ} is the unique solution of this equation; it belongs to the interval $[\bar{c}_s, c_{sm}]$.

We are now in position to complete Lemma 4.6. Having checked that $\gamma_s \geq 0$ and $S(t) \geq S_{\widetilde{y}}$ when Phase R is involved, we have by Lemma 4.5:

Lemma 4.8. *Assume that Assumption 1 holds, that $u'(\cdot)$ is convex, and that $\hat{c}_s \leq c_s \leq c_{sm}$. Denote with S^{QP} the unique solution to Equation (4.3.4) (it exists according to Lemma B.2). Then:*

if $\hat{c}_s \leq c_s \leq c_{sQ}$ *for every $S^0 \in [S^{QP}, S_m]$, the trajectory starting in Phase Q as described in Lemma 4.6 is optimal; there is no Phase R.*

if $c_{sQ} < c_s \leq c_{sm}$ *: Let S^{RQ} be defined by (4.3.15). Then $S^{RQ} \leq S_{\widetilde{y}}$, and*

- **for** $S^0 \in [S^{QP}, S^{RQ}]$**:** *for every $S^0 \in [S^{QP}, S_m]$, the trajectory starting in Phase Q as described in Lemma 4.6 is optimal; there is no Phase R.*
- **for** $S^0 \in [S^{RQ}, S_m]$**:** *the trajectory starting from $S(t^0) = S^0$, staying in Phase R for $t \in [t^0, t^{RQ}]$ (where $t^{RQ} = t^0 - \beta^{-1} \log(S^{RQ}/S^0)$), and continuing in Phase Q for $t \in [t^{RQ}, \infty)$ from $S(t^{RQ}) = S^{RQ}$, as described in Lemma 4.6, is optimal.*

Concluding this paragraph, we observe that controls are discontinuous at time t^{RQ}. Indeed, we have, from the values of control in phases Q and R, and the fact that $\lambda_Z(t^{RQ}) = (c_x - c_y)/\zeta$:

$$
\begin{array}{lll}
x(t^{RQ-}) = \dfrac{\beta}{\zeta}(S_m - S^{RQ}) & s(t^{RQ-}) = 0 & y(t^{RQ-}) = \dfrac{\beta}{\zeta}(S^{RQ} - S_{\widetilde{y}}) \\
x(t^{RQ+}) = \widetilde{y} & s(t^{RQ+}) = \zeta\widetilde{y} - \beta(S_m - S^{RQ}) & y(t^{RQ+}) = 0.
\end{array}
$$

On the other hand, the current-value Hamiltonian $H(t) := u(x(t) + y(t)) - c_s s(t) - c_x x(t) - c_y y(t) + \lambda_Z(t)[-\alpha Z(t) + \beta S(t) + \zeta x(t) - s(t)] + \lambda_S(t)[-\beta S(t) + s(t)]$ is continuous at time t^{RQ}, which we check now. Since the trajectory is such that $Z(t) = \overline{Z}$, the value of $[-\alpha Z + \beta S + \zeta x - s]$ is identically 0. Next, we have $\lambda_S(t^{RQ}) = c_s + (c_x - c_y)/\zeta$. Also, the total energy consumption $x(t) + y(t)$ is continuous at $t = t^{RQ}$ with value $\widetilde{y}$. Then,

$$
\begin{aligned}
H(t^{RQ-}) &= u(\widetilde{y}) - c_x\frac{\beta}{\zeta}(S_m - S^{RQ}) - c_y\frac{\beta}{\zeta}(S^{RQ} - S_{\widetilde{y}}) - \beta S^{RQ}\left(c_s + \frac{c_x - c_y}{\zeta}\right) \\
&= u(\widetilde{y}) - c_y\widetilde{y} + \frac{c_y - c_x}{\zeta}\beta(S_m - S^{RQ}) - \beta S^{RQ}\left(c_s + \frac{c_x - c_y}{\zeta}\right) \\
&= u(\widetilde{y}) - c_y\widetilde{y} + \frac{c_y - c_x}{\zeta}\beta S_m - c_s\beta S^{RQ} \\
H(t^{RQ+}) &= u(\widetilde{y}) - c_x\widetilde{y} - c_s s(t^{RQ+}) + \left(c_s + \frac{c_x - c_y}{\zeta}\right)(-\beta S^{RQ} + s(t^{RQ+})) \\
&= u(\widetilde{y}) - c_x\widetilde{y} + \frac{c_x - c_y}{\zeta}\left(\zeta\widetilde{y} - \beta(S_m - S^{RQ})\right) - \beta S^{RQ}\left(c_s + \frac{c_x - c_y}{\zeta}\right) \\
&= u(\widetilde{y}) - c_y\widetilde{y} + \frac{c_y - c_x}{\zeta}\beta S_m - c_s\beta S^{RQ}\ .
\end{aligned}
$$

4.3.2 Synthesis on the boundary $Z = \widetilde{Z}(S)$, large c_s

At this point, we have a complete description of optimal trajectories starting from initial points on the boundary $Z = \widetilde{Z}(S)$.

The situation of phases is summarized in Figure 4.1 (page 42). This figure depicts the optimal consumption $x(t)$, $y(t)$ and capture $s(t)$ as a *state feedback*. As a function of time, $S(t)$ is decreasing (or constant if $c_s = \hat{c}_s$) so that the evolution occurs from right to left. Capture is represented as $s(t)/\zeta$ in order to make an easier comparison with its maximum value $x(t)$.

The different cases are detailed as follows. The trajectory of interest is starting at $S = S_M$ and $Z = 0$. In all situations, the optimal trajectory is in Phase U (see Section 3.5.6) as long as $S > S_m$. What happens next depends on c_s.

Case $c_s \geq c_{sm}$. In this situation, the sequence of phases is $U/R/P$ (Lemma 4.7 on Page 39). Capture $s(t)$ is zero at all times. Consumption $x(S)$ is a straight line with slope $-\overline{x}/S_m = -\beta/\zeta$, for $S \leq S_m$, see (3.5.2) and (3.5.17). Consumption $y(S)$ is a straight line with slope $\overline{x}/S_m$ for $S_{\widetilde{y}} \leq S \leq S_m$, see (3.5.18). Both paths $x(t)$ and $y(t)$ are continuous.

Case $c_{sm} \geq c_s \geq c_{sQ}$. In this situation, the sequence of phases is $U/R/Q/P$ (Lemma 4.8 on Page 40). As observed at the end of Section 4.3.1.3, the consumption/capture paths $x(t)$, $s(t)$ and $y(t)$ are continuous except for a discontinuity at $t = t^{RQ}$ (i.e. when $S = S^{RQ}$). The function $x(t) + y(t)$ is continuous everywhere. When in Phase Q, $x(t)$ and $s(t)$ are not straight lines, contrary to what the figure suggests for ease of representation. See also Appendix E.3 on page 105. However, $x(t) - s(t)/\zeta$ is linear, according to (4.2.3).

Case $c_{sQ} \geq c_s > \hat{c}_s$. In this situation, the sequence of phases is reduced to $U/Q/P$ (Lemma 4.8 on Page 40). The paths $x(t)$, $s(t)$ and $y(t)$ are continuous except for a discontinuity at $t = t^{UQ}$ (i.e. when $S = S^{UQ} = S_m$). The function $x(t) + y(t)$ is also discontinuous at that point.

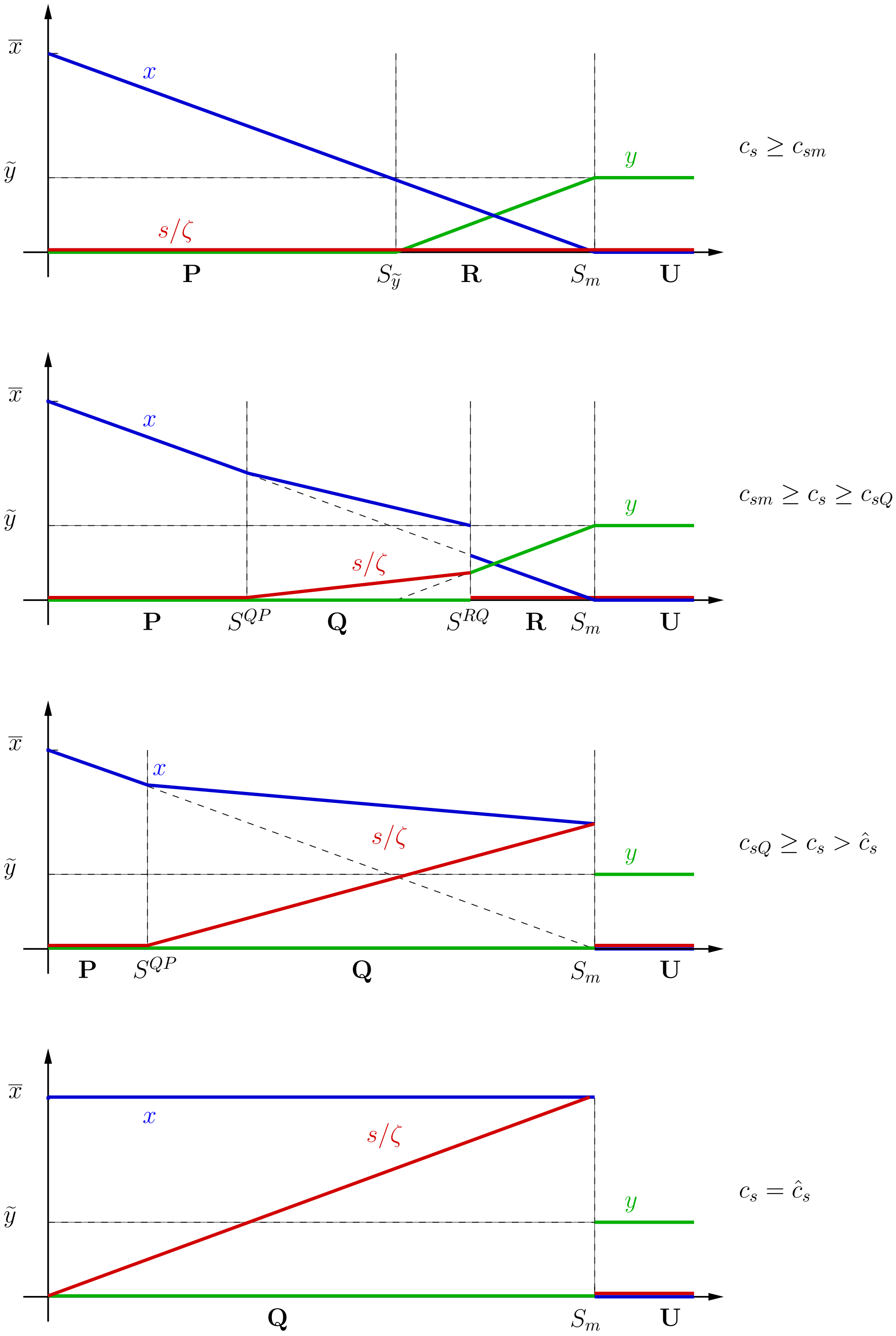

Figure 4.1: Phases on the boundary for $c_s \geq \hat{c}_s$: optimal controls as state feedback

Case $c_s = \hat{c}_s$**.** In this particular situation, the sequence of phases is U/Q, but all points in phase Q are stationary (Lemma 4.4 on Page 34). The paths $x(t)$, $s(t)$ and $y(t)$ are continuous except for a discontinuity at $t = t^{UQ} = S_m$. The function $x(t) + y(t)$ is also discontinuous at that point. The function $s(t)$ is a straight line as a function of $S(t)$.

4.4 Junction with the boundary

The previous section has addressed pieces of optimal trajectories included in the boundary $Z = \widetilde{Z}(S)$. We now study how optimal trajectories located inside the domain join this boundary. It turns out that, depending on the value of the parameters, two types of junctions take place. One is a "regular" junction, with continuity of state and adjoint variables: we will show in Section 4.4.2 that it takes place with the boundary phases called P, Q, R and U. The second one is a junction at the particular location $(S_m, \overline{Z})$, with a discontinuity in the adjoint variable λ_Z. We term these "singular" junctions and analyze them in Section 4.4.3.

We start the analysis with the introduction in Section 4.4.1 of useful properties of adjoint variables, and a very convenient graphical representation.

4.4.1 Evolution of adjoint variables

Figure 4.2 represents the phase diagram of adjoint variables (λ_Z, λ_S) governed by equations (3.2.2), together with several particular values, curves, zones and locations.

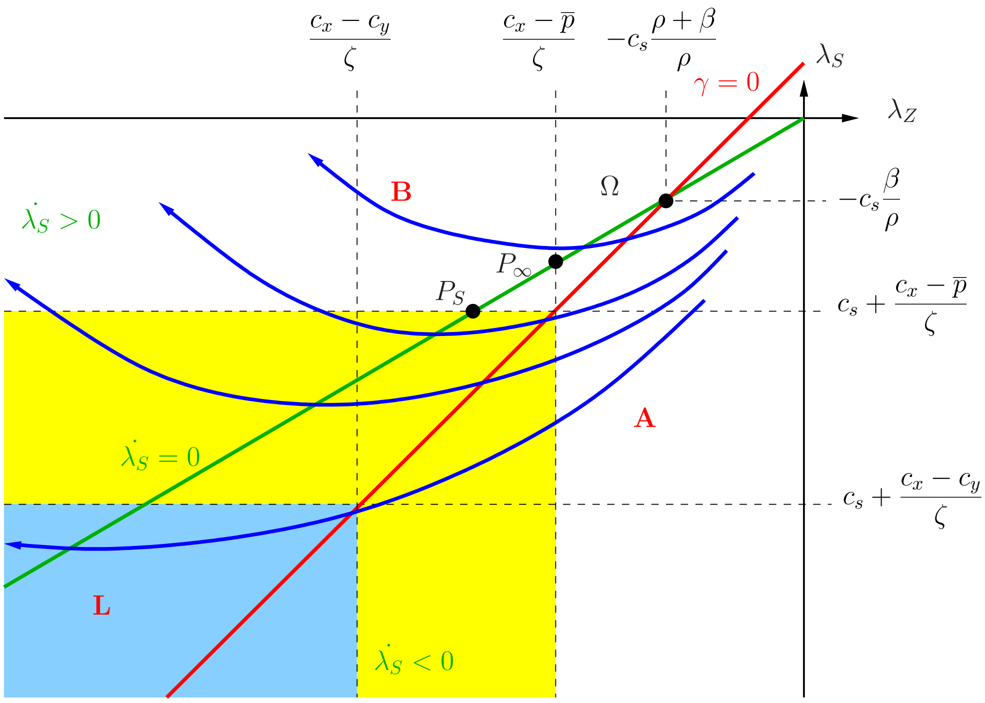

Figure 4.2: Trajectories of adjoint variables through phases A, B and L

Trajectories of $(\lambda_Z(t), \lambda_S(t))$ are represented as blue lines. They all reach their minimal value on the green line of equation $(\rho + \beta)\lambda_S = \beta\lambda_Z$, which is the locus of points where $\dot{\lambda}_S = 0$. When above this curve, $\lambda_S(t)$ increases, and it decreases below. In all cases, $\lambda_Z(t)$ is decreasing.

The zones corresponding to Phases A and B are delimited by the red line $\gamma = \lambda_S - \lambda_Z - c_s = 0$. Phase A is below the line, Phase B is above it. The zone corresponding to Phase L is represented in blue. It is separated from Phase A by the line $\lambda_Z = (c_x - c_y)/\zeta$ (corresponding to

$x^{(A)} = q^d(c_x - \zeta\lambda_Z) = \widetilde{y}$) and from Phase B by the line $\lambda_S = c_s + (c_x - c_y)/\zeta$ (corresponding to $x^{(B)} = q^d(c_x + \zeta c_s - \zeta\lambda_S) = \widetilde{y}$). Inside this blue zone, the value of $x(t)$ as given by first-order conditions of Phase A or Phase B is less than $\widetilde{y}$. As a consequence of Lemma 3.2, this means that $y = \widetilde{y}$, that is, Phase L, is optimal.[1]

Dashed lines $\lambda_Z = (c_x - \overline{p})/\zeta$ and $\lambda_S = c_s + (c_x - \overline{p})/\zeta$ correspond to values of the control x equal to $\overline{x}$, in Phase A and Phase B respectively. The yellow zone represents values where the optimal control is $\widetilde{y} \le x \le \overline{x}$, whatever the phase. In the zone outside it, $x > \overline{x}$.

Figure 4.2 also features some particular points:

$$P_S = \left(\frac{\rho+\beta}{\beta}\left(c_s + \frac{c_x - \overline{p}}{\zeta}\right), c_s + \frac{c_x - \overline{p}}{\zeta}\right) \qquad P_\infty = \left(\frac{c_x - \overline{p}}{\zeta}, \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta}\right).$$

These correspond, respectively, to the stationary Phase S (Lemma 4.2), and to the limiting values in terminal Phase P when $t \to \infty$ (see (4.1.9) on page 31).

Finally, point Ω has been introduced in (3.5.12). It is shown in Lemma 4.4 that this point corresponds to a stationary solution in the specific case $c_s = \hat{c}_s$. In every case, it is a *repulsive* point for the dynamics of (λ_Z, λ_S) in Phase Q: in this phase the adjoint variables move on the red line $\gamma = 0$ away from point Ω. See the discussion of Section 3.5.2 on page 24.

Observe that the elements in black and red on Figure 4.2 depend only on cost parameters (c_x, c_s, c_x, $\overline{p}$) and ζ, whereas blue and green elements depend only on α, β and ρ. This separation is not perfect though, because $\overline{p}$ is determined by the cost function $u(\cdot)$ and the special consumption value $\overline{x}$, which itself is defined with $\overline{Z}$, β and α.

Depending on values of the parameters, the green line $\dot{\lambda}_S = 0$ may enter the blue zone Phase L either by its horizontal boundary, or by its vertical one. In the first case, the corner of the Phase L zone is below the line, which translates as:

$$(\rho+\beta)\left(c_s + \frac{c_x - c_y}{\zeta}\right) \le \beta\,\frac{c_x - c_y}{\zeta} \qquad \Longleftrightarrow \qquad c_s \le \bar{c}_s,$$

where $\bar{c}_s$ has been defined in (3.5.22). This is the situation represented in Figure 4.2, see also Figure 4.5. The other situation is represented in Figure 4.3 on page 49.

4.4.2 Regular junctions

Depending on the value of c_s, optimal trajectories in the interior can be in Phase A and join continuously (both for state and adjoint variables) the boundary. We review these cases in this section. In all of them, imposing the continuity of λ_S at the junction point is sufficient for obtaining a solution.

4.4.2.1 Junction with Phase P

Lemma 4.9. *Assume that Assumption 1 holds, that $u'(\cdot)$ convex and that $c_s > \hat{c}_s$. Then the following trajectory is optimal. The trajectory is in Phase A, characterized by $x(t) = q^d(c_x - \zeta\lambda_Z(t))$, $s(t) = y(t) = 0$, $S(t) = S^0 e^{-\beta(t-t^0)}$, and*

$$Z(t) = Z^0 e^{-\alpha(t-t^0)} + S^0\frac{\beta}{\alpha-\beta}\left(e^{-\beta(t-t^0)} - e^{-\alpha(t-t^0)}\right) + \zeta\int_{t^0}^{t} e^{-\alpha(t-u)} q^d(c_x - \zeta\lambda_Z(u))du \qquad (4.4.1)$$

and λ_S, λ_Z given by Equations (3.2.4) *and* (3.2.5)*, for $t \in [t^0, t^{AP}]$, where t^{AP} solves the equation $Z(t^{AP}) = \overline{Z}$. Then the trajectory continues in Phase P as described in Lemma 4.1.*

[1] In terms of "cleaned" and "dirty" carbon consumption (see Footnote 4 of Chapter 2 on page 7), the blue zone corresponds to the consumption of only renewable energy ($x_c = x_d = 0$), the zone labelled "A" (below the red line and to the right of the blue zone) corresponds to only dirty carbon consumption ($x_c = y = 0$) and the zone labelled "B" corresponds to only cleaned carbon consumption ($x_d = y = 0$). Mixed consumption is possible only when costate variables move on the boundary of these zones.

Such trajectories are illustrated for instance in Figures 4.11 and 4.12 on page 58.

Proof. Since the trajectory described here is continuous, with piecewise continuously differentiable λ_S and λ_Z, we will apply Corollary 3.1. It is necessary to check that the constraints $\gamma_s(t) \geq 0$ and $\underline{\gamma}_y(t) \geq 0$ are satisfied on the trajectory.

Observe that in both Phases A and P, $\underline{\gamma}_y = c_y - c_x + \zeta\lambda_Z$. It is therefore a continuous function on the trajectory. In Phase P, $\underline{\gamma}_y \geq 0$ so that $\underline{\gamma}_y(t^{AP}) \geq 0$. Since λ_Z is decreasing in Phase A, so is $\underline{\gamma}_y$ and we have for all $t \in [t^0, t^{AP}]$: $\underline{\gamma}_y(t) \geq 0$.

Likewise, $\gamma_s = \lambda_Z - \lambda_S + c_s$ in both phases A and P, and it is positive in Phase P, hence at time $t = t^{AP}$. A straightforward variation analysis based on observations in Section 3.2.1 reveals that the general behavior of $\gamma_s(t)$ is as follows. Starting from $t \to -\infty$, γ_s starts from c_s then increases, then decreases, goes through 0 and tends to $-\infty$ when $t \to +\infty$. Therefore, it is necessarily positive on the interval $[t^0, t^{PA}]$ since it is positive at the end of the interval. □

Observe also that when in Phase A, we have $\dot{Z} > 0$, which justifies the idea that there are initial values (S^0, Z^0), $Z^0 < \overline{Z}$, such that $Z(t) = \overline{Z}$ ($Z(t)$ given by (4.4.1)) has actually a solution. Since in Phase A we have $x(t) = q^d(c_x - \zeta\lambda_Z(t))$ and λ_Z is decreasing, $x(t)$ is decreasing as well. Its value at t^{AP} is $x^{(P)}(t^{AP}) = \overline{x} - \beta S^{AP}/\zeta$. Then we can write (remember that $\alpha\overline{Z} = \zeta\overline{x}$):

$$\dot{Z} = -\alpha Z + \beta S + \zeta x = \alpha(\overline{Z} - Z) + \beta(S - S^{AP}) + \zeta(x - \overline{x} + \beta S^{AP}/\zeta) \ .$$

Since S is also decreasing in Phase A, it is always larger than S^{AP}. Then all three terms in this expression are positive. They all vanish at t^{AP}, which means that $\dot{Z}(t^{AP}) = 0$: the trajectory joins the ceiling $Z = \overline{Z}$ tangentially. See also Section D.1 in the Appendix, page 96.

The set of initial positions (S^0, Z^0) of trajectories which satisfy Lemma 4.9 is limited by the particular trajectory which joins point $(S^{QP}, \overline{Z})$ (when $c_s \in (\hat{c}_s, c_{sm}]$) or point $(S_{\widetilde{y}}, \overline{Z})$ (when $c_s \geq c_{sm}$).

4.4.2.2 Junction with Phase Q

The geometric position of $(\lambda_Z^{(Q)}(t), \lambda_S^{(Q)}(t))$ also allows to construct consistent continuous trajectories where Phase A joins Phase Q.

Lemma 4.10. *Assume that Assumption 1 holds, that $u'(\cdot)$ convex and that $\hat{c}_s \leq c_s \leq c_{sm}$. Then the following trajectory is optimal. The trajectory is in Phase A (see its equations in Lemma 4.9) for $t \in [t^0, t^{AQ}]$, where t^{AQ} solves the equation $Z^{(A)}(t^{AQ}) = \overline{Z}$. Then the trajectory continues in Phase Q as described in Lemma 4.4 (if $c_s = \hat{c}_s$), or Lemmas 4.6 and 4.8 (if $\hat{c}_s < c_s \leq c_{sm}$).*

Such trajectories are illustrated for instance in Figure 4.8 page 54, Figure 4.12 page 58 or Figure 4.17 page 62.

Proof. The proof is similar to that of Lemma 4.9, with the difference that $\gamma_s(t^{AQ}) = 0$ instead of being positive. One concludes nevertheless that $\underline{\gamma}_y$ and γ_s are both positive. □

4.4.2.3 Junction with Phase R

It is also possible to construct consistent continuous trajectories where Phase A joins Phase R.

Lemma 4.11. *Assume that Assumption 1 holds, that $u'(\cdot)$ convex and that $c_s \geq c_{sQ}$. Then the following trajectory is optimal. The trajectory is in Phase A (see its equations in Lemma 4.9) for $t \in [t^0, t^{AR}]$, where t^{AR} solves the equation $Z(t^{AR}) = \overline{Z}$. Then the trajectory continues in Phase R as described in Lemma 4.7 or Lemma 4.8.*

Such trajectories are illustrated for instance in Figure 4.17, page 62 or Figure 4.20, page 64.

Proof. In Phase R, λ_Z is constant and λ_S, given by Equation (4.3.14), is increasing. According to Lemmas 4.7 and 4.8, the trajectory in Phase R finishes either in Phase Q of in Phase P, with the continuity of λ_S and λ_Z, and therefore of $\gamma_s = \lambda_Z - \lambda_S + c_s$. This function either vanishes at $t = t^{RQ}$ or is positive at $t = t^{RP}$. It is decreasing, therefore it is positive in Phase R (see an illustration in Figure 4.16 or Figure 4.19). The same reasoning as for the proof of Lemma 4.9 can be applied, to conclude that $\underline{\gamma}_y$ and γ_s are both positive. $\square$

The following observation will be useful later on: for all t in Phase R, and in particular for $t = t^{AR}$, $\lambda_Z^{(R)} + c_s \geq \lambda_S(t) \geq \beta\lambda_Z^{(R)}/(\rho+\beta)$, where $\lambda_Z^{(R)} = (c_x - c_y)/\zeta$. This is a consequence of the explicit formulas we have obtained for $\lambda_S(t)$, in (4.3.11) when $c_s \geq c_{sm}$, or (4.3.14) when $c_{sQ} \leq c_s \leq c_{sm}$.

4.4.2.4 Junction with Phase U

It is also possible to construct consistent continuous trajectories which are in Phase A, then hit the curve $Z = Z_M(S)$, then follow this curve in Phase U. We have two ways to prove that such a trajectory is optimal: one with the constraint $Z \leq Z_M(S)$ explicitly taken into account, one without it. Several features are common to both cases.

In both situations, suppose that an optimal trajectory follows the curve $Z = Z_M(S)$ during the time interval $t \in [t^{AU}, t^{UR}]$. At time t^{UR}, the value of S is $S(t^{UR}) = S_m$. At time t^{AU}, a trajectory coming from the interior in Phase A hits the curve $Z = Z_M(S)$. Considering for instance the related finite-horizon problem (see Section C.1.2.2 on page 92) allows to "guess" that

$$\lambda_Z(t^{AU}) \quad = \quad \lambda_Z^{(L)} \;:=\; \frac{c_x - c_y}{\zeta} \; . \tag{4.4.2}$$

In addition, the dynamics of $S(\cdot)$ provide the duration $t^{UR} - t^{AU} = \beta^{-1}\log(S^{AU}\;/S_m)$, which is the time it takes for the trajectory to reach $S = S_m$ starting at $S^{AU} = S(t^{AU})$.

Also, since $x(t) + y(t) = \widetilde{y}$, (2.3.4) implies that $\underline{\gamma}_y(t) = 0$ for all $t \in [t^{AU}, t^{UR}]$, in accordance with (2.3.7). With (2.3.3), this implies

$$\lambda_Z + \gamma_{sx} \quad = \quad \frac{c_x - c_y}{\zeta} \; . \tag{4.4.3}$$

Finally, we have observed at the end of Section 4.4.2.3 that $\beta\lambda_Z^{(L)}/(\rho+\beta) \leq \lambda_S(t^{UR}) \leq \lambda_Z^{(L)} + c_s$.

Solution with an explicit constraint. In this specific situation, we make use of Corollary 3.2. In this case, the dynamics of the adjoint variables during Phase U *do not* obey (2.3.10) and (2.3.11), but rather (2.3.10) and (2.3.30). Let us first develop the computations relevant to this case.

The dynamics of the adjoint variables (2.3.10) and (2.3.30) are then:

$$\begin{aligned}
\dot{\lambda}_Z &= (\rho+\alpha)\lambda_Z + \nu_Z \\
\dot{\lambda}_S &= (\rho+\beta)\lambda_S - \beta\lambda_Z - \nu_Z \widetilde{Z}'_M(S) \; .
\end{aligned}$$

Looking for continuous trajectories, we deduce from (4.4.2) that λ_Z should be constant on the interval $I = [t^{AU}, t^{UR}]$. Equation (4.4.3) is satisfied with $\gamma_{sx} = 0$. Then we must have (as we had in Phase R), $\nu_Z = -(\rho+\alpha)\lambda_Z$ which is positive as required by condition (2.3.29). When replaced in the second equation, we have:

$$\dot{\lambda}_S(t) \;=\; (\rho+\beta)\lambda_S(t) - \beta\frac{c_x - c_y}{\zeta} + (\rho+\alpha)\frac{c_x - c_y}{\zeta}\widetilde{Z}'_M(S(t)) \; , \tag{4.4.4}$$

and $S(t) = S(t^{AU})e^{-\beta(t - t^{AU})}$.

We now show that $\gamma_s(t) \geq 0$ for $t \in I$. Since $\gamma_s(t) = \lambda_Z^{(L)} - \lambda_S(t) + c_s$, we have $\dot{\gamma}_s = -\dot{\lambda}_S$. But because $c_x - c_y < 0$ and $\widetilde{Z}'(S) \leq 0$, we deduce from (4.4.4) that for all $t \in I$,

$$\dot{\lambda}_S(t) \;\geq\; (\rho+\beta)\lambda_S(t) - \beta\lambda_Z^{(L)} \; .$$

We also know that $\lambda_S(t^{UR}) \geq \beta\lambda_Z^{(L)}/(\rho+\beta)$. Thanks to Grönwall's lemma, we deduce that $\lambda_S(t) \geq \beta\lambda_Z^{(L)}/(\rho+\beta)$ for all $t \in I$, and that $\lambda_S(t)$ is increasing. This implies in turn that γ_s is decreasing. Since it is positive at $t = t^{UR}$, it is positive for all $t \in I$.

We now have the elements to prove the following result.

Lemma 4.12. *Assume that Assumption 1 holds, that $u'(\cdot)$ convex and that $c_s > c_{sQ}$. Let t^{AU} be an arbitrary time instant. Then for every $S^{AU} \in [S_m, S_M]$, there exists an optimal trajectory with $S(t^{AU}) = S^{AU}$, $Z(t^{AU}) = Z_M(S^{AU})$, which runs as follows.*

The trajectory is in Phase A (see its equations in Lemma 4.9) for $t \in [t^0, t^{AU}]$, where t^0 solves the equation $Z(t^0) = 0$.

Then the trajectory continues in Phase U, with $\lambda_Z(t) = \lambda_Z^{(L)}$ for all $t \in [t^{AU}, t^{UR}]$, and $\lambda_S(t)$ given by the solution to the differential equation (4.4.4) *with boundary condition at $t = t^{UR}$.*

Then the trajectory continues in Phase R and Phase P, or Phases R, Q, P, as described in Lemma 4.7 or Lemma 4.8 respectively.

Proof. In order to use Corollary 3.2, we construct a trajectory with continuous functions $\lambda_Z(\cdot)$ and $\lambda_S(\cdot)$.

Fix some arbitrary t^{AU} with $S(t^{AU}) = S^{AU}$, $Z(t^{AU}) = Z_M(S^{AU})$.

Continuous trajectories for λ_S and λ_Z are provided for $t \geq t^{UR}$ by Lemma 4.7 or Lemma 4.8, depending on the value of c_s. In particular, from the analysis of Phase R (Section 4.2.3), we know the property that $\lambda_Z(t^{UR}) = \lambda_Z^{(L)}$. Then, defining $\lambda_S(t)$ for $t \in I$ as the solution of (4.4.4) with the boundary condition at $t = t^{UR}$ provides a continuous function over the interval $t \in [t^{AU}, \infty)$. We have checked in the computation above that $\gamma_s(t)$ and $\underline{\gamma}_y(t)$ are both positive for all $t \in [t^{AU}, t^{UR}]$, and therefore for all $t \geq t^{AU}$.

When $t < t^{AU}$, the trajectory is in Phase A, with adjoint variables continuous at $t = t^{AU}$. The proof that λ_S, λ_Z, γ_s and $\underline{\gamma}_y$ are consistent is as in Lemma 4.9. The existence of a time t^0 at which $Z(t^0) = 0$ is implied by the fact that $\dot{Z} = -\alpha Z + \beta S + \zeta x$, with $-\alpha Z + \beta S > 0$ and $x \geq \widetilde{y}$: since $\dot{Z} > \zeta\widetilde{y}$, trajectories in Phase A (taken backwards) necessarily exit the domain in finite time. $\square$

Solution without an additional constraint. We consider here the framework of Corollary 2.1, so that the unique state constraint explicitly enforced is $Z \leq \overline{Z}$. The dynamics of the adjoint variables are then that of Phase L, see Section 3.4.3. Accordingly, we have the following relationship between adjoint variables at the beginning and at the end of the phase:

$$\begin{aligned}
\lambda_Z(t^{UR}) &= \lambda_Z(t^{AU})e^{(\rho+\alpha)(t^{UR}-t^{AU})} \\
\lambda_S(t^{UR}) &= \lambda_S(t^{AU})e^{(\rho+\beta)(t^{UR}-t^{AU})} - \frac{\beta}{\alpha-\beta}\lambda_Z(t^{AU})\left(e^{(\rho+\alpha)(t^{UR}-t^{AU})} - e^{(\rho+\beta)(t^{UR}-t^{AU})}\right).
\end{aligned}$$

Two boundary conditions are enforced if we mean to use Corollary 3.1: λ_S is continuous at $t = t^{UR}$ and λ_Z is continuous at $t = t^{AU}$ because at this point, no state constraint becomes or ceases to be active. The value $\lambda_Z(t^{AU})$ is "guessed" to be as in (4.4.2). Accordingly:

$$\begin{aligned}
\lambda_Z(t^{UR}) &= \frac{c_x - c_y}{\zeta}\left(\frac{S^{AU}}{S_m}\right)^{\frac{\rho+\alpha}{\beta}} \\
\lambda_S(t^{UR}) &= \lambda_S(t^{AU})\left(\frac{S^{AU}}{S_m}\right)^{\frac{\rho+\beta}{\beta}} - \frac{\beta}{\alpha-\beta}\frac{c_x - c_y}{\zeta}\left(\left(\frac{S^{AU}}{S_m}\right)^{\frac{\rho+\alpha}{\beta}} - \left(\frac{S^{AU}}{S_m}\right)^{\frac{\rho+\beta}{\beta}}\right).
\end{aligned}$$

This last equation allows to express the value of $\lambda_S(t^{AU})$ in function of $\lambda_S(t^{UR})$ which is known from Lemma 4.7 or Lemma 4.8:

$$\lambda_S(t^{AU}) \;=\; \lambda_S(t^{UR}) \left(\frac{S_m}{S^{AU}}\right)^{\frac{\rho+\beta}{\beta}} \;+\; \frac{\beta}{\alpha-\beta}\frac{c_x-c_y}{\zeta}\left(\left(\frac{S^{AU}}{S_m}\right)^{\frac{\alpha-\beta}{\beta}}-1\right) .$$

Next, we check the first-order conditions (2.3.2)–(2.3.3) over the time interval $[t^{AU}, t^{UR}]$, Condition (2.3.4) having been checked in the preliminaries. We clearly have $\lambda_Z(t) \leq (c_x - c_y)/\zeta$, so that $\gamma_{sx} \geq 0$, in accordance with (2.3.5). Finally, from (2.3.2), we have

$$\gamma_s \;=\; c_s + \lambda_Z + \gamma_{sx} - \lambda_S \;=\; c_s + \frac{c_x-c_y}{\zeta} - \lambda_S$$

and indeed, since $\lambda_S(\cdot)$ is increasing on interval $[t^{AU}, t^{UR}]$ and $\lambda_S(t^{UR}) \leq c_s + \lambda_Z^{(L)}$ as shown above, we have $\gamma_s \geq 0$ in accordance with (2.3.6).

In summary, we have constructed a trajectory as described in Lemma 4.12, with the difference that λ_Z is not continuous but has a jump at time t^{UR} of magnitude

$$\lambda_Z(t^{UR+}) - \lambda_Z(t^{UR-}) \;=\; \frac{c_x-c_y}{\zeta}\left(1-\left(\frac{S^{AU}}{S_m}\right)^{\frac{\rho+\alpha}{\beta}}\right) \;\geq\; 0 .$$

Such a jump is compatible with condition (2.3.22). It can be checked that the trajectories $\lambda_S(t)$ are the same in both constructions.

Such trajectories are illustrated for instance in Figures 4.16 and 4.17 on page 62.

4.4.3 Singular junctions

The location $(S_m, \overline{Z})$ of the boundary has a particular status. For one thing, we have seen in Lemma 4.2 (page 32) that there exist stationary optimal trajectories staying at that point: what we have called Phase S. This situation happens if and only if $c_s \leq \hat{c}_s$. When $c_s > \hat{c}_s$, this location is not stationary anymore, but may retain its "non-standard" character.

This singular character lies in the fact that λ_Z may have jumps at the time when state $(S_m, \overline{Z})$ is attained, say, at time T. This feature can be "guessed" using the finite-horizon arguments developed in Appendix C.1. When the "final" state of such an optimization problem is constrained to be $(S_m, \overline{Z})$, λ_S is necessarily continuous, but λ_Z is not determined by sufficient conditions. There is however a lower bound on it.

In order to prove the optimality of such trajectories, we shall indeed invoke Corollary 3.1. Accordingly, we shall construct trajectories where the adjoint variable λ_S is continuous and where λ_Z may have one jump at time T. According to (3.1.1), such jumps may occur only *upwards*, that is, $\lambda_Z(T^-) \leq \lambda_Z(T^+)$ where $\lambda_Z(T^+)$ is determined by the remainder of the trajectory.

When setting $\lambda_Z(T)$ to all possible values in $(-\infty, \lambda_Z(T^+))$, we obtain a family of trajectories. For all of them, the state variables end up at point $(S_m, \overline{Z})$ in Phase B, and the adjoint variables end up at point $(\lambda_Z(T), \lambda_S(T))$. During their evolution before time T, these trajectories may actually be in one of three possible phases, according to the sign of $\lambda_S - \lambda_Z - c_s$ (Phases A or B), and to whether the consumption x is larger or smaller than $\widetilde{y}$ (Phase L). Again, two cases must be distinguished: whether $c_s > \hat{c}_s$ or $c_s \leq \hat{c}_s$. We investigate the first situation in Section 4.4.3.1, and the second one in Section 4.4.3.2.

4.4.3.1 Junctions passing through point $(S_m, \overline{Z})$

The situation of adjoint variables when $c_s > \hat{c}_s$ is represented in Figure 4.3. It is assumed that a family of trajectories of $(\lambda_Z(t), \lambda_S(t))$ terminate at some time T with the same value of $\lambda_S(T)$, represented as a horizontal dashed line.

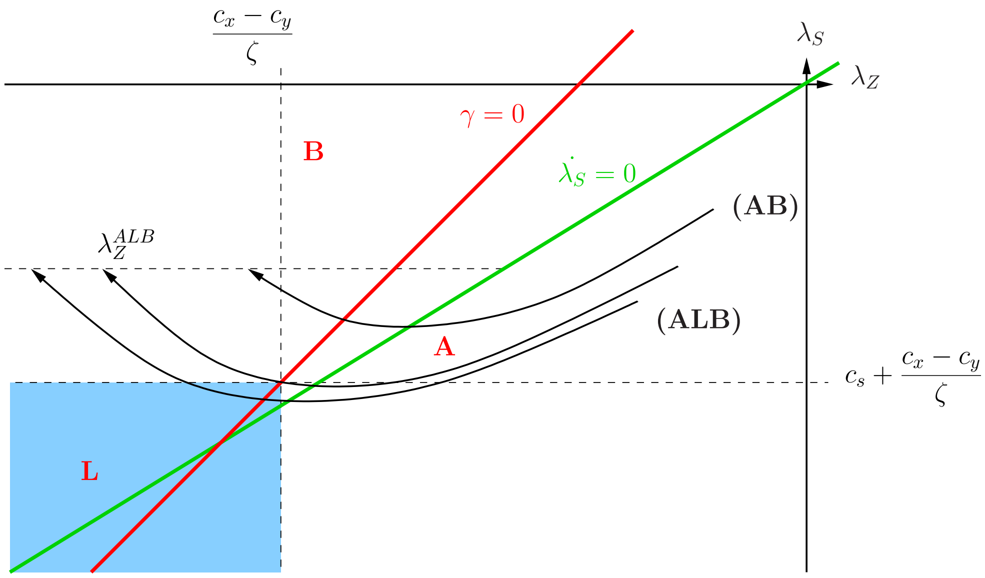

Figure 4.3: Trajectories of adjoint variables through phases A, B and L, $c_s > \hat{c}_s$

As observed in Section 4.4.1, in the situation where $c_s > \bar{c}_s$, the green line $\dot{\lambda}_S = 0$ enters the Phase L zone by intersecting its vertical boundary. In that case, whenever the point $(\lambda_Z(t), \lambda_S(t))$ is in Phase B, $\lambda_S(t)$ is increasing.

Figure 4.3 displays a particular value λ_Z^{ABL} which is such that when $\lambda_Z(T) = \lambda_Z^{ALB}$, the trajectory of (λ_Z, λ_S) goes precisely through the corner of Phase L. Two types of trajectories are possible: either $\lambda_Z(T) > \lambda_Z^{ALB}$ and Phase A is followed by Phase B (tagged as **(AB)** in the figure), or $\lambda_Z(T) < \lambda_Z^{ALB}$ and the phases are A, then L, then B (tagged as **(ALB)**). In the limiting case $\lambda_Z(T) = \lambda_Z^{ABL}$, Phase L is just "touched" at a single point in time.

These observations can be used to prove the following result.

Lemma 4.13. *Assume that Assumption 1 holds, that $u'(\cdot)$ convex and that $\hat{c}_s < c_s < c_{sQ}$. Let t^0 be an arbitrary time instant. Let $\lambda_S^0 = \lambda_S(t^0)$ be the value of λ_S when the optimal trajectory described in Lemma 4.8 starts from point $(S_m, \overline{Z})$.*

Then for every $\lambda_Z^0 \in (-\infty, \lambda_S^0 - c_s)$, there exists an optimal trajectory in the interior of Domain $\mathcal{D}$ which ends up at $(S_m, \overline{Z})$ at time t^0, and such that $\lambda_Z(t^{0-}) = \lambda_Z^0$.

Proof. Once again, we use Corollary 3.1 by constructing adjoint variable functions λ_S continuous and λ_Z continuous except at $t = t^0$, with a jump in the positive direction at $t = t^0$.

The phase this trajectory is in depends on the value of $(\lambda_Z(t), \lambda_Z(t))$ as explained in Section 4.4.1. This guarantees the consistency of multiplier $\underline{\gamma}_y$. The consistency of multipliers γ_s and γ_{sx} can be deduced from the graphical configuration of Figure 4.3.

There remains to prove that the state trajectory is consistent. The argument is that both Z and S are decreasing along these trajectories, which we prove by considering, backwards, the successive phases possible. First of all, we observe that $\lambda_S^0 \leq c_s + (c_x - \overline{p})/\zeta$. Indeed, it can be gathered from the proofs of Lemma 4.1, Lemma 4.6, Lemma 4.7 and Lemma 4.8 that the function $\lambda_S(t)$ is increasing for the optimal trajectory lying on the boundary $Z = \overline{Z}$ for $t \geq t^0$. But its limit when $t \to \infty$ is $\lambda_S^{(P)}(\infty) = \beta(c_x - \overline{p})/\zeta(\rho + \beta)$ (see the definition of P_∞ in (4.1.9)). This value is smaller than $c_s + (c_x - \overline{p})/\zeta$ when $c_s > \hat{c}_s$. The statement on λ_S^0 is therefore proved.

Consider then the trajectory just before it reaches $(S_m, \overline{Z})$. It is necessarily in Phase B. In view of Figure 4.3, $\lambda_S(t) \leq \lambda_S^0$ as long as t is in Phase B. Therefore, $x(t) = q^d(c_x + \zeta c_x - \zeta \lambda_S(t)) < \overline{x}$ (see also the description of Figure 4.2 in Section 4.4.1, p. 43). Since $\dot{S} = -\beta S + \zeta x$, having $\dot{S}(t) = 0$ for some $t < t^0$ would require $\beta S(t) = \zeta x(t) < \zeta \overline{x}$, in other words, $S(t) < S_m$. But $S(t^0) = S_m$ with

$\dot{S}(t^0) > 0$, so this is not possible. Then S is necessarily decreasing and larger than S_m. Finally, $\dot{Z} = -\alpha Z + \beta S$ is also negative for $S \geq S_m$.

The piece of trajectory in Phase B may be preceded by a piece in Phase L or Phase A. In both phases, S is decreasing. Therefore it is always larger than S_m. Also in both phases, $\dot{Z} \geq -\alpha Z + \beta S$. This is necessarily negative if $S > S_m$.

We have therefore proved the claim that S and Z decrease over all optimal trajectories ending at $(S_m, \overline{Z})$. They are therefore consistent as long as $Z > 0$. The condition $Z = 0$ determines the starting date and location of the optimal trajectory. Such a date necessarily exists, following the argument used in the proof of Lemma 4.12, whether trajectories exit (backwards) the domain when in Phase A or Phase B. □

4.4.3.2 Junction with Phase S

For values of c_s less than the threshold $\hat{c}_s$, the final phase is Phases S, stationary at point $(S_m, \overline{Z})$ (Lemma 4.2 on page 32). The principle, used in Section 4.3, that optimal trajectories follow the boundary $Z = \widetilde{Z}(S)$ until the final phase, does not hold anymore. It turns out that optimal trajectory may leave the boundary and return to it.

The difficulty for proving the consistency of candidate optimal trajectories is here that optimal trajectories of the state happen to "turn around" the final point $(S_m, \overline{Z})$ instead of reaching it in a straight, monotonous way. In particular, trajectories that start at a point $(S^0, \overline{Z})$ will *leave* the line $Z = \overline{Z}$ and later return to it. We discuss in Section 4.5.5.2 (p. 65) why going directly to $S = S_m$ while maintaining $Z = \overline{Z}$ is not optimal. In addition to this odd behavior, the trajectories join the final point very smoothly, with order 2 or order 3 junctions. This makes the local analysis unusually involved.

The typical situation is depicted in Figure 4.4 (top), which represents an evolution of the system supposed to be in Phase B. The three curves represented are $s(t) = \zeta x(t)$, $\beta S(t)$ and $\alpha Z(t)$. These three functions take the same value at time $t^0 = T$ where the point $(S_m, \overline{Z})$ is reached. In the state space (S, Z), the evolution is as in Figure 4.4 (bottom).

According to the diagram, going backwards under these dynamics, the state (S, Z) possibly exits the domain $\mathcal{D}$. Such a trajectory cannot be entirely consistent. Indeed, depending on the value of $\lambda_Z(T)$, the phase is limited by one of the events: (a) $Z = \overline{Z}$; (b) $Z = 0$; (c) $\gamma_{sx} = 0$; (d) $x(t) = \widetilde{y}$. We address these possibilities below.

The proof that the general scheme of Figure 4.4 is correct, at least for a set of trajectories "close" to the point $(S_m, \overline{Z})$, is to be found in Appendix D.2.2, p. 99.

The following lemma explains the form of optimal trajectories for the case $c_s < \hat{c}_s$, just before they reach the point $(S_m, \overline{Z})$. The statement refers to the values of adjoint variables in Phase S and the jump conditions which are in this case:

$$\lambda_S(T) \;=\; \lambda_S^{(S)} \;:=\; c_s + \frac{c_x - \overline{p}}{\zeta}\;, \qquad \lambda_Z(T^-) \;\leq\; \lambda_Z^{(S)} := \frac{\rho + \beta}{\beta}\, \lambda_S^{(S)}\;.$$

Lemma 4.14. *Assume that Assumption 1 holds and that $c_s < \hat{c}_s$. Let T be an arbitrary time instant. Then there exists a positive constant $\overline{\ell}$ such that for all $\ell \in (0, \overline{\ell}]$, the following trajectories are optimal:*

in a time interval $[\tau_1, T)$, the system is in Phase B, with $S(\tau_1) \in (0, S_m)$, $Z(\tau_1) = \overline{Z}$, $S(T) = S_m$, $Z(T) = \overline{Z}$, $\lambda_S(T) = \lambda_S^{(S)}$ and $\lambda_Z(T^-) = \lambda_Z^{(S)} - \ell$,

in the time interval $[T, \infty)$, the trajectory is stationary with $S(t) = S_m$, $Z(t) = \overline{Z}$, $\lambda_S(t) = \lambda_S^{(S)}$ and $\lambda_Z(t) = \lambda_Z^{(S)}$ (Phase S).

Proof. By construction, both pieces of this trajectory satisfy the differential equations of the first-order conditions, and the control constraints on x, s and y. Also by construction, the trajectories are continuous everywhere, except for $\lambda_Z(\cdot)$ which has a discontinuity at $t = T$. The jump at $t = T$ satisfies the condition (2.3.18). There remains to check the constraints on states and multipliers.

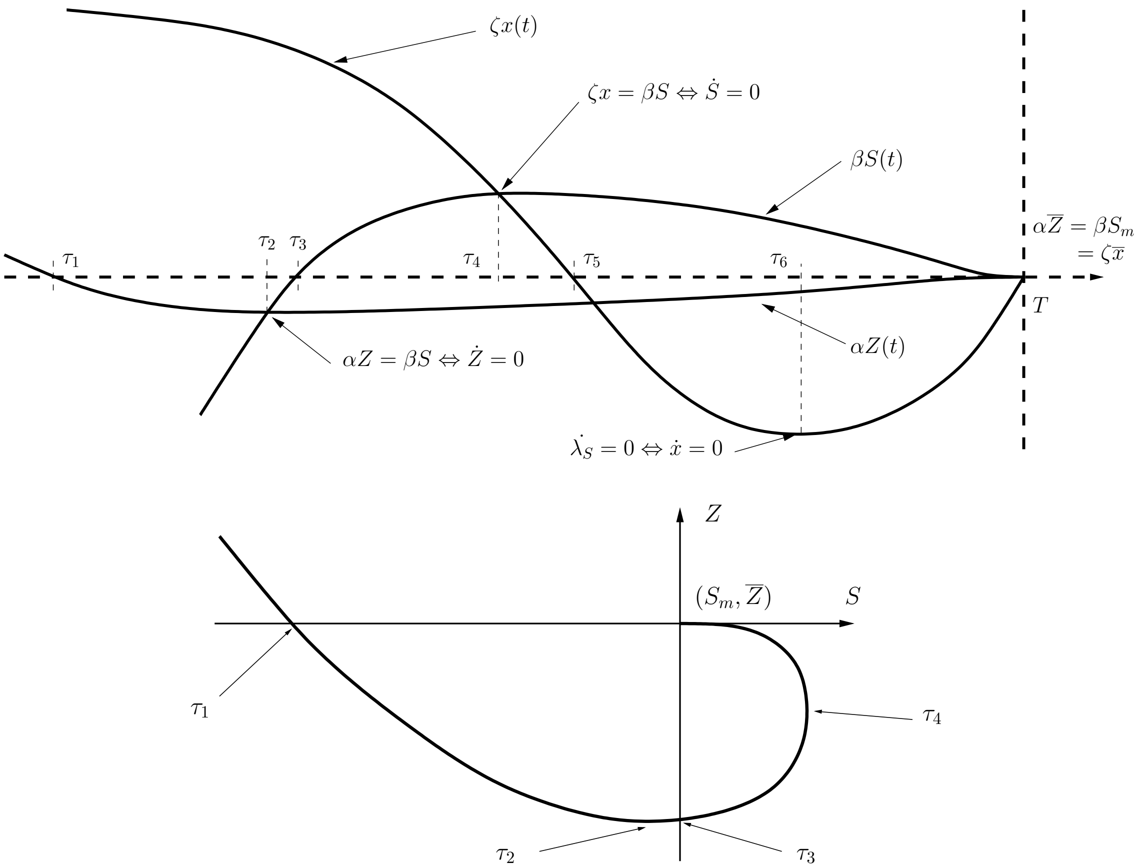

Figure 4.4: Trajectories of state and control in Phase B just before joining Phase S: over time (top); in the state space (S, Z) (bottom)

Using Lemma D.1 (p. 99), for each $\ell \in (0, \overline{\ell}]$ there exists τ_1 such that $Z(\tau_1) = \overline{Z}$ and $Z(t) < \overline{Z}$ for $t \in (\tau_1, T)$. There also exists τ_3 such that $S(\tau_3) = S_m$. Applying Grönwall's lemma to the differential equation $\dot{S} = -\beta S + \zeta x$, $S(0) = S_m$, with the bound $x(t) \leq \hat{x} := q^d(c_x + \zeta c_s)$, we conclude that $S(t) > 0$ for all $t > \tau_3 + \beta^{-1} \log(1 - \overline{x}/\hat{x})$. Likewise, since $\dot{Z} = -\alpha Z + \beta S \leq \beta S_M$, $Z(t) > 0$ for all $t > T - S_m/(\beta S_M)$. Since τ_1 can be bounded by $C_3 \ell$ (Lemma D.2), we conclude that $\overline{\ell}$ can be chosen so that the trajectory in Phase B satisfies all state constraints $Z \leq \overline{Z}$, $Z \geq 0$ and $S \geq 0$ in the interval $[\tau_1, T]$.

We now turn to the constraints on multipliers. Clearly, in Phase B, $x > 0$ and $s > 0$ so that $\gamma_s = 0$ and it remains to check that $\gamma_{sx} \geq 0$ and $\underline{\gamma}_y \geq 0$. These are respectively equivalent to

$$
\begin{aligned}
\lambda_S - \lambda_Z - c_s &\geq 0 \\
\lambda_S &\geq c_s + \frac{c_x - c_y}{\zeta} .
\end{aligned}
$$

This second inequality is satisfied when $\lambda_Z(T^-) \in [\lambda_S^L, \lambda_Z^{(S)}]$ (see the definition of of λ_S^L below). The constant $\overline{\ell}$ can be chosen such that this is the case for all ℓ. For the first inequality, it is easily shown that the function $\gamma(t) = \lambda_S(t) - \lambda_Z(t) - c_s$ is increasing. Therefore, the instant τ at which $\gamma(\tau) = 0$ is an increasing function of ℓ. Since the value of $\gamma(0)$ is strictly positive in Phase (see Lemma 4.2 on page 32), the value of τ can never approach 0. Hence, the value of $\overline{\ell}$ can be chosen so that, for all $\ell \leq \overline{\ell}$, $\gamma_{sx}(t) = \gamma(t) > 0$ for $t \in [\tau_1, T]$. □

Informally, we now describe what happens on optimal trajectories *before* they enter Phase B,

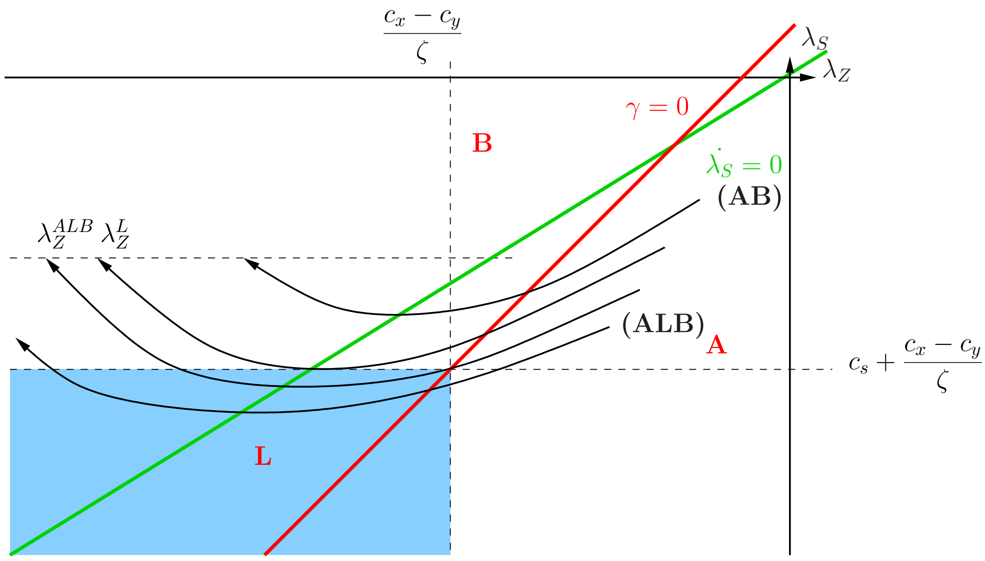

Figure 4.5: Trajectories of adjoint variables through phases A, B and L, $c_s \leq \hat{c}_s$

or when $\lambda_Z(T)$ is outside the range specified in Lemma 4.14. The complete picture is shown in Figure 4.7 on Page 54.

On Figure 4.5, we have represented the general situation for adjoint variables. As mentioned in Section 4.4.1, the curve $\dot{\lambda}_S = 0$ enters the "Phase L" zone by intersecting its horizontal boundary. There exists here two critical values $\lambda_Z^L > \lambda_Z^{ABL}$ with the following properties. (1) for all $\lambda_Z(T^-) > \lambda_Z^L$, the trajectory of $(\lambda_Z(t), \lambda_S(t))$ for $t \leq T$ never enters the L zone: it simply passes from Phase A to Phase B; (2) for $\lambda_Z(T^-) = \lambda_Z^L$, it just "touches" Phase L; (3) for $\lambda_Z^{ALB} < \lambda_Z(T^-) < \lambda_Z^L$, it goes through phases A, B, L then B again; (4) for $\lambda_Z(T^-) \leq \lambda_Z^{ABL}$, these trajectories go through phases A, L and B.

When $\lambda_Z(T^-) \in (\lambda_Z^{ABL}, \lambda_Z^L)$, the minimum of the curve $x(t)$ (Figure 4.4) is below $\widetilde{y}$. There is a period in Phase L inserted inside Phase B, between time instants τ_5 and T. This essentially does not modify the behavior in Figure 4.4 because $S(t)$ is decreasing and $Z(t)$ is increasing in Phase L, given that $\beta S(t) > \alpha Z(t)$. A sketch of the corresponding trajectory is drawn in Figure 4.6.

In all situations where $\lambda_Z(T^-) > \lambda_Z^{ABL}$, the last (or unique) Phase B is always preceded by a Phase A. When $\lambda_Z(T^-) < \lambda_Z^{ABL}$, this is different since the Phase L is directly preceded by a Phase A. Observe that in Phase A, $Z(t)$ is increasing and $S(t)$ is decreasing (see Section 4.4.2.1). Therefore, the behavior represented in Figure 4.4 is not possible in Phase A. There exists a critical value of λ_Z which the trajectory $(\lambda_Z(t), \lambda_S(t))$ switches from Phase A to Phase B at the exact moment where $Z(t)$ reaches $\overline{Z}$. This critical value may or may not be larger than λ_Z^L.

4.5 Description and classification of optimal trajectories

We are now in position to describe the optimal trajectories in the different cases.

First of all, compiling the optimality results stated in Lemmas 4.1–4.14, we see that several threshold values for c_s have been identified:

$$\hat{c}_s \ < \ c_{sQ} \ < \ c_{sm} \ .$$

These are respectively defined by (3.5.27) (see also Section 4.1), by the solution of (4.3.16), and in (4.3.6).

We call these situations respectively: "c_s small", "c_s medium-inf", "c_s medium-sup" and "c_s large". Some qualitative features of the optimal trajectories are summarized in the following table,

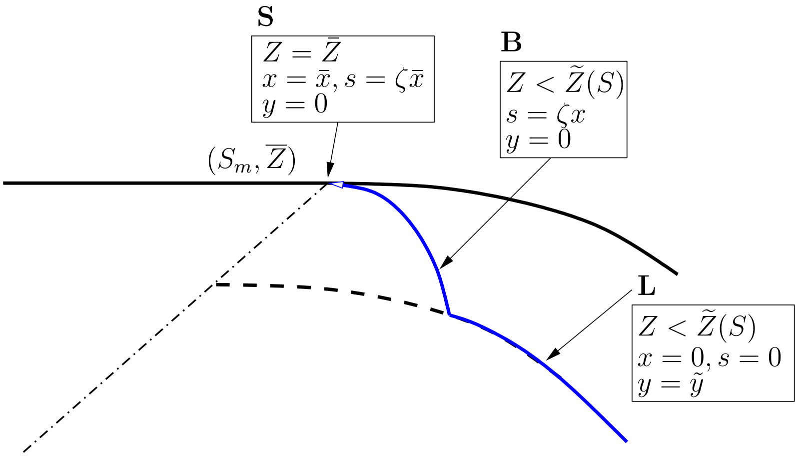

Figure 4.6: Trajectories of the state through phases L, B and S

according to the intervals where c_s lies. For instance, when c_s goes from 0 to infinity: Phase S is replaced by Phase P when c_s goes through $\hat{c}_s$. Phases Q and L disappear and are replaced by Phase R when c_s goes through c_{sQ}. Finally, Phase Q disappears when c_s goes beyond c_{sm}.

Range of c_s	0 – $\hat{c}_s$	$\hat{c}_s$ – c_{sQ}	c_{sQ} – c_{sm}	c_{sm} – ∞
Value of $S(\infty)$	0	S_m	S_m	S_m
Continuity of λ_Z	n	n	y	y
Simultaneous use of x and y	n	n	possible	possible
Use of y inside the domain	n	n	possible	possible
Use of capture s:				
in every optimal trajectory	y	n	n	n
inside the domain	possible	possible	n	n
on the boundary	y	possible	possible	n
Phases present	A, B, L, Q, S, U	A, B, L, P, Q, U	A, P, Q, R, U	A, P, R, U
Succession of phases (not exhaustive)	U/S, Q/B/S, A/B/(L/B/)S, A/Q/B/(L/B/)S	L/Q/P, A/B/Q/P, A/B/(L/B/)Q/P, A/Q/P, A/P	A/U/R/Q/P, A/R/Q/P, A/Q/P, A/P	A/P, A/R/P, A/U/R/P

We describe these four cases next, with the help of diagrams in the state space (S, Z) and in the space of adjoint variables (λ_Z, λ_S). See Section 4.4.1 on page 43 for the general description of such diagrams. In addition, we make the convention that, for some phases x and y, point S^{xy} on a state space diagram generally mark where the state moves from Phase x to Phase y. They correspond to points P_{xy} on the corresponding adjoint variable diagram: these represent the location of the adjoint variables when the state is $(S^{xy}, \overline{Z})$. Point P_S represents the location of the adjoint variables when the state trajectory passes through $(S_m, \overline{Z})$ or stays at this point.

4.5.1 Small c_s ($c_s < \hat{c}_s$)

When $0 < c_s < \hat{c}_s$, the situation is represented in Figure 4.7 on page 54 (for the evolution of $(\lambda_Z(t), \lambda_S(t))$ over time), Figure 4.8 (for the evolution of $(S(t), Z(t))$ over time) and Figure 4.9 on page 56 for the correspondence between the evolution of λ_Z, λ_S and that of consumption.[2] See also Figure 4.10 for the particular case $c_s = \hat{c}_s$. The results relevant to these figures are Lemma 4.2 (p. 32) and Lemma 4.14 (p. 50) and the discussion following it.

[2]The evolution of adjoint variables is depicted through their opposite values $-\lambda_Z$ and $-\lambda_S$ which have interpretation in Economics as shadow prices.

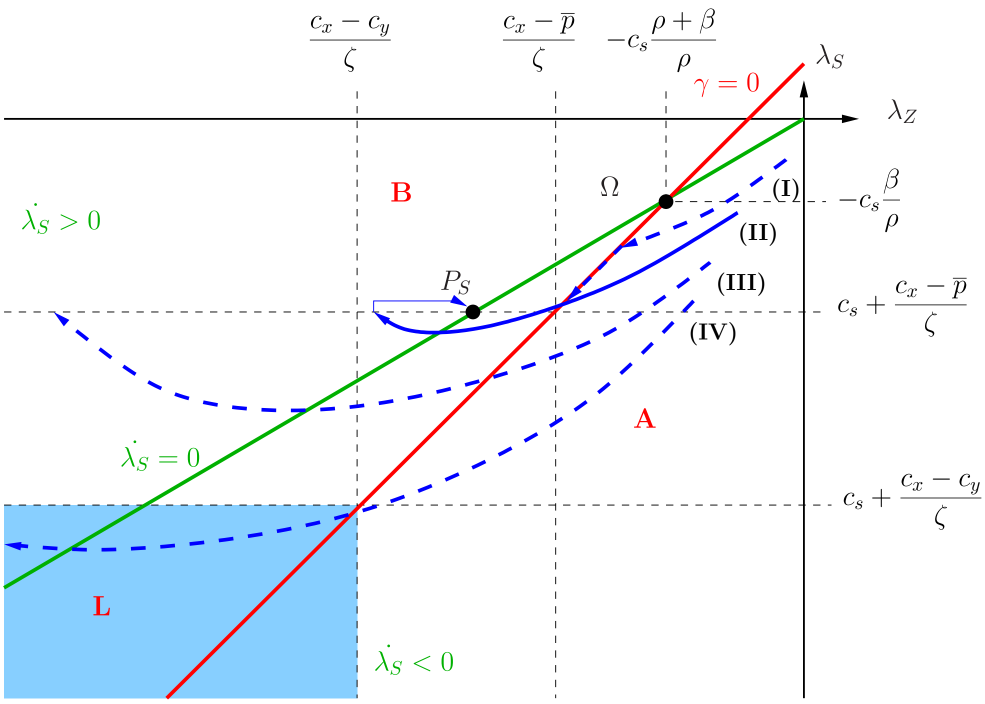

Figure 4.7: Evolution of (λ_Z, λ_S), case $c_s < \hat{c}_s$

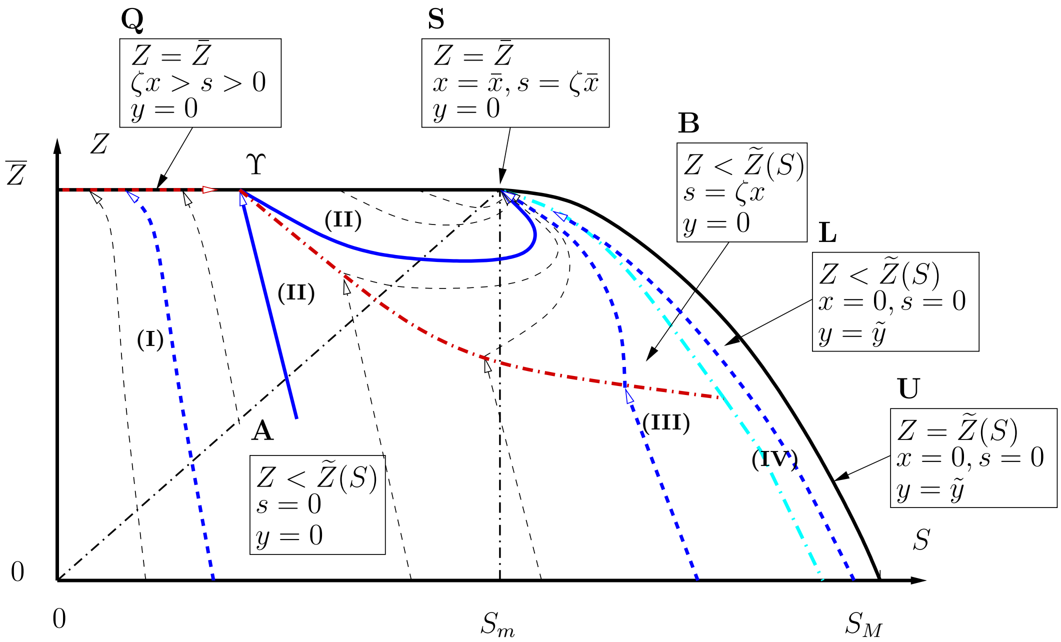

Figure 4.8: Evolution of (S, Z), case $c_s < \hat{c}_s$

The typical situation can be summarized as follows:

Phase A A trajectory starting with $S(0)$ small enough will follow a state and a adjoint path as the ones labeled with **I** in Figures 4.7 and 4.8. The adjoint path and the consumption/capture path is represented in Figure 4.9. Capture is 0, and Z will increase and S decrease, until Z hits the ceiling. Both λ_Z and λ_S are decreasing in this phase. At some point in time, simultaneously, $Z(t) = \overline{Z}$ and $\lambda_S(t) = \lambda_Z(t) + c_s$. The trajectory enters Phase Q.

Phase Q Next, the trajectory stays at the ceiling in Phase Q: capture occurs according to Equation (4.2.3): $s = \zeta x - \beta(S_m - S)$. Since $S(t)$ increases towards S_m, the gap between $\zeta x(t)$ and $s(t)$ decreases over time. It is not possible for the optimal trajectory to stay on the boundary $Z = \overline{Z}$ until $S = S_m$, as explained in Section 4.5.5.2, page 65. There exists therefore a point (labeled Υ in Figure 4.8) where the trajectory leaves the boundary and enters Phase B.

This particular trajectory is labeled as **(II)** and represented as a continuous blue line in Figures 4.7 and 4.8.

Phase B A trajectory with initial position $(S(0), Z(0))$ close enough to $(S_m, \overline{Z})$ but not too close to the curve $Z = Z_M(S)$ will be in Phase B. In this phase, capture is maximum, and the dynamics of Z is given by $\dot{Z} = \beta S - \alpha Z$. Initially, $S(t)$ is increasing and $Z(t)$ is decreasing, until $\beta S = \alpha Z$. Then $Z(t)$ is increasing again. The adjoint variable $\lambda_S(t)$ is decreasing then increasing, and so is the consumption $x(t)$. There happens a time at which $\dot{S} = \zeta x - \beta S$ becomes null then negative, and $S(t)$ decreases. The trajectory ends up at point $(S_m, \overline{Z})$ in Phase S. See also Figure 4.4 on page 51.

Some trajectories, as the one labeled **(III)** in the figures, follow a sequence of phases A/B/S. They do not reach the ceiling $Z = \overline{Z}$ before the final phase S.

Phase L If the initial state $(S(0), Z(0))$ is close to the curve $Z = Z_M(S)$, then consumption as it would be in Phase B falls below $\widetilde{y}$, or equivalently that λ_S falls below $c_s + (c_x - \overline{p})/\zeta$. In that case, the trajectory is in Phase L, which is typically inserted between two periods in Phase B. This situation not represented in Figure 4.9), but in Figure 4.7, it corresponds to trajectories of the adjoint variable entering the zone colored in light blue. During this Phase L, $x = 0$ and $y = \widetilde{y}$.

Some trajectories, as the one labeled **(IV)** in the figures, follow a sequence of phases L/B/S.

Phase S All trajectories terminate at the point $(S_m, \overline{Z})$, where they stay forever. The values of (λ_Z, λ_S), as well as x, y and s are constant in that phase: they are given in Section 4.1.2. These terminal values correspond to the point marked as P_S in Figure 4.7.

4.5.2 Medium-inf c_s ($\hat{c}_s < c_s < c_{sQ}$)

When $\hat{c}_s \leq c_s < \bar{c}_s$, the situation is represented in Figures 4.11 and Figures 4.14 (for the evolution of $(\lambda_Z(t), \lambda_S(t))$ over time) and 4.12 (for the evolution of $(S(t), Z(t))$ over time). See also Figure 4.10 for the boundary case $c_s = \hat{c}_s$ and Figure 4.15 for the boundary case $c_s = c_{sQ}$. Also relevant to this range of c_s values is the particular value $\bar{c}_s$ defined in (3.5.22), which is such that $\hat{c}_s < \bar{c}_s < c_{sQ}$. This case is represented in Figure 4.13. Relevant results are Lemmas 4.1, 4.8, 4.9, 4.10 and 4.13.

The distinction between cases $c_s < \bar{c}_s$ (Figure 4.11) and $c_s > \bar{c}_s$ (Figure 4.14) lies in the geometric position of the point Ω. When $c_s = \hat{c}_s$, points Ω and P_S coincide and are located outside the zone labelled as "**L**". When $c_s = \bar{c}_s$, the point Ω enters this zone, and it lies inside it when $c_s > \bar{c}_s$. In that case, it becomes geometrically possible for the point P_S to move on the line $\gamma = 0$ to a position where $\lambda_Z = (c_y - c_x)/\zeta$. However, it does not do so as long as $c_s < c_{sQ}$. Indeed, the value of c_{sQ} is defined in Section 4.3.2 on p. 41 as the value of c_s such that point P_S is located both on the line $\gamma = 0$ and the boundary $\lambda_Z = (c_x - c_y)/\zeta$.

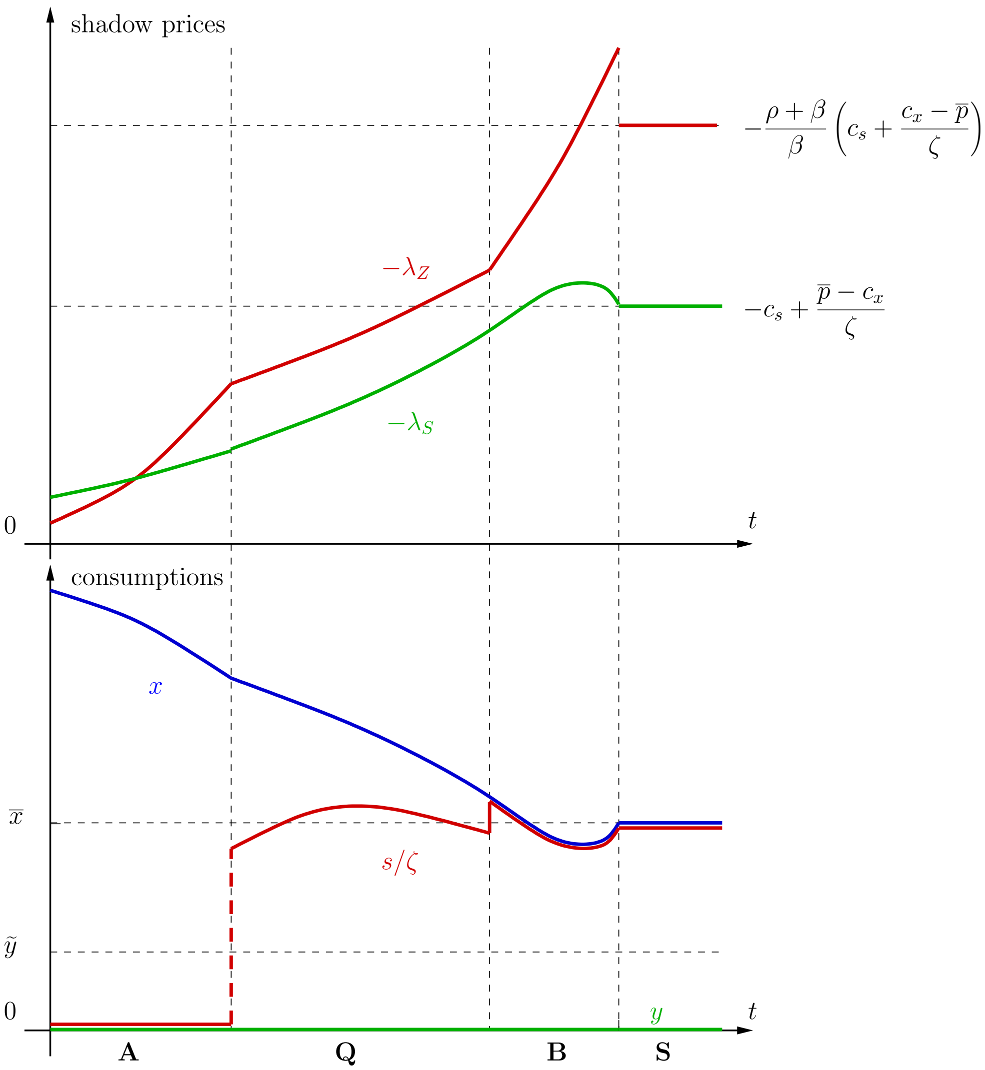

Figure 4.9: Evolution of λ_Z, λ_S, x, y and s, case $c_s < \hat{c}_s$

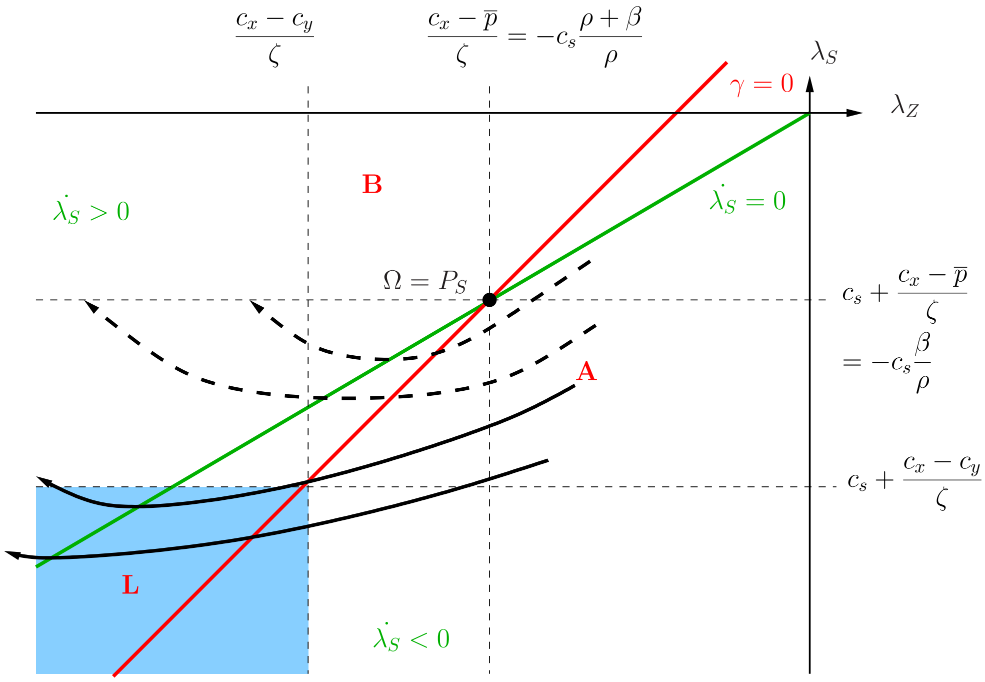

Figure 4.10: Evolution of (λ_Z, λ_S), case $c_s = \hat{c}_s$

Figures 4.11 and 4.12 exhibit four trajectories, labeled as **(I)** to **(IV)**. These trajectories go, respectively through phases A/P, A/Q/P, A/S/Q/P where Phase S is limited to a passage through point $(S_m, \overline{Z})$, and phases A/B/S/Q/P. The possibilities A/B/L/B/S/Q/P and A/L/B/S/Q/P also exist (as explained in Section 4.4.3.2) but are not represented. We now describe these curves.

A typical trajectory starting with a moderate value of $S(0)$ (labeled as **(II)**) has the following features.

Phase A It starts in the interior of the domain in Phase A. The evolution of $(\lambda_Z(t), \lambda_S(t))$ is that of the "free" trajectories (3.2.4)–(3.2.5). While λ_Z always decreases, λ_S decreases, then increases again.

Phase Q If the initial value of S is large enough, the value of $\gamma(t) = \lambda_S(t) - \lambda_Z(t) - c_s$, which is negative in Phase A, eventually vanishes. At that moment, the value of $Z(t)$ hits the ceiling $\overline{Z}$. The trajectory then continues in Phase Q: atmospheric stock at the ceiling, with some capture $s(t)$.

In Figure 4.11, the point moves on the red line which represents $\gamma = 0$. It moves *upwards* because $\dot{\lambda}_S > 0$ since the point is located above the green line which represents $\dot{\lambda}_S = (\rho + \beta)\lambda_S - \beta\lambda_Z = 0$.

Eventually, the value of $s(t)$ vanishes and the trajectory enters Phase P.

Phase P Phase P is terminal: the states moves asymptotically to point $(0, \overline{Z})$; the adjoint variables move the point materialized as P_∞. At that location, we have simultaneously $\dot{\lambda}_S = 0$ and $\lambda_Z = (c_x - \overline{p})/\zeta$, corresponding to a consumption of $\overline{x}$ (see also Figure 4.1).

The dashed line which passes through P_{QP} and P_∞ in Figure 4.11 is the trajectory of the adjoint variables in Phase P, which is actually independent of c_s. It need not be a straight line in general, but it is indeed so in the "linear-quadratic" case developed in Appendix E.

A trajectory which starts with smaller values of $S(0)$ (labeled as **(I)** on the figures) will follow Phase A in the interior of the domain, but will enter directly Phase P. At the contact point with the boundary $Z = \overline{Z}$, the trajectory is tangent, as explained in Appendix D and Section 4.4.2.1.

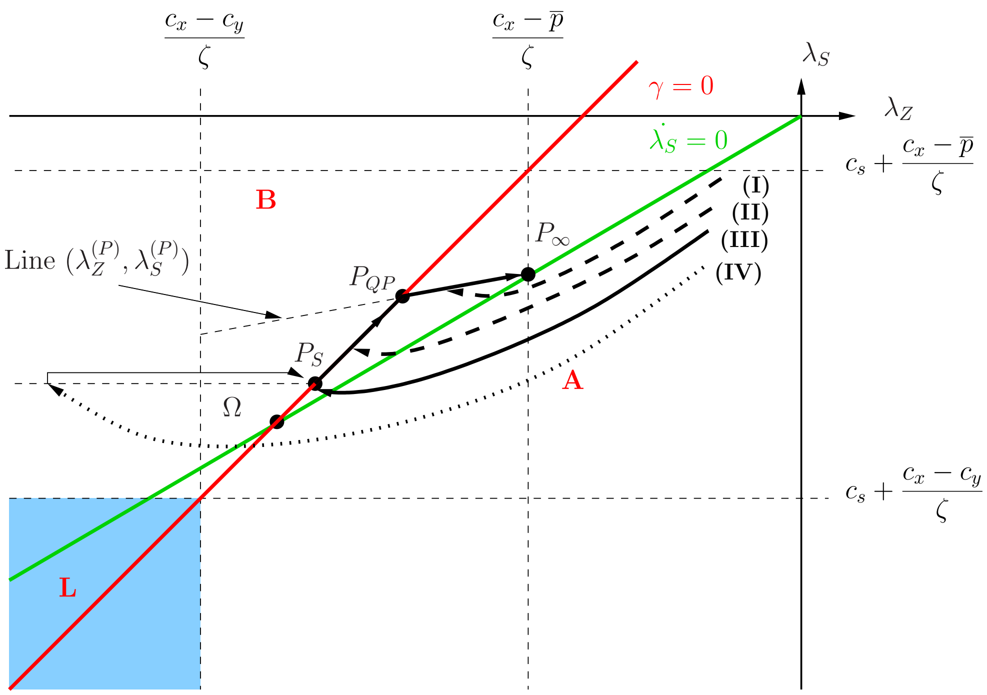

Figure 4.11: Evolution of (λ_Z, λ_S), case $\hat{c}_s \leq c_s \leq \bar{c}_s$

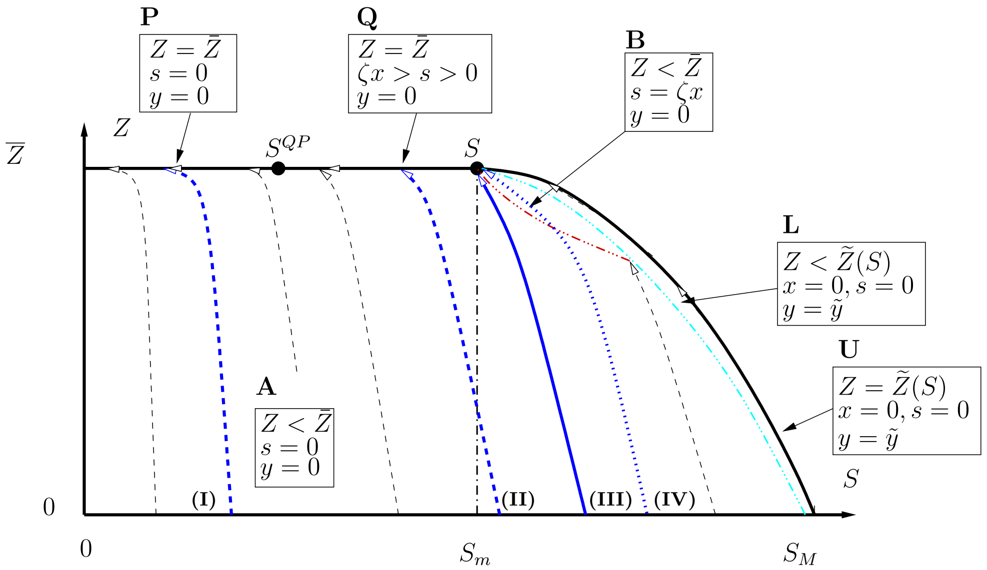

Figure 4.12: Evolution of (S, Z), case $\hat{c}_s \leq c_s \leq c_{sQ}$

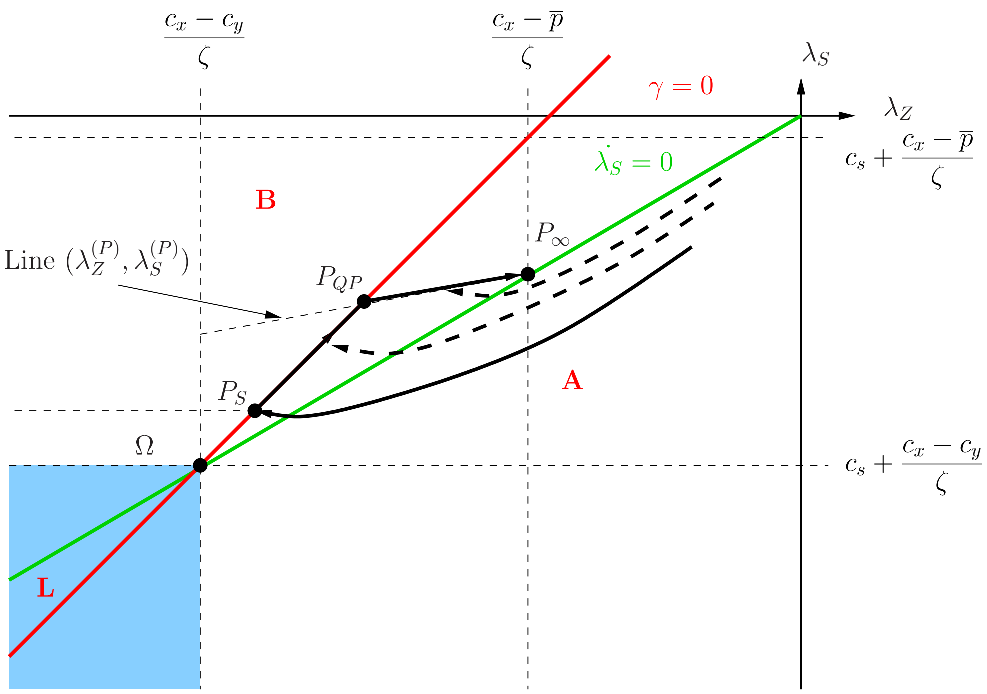

Figure 4.13: Evolution of (λ_Z, λ_S), case $c_s = \bar{c}_s$

On the other hand, a trajectory starting with a large value of $S(0)$ (labeled as **(IV)** on the figures) will get close to the boundary $Z = Z_M(S)$ and has the following features.

Phase A It starts in the interior of the domain in Phase A as before. However, either λ_Z reaches the critical value $(c_x - c_y)/\zeta$ or λ_S reaches the critical value $c_s + (c_x - c_y)/\zeta$. In the first event, the trajectory enters Phase L; in the second event, it enters directly Phase B.

Phase L Consumption $x(t)$ falls below the level $\widetilde{y}$. Consistent with Lemma 3.2 on page 21, it becomes optimal to set $x = 0$ and consume $y(t) = \widetilde{y}$. The state variables evolve along "free" trajectories, as well as adjoint variables. Eventually, $\gamma(t)$ becomes positive and λ_S increases to become equal to $c_s + (c_x - c_y)/\zeta$. At that moment, the trajectory enters Phase B.

Phase B Capture $s(t) = \zeta x(t)$ is maximal. This piece of trajectory ends up at point $(S_m, \overline{Z})$ with a value of $\lambda_S = (c_x - \overline{p})/\zeta$ corresponding to a consumption $x = \overline{x}$. The value of λ_Z however depends on the trajectory. The smaller it is, the closer the trajectory gets to the limit $Z = Z_M(S)$.

Phases Q and P From the point $(S_m, \overline{Z})$, the trajectory enters Phase Q. There is a *discontinuity* in the value of λ_Z (represented as a thin line in Figure 4.11) so that $\gamma(t) = \lambda_S(t) - \lambda_Z(t) - c_s$, which is negative in Phase B, becomes null in Phase Q. The evolution is similar to the situation described previously. Eventually, the value of $s(t)$ vanishes and the trajectory enters terminal Phase P.

One particular trajectory (labeled as **(III)** on the figures) joins with the boundary precisely at point $(S_m, \overline{Z})$. On this trajectory, the adjoint variables are continuous.

4.5.3 Medium-sup c_s $(c_{sQ} < c_s < c_{sm})$

The situation is represented in Figure 4.17. In that case, the point P_S is located on the boundary $\lambda_Z = (c_x - c_y)/\zeta$ of the zone **L**, which corresponds to the fact that a Phase R appears on the

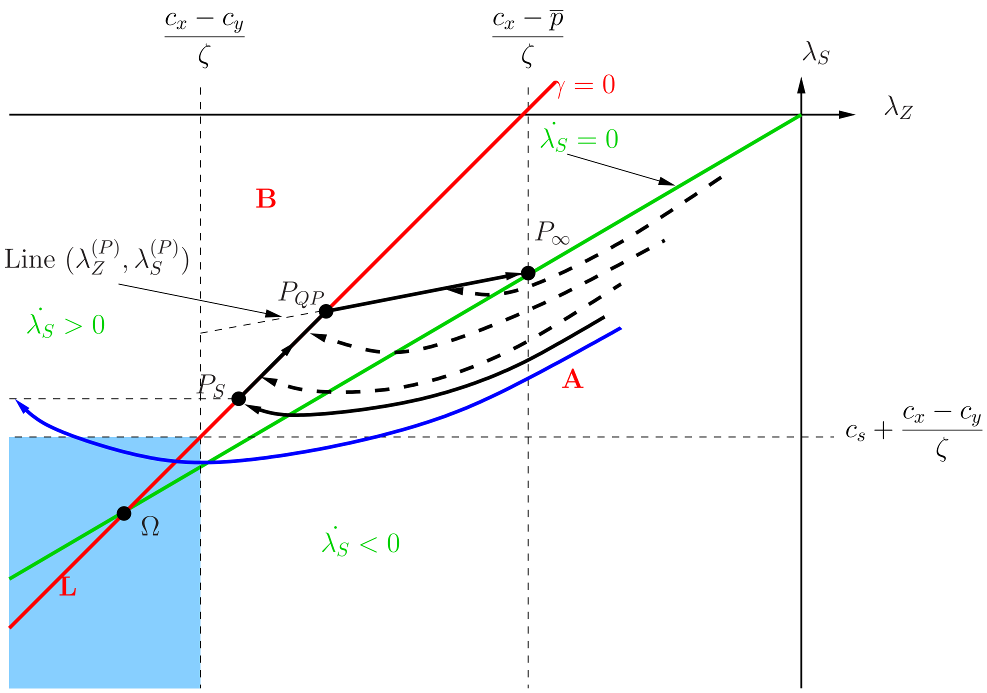

Figure 4.14: Evolution of (λ_Z, λ_S), case $\bar{c}_s \leq c_s \leq c_{sQ}$

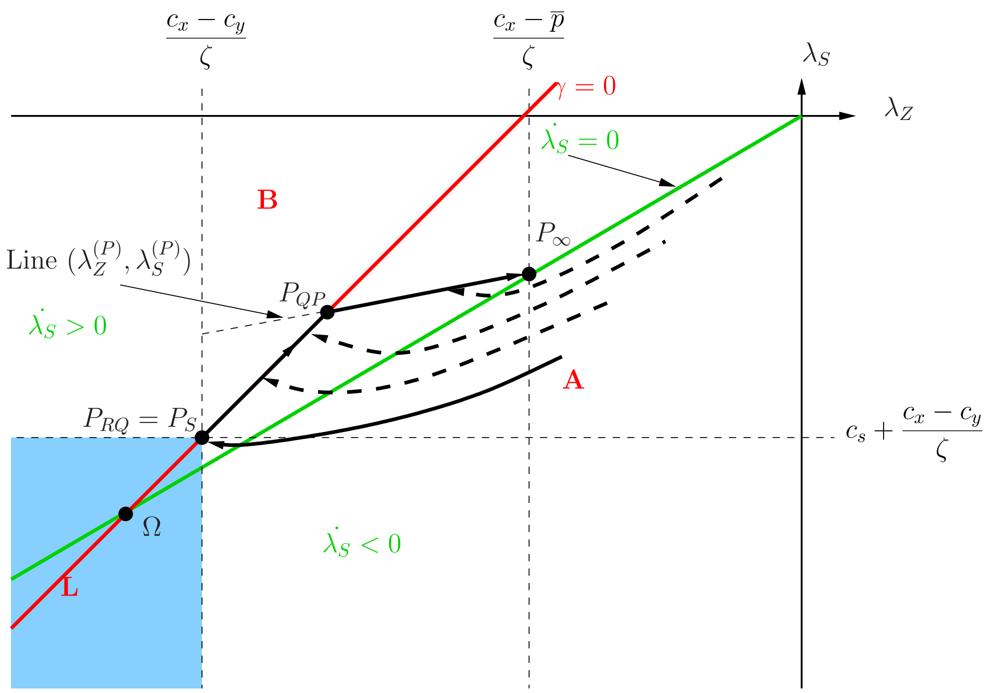

Figure 4.15: Evolution of (λ_Z, λ_S), case $c_s = c_{sQ}$

boundary $Z = \overline{Z}$. In Figure 4.16, a point P_{RQ} appears. Relevant results are Lemmas 4.1, 4.8, 4.9, 4.10, 4.11 and 4.12.

In that case, the scenario above is modified as follows, for initial values of S large enough:

Phase A ends when λ_Z reaches $(c_y - c_x)/\zeta$ first. At that moment, $Z(t)$ reaches $Z_M(S(t))$ and consumption $x(t)$ reaches $\widetilde{y}$. Depending on whether $S(t)$ is larger or smaller than S_m, the trajectory continues in Phase U, or one of Phases R, Q or P, respectively.

Phase U The trajectory continues along $Z = Z_M(S)$ with $x = 0$ and $y = \widetilde{y}$. The value of λ_S is increasing and λ_Z is constant at $(c_y - c_x)/\zeta$. The point $(\lambda_Z(t), \lambda_Z(t))$ therefore moves up in Figure 4.16 on the line $\lambda_Z = (c_y - c_x)/\zeta$. The state trajectory eventually reaches $(S_m, \overline{Z})$. The location of (λ_Z, λ_S) corresponding to this time instant is labeled as P_S in Figure 4.16.

Phase R The trajectory in Phase R has been described in Figure 4.1: as S decreases from S_m to $S_{\widetilde{y}}$, consumption x increases from 0 to $\widetilde{y}$ while y decreases from $\widetilde{y}$ to 0, their sum being always $x+y = \widetilde{y}$. The point $(\lambda_Z(t), \lambda_S(t))$ continues to move up on the line $\lambda_Z = (c_y - c_x)/\zeta$. Eventually, $\gamma(t) = 0$ and the trajectory enters Phase Q at point $(S^{RQ}, \overline{Z})$, see Figure 4.1.

Phase Q and P It becomes optimal to use capture. As $S(t)$ decreases, capture $s(t)$ decreases also and eventually vanishes: the trajectory enters terminal Phase P at point $(S^{QP}, \overline{Z})$.

See Figure 4.15 for the boundary case $c_s = c_{sQ}$. In this last case, the points P_S and P_{RQ} coincide. Phase R just vanishes.

4.5.4 Large c_s ($c_s \geq c_{sm}$)

When $c_s > c_{sm}$, Phase Q disappears completely, as well as Phase B. Actually, capture is so expensive in this case that $s(t) = 0$ at all times. The model is equivalent to one where capture is not possible at all.

The situation is represented in Figures 4.19 (for the evolution of $(\lambda_Z(t), \lambda_S(t))$ over time) and 4.20 (for the evolution of $(S(t), Z(t))$ over time). See also Figure 4.18 for the boundary case $c_s = c_{sm}$. Relevant results are Lemmas 4.1, 4.7, 4.9, 4.11 and 4.12.

The description of a typical trajectory is quite similar to the case $c_{sQ} < c_s < c_{sm}$ ("medium-sup c_s"), except that there is no Phase Q. When in Phase R, as S decreases from S_m to $S_{\widetilde{y}}$, consumption x increases from 0 to $\widetilde{y}$ while y decreases from $\widetilde{y}$ to 0, their sum being always $x + y = \widetilde{y}$. The trajectory then continues in Phase P as before.

Trajectories starting from smaller values of $S(0)$ will have a succession of phases A/R/P or just A/P.

4.5.5 Complements on the case $c_s \leq \hat{c}_s$

This section gathers additional observations on the case where c_s is small. This is the case where Phase S is the terminal phase, and we develop in Section 4.5.5.1 an elementary argument for this (elementary in the sense that it does not use adjoint variables). It is also the situation where optimal trajectories may leave the boundary $Z = \overline{Z}$, and we develop an argument for this in Section 4.5.5.2.

4.5.5.1 An interpretation of threshold $\hat{c}_s$ through a perturbation analysis

An interpretation of the value $\hat{c}_s$ derives from a local perturbation of trajectories close to the point $(S_m, \overline{Z})$, as follows.

Consider the reference situation where $Z(t) = \overline{Z}$, $S(t) = S_m$, $x(t) = \overline{x}$ and $s(t) = \zeta\overline{x}$ (see Section 4.1.2). Assume that on the time interval $[0, \Delta t]$, the consumption is modified into $x(t) = \overline{x} - \Delta x$ (constant over time) and the capture computed so that the constraint $Z(t) = \overline{Z}$ still holds. Then since $\dot{Z} = 0$, we must have:

$$0 \;=\; -\alpha\overline{Z} + \beta S(t) + \zeta(\overline{x} - \Delta x) - s(t) \quad\Longrightarrow\quad s(t) \;=\; \beta S(t) - \zeta\Delta x\ .$$

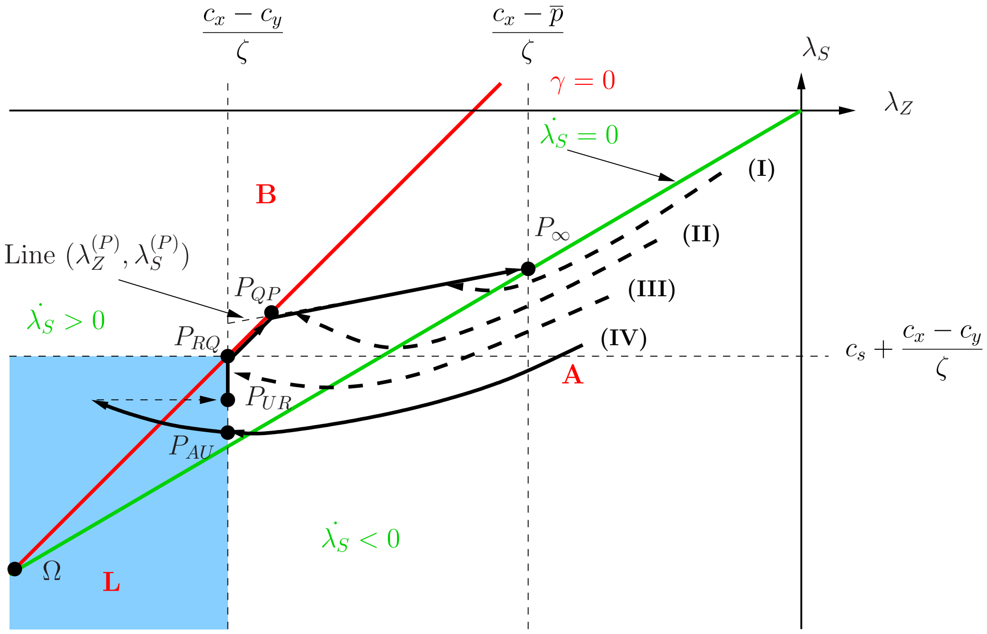

Figure 4.16: Evolution of (λ_Z, λ_S), case $c_{sQ} \leq c_s \leq c_{sm}$

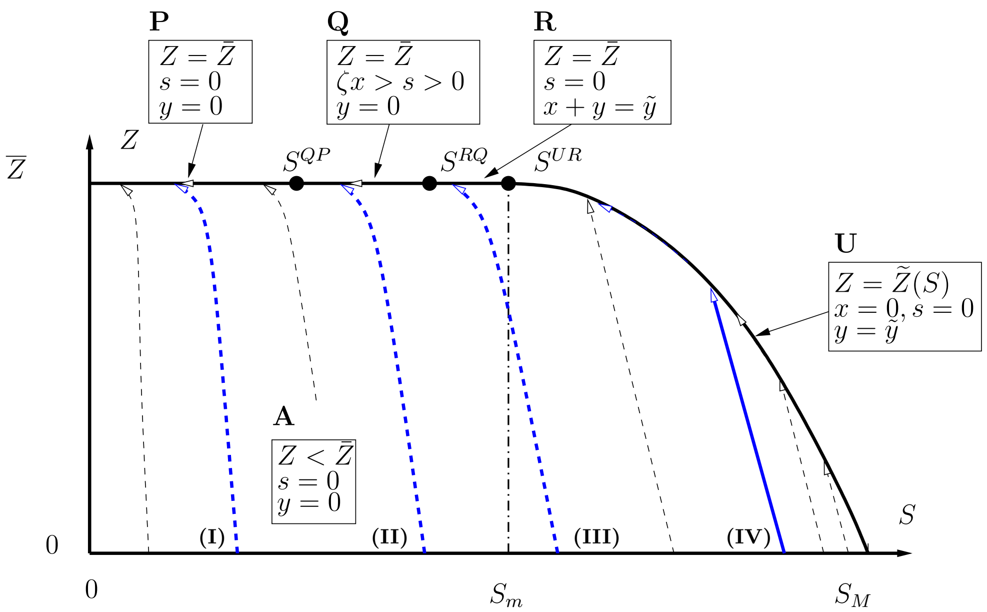

Figure 4.17: Evolution of (S, Z), case $c_{sQ} \leq c_s \leq c_{sm}$

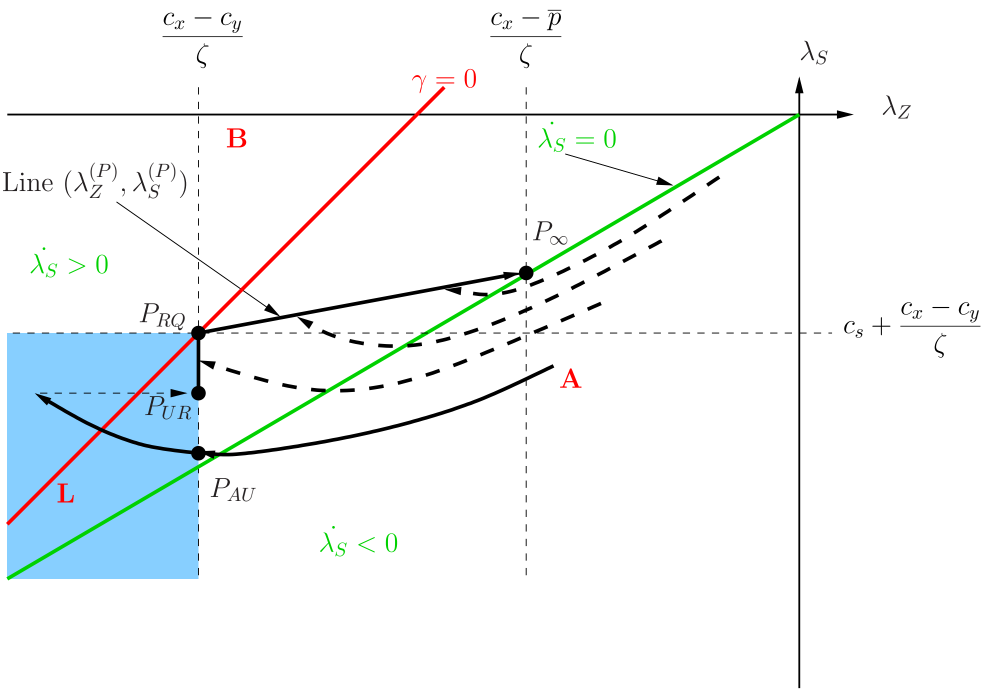

Figure 4.18: Evolution of (λ_Z, λ_S), case $c_s = c_{sm}$

As a consequence, we have $\dot{S} = -\beta S + s = -\zeta \Delta x$ is constant on the interval, and $S(t) = S_m - \zeta \Delta x \; t$.

On interval $[\Delta t, \infty)$, capture is restored to the nominal level $\zeta \overline{x}$, and consumption is such that $Z = \overline{Z}$: it is therefore

$$x(t) \;=\; \overline{x} \;+\; \beta(S_m - S(t))/\zeta \; .$$

As a consequence, $\dot{S} = \beta(S_m - S)$ on the interval, and $S(t) = S_m + (S(\Delta t) - S_m)e^{-\beta(t-\Delta t)} = S_m - \zeta \Delta x \Delta t e^{-\beta(t-\Delta t)}$.

On the interval $[0, \Delta t]$, the difference in the sum of discounted net surplus between both trajectories is

$$\begin{aligned} D_1 &= \int_0^{\Delta t} e^{-\rho t}[u(\overline{x}) - u(\overline{x} - \Delta x) - c_x \Delta x - c_s(\zeta \overline{x} - \beta S + \zeta \Delta x)]\mathrm{d}t \\ &= \frac{1 - e^{-\rho \Delta t}}{\rho}[u(\overline{x}) - u(\overline{x} - \Delta x) - (c_x + \zeta c_s)\Delta x] - c_s \int_0^{\Delta t} e^{-\rho t} \beta(S_m - S(t))\mathrm{d}t \\ &= \frac{1 - e^{-\rho \Delta t}}{\rho}[u(\overline{x}) - u(\overline{x} - \Delta x) - (c_x + \zeta c_s)\Delta x] - \beta c_s \zeta \Delta x \int_0^{\Delta t} t e^{-\rho t}\mathrm{d}t \; . \end{aligned}$$

On the interval $[\Delta t, \infty)$, this difference is:

$$\begin{aligned} D_2 &= \int_{\Delta t}^{\infty} e^{-\rho t}[u(\overline{x}) - u(\overline{x} + \beta(S_m - S(t))/\zeta) + c_x \beta(S_m - S(t))/\zeta]\mathrm{d}t \\ &= \int_{\Delta t}^{\infty} e^{-\rho t}[u(\overline{x}) - u(\overline{x} + \beta \Delta t \Delta x e^{-\beta(t-\Delta t)}) + \beta c_x \Delta t \Delta x e^{-\beta(t-\Delta t)}]\mathrm{d}t \; . \end{aligned}$$

When Δt tends to 0, we have

$$D_2 \;=\; \int_{\Delta t}^{\infty} e^{-\rho t}[-\overline{p}\beta \Delta t \Delta x e^{-\beta(t-\Delta t)} + \beta c_x \Delta t \Delta x e^{-\beta(t-\Delta t)}]\mathrm{d}t \;+\; o(\Delta t)$$

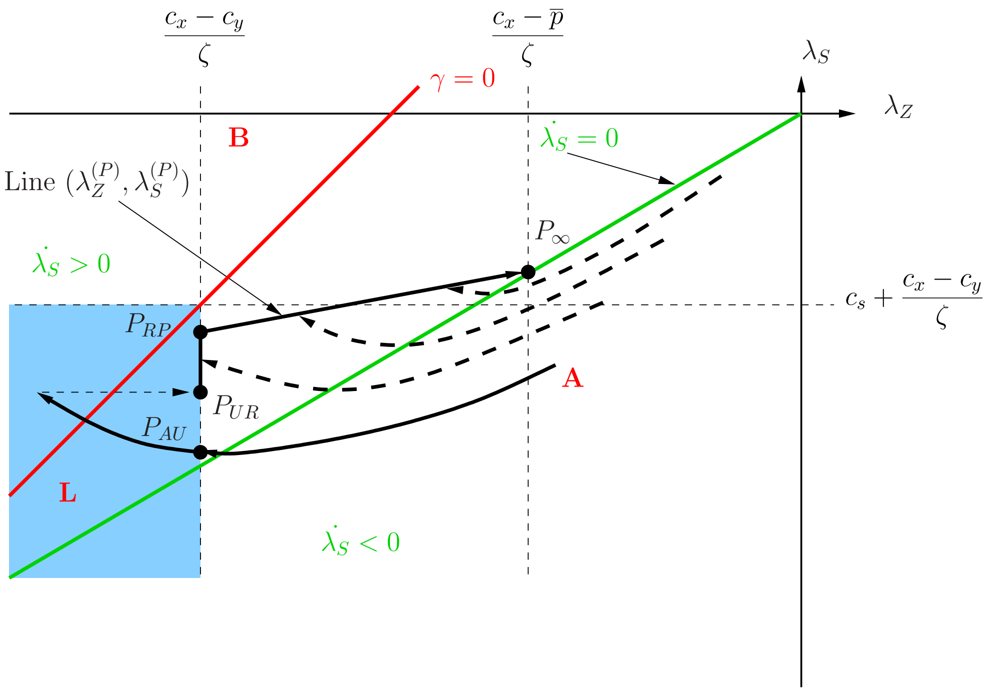

Figure 4.19: Evolution of (λ_Z, λ_S), case $c_s \geq c_{sm}$

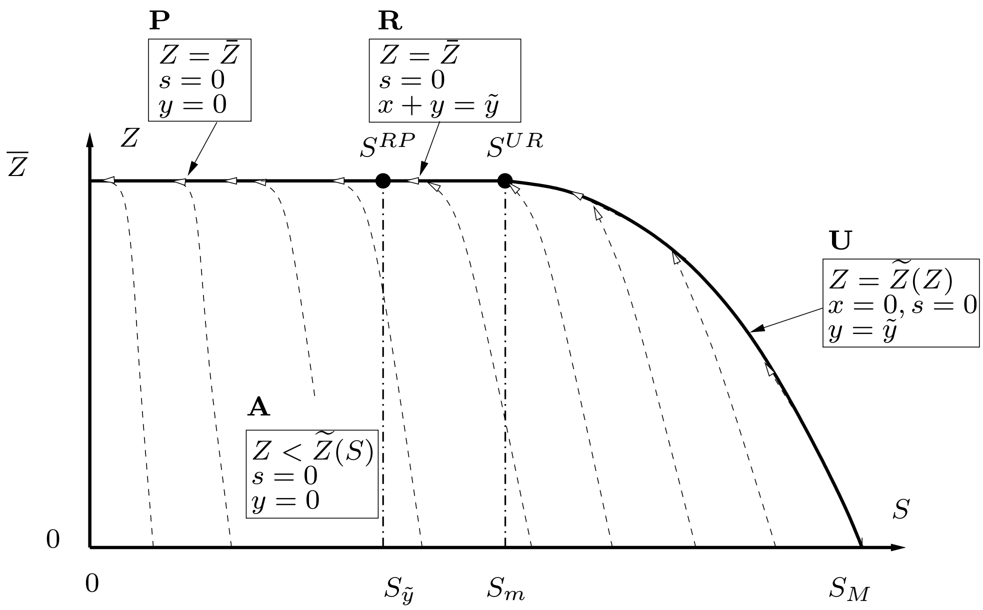

Figure 4.20: Evolution of (S, Z), case $c_s \geq c_{sm}$

$$= \frac{\beta}{\rho+\beta}\Delta t \Delta x (c_x - \overline{p}) \; + \; o(\Delta t) \; .$$

On the other hand, assuming that Δx is also small,

$$\begin{aligned} D_1 &= [u(\overline{x}) - u(\overline{x} - \Delta x) - (c_x + \zeta c_s)\Delta x]\Delta t \; + \; o(\Delta t) \\ &= [\overline{p}\Delta x - (c_x + \zeta c_s)\Delta x + o(\Delta x)]\Delta t \; + \; o(\Delta t) \\ &= (\overline{p} - c_x - \zeta c_s)\Delta x \Delta t + o(\Delta x)\Delta t \; + \; o(\Delta t) \; . \end{aligned}$$

If the reference trajectory is optimal, then $D_1 + D_2$ must be positive. Asymptotically when Δt and Δx tend to 0, this means:

$$0 \;\leq\; (\overline{p} - c_x - \zeta c_s)\Delta x \Delta t + \frac{\beta}{\rho+\beta}\Delta t \Delta x (c_x - \overline{p}) \;=\; \Delta t \Delta x (\hat{c}_s - c_s) \; .$$

The value of Δt has been chosen positive, and the value of Δx *must* be positive also: otherwise the trajectory would not be admissible. We conclude that necessarily, $c_s \leq \hat{c}_s$.

4.5.5.2 Non-optimality of joining Phase Q and Phase S: a necessity argument

In this paragraph, we develop the argument that no optimal trajectory consists in Phase Q joining Phase S. We have seen that Phase S can be terminal only if $c_s \leq \hat{c}_s$. Moreover, we know that if $c_s = \hat{c}_s$, every point $(S, \overline{Z})$ with $S \leq S_m$ is stationary, so that Phase Q cannot be followed by Phase S. We therefore assume that $c_s < \hat{c}_s$.

The argument is based on the fact that when the adjoint variables of optimal trajectories have jumps, the sign of these jumps is related to the constraints that become binding or cease to be so. A relevant result is (4.19) in Sethi & Thompson (2000, p. 107), which provides *necessary* conditions for an optimal trajectory. This theorem applies to pure state constraints $g(\mathbf{x}) \geq 0$ (we use the notation of Section 2.1 instead of that of Sethi & Thompson (2000)), with g "as many times continuously differentiable as necessary". It states that there must exist a vector of costate variables $p(\cdot)$ and a positive function $\zeta(\cdot)$ such that in particular, at each entry or contact time τ of an optimal trajectory with the constraint $g(\cdot)$, the the "jump condition" (see (2.3.18)) is:

$$p_i(\tau^-) - p_i(\tau^+) \;=\; \zeta(\tau)\frac{\partial g}{\partial x_i}(\mathbf{x}(\tau)) \; .$$

We can apply the theorem to the constraint $g(S,Z) := \overline{Z} - Z \geq 0$ which is C^∞. Consider an optimal trajectory with initial condition $(S^0, \overline{Z})$ at some arbitrary time t^0, with $S^0 < S_m$. Let t^{QS} be such that $S^{(Q)}(t^{QS}) = S_m$. Since the constraint is $g(S,Z) = \overline{Z} - Z \geq 0$ then there exits $\beta := \zeta(t^{QS}) \geq 0$ such that:

$$\begin{aligned} \lambda_S(t^{QS-}) - \lambda_S(t^{QS}) &= \beta \; \frac{\partial g}{\partial S}(S(t^{QS}), Z(t^{QS})) \;=\; 0 \\ \lambda_Z(t^{QS-}) - \lambda_Z(t^{QS}) &= \beta \; \frac{\partial g}{\partial Z}(S(t^{QS}), Z(t^{QS})) \;=\; -\beta \; . \end{aligned}$$

The function λ_S *must* therefore be continuous at $t = t^{QS}$. However, we have the possibility that λ_Z has a jump, with:

$$\lambda_Z(t^{QS-}) \; - \; \lambda_Z(t^{QS}) \;\leq\; 0 \; . \tag{4.5.1}$$

On the other hand, we have in Phase Q:

$$\lambda_Z(t^{QS-}) \; - \; \lambda_S(t^{QS-}) \; + \; c_s \;=\; \lambda_Z^{(Q)}(t^{QS-}) \; - \; \lambda_S^{(Q)}(t^{QS-}) \; + \; c_s \;=\; 0 \; . \tag{4.5.2}$$

In Phase S, given the values (4.1.15) and (4.1.16), we have:

$$\lambda_Z(t^{QS+}) \; - \; \lambda_S(t^{QS+}) \; + \; c_s \;=\; \lambda_Z^{(S)} \; - \; \lambda_S^{(S)} \; + \; c_s$$

$$= \frac{\rho+\beta}{\beta}\left(c_s + \frac{c_x - \overline{p}}{\zeta}\right) - \left(c_s + \frac{c_x - \overline{p}}{\zeta}\right) + c_s$$
$$= \frac{\rho+\beta}{\beta} c_s + \frac{\rho}{\beta}\frac{c_x - \overline{p}}{\zeta} = \frac{\rho+\beta}{\beta}(c_s - \hat{c}_s) . \quad (4.5.3)$$

However, using the continuity of $\lambda_S(t)$ in (4.5.2) and (4.5.3) we obtain, by difference,

$$\lambda_Z(t^{QS-}) - \lambda_Z(t^{QS+}) = -\frac{\rho+\beta}{\beta}(c_s - \hat{c}_s) > 0 ,$$

a contradiction with (4.5.1). Such an optimal trajectory cannot exist.

4.5.5.3 Non-optimality of joining Phase Q and Phase S: a comparison argument

In this section, we provide an argument for the non-optimality of a trajectory following the ceiling $Z = \overline{Z}$, through the comparison of trajectories.

Specifically, we evaluate the value $V^{(Q)}(S^0)$ of a trajectory constrained to stay on $Z = \overline{Z}$, starting from some $S = S^0 < S_m$ and ending in $S = S_m$, and optimal within this set of constraints. Next, we evaluate the value $V^{(C)}(S^0)$ of a specifically constructed trajectory, also starting from $S = S^0$ and ending in $S = S_m$, but which behaves much like trajectories in Phase B, without being necessarily optimal. When S^0 is close to S_m, we arrive at the following asymptotic expansions:

$$V^{(Q)}(S^0) = V(S_m) + \lambda_S^{(S)}(S^0 - S_m) + \alpha^{(Q)}(S_m - S^0)^{3/2} + o((S_m - S^0)^{3/2}) \quad (4.5.4)$$
$$V^{(C)}(S^0) = V(S_m) + \lambda_S^{(S)}(S^0 - S_m) + O((S_m - S^0)^{5/3}) , \quad (4.5.5)$$

where $\lambda_S^{(S)} = c_s + (c_x - \overline{p})/\zeta$ and $\alpha^{(Q)} < 0$. Since $(S_m - S^0)^{5/3}$ is asymptotically smaller than $(S_m - S^0)^{3/2}$ when $S^0 \to S_m$, it follows that there exists a range of values for S^0 such that $V^{(C)}(S^0) > V^{(Q)}(S^0)$. Indeed, from the expressions above, we have, as $S^0 \uparrow S_m$,

$$V^{(C)}(S^0) - V^{(Q)}(S^0) = -\alpha^{(Q)}(S_m - S^0)^{3/2} + o((S_m - S^0)^{3/2})$$
$$= (S_m - S^0)^{3/2}\left[-\alpha^{(Q)} + o(1)\right] .$$

The term in brackets is necessarily strictly positive on some interval for S^0. For initial values of S^0 in this interval, the trajectory in Phase Q is outperformed by the special trajectory "(C)".

Preliminary: value of a trajectory. All trajectories considered in this paragraph start from some state $(S^0, \overline{Z})$ at time t^0, and eventually reach the state $(S_m, \overline{Z})$ at time T. The value of this trajectory is then:

$$V(S^0) = \int_{t^0}^{T} [u(x(t)) - c_x x(t) - c_s s(t)] e^{-\rho(t-t^0)} \mathrm{d}t + e^{-\rho(T-t^0)} \frac{u(\overline{x}) - (c_x + \zeta c_s)\overline{x}}{\rho}$$
$$= \int_{t^0}^{T} [u(x(t)) - u(\overline{x}) - c_x(x(t) - \overline{x}) - c_s(s(t) - \zeta\overline{x})] e^{-\rho(t-t^0)} \mathrm{d}t + \frac{u(\overline{x}) - (c_x + \zeta c_s)\overline{x}}{\rho}$$
$$= V(S_m) + \int_{t^0}^{T} [u(x(t)) - u(\overline{x}) - c_x(x(t) - \overline{x}) - c_s(s(t) - \zeta\overline{x})] e^{-\rho(t-t^0)} \mathrm{d}t . \quad (4.5.6)$$

The value $V(S_m)$ of state $(S_m, \overline{Z})$ is the same for all trajectories.

Optimal trajectory on the ceiling. Finding the optimal control when the constraint $Z = \overline{Z}$ is enforced is the topic of Appendix C.1.3 on page 93.

The value of an optimal trajectory is given by (cf. (4.5.6)):

$$V^{(Q)}(S^0) = V(S_m) + h(t^0)$$

where

$$h(t^0) \;=\; \int_{t^0}^{T} \left[u(x(t)) - u(\overline{x}) - (c_x + \zeta c_s)(x(t) - \overline{x}) + \beta c_s (S_m - S(t))\right] e^{-\rho(t-t^0)} \mathrm{d}t \;. \qquad (4.5.7)$$

The difficulty is that the value is expressed as a function of t^0 and we need it as a function of S^0. The relationship between both variables is not explicit: we will need to approximately express S^0 as a function of t^0 when t^0 is close to T.

We first expand $h(\cdot)$ as a Talyor series at $t^0 = T$. Differentiating, we get successively:

$$\begin{aligned}
h'(t) &= -\left[u(x(t)) - u(\overline{x}) - (c_x + \zeta c_s)(x(t) - \overline{x}) + \beta c_s (S_m - S(t))\right] \;+\; \rho h(t) \\
h''(t) &= -\left[\dot{x}(t) u'(x(t)) - (c_x + \zeta c_s)\dot{x}(t) - \beta c_s \dot{S}(t)\right] \;+\; \rho h'(t) \\
h'''(t) &= -\left[\ddot{x}(t) u'(x(t)) + (\dot{x}(t))^2 u''(x(t)) - (c_x + \zeta c_s)\ddot{x}(t) - \beta c_s \ddot{S}(t)\right] \;+\; \rho h''(t) \;.
\end{aligned}$$

Since $x(T) = \overline{x}$, and $\dot{S} = \zeta(x - \overline{x})$, these derivative evaluate at $t^0 = T$ as:

$$\begin{aligned}
h'(T) &= 0 \\
h''(T) &= -\left[\dot{x}(T)\overline{p} - (c_x + \zeta c_s)\dot{x}(T)\right] \;=\; \zeta \lambda_S^{(S)} \dot{x}(T) \\
h'''(T) &= -\left[\ddot{x}(T)\overline{p} + (\dot{x}(T))^2 u''(\overline{x}) - (c_x + \zeta c_s)\ddot{x}(T) - \beta c_s \zeta \dot{x}(T)\right] \;+\; \rho h''(T) \\
&= \zeta \lambda_S^{(S)} \ddot{x}(T) \;+\; \dot{x}(T)\left(-\dot{x}(T) u''(\overline{x}) + \zeta(\beta c_s + \rho \lambda_S^{(S)})\right) \;.
\end{aligned}$$

The Taylor expansion of $h(\cdot)$ writes then as:

$$\begin{aligned}
h(t^0) &= \frac{1}{2}\zeta \lambda_S^{(S)} \dot{x}(T)(t^0 - T)^2 \;+\; \frac{1}{6}\zeta \lambda_S^{(S)} \ddot{x}(T)(t^0 - T)^3 \\
&\quad + \frac{1}{6}\dot{x}(T)\left(-\dot{x}(T) u''(\overline{x}) + \zeta(\beta c_s + \rho \lambda_S^{(S)})\right)(t^0 - T)^3 \;+\; O((t^0 - T)^4) \;. \qquad (4.5.8)
\end{aligned}$$

Using again the fact that $\dot{S} = \zeta(x - \overline{x})$, a Taylor expansion for $S(t^0)$ is:

$$S(t^0) \;=\; S_m \;+\; \frac{1}{2}\zeta \dot{x}(T)(t^0 - T)^2 \;+\; \frac{1}{6}\zeta \ddot{x}(T)(t^0 - T)^3 \;+\; O((t^0 - T)^4) \;. \qquad (4.5.9)$$

Substituting (4.5.9) into (4.5.8), we get:

$$h(t^0) \;=\; \lambda_S^{(S)}(S(t^0) - S_m) \;+\; \frac{1}{6}\dot{x}(T)\left(-\dot{x}(T) u''(\overline{x}) + \zeta(\beta c_s + \rho \lambda_S^{(S)})\right)(t^0 - T)^3 \;+\; O((t^0 - T)^4) \;. \qquad (4.5.10)$$

The value $\dot{x}(T)$ is obtained from (C.1.13) and (C.1.14) as:

$$\begin{aligned}
\dot{x}(T) &= -\zeta \dot{\mu}_S(T)(q^d)'(c_x + \zeta c_s - \zeta \mu_S(T)) \;=\; -\,\zeta\rho\left(\lambda_S^{(S)} + \frac{\beta c_s}{\rho}\right)(q^d)'(\overline{p}) \\
&= \frac{\zeta}{W}(\rho \lambda_S^{(S)} + \beta c_s) \;=\; \frac{\zeta}{W}(\rho + \beta)(c_s - \hat{c}_s) \;.
\end{aligned}$$

We have introduced the notation $W = -u''(\overline{x}) = -1/(q^d)'(\overline{p})$. By assumption on $u(\cdot)$, $W > 0$. Since $c_s < \hat{c}_s$ by assumption, we have $\dot{x}(T) < 0$.

On the other hand, from (4.5.9), we can solve "approximately" this equation for t^0 as a function of $S^0 = S(t^0)$. We obtain, remembering that $t^0 - T < 0$ and since $\dot{x}(T) < 0$:

$$t^0 - T \;=\; -\sqrt{-\frac{2(S_m - S^0)}{\zeta \dot{x}(T)}}\,(1 + o(1)) \;.$$

When replaced in (4.5.10), we obtain after simplification:

$$h(t^0(S^0)) \;=\; -\lambda_S^{(S)}(S_m - S^0) \;-\; \frac{1}{6}\;\dot{x}(T)\;2W\dot{x}(T)\;\left(-\frac{2(S_m - S^0)}{\zeta \dot{x}(T)}\right)^{3/2}(1 + o(1))$$

$$= \quad -\lambda_S^{(S)}(S_m - S^0) \; - \; \frac{2}{3\zeta}\left(2W(\rho+\beta)(\hat{c}_s - c_s)\right)^{1/2}(S_m - S^0)^{3/2}(1+o(1)) \; .$$

Summing up, we have proved the expansion (4.5.4) with the constant

$$\alpha^{(Q)} \quad = \quad - \; \frac{2}{3\zeta}\left(2W(\rho+\beta)(\hat{c}_s - c_s)\right)^{1/2}$$

which is strictly negative as announced.

A trajectory leaving the ceiling. We now construct and analyze a specific trajectory generated by a control as in Phase B, that is, where $s(t) = \zeta x(t)$. Throughout the computations, we repeatedtly use the identity $\alpha\overline{Z} = \beta S_m = \zeta\overline{x}$. We start with the choice of $Z(t)$ and we successively deduce $S(t)$ and $s(t) = \zeta x(t)$. We begin with defining:

$$\varepsilon = S_m - S^0 \qquad u_0 \;=\; \left(\frac{24W\varepsilon}{\zeta^2(\rho+\alpha)(\rho+\beta)\lambda_S^{(S)}}\right)^{1/3} ,$$

a *negative* time value since $\lambda_S^{(S)} < 0$. Then, define, for $u \in [u_0, 0]$,

$$Z(T+u) \;=\; \overline{Z} \;+\; \frac{\beta\varepsilon}{u_0^2}u^3\left(1 - \frac{u}{u_0}\right) \; . \tag{4.5.11}$$

Clearly, $Z(T+u_0) = Z(T) = \overline{Z}$. On the interval $u \in [u_0, 0]$, $Z(T+u) \le \overline{Z}$. Since ε/u_0^2 is of order $\varepsilon^{1/3}$, the function $Z(\cdot)$ is positive on the interval $[u_0, 0]$ when ε is sufficiently small. According to the dynamics of $Z(\cdot)$, we must have

$$\beta S(T+u) \;=\; \dot{Z}(T+u) + \alpha Z(T+u) \;=\; \beta S_m \;+\; \frac{\beta\varepsilon u^2}{u_0^3}\left(3u_0 - (4-\alpha u_0)u - \alpha u^2\right) \; .$$

It is readily verified that $S(T+u_0) = S^0$ and $S(T) = S_m$.

Next, the dynamics on S provide the value of the control s:

$$\begin{aligned} s(T+u) \;&=\; \dot{S}(T+u) + \beta S(T+u) \\ &=\; \zeta\overline{x} + \frac{\varepsilon u}{u_0^3}\left(6u_0 - 12u + 3(\alpha+\beta)u_0 u - \beta(4-\alpha u_0)u^2 - 4\alpha u^2 - \alpha\beta u^3\right) \; . \end{aligned}$$

Since $u \in [u_0, 0]$ and u_0 is of order $\varepsilon^{1/3}$, it follows that the term in parentheses can be made as small as needed by a proper choice of ε. Therefore, for ε sufficiently close to 0, the function $s(T+u)$ is positive for $u \in [u_0, 0]$ and the control is admissible.

At this point, we have verified that the proposed trajectory is feasible: it stays in the domain of constraints, and the control associated to it is valid. Moreover, it starts in state $(S^0, \overline{Z})$ at time $t^0 = T + u_0$ and ends up in state $(S_m, \overline{Z})$ at time T.

We now turn to the evaluation of the value $V^{(C)}(S^0)$ of this trajectory. It is (cf. (4.5.6)):

$$V^{(C)}(S^0) \;=\; V(S_m) \;+\; h(t^0)$$

where $t^0 = T + u_0$ and

$$h(t^0) \;=\; \int_{t^0}^{T}\left[u(x(t)) - u(\overline{x}) - (c_x + \zeta c_s)(x(t) - \overline{x})\right]e^{-\rho(t-t^0)}\mathrm{d}t \; .$$

Computing the Taylor expansion of $h(\cdot)$ turns out to be cumbersome, because an expansion to order 5 is necessary. We use a different approach, after a small preparation aimed at shortening later computations. Let us introduce the function u_2 such that:

$$u(x) - u(\overline{x}) - \overline{p}(x - \overline{x}) = u_2(x) \; .$$

Note that $\overline{p} = u'(\overline{x})$. Replacing in the expression for $h(t^0)$, we obtain successively,

$$\begin{aligned} h(t^0) &= \int_{t^0}^{T} [u_2(x(t)) + (\overline{p} - c_x - \zeta c_s)(x(t) - \overline{x})] \, e^{-\rho(t-t^0)} \mathrm{d}t \\ &= \int_{t^0}^{T} \left[u_2(x(t)) - \lambda_S^{(S)} \zeta(s(t) - \zeta\overline{x})\right] e^{-\rho(t-t^0)} \mathrm{d}t \\ &= \int_{t^0}^{T} \left[u_2(x(t)) - \lambda_S^{(S)} \dot{S}(t) - \lambda_S^{(S)} \beta(S(t) - S_m)\right] e^{-\rho(t-t^0)} \mathrm{d}t \ . \end{aligned} \tag{4.5.12}$$

In the last expression, we have used the fact that $s = \dot{S} + \beta S$, which implies $s - \zeta\overline{x} = \dot{S} + \beta(S - S_m)$. Integrating by parts, we have:

$$\int_{t^0}^{T} \dot{S}(t) e^{-\rho(t-t^0)} \mathrm{d}t \ = \ S_m - S^0 \ + \ \rho \int_{t^0}^{T} (S(t) - S_m) e^{-\rho(t-t^0)} \mathrm{d}t \ . \tag{4.5.13}$$

In the same vein, we have the following property, which we shall use later:

$$\int_{t^0}^{T} \dot{Z}(t) e^{-\rho(t-t^0)} \mathrm{d}t \ = \ \rho \int_{t^0}^{T} (Z(t) - \overline{Z}) e^{-\rho(t-t^0)} \mathrm{d}t \ . \tag{4.5.14}$$

Replacing in (4.5.12), we obtain the new expression:

$$h(t^0) \ = \ \lambda_S^{(S)}(S^0 - S_m) \ + \ \int_{t^0}^{T} u_2(x(t)) e^{-\rho(t-t^0)} \mathrm{d}t \ - \ (\rho + \beta)\lambda_S^{(S)} \int_{t^0}^{T} (S(t) - S_m) e^{-\rho(t-t^0)} \mathrm{d}t \ .$$

We now proceed with the expansion of the integrals in this expression as $\varepsilon \to 0$, remembering that $u_0 = t^0 - T$ is of order $\varepsilon^{1/3}$. First, since $u_2(x) = -(W/2)(x - \overline{x})^2 + O((x - \overline{x})^3)$, we get:

$$\begin{aligned} &\int_{t^0}^{T} u_2(x(t)) e^{-\rho(t-t^0)} \mathrm{d}t \\ &= -\frac{W}{2\zeta^2} \int_{u_0}^{0} \left[(s(T+u) - \zeta\overline{x})^2 + O((s(T+u) - \zeta\overline{x})^3)\right] e^{-\rho(u-u_0)} \mathrm{d}u \\ &= -\frac{W}{2\zeta^2} \int_{u_0}^{T} \frac{\varepsilon^2}{u_0^6} 36u^2 (u_0 - 2u + O(u_0^2))^2 e^{-\rho(u-u_0)} \mathrm{d}u \\ &= -\frac{W}{2\zeta^2} \frac{\varepsilon^2}{u_0^6} \left(-\frac{24}{5} u_0^5 + O(u_0^6)\right) \ = \ \frac{12W}{5\zeta^2} \, \frac{\varepsilon^2}{u_0} \ + \ O(\varepsilon^2) \ . \end{aligned}$$

For the second integral, from the value of $S(t)$ we have:

$$\begin{aligned} &\int_{t^0}^{T} (S(t) - S_m) e^{-\rho(t-t^0)} \mathrm{d}t \\ &= \int_{t^0}^{T} \frac{1}{\beta} (\dot{Z}(t) + \alpha(Z(t) - \overline{Z})) e^{-\rho(t-t^0)} \mathrm{d}t \ = \ \frac{\rho + \alpha}{\beta} \int_{u_0}^{0} (Z(T+u) - \overline{Z})) e^{-\rho(u-u_0)} \mathrm{d}u \\ &= (\rho + \alpha) \frac{\varepsilon}{u_0^2} \int_{u_0}^{0} u^3 \left(1 - \frac{u}{u_0}\right) e^{-\rho(u-u_0)} \mathrm{d}u \ = \ - \ \frac{\rho + \alpha}{20} \, \varepsilon u_0^2 \ + \ O(\varepsilon^2) \ . \end{aligned}$$

We have used (4.5.14) in the derivation. Gathering the different parts and replacing u_0 with $-L\varepsilon^{1/3}$ and ε with $S_m - S^0$, we obtain:

$$V^{(C)}(S^0) \ = \ V(S_m) + \lambda_S^{(S)}(S^0 - S_m) + \left((\rho + \alpha)(\rho + \beta)\lambda_S^{(S)} \frac{L^2}{20} - \frac{12W}{5\zeta^2 L}\right) (S_m - S^0)^{5/3} + O((S_m - S^0)^2) \ .$$

This expansion is the one announced in (4.5.5) with the term $O((S_m - S^0)^{5/3})$ made more precise.

Final comments. Some observations resulting from the analysis:

1. The time each trajectory takes to reach the final state is expressed as a function of $\varepsilon = S_m - S^0$. When the ceiling is followed, we have $T - t^0 = O(\varepsilon^{1/2})$, whereas for the constructed trajectory, $T - t^0 = -u_0 = O(\varepsilon^{1/3})$. This last trajctory takes therefore *more time* to reach the final state, during which slightly more utility is accrued.

2. Introducing the function u_2 is also possible in the analysis of Phase Q trajectories. Introducing it in (4.5.7), we write successively, using (4.5.13),

$$\begin{aligned} h(t^0) &= \int_{t^0}^{T} \left[u_2(x(t)) + (\overline{p} - c_x - \zeta c_s)(x(t) - \overline{x}) + \beta c_s(S_m - S(t))\right] e^{-\rho(t-t^0)}\mathrm{d}t \\ &= \int_{t^0}^{T} \left[u_2(x(t)) - \lambda_S^{(S)}\dot{S}(t) + \beta c_s(S_m - S(t))\right] e^{-\rho(t-t^0)}\mathrm{d}t \\ &= \lambda_S^{(S)}(S^0 - S_m) \ + \ \int_{t^0}^{T} \left[u_2(x(t)) + (\beta c_s + \zeta\lambda_S^{(S)})(S_m - S(t))\right] e^{-\rho(t-t^0)}\mathrm{d}t \ . \end{aligned}$$

   A Taylor expansion of the integral leads to (4.5.4).

3. We have selected the relationship between u_0 and ε as $u_0 = -L\varepsilon^{1/3}$, being guided by the analysis of Phase B in Appendix D.2.2. Alternately, one may choose a general relationship $u_0 = -C\varepsilon^{\gamma}$, and conclude that the optimal choice of parameters is $\gamma = 1/3$ and $C = L$ as above.

4.6 Extensions and concluding remarks

We conclude this analysis with several comments related to particular values, limiting cases or extensions of the results.

4.6.1 Finite storage capacity.

If the sequestered stock is assumed to have a maximal capacity $\overline{S}$, additional phases appear when $S = \overline{S}$. Assume that $\overline{S} > 0$ so that sequestration is effectively possible. We briefly sketch the construction of solutions in this case, under the condition $c_s \le \hat{c}_s$. When $c_s > \hat{c}_s$, optimal trajectories are obtained by simply restricting the trajectories previously obtained to $S \le \overline{S}$.

Stationary states. Assuming that S and Z are constant, and solving the system of equations $\dot{S} = \dot{Z} = 0$, we obtain in general the constant consumptions

$$s \ = \ \beta S \qquad\qquad x \ = \ \frac{\alpha Z}{\zeta} \ .$$

Since we must have $s \le \zeta x$, this is possible only when $\beta S \le \alpha Z$. Imposing $\dot{\lambda}_Z = \dot{\lambda}_S = 0$, we get $\nu_Z = -(\rho + \alpha)\lambda_Z$ and $(\rho + \beta)\lambda_S = \beta\lambda_Z$. If the stationary point is such that $Z < \overline{Z}$, then $\nu_Z = 0$ and $\lambda_Z = \lambda_S = 0$. This cannot satisfy the first-order conditions.

Therefore, when $\overline{S} \le S_m$, the point $(\overline{S}, \overline{Z})$ is the only candidate stationary point. From the state dynamics (2.2.2), we get the constant consumption and capture values:

$$s \ = \ \bar{\bar{s}} \ := \ \beta\overline{S}, \qquad\qquad x \ = \ \overline{x} \ .$$

We have indeed $\bar{\bar{s}} \le \zeta\overline{x}$, with equality if and only if $\overline{S} = S_m$.

In the case $0 < \overline{S} < S_m$, inequality is strict and we have $\gamma_s = \gamma_{sx} = 0$. Consequently, from (2.3.3) then (2.3.2), assuming $y = 0$, we obtain the constant values for the costate variables:

$$\lambda_S \ = \ c_s + \frac{c_x - \overline{p}}{\zeta} \qquad\qquad \lambda_Z \ = \ \frac{c_x - \overline{p}}{\zeta} \ .$$

From (2.3.7), we obtain $\underline{\gamma}_y = c_y - \overline{p}$, positive by assumption.

Finally, the value of ν_S is obtained from (2.3.14) as:

$$\nu_S \;=\; \beta\lambda_Z - (\rho+\beta)\lambda_S \;=\; -\,\rho\,\frac{c_x - \overline{p}}{\zeta} - (\rho+\beta)c_s \;=\; (\rho+\beta)(\hat{c}_s - c_s)$$

and $\nu_S \geq 0$ under the condition $c_s \leq \hat{c}_s$.

However, in the case $\overline{S} = S_m$, $\gamma_s = 0$ but γ_{sx} is possibly a positive number. From first-order conditions (2.3.3) then (2.3.2) we obtain:

$$\lambda_S \;=\; c_s + \frac{c_x - \overline{p}}{\zeta} \qquad\qquad \lambda_Z \;=\; \frac{c_x - \overline{p}}{\zeta} - \gamma_{sx}\;,$$

The dynamics (2.3.11) of λ_S imply:

$$0 \;=\; (\rho+\beta)\lambda_S - \beta\lambda_Z + \nu_S \qquad \Longrightarrow \qquad \nu_S \;=\; \beta\lambda_Z - (\rho+\beta)\lambda_S\;.$$

Finally, we have from (2.3.10): $\nu_Z = \dot{\lambda}_Z - (\rho+\alpha)\lambda_Z$. The constraints $\gamma_{sx} \geq 0$, $\nu_Z \geq 0$ and $\nu_S \geq 0$ are satisfied when λ_Z is chosen as any constant such that

$$\frac{\rho+\beta}{\beta}\left(c_s + \frac{c_x - \overline{p}}{\zeta}\right) \;\leq\; \lambda_Z \;\leq\; \frac{c_x - \overline{p}}{\zeta}\;,$$

this interval being nonempty whenever $c_s \leq \hat{c}_s$. Other, non-constant choices of $\lambda_Z(t)$ are possible. The value of λ_Z is not uniquely determined in this case but the value of $\lambda_Z + \gamma_{sx}$ is determined.

Non-stationary phase. Assume now that $S = \overline{S}$ and $Z < \overline{Z}$. Since $\dot{S} = 0$, we have the constant capture rate $s = \bar{\bar{s}}$ as above. The value $x(t)$ may be either $\bar{\bar{s}}/\zeta$ (as in Phase B) or strictly smaller (as in Phase Q). We first rule out the second possibility.

Assume indeed that $\bar{\bar{s}} < \zeta x$. Then $\gamma_s = \gamma_{sx} = 0$ and $\lambda_S = \lambda_Z + c_s$. This implies $\dot{\lambda}_S = \dot{\lambda}_Z$. Using the dynamics (2.3.11) and (2.3.10), we get:

$$(\rho+\alpha)\lambda_Z \;=\; (\rho+\beta)\lambda_S - \beta\lambda_Z + \nu_S \qquad \Longrightarrow \qquad \nu_S \;=\; \alpha\lambda_Z - (\rho+\beta)c_s\;.$$

Since $\lambda_Z < 0$, this implies $\nu_S < 0$, a contradiction.

Therefore, if such a phase exists, we must have a constant consumption.

$$x \;=\; \bar{\bar{x}} \;:=\; \frac{\beta\overline{S}}{\zeta}\;.$$

As in Phase B, the first-order conditions imply $\lambda_Z + \gamma_{sx} = \lambda_S - c_s$, then

$$u'(\bar{\bar{x}}) \;=\; c_x + \zeta c_s - \zeta\lambda_S\;.$$

Accordingly, with the notation $\bar{\bar{p}} := u'(\bar{\bar{x}})$, the value of λ_S is constant at:

$$\lambda_S \;=\; c_s + \frac{c_x - \bar{\bar{p}}}{\zeta}\;.$$

As previously, since λ_S is constant, we must have:

$$\nu_S \;=\; \beta\lambda_Z - (\rho+\beta)\lambda_S\;.$$

The constraints $\nu_S \geq 0$ and $\gamma_{sx} \geq 0$ are satisfied as long as λ_Z is in the interval defined by:

$$\frac{\rho+\beta}{\beta}\left(c_s + \frac{c_x - \bar{\bar{p}}}{\zeta}\right) \;\leq\; \lambda_Z \;\leq\; \frac{c_x - \bar{\bar{p}}}{\zeta}\;.$$

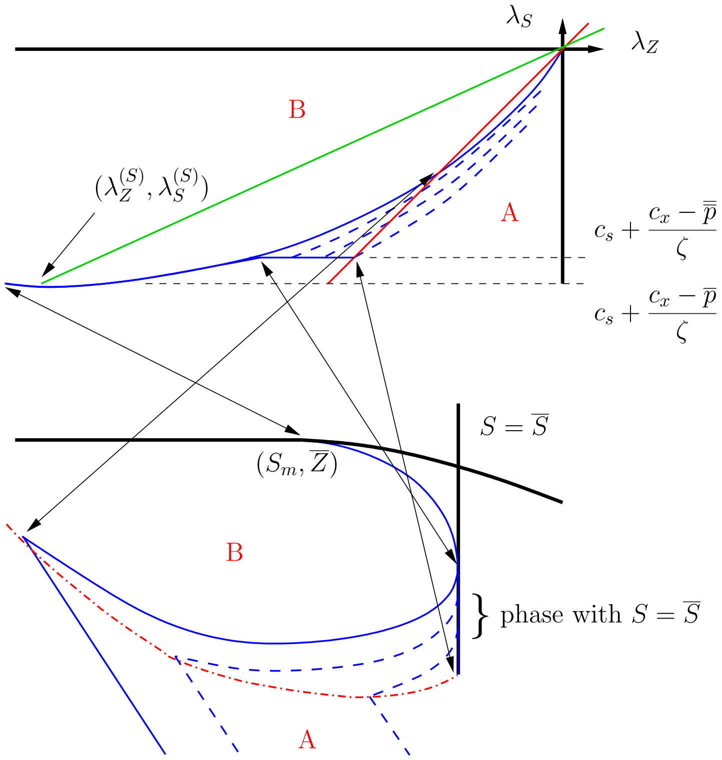

Figure 4.21: Trajectories of (λ_Z, λ_S) (top) and (S, Z) (bottom) in presence of a limit on sequestered stock

Connection with other phases. When $\overline{S} > S_m$, the phase where $S = \overline{S}$ cannot be terminal. It must therefore be connected with other phases.

Obviously, the optimal trajectories we have described in the case $\overline{S} = +\infty$ are still optimal if they lie entirely in the domain $\mathcal{D} \cap \{S \leq \overline{S}\}$. Consider the case $c_s < \hat{c}_s$: these trajectories are represented on Figures 4.8 and 4.7 on page 54. We focus on trajectories in Phase B that end up at point $(S_m, \overline{Z})$ at some time instant T. Those are parametrized by the value $\lambda_Z(T^-)$.

Consider some value $\overline{S} > S_m$. To it corresponds a consumption $\bar{\bar{x}} > \overline{x}$, a price $\bar{\bar{p}} < \overline{p}$ and a value $\lambda_S = \overline{\lambda}_S = c_s + (c_x - \bar{\bar{p}})/\zeta > c_s + (c_x - \overline{p})/\zeta = \lambda_S^{(S)}$. Pick a value $\lambda_Z(T^-)$, and follow backwards the corresponding trajectory. There exists a time $\bar{\bar{t}} < T$ such that $\lambda_S(\bar{\bar{t}}) = \overline{\lambda}_S$. Then, if $\overline{S}$ is not too large, there exists some $\lambda_Z(T^-)$ for which $S(\bar{\bar{t}}) = \overline{S}$. Since $\beta\overline{S} = \zeta\bar{\bar{x}}$, the trajectory is such that $\dot{S}(\bar{\bar{t}}) = 0$: it is therefore tangent to the line $S = \overline{S}$. It is possible to glue the piece of trajectory with $S = \overline{S}$ with the trajectory in Phase B at this point.

Going backwards in time, the trajectory of adjoint variables (λ_Z, λ_S) moves on the line $\lambda_S = \overline{\lambda}_S$ until it reaches the line $\lambda_S = \lambda_Z + c_s$. At this point the first-order conditions cease to be satisfied, and the trajectory with $S = \overline{S}$ cannot be prolongated.

The situation is represented in Figure 4.21. Some trajectories (not represented) are the same as in the case $\overline{S} = +\infty$: those located inside the "loop" and those located above the loop but to the left of $S = \overline{S}$. Trajectories located below the loop are different from those of the case $\overline{S} = +\infty$.

When $\overline{S}$ is too large, such a diagram is not feasible because: either the condition $\lambda_S(\bar{\bar{t}}) = \bar{\lambda}_S$ cannot be met while the point (λ_Z, λ_S) is in Phase B; or else because the condition $S(\bar{\bar{t}}) = \overline{S}$ cannot be met. The limiting situation will be when an optimal trajectory has a vertical tangent ($\dot{S} = 0$) while at the same time passing from Phase A to Phase B. See the trajectory numbered as **(III)** in Figure 4.8. This defines a limiting value $\overline{S}_{max}$. For all values of $\overline{S} > \overline{S}_{max}$, the optimal trajectories are just the same as when $\overline{S} = +\infty$.

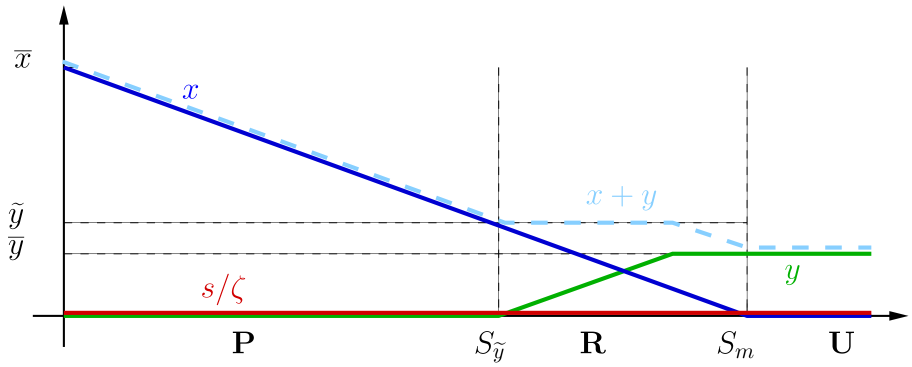

Figure 4.22: Optimal control as a state feedback of $(S, \overline{Z})$, in the case where a limit $\overline{y}$ exists, and in the case $c_s \geq c_{sm}$

4.6.2 Bounded clean energy production capacity

If there is a bound $\overline{y}$ on the consumption of renewable energy, and if $\overline{y} < \widetilde{y}$, then some optimal trajectories are modified. Denote $c_{\overline{y}} = u'(\overline{y})$. Since u' is decreasing by Assumption 1, we have $c_{\overline{y}} > c_y$.

The first-order equations related to y are now (2.3.4), (2.3.7) and (2.3.8):

$$u'(x+y) \;=\; c_y - \underline{\gamma}_y + \overline{\gamma}_y, \qquad \underline{\gamma}_y y = 0 \qquad \overline{\gamma}_y[\overline{y} - y] = 0$$

and $\underline{\gamma}_y \geq 0$, $\overline{\gamma}_y \geq 0$. In every situation where $y = 0$ was found to be optimal, it is still now with the choice $\overline{\gamma}_y = 0$. When $y = \widetilde{y}$, together with $x = s = 0$ was found to be optimal (Phase L or Phase U), then it has to be replaced with $y = \overline{y}$ and we must set $\overline{\gamma}_y = c_{\overline{y}} - c_y$ which is indeed positive. But then along such optimal trajectories, $u'(x+y) = c_{\overline{y}}$ instead of c_y. Therefore, when "glueing" pieces of trajectories, we must substitute $c_{\overline{y}}$ to c_y. For instance, in Figure 4.3 representing adjoint variables, the zone "L" must now be defined by $\lambda_Z \leq (c_{\overline{y}} - c_x)/\zeta$ and $\lambda_S \leq c_s + (c_{\overline{y}} - c_x)/\zeta$.

Finally, there is the situation of Phase R, where $x > 0$ and $y > 0$ was found to be optimal. If $0 < y < \overline{y}$, we must have $\underline{\gamma}_y = \overline{\gamma}_y = 0$ and the situation is the same as described in Section 3.5.3, with $x + y = \widetilde{y}$. Since y is determined by (3.5.18), the constraint $0 \leq y \leq \overline{y}$ imposes now

$$S_{\widetilde{y}} \leq S \leq S_m - \frac{\zeta}{\beta}(\widetilde{y} - \overline{y}) \;.$$

If $y = \overline{y}$, then we have $u'(x + \overline{y}) = c_y + \overline{\gamma}_y$ with, by Equation (3.5.18), $x = (\beta/\zeta)(S_m - S)$. For $S \in [S_m - (\zeta/\beta)(\widetilde{y} - \overline{y}), S_m]$, which amounts to say: $S_m - S \in [0, (\zeta/\beta)(\widetilde{y} - \overline{y})]$, we see that $x + \overline{y} \in [\overline{y}, \widetilde{y}]$ so that $\overline{\gamma}_y = u'(x + \overline{y}) - c_y \geq 0$.

Figure 4.22 represents the new situation in the case c_s "large", to be compared with Figure 4.1. The value of the total energy consumption $x + y$ has also been represented: this value is not bounded below by $\widetilde{y}$ anymore, but instead by $\overline{y}$. The situation for other values of c_s follows. Observe that the threshold values c_{sQ} and c_{sm} identified in the analysis are changed also.

4.6.3 Storage costs

Assume that the cost function includes a constant storage management cost c_m per unit of stored carbon and per unit time. The maximization problem (2.2.1) becomes:

$$\max_{s(\cdot),x(\cdot),y(\cdot)} \int_0^\infty [u(x(t) + y(t)) - c_s s(t) - c_x x(t) - c_y y(t) - c_m S(t)]\, e^{-\rho t} \mathrm{d}t \;.$$

The modification in the first-order conditions is limited to (2.3.11) which becomes (assuming that $\overline{S} = +\infty$, hence $\nu_S = 0$):

$$\dot{\lambda}_S \;=\; (\rho + \beta)\lambda_S - \beta\lambda_Z + c_m \;.$$

Accordingly, on diagrams representing the the evolution of (λ_Z, λ_S) (see the generic Figure 4.2 on page 43 and its specialization in different cases), the line $\dot{\lambda}_S = 0$ must be translated down (assuming logically that the cost c_m is positive) since its equation is now $(\rho+\beta)\lambda_S = \beta\lambda_Z - c_m$. The geometric locations P_S, P_∞ and Ω are therefore changed, providing respectively the new stationary values for Phase S, Phase P and Phase Q:

$$P_S = \left(\frac{c_m}{\beta} + \frac{\rho+\beta}{\beta}\left(c_s + \frac{c_x - \overline{p}}{\zeta}\right), c_s + \frac{c_x - \overline{p}}{\zeta} \right), \qquad P_\infty = \left(\frac{c_x - \overline{p}}{\zeta}, \frac{\beta}{\rho+\beta}\frac{c_x - \overline{p}}{\zeta} - \frac{c_m}{\rho+\beta} \right),$$

$$\Omega = \left(-\, c_s \frac{\rho+\beta}{\rho} - \frac{c_m}{\rho}, \; -\, c_s \frac{\beta}{\rho} - \frac{c_m}{\rho} \right).$$

The discussion of the different parametric cases then goes along the same lines as when $c_m = 0$. In particular, the value of c_s which separates the case where Phase S is terminal and where it is not, is determined by the condition $\Omega = P_S$ (see Figure 4.10 on page 57). We obtain now:

$$\hat{c}_s = \frac{\rho}{\rho+\beta}\left(\frac{\overline{p} - c_x}{\zeta} - \frac{c_m}{\rho} \right).$$

4.6.4 Complements on capture costs

Assumption 1 includes the assumption that $c_s > 0$. The case $c_s = 0$ is also interesting from the Economics standpoint, in the sense that it allows to concentrate on the impact of the externality provoked by the leakage of sequestered CO_2. Analyzing the case $c_s < 0$ is also relevant, be it for the sake of completeness.

When $c_s = 0$, Lemma 3.1 does not apply. It was concluded that, when $\alpha > 0$, the fact that $0 < s < \zeta x$ implies $\lambda_S = \lambda_Z = 0$. Consumptions must then be $x = \widetilde{x}$ and $y = 0$. The value of s is not determined by first-order equations. However, whatever its value, the dynamics (2.2.2) imply that

$$\dot{S} + \dot{Z} = -\,\alpha Z + \zeta\widetilde{x} = \alpha(\overline{Z} - Z) + \zeta(\widetilde{x} - \overline{x}) \geq \zeta(\widetilde{x} - \overline{x}).$$

As a consequence, the state of the system must exit the interior of domain $\mathcal{D}$ in finite time. Assuming that the boundary is hit where $Z = \overline{Z}$, the system must continue in Phase Q, still with $x(t) = \tilde{x}$, until it reaches $S = S_m$ where it must stop, at some time T, in Phase S. However, such a trajectory has a value equal to:

$$\frac{u(\widetilde{x}) - c_x\widetilde{x}}{\rho}(1 - e^{-\rho T}) + e^{-\rho T}\frac{u(\overline{x}) - c_x\overline{x}}{\rho}$$

and it is outperformed by any trajectory with a constant consumption $x = \overline{x}$. It cannot be optimal, and the initial assumption that $0 < s < \zeta x$ must be wrong. The bang-bang principle of Lemma 3.1 applies also when $c_s = 0$.

Let us now turn to the case where $c_s < 0$. The synthesis of Section 4.5.1 (p. 74) essentially applies, with the following difference in the interior of the domain. Recall from, *e.g.* Section 4.4.1 on page 43, that the sign of $\gamma(t) = \lambda_Z(t) - \lambda_S(t) - c_s$ determines whether Phase A or Phase B prevails. When $t \to -\infty$, both $\lambda_S(t)$ and $\lambda_Z(t)$ tend to 0. Therefore, $\gamma(t) \to -c_s > 0$. Then, going backwards in time, it is now possible to exit Phase A and switch to Phase B. So, in forward time, some trajectories will start in Phase B, switch temporarily to Phase A, then switch back to Phase A. When c_s is a large negative cost, all trajectories will eventually stay in Phase B.

4.6.5 Comparison with non-leaky reservoirs

The situation where $\beta = 0$ is the one studied in Lafforgue et al. (2008*a*) and Lafforgue et al. (2008*b*), where it is also assumed that $\overline{S}$ is finite. This situation is not a special case of the analysis above (which requires $\beta > 0$) but can be analyzed directly.

It turns out that in this case, $\lambda_S \equiv 0$, which is economically interpreted as S being "free". Then there are three cases for c_s. Observe that $\hat{c}_s = (\overline{p} - c_x)/\zeta$ when $\beta = 0$ and it does not depend on ρ anymore.

$c_s \geq \hat{c}_s$: there no capture, $x = \overline{x}$, S is constant, $Z = \overline{Z}$;

$0 \leq c_s < \hat{c}_s$: consumption is $x = q^d(c_x + \zeta c_s)$, capture is $s = x - \overline{x}$, $Z = \overline{Z}$;

$c_s < 0$: there is full capture $s = \zeta x$, with consumption $x = q^d(c_x + \zeta c_s)$, $Z < \overline{Z}$.

When comparing with the situation where $\beta > 0$, we see that both points S_m and $S_{\widetilde{y}}$ go to infinity. Phases R, S and U vanish. There is no possibility of having simultaneously trajectories with and without capture, nor of having consumption of the renewable resource.

Appendix A

Synthetic description of the different phases

A.1 Phase A (free extraction of the NRR; no sequestration)

Specification

s	x	y	X	Z	S
$=0$	>0	$=0$	>0	$<\overline{Z}$	≥ 0

Dynamical system

$$\begin{cases} \dot{X} &= -x \\ \dot{Z} &= -\alpha Z + \beta S + \zeta x \\ \dot{S} &= -\beta S \\ \dot{\lambda}_X &= \rho \lambda_X \\ \dot{\lambda}_Z &= (\rho + \alpha)\lambda_Z \\ \dot{\lambda}_S &= (\rho + \beta)\lambda_S - \beta \lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S &=& c_s + \lambda_Z - \gamma_s & \Longrightarrow & \gamma_s &=& c_s + \lambda_Z - \lambda_S \\ u'(x) &=& c_x + \lambda_X - \zeta\lambda_Z & \Longrightarrow & x &=& q^d(c_x + \lambda_X - \zeta\lambda_Z) \\ u'(x) &=& c_y - \gamma_y & \Longrightarrow & \gamma_y &=& c_y - c_x - \lambda_X + \zeta\lambda_Z \end{array}$$

Constraints

$$\begin{array}{cl} X & X > 0 \\ Z & Z < \overline{Z} \\ s & s = 0 \\ x & c_x + \lambda_X - \zeta\lambda_Z \geq 0 \\ y & y = 0 \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z = 0 \\ \gamma_s & c_s + \lambda_Z - \lambda_S \geq 0 \\ \gamma_{sx} & \gamma_{sx} = 0 \\ \gamma_y & c_y - c_x - \lambda_X + \zeta\lambda_Z \geq 0 \end{array}$$

A.2 Phase B (free extraction of the NRR; maximal sequestration)

Specification

s	x	y	X	Z	S
$=\zeta x$	>0	$=0$	>0	$<\overline{Z}$	≥ 0

Dynamical system

$$\begin{cases} \dot{X} &= -x \\ \dot{Z} &= -\alpha Z + \beta S \\ \dot{S} &= -\beta S + \zeta x \\ \dot{\lambda}_X &= \rho \lambda_X \\ \dot{\lambda}_Z &= (\rho+\alpha)\lambda_Z \\ \dot{\lambda}_S &= (\rho+\beta)\lambda_S - \beta\lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rcl} \lambda_S &=& c_s + \lambda_Z + \gamma_{sx} \\ u'(x) &=& c_x + \lambda_X - \zeta\lambda_Z - \zeta\gamma_{sx} \\ u'(x) &=& c_y - \gamma_y \end{array} \quad \begin{array}{c} \Longrightarrow \\ \Longrightarrow \\ \Longrightarrow \end{array} \quad \begin{array}{rcl} \gamma_{sx} &=& \lambda_S - c_s - \lambda_Z \\ x &=& q^d(c_x + \lambda_X - \zeta\lambda_S + \zeta c_s) \\ \gamma_y &=& c_y - c_x - \lambda_X + \zeta\lambda_S - \zeta c_s \end{array}$$

Constraints

$$\begin{array}{cl} X & X > 0 \\ Z & Z < \overline{Z} \\ s & s = \zeta x \\ x & c_x + \lambda_X - \zeta\lambda_S + \zeta c_s \geq 0 \\ y & y = 0 \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z = 0 \\ \gamma_s & \gamma_s = 0 \\ \gamma_{sx} & \lambda_S - c_s - \lambda_Z \geq 0 \\ \gamma_y & c_y - c_x - \lambda_X + \zeta\lambda_S - \zeta c_s \geq 0 \end{array}$$

A.3 Phase L (zero extraction of the NRR)

Specification

s	x	y	X	Z	S
$= 0$	$= 0$	> 0	> 0	$< \overline{Z}$	≥ 0

Dynamical system

$$\begin{cases} \dot{X} &= -x \\ \dot{Z} &= -\alpha Z + \beta S \\ \dot{S} &= -\beta S \\ \dot{\lambda}_X &= \rho \lambda_X \\ \dot{\lambda}_Z &= (\rho + \alpha)\lambda_Z \\ \dot{\lambda}_S &= (\rho + \beta)\lambda_S - \beta \lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} 0 & = & \lambda_S - c_s - \lambda_Z + \gamma_s - \gamma_{sx} & \Longrightarrow & \gamma_s - \gamma_{sx} & = & \lambda_Z - \lambda_S + c_s \\ u'(y) & = & c_x + \lambda_X - \zeta\lambda_Z - \zeta\gamma_{sx} & & & & \\ u'(y) & = & c_y & \Longrightarrow & y & = & \tilde{y} \end{array}$$

Constraints

$$\begin{array}{cl} X & X > 0 \\ Z & Z \leq \overline{Z} \\ s & s = 0 \\ x & x = 0 \\ y & y > 0 \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z = 0 \\ \gamma_s & \gamma_s \geq 0 \\ \gamma_{sx} & \gamma_{sx} \geq 0 \\ \gamma_y & \gamma_y = 0 \end{array}$$

A.4 Phase P (ceiling; no sequestration)

Specification

s	x	y	X	Z	S
$=0$	>0	$=0$	>0	$=\overline{Z}$	≥ 0

Ceiling constraint:

$$x \ = \ \overline{x} - \frac{\beta}{\zeta} S \ = \ \frac{\beta}{\zeta}(S_m - S) \ .$$

Dynamical system

$$\begin{cases} \dot{X} & = & -x \\ \dot{Z} & = & 0 \\ \dot{S} & = & -\beta S \\ \dot{\lambda}_X & = & \rho \lambda_X \\ \dot{\lambda}_Z & = & (\rho + \alpha)\lambda_Z + \nu_Z \\ \dot{\lambda}_S & = & (\rho + \beta)\lambda_S - \beta \lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S & = & c_s + \lambda_Z - \gamma_s & \Longrightarrow & \gamma_s & = & c_s + \lambda_Z - \lambda_S \\ u'(x) & = & c_x + \lambda_X - \zeta\lambda_Z & \Longrightarrow & \lambda_Z & = & \frac{1}{\zeta}(c_x + \lambda_X - u'(\overline{x} - \frac{\beta S}{\zeta})) \\ u'(x) & = & c_y - \gamma_y & \Longrightarrow & \gamma_y & = & c_y - c_x - \lambda_X + \zeta\lambda_Z \end{array}$$

Constraints

$$\begin{array}{cl} X & X > 0 \\ Z & Z = \overline{Z} \\ s & s = 0 \\ x & S \leq S_m \text{ et } c_x + \lambda_X - \zeta\lambda_Z \geq 0 \\ y & y = 0 \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z \geq 0 \\ \gamma_s & c_s + \lambda_Z - \lambda_S \geq 0 \\ \gamma_{sx} & \gamma_{sx} = 0 \\ \gamma_y & c_y - c_x - \lambda_X + \zeta\lambda_Z \geq 0 \iff u'(x) \leq c_y \\ & \iff x \geq \widetilde{y} \iff S \leq S_{\widetilde{y}}. \end{array}$$

A.5 Phase Q (ceiling; sequestration)

Specification

s	x	y	X	Z	S
> 0 and $< \zeta x$	> 0	$= 0$	> 0	$= \overline{Z}$	≥ 0

Ceiling constraint:

$$s \;=\; \zeta(x - \overline{x}) + \beta S \;=\; \zeta x - \beta(S_m - S) \;.$$

Dynamical system

$$\begin{cases} \dot{X} &= -x \\ \dot{Z} &= 0 \\ \dot{S} &= \zeta(x - \overline{x}) \\ \dot{\lambda}_X &= \rho\lambda_X \\ \dot{\lambda}_Z &= (\rho+\alpha)\lambda_Z + \nu_Z \\ \dot{\lambda}_S &= (\rho+\beta)\lambda_S - \beta\lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S &=& c_s + \lambda_Z & & & & \\ u'(x) &=& c_x + \lambda_X - \zeta\lambda_Z & \Longrightarrow & x &=& q^d(c_x + \lambda_X - \zeta\lambda_Z) \\ u'(x) &=& c_y - \gamma_y & \Longrightarrow & \gamma_y &=& c_y - c_x - \lambda_X + \zeta\lambda_Z \end{array}$$

Constraints

$$\begin{array}{cl} X & X > 0 \\ Z & Z = \overline{Z} \\ s & S \leq S_m \text{ and } x \geq \frac{\beta}{\zeta}(S_m - S) \\ x & c_x + \lambda_X - \zeta\lambda_Z \geq 0 \\ y & y = 0 \\ \nu_X & \nu_X = 0 \\ \nu_Z & \lambda_Z \leq \frac{\rho+\beta}{\alpha}c_s \text{ or } \lambda_S \leq \frac{\rho+\alpha+\beta}{\alpha}c_s \\ \gamma_s & c_s + \lambda_Z - \lambda_S = 0 \\ \gamma_{sx} & \gamma_{sx} = 0 \\ \gamma_y & c_y - c_x - \lambda_X + \zeta\lambda_Z \geq 0 \end{array}$$

Observations. Conditions $\lambda_Z \leq 0$ and $c_s + \lambda_Z - \lambda_S \geq 0$ imply Conditions $\lambda_Z \leq \frac{\rho+\beta}{\alpha}c_s$ or $\lambda_S \leq \frac{\rho+\alpha+\beta}{\alpha}c_s$.

If $c_s = 0$, then λ_Z cannot change sign.

A.6 Phase R (ceiling; no sequestration, double extraction)

Specification

s	x	y	X	Z	S
$= 0$	> 0	> 0	> 0	$= \overline{Z}$	≥ 0

Ceiling constraint:

$$x \;=\; \overline{x} - \frac{\beta}{\zeta} S \;=\; \frac{\beta}{\zeta}(S_m - S) \; .$$

Dynamical system

$$\begin{cases} \dot{X} &= -x \\ \dot{Z} &= 0 \\ \dot{S} &= -\beta S \\ \dot{\lambda}_X &= \rho \lambda_X \\ \dot{\lambda}_Z &= (\rho + \alpha)\lambda_Z + \nu_Z \\ \dot{\lambda}_S &= (\rho + \beta)\lambda_S - \beta \lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S &=& c_s + \lambda_Z - \gamma_s & \Longrightarrow & \gamma_s &=& c_s + \lambda_Z - \lambda_S \\ u'(x+y) &=& c_x + \lambda_X - \zeta \lambda_Z & \Longrightarrow & \lambda_Z &=& \frac{1}{\zeta}(c_x + \lambda_X - c_y) \\ u'(x+y) &=& c_y & \Longrightarrow & y &=& \widetilde{y} - \frac{\beta}{\zeta}(S_m - S) \;=\; \frac{\beta}{\zeta}(S - S_{\widetilde{y}}) \; . \end{array}$$

Constraints

$$\begin{array}{ll} X & X > 0 \\ Z & Z = \overline{Z} \\ s & s = 0 \\ x & S \geq S_m \\ y & y > 0 \iff S(t) \geq S_{\widetilde{y}} \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z \geq 0 \\ \gamma_s & c_s + \lambda_Z - \lambda_S \geq 0 \\ \gamma_{sx} & \gamma_{sx} = 0 \\ \gamma_y & \gamma_y = 0 \end{array}$$

A.7 Phase S (ceiling for Z et S; maximal sequestration)

Specification

s	x	y	X	Z	S
$=\zeta\overline{x}$	$=\overline{x}$	$=0$	>0	$=\overline{Z}$	$=S_m$

Ceiling constraint: satisfied by construction.

Dynamical system

$$\begin{cases} \dot{X} &= -\overline{x} \\ \dot{Z} &= 0 \\ \dot{S} &= 0 \\ \dot{\lambda}_X &= \rho\lambda_X \\ \dot{\lambda}_Z &= (\rho+\alpha)\lambda_Z + \nu_Z \\ \dot{\lambda}_S &= (\rho+\beta)\lambda_S - \beta\lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S &=& c_s + \lambda_Z + \gamma_{sx} & \Longrightarrow & \gamma_{sx} + \lambda_Z &=& \lambda_S - c_s \\ u'(\overline{x}) &=& c_x + \lambda_X - \zeta\lambda_Z - \zeta\gamma_{sx} & \Longrightarrow & \gamma_{sx} + \lambda_Z &=& \frac{1}{\zeta}(c_x + \lambda_X - \overline{p}) \\ u'(\overline{x}) &=& c_y - \gamma_y & \Longrightarrow & \gamma_y &=& c_y - \overline{p}\ . \end{array}$$

Constraints

$$\begin{array}{ll} X & X > 0 \\ Z & Z = \overline{Z} \\ s & s = \zeta x \\ x & S \leq S_m \\ y & y > 0 \iff S(t) \geq S_{\tilde{y}} \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z \geq 0 \\ \gamma_s & \gamma_s = 0 \\ \gamma_{sx} & \gamma_{sx} \geq 0 \\ \gamma_y & \gamma_y \geq 0 \end{array}$$

A.8 Phase T (no NRR; extraction of the NRR; extraction of the RR)

Specification

s	x	y	X	Z	S
$=0$	$=0$	>0	$=0$	$<\widetilde{Z}(S)$	≥ 0

Dynamical system

$$\begin{cases} \dot{X} &=& 0 \\ \dot{Z} &=& -\alpha Z + \beta S \\ \dot{S} &=& -\beta S \\ \dot{\lambda}_X &=& \rho\lambda_X - \nu_X \\ \dot{\lambda}_Z &=& (\rho+\alpha)\lambda_Z \\ \dot{\lambda}_S &=& (\rho+\beta)\lambda_S - \beta\lambda_Z \end{cases}$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S &=& c_s + \lambda_Z - \gamma_s + \gamma_{sx} & \Longrightarrow & \gamma_s &=& \frac{1}{\zeta}(c_x + \zeta c_s + \lambda_X) \\ u'(y) &=& c_x + \lambda_X - \zeta\lambda_Z - \zeta\gamma_{sx} & \Longrightarrow & \gamma_{sx} &=& \frac{1}{\zeta}(c_x + \lambda_X) - \lambda_Z \\ u'(y) &=& c_y & \Longrightarrow & y &=& \widetilde{y} \end{array}$$

Constraints

$$\begin{array}{ll} X & X = 0 \\ Z & Z < \widetilde{Z}(S) \\ s & s = 0 \\ x & x = 0 \\ y & y > 0 \\ \nu_X & \nu_X \geq 0 \\ \nu_Z & \nu_Z = 0 \\ \gamma_s & \gamma_s \geq 0 \\ \gamma_{sx} & \gamma_{sx} \geq 0 \\ \gamma_y & \gamma_y = 0 \end{array}$$

A.9 Phase U (no extraction of the NRR; extraction of the RR; NRR available)

Specification

s	x	y	X	Z	S
$=0$	$=0$	>0	>0	$=\widetilde{Z}(S)$	≥ 0

Dynamical system

$$\left\{\begin{array}{lcl} \dot{X} & = & 0 \\ \dot{Z} & = & -\alpha Z + \beta S \\ \dot{S} & = & -\beta S \\ \dot{\lambda}_X & = & \rho\lambda_X \\ \dot{\lambda}_Z & = & (\rho+\alpha)\lambda_Z \\ \dot{\lambda}_S & = & (\rho+\beta)\lambda_S - \beta\lambda_Z + \nu_Z \widetilde{Z}'(S) \end{array}\right.$$

First order conditions

$$\begin{array}{rclcrcl} \lambda_S & = & c_s + \lambda_Z - \gamma_s + \gamma_{sx} & \Longrightarrow & \gamma_s & = & \frac{1}{\zeta}(c_x + \zeta c_s + \lambda_X) \\ u'(y) & = & c_x + \lambda_X - \zeta\lambda_Z - \zeta\gamma_{sx} & \Longrightarrow & \gamma_{sx} & = & \frac{1}{\zeta}(c_x + \lambda_X) - \lambda_Z \\ u'(y) & = & c_y & \Longrightarrow & y & = & \widetilde{y} \end{array}$$

Constraints

$$\begin{array}{cl} X & X > 0 \\ Z & Z = \widetilde{Z}(S) \\ s & s = 0 \\ x & x = 0 \\ y & y > 0 \\ \nu_X & \nu_X = 0 \\ \nu_Z & \nu_Z \geq 0 \\ \gamma_s & \gamma_s \geq 0 \\ \gamma_{sx} & \gamma_{sx} \geq 0 \\ \gamma_y & \gamma_y = 0 \end{array}$$

Appendix B

Properties of auxiliary functions L and M

The functions $L(\cdot)$ and $M(\cdot)$ are defined in (4.1.4) and (4.1.5) as:

$$\begin{aligned} L(S) &= \beta \int_0^\infty e^{-(\rho+\beta)v} \left(c_x - u'(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) \right) \mathrm{d}v \\ M(S) &= \beta \int_0^\infty e^{-(\rho+\beta)v} \left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) \right) \mathrm{d}v \ . \end{aligned}$$

They differ by a constant and negative additive factor:

$$L(S) \ = \ \frac{\beta}{\rho+\beta}(c_x - \overline{p}) \ + \ M(S) \ .$$

The function M is clearly negative with $M(0) = 0$. It is decreasing: differentiating in its definition, one gets:

$$L'(S) \ = \ M'(S) \ = \ \frac{\beta^2}{\zeta} \int_0^\infty e^{-(\rho+2\beta)v} u'' \left(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v} \right) \mathrm{d}v \ , \tag{B.0.1}$$

which is negative because $u'' \leq 0$. The function L is therefore decreasing as well.

Lemma B.1. *Under Assumption 1, we have the bounds, for all $S \geq 0$:*

$$L(S) \ \geq \ \frac{\beta}{\rho+\beta} \ \left(c_x - u' \left(\overline{x} - \frac{\beta}{\zeta} S \right) \right) \tag{B.0.2}$$

$$M(S) \ \geq \ \frac{\beta}{\rho+\beta} \ \left(\overline{p} - u' \left(\overline{x} - \frac{\beta}{\zeta} S \right) \right) \ , \tag{B.0.3}$$

with equality if and only if $S = 0$.

Proof. This bound is proven with the following sequence of inequalities. Given that $u'(\cdot)$ is decreasing, then for all $v \geq 0$,

$$\begin{aligned} \frac{\beta}{\zeta} S e^{-\beta v} &\leq \frac{\beta}{\zeta} S \\ \overline{x} - \frac{\beta}{\zeta} S e^{-\beta v} &\geq \overline{x} - \frac{\beta}{\zeta} S \\ u' \left(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v} \right) &\leq u' \left(\overline{x} - \frac{\beta}{\zeta} S \right) \end{aligned}$$

$$
\begin{aligned}
\overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}\right) &\geq \overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta} S\right) \\
\int_0^\infty e^{-(\rho+\beta)v} \left[\overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}\right)\right] \mathrm{d}v &\geq \frac{1}{\rho+\beta} \left[\overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta} S\right)\right] .
\end{aligned}
$$

□

As a corollary, from the definition of c_{sm} in Equation (4.3.6), we have the inequality:

$$
\begin{aligned}
c_{sm} &= \frac{c_y - c_x}{\zeta} + L(S_{\widetilde{y}}) \\
&\geq \frac{c_y - c_x}{\zeta} + \frac{\beta}{\rho+\beta} \left(c_x - u'\left(\overline{x} - \frac{\beta}{\zeta} S_{\widetilde{y}}\right)\right) \\
&= \frac{c_y - c_x}{\zeta} + \frac{\beta}{\rho+\beta} \frac{c_x - c_y}{\zeta} = \bar{c}_s .
\end{aligned}
\tag{B.0.4}
$$

The following refines this reasoning. According to the definition of c_{sm} in Equation (4.3.6), and that of $L(S)$ in Equation (4.1.4), we have actually:

$$
\begin{aligned}
c_{sm} - \bar{c}_s &= \frac{c_y - c_x}{\zeta} + L(S_{\widetilde{y}}) - \left(\frac{c_y - c_x}{\zeta} + \frac{\beta}{\rho+\beta} \frac{c_x - c_y}{\zeta}\right) \\
&= \beta \int_0^\infty e^{-(\rho+\beta)v} \left(u'\left(\overline{x} - \frac{\beta}{\zeta} S_{\widetilde{y}} e^{-\beta v}\right) - u'\left(\overline{x} - \frac{\beta}{\zeta} S_{\widetilde{y}}\right)\right) \mathrm{d}v .
\end{aligned}
\tag{B.0.5}
$$

This is positive, because u' is decreasing.

Alternate expressions exist for $L(\cdot)$ and $M(\cdot)$. For instance:

$$
L(S) = \frac{\beta c_x}{\rho+\beta} + \frac{\zeta}{\beta S} u(\overline{x} - \frac{\beta}{\zeta} S) - \frac{\zeta\rho}{\beta S} \int_0^\infty e^{-\rho t} u(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta t}) \mathrm{d}t . \tag{B.0.6}
$$

This expression is obtained from the definition in (4.1.4) and integration by parts as:

$$
\begin{aligned}
L(S) &= \beta \int_0^\infty e^{-(\rho+\beta)v} \left(c_x - u'(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v})\right) \mathrm{d}v \\
&= \frac{\beta c_x}{\rho+\beta} - \frac{\zeta}{\beta S} \int_0^\infty e^{-\rho v} \frac{\beta^2 S}{\zeta} e^{-\beta v} u'(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) \mathrm{d}v \\
&= \frac{\beta c_x}{\rho+\beta} - \frac{\zeta}{\beta S} \left\{ \left[e^{-\rho v} u(\overline{x} - \frac{\beta}{\zeta} S)\right]_0^\infty + \int_0^\infty \rho e^{-\rho v} u(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) \mathrm{d}v \right\} \\
&= \frac{\beta c_x}{\rho+\beta} + \frac{\zeta}{\beta S} u(\overline{x} - \frac{\beta}{\zeta} S) - \frac{\zeta\rho}{\beta S} \int_0^\infty e^{-\rho v} u(\overline{x} - \frac{\beta}{\zeta} S e^{-\beta v}) \mathrm{d}v .
\end{aligned}
$$

We now prove results concerning the resolution of Equation (4.3.4), that is:

$$
\zeta(c_s - \hat{c}_s) + \overline{p} - u'(\overline{x} - \frac{\beta}{\zeta} S) = M(S) . \tag{B.0.7}
$$

Lemma B.2. *We have the following properties, under Assumption 1:*

(i) if $c_s < \hat{c}_s$, then Equation (4.3.4)/(B.0.7) *has no positive solution;*

(ii) if in addition $u'(\cdot)$ is convex, and if

$$\hat{c}_s \leq c_s \leq c_{sm}$$

then Equation (4.3.4)/(B.0.7) *has a unique solution $S^{QP} \in [0, S_{\widetilde{y}}]$;*

(iii) if in addition $u'(\cdot)$ is convex, and if $c_s > c_{sm}$, then Equation (4.3.4)/(B.0.7) *has no solution in* $[0, S_{\widetilde{y}}]$.

Proof.

(i) Denote with $\phi(S)$ the left-hand side of the equation. According to the bound (B.0.3), we have

$$M(S) \;\geq\; \frac{\beta}{\rho+\beta}\;\left(\overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta}S\right)\right)\;.$$

If the right-hand side of this inequality is strictly larger than $\phi(S)$, then the statement is proved. This sufficient condition writes as:

$$\begin{aligned}
\frac{\beta}{\rho+\beta}\;\left(\overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta}S\right)\right) \;&>\; \zeta(c_s - \hat{c}_s) \;+\; \overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta}S\right) \\
\iff\quad \zeta(c_s - \hat{c}_s) \;&<\; -\;\frac{\rho}{\rho+\beta}\left(\overline{p} - u'\left(\overline{x} - \frac{\beta}{\zeta}S\right)\right)\;.
\end{aligned}$$

This last inequality indeed holds since $c_s - \hat{c}_s < 0$ by assumption, and the right-hand side is positive for $S \geq 0$.

(ii) We first show that the function

$$h(S) \;=\; \zeta(c_s - \hat{c}_s) \;+\; \overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S) \;-\; M(S)$$

is decreasing. As a consequence, there is at most one solution to Equation (B.0.7) for $S \in [0, S_{\widetilde{y}}]$.

If $u'(\cdot)$ is convex, then $u''(\cdot)$ is increasing. Then we have:

$$\begin{aligned}
\frac{\beta}{\zeta}Se^{-\beta v} \;&\leq\; \frac{\beta}{\zeta}S \\
\overline{x} - \frac{\beta}{\zeta}Se^{-\beta v} \;&\geq\; \overline{x} - \frac{\beta}{\zeta}S \\
u''\left(\overline{x} - \frac{\beta}{\zeta}Se^{-\beta v}\right) \;&\geq\; u''\left(\overline{x} - \frac{\beta}{\zeta}S\right)\;.
\end{aligned}$$

Given Equation (B.0.1) for $M'(S)$, we have for all $S \geq 0$,

$$M'(S) \;\geq\; \frac{\beta}{\rho+2\beta}\;\frac{\beta}{\zeta}\;u''\left(\overline{x} - \frac{\beta}{\zeta}S\right)\;.$$

On the other hand, we have

$$\begin{aligned}
h'(S) \;&=\; \frac{\beta}{\zeta}\;u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) \;-\; M'(S) \\
&\leq\; \frac{\beta}{\zeta}\;u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) \;-\; \frac{\beta}{\rho+2\beta}\;\frac{\beta}{\zeta}\;u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) \\
&=\; \frac{\beta}{\zeta}\;\frac{\rho+\beta}{\rho+2\beta}u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) \;\leq\; 0\;.
\end{aligned}$$

Therefore, h is decreasing. The solutions to (B.0.7) are the zeroes of $h(\cdot)$.

The uniqueness remains to be proved. When $S^{QP} = 0$, the left-hand side of (4.3.4) is $\zeta(c_s - \hat{c}_s) \geq 0$ whereas the right-hand side is 0. There will necessarily be a solution in the interval $[0, S_{\widetilde{y}}]$ if the left-hand side evaluated at $S^{QP} = S_{\widetilde{y}}$, that is, $\zeta(c_s - \hat{c}_s) + \overline{p} - c_y$, is smaller than the right-hand side evaluated at the same point, that is, $M(S_{\widetilde{y}})$. This condition is exactly $c_s \leq c_{sm}$. We have therefore existence and uniqueness in this case.

(*iii*) As proved above, the function $h(\cdot)$ is decreasing, and its value at $S = S_{\tilde{y}}$ is strictly positive if $c_s > c_{sm}$. Therefore, this function has no zero, that is, Equation (4.3.4) has no solution.

$\square$

Lemma B.3. *Assume that $\hat{c}_s \leq c_s \leq c_{sm}$ and that $u'(\cdot)$ is convex. Then the unique solution S^{QP} of Equation* (B.0.7) *in the interval $[0, S_{\tilde{y}}]$ is an increasing function of c_s, and the term*

$$\phi(c_s) \; := \; \frac{\rho+\beta}{\rho}(c_s - \hat{c}_s) \; + \; \frac{1}{\zeta}\left(\overline{p} - u'(\overline{x} - \frac{\beta}{\zeta}S^{QP})\right)$$

is positive.

Proof. Denote with $\sigma(c_s)$ the solution S^{QP} of Equation (B.0.7). By implicit differentiation with respect to c_s, we get

$$\zeta + \frac{\beta}{\zeta}\sigma'(c_s)u''\left(\overline{x} - \frac{\beta}{\zeta}\sigma(c_s)\right) \; = \; \sigma'(c_s)M'(\sigma(c_s))$$

hence

$$\sigma'(c_s) \; = \; \zeta\left(M'(\sigma(c_s)) - \frac{\beta}{\zeta}u''\left(\overline{x} - \frac{\beta}{\zeta}\sigma(c_s)\right)\right)^{-1} .$$

The denominator is $-h'(\sigma(c_s))$ in the notation of the proof of Lemma B.2. It is therefore positive, and it has been proved that

$$M'(\sigma(c_s)) - \frac{\beta}{\zeta}u''\left(\overline{x} - \frac{\beta}{\zeta}\sigma(c_s)\right) \; \geq \; - \; \frac{\beta}{\zeta}\,\frac{\rho+\beta}{\rho+2\beta}u''\left(\overline{x} - \frac{\beta}{\zeta}S\right) \, .$$

Therefore, σ' is positive and σ is increasing. Moreover,

$$\sigma'(c_s) \; \geq \; \zeta\left(-\frac{\beta}{\zeta}\,\frac{\rho+\beta}{\rho+2\beta}u''\left(\overline{x} - \frac{\beta}{\zeta}\sigma(c_s)\right)\right)^{-1} .$$

The function $\phi(c_s)$ is such that $\phi(0) = 0$ and

$$\begin{aligned} \phi'(c_s) \; &= \; \frac{\rho+\beta}{\beta} \; + \; \frac{\beta}{\zeta^2}\sigma'(c_s)u''\left(\overline{x} - \frac{\beta}{\zeta}\sigma(c_s)\right) \\ &\geq \; \frac{\rho+\beta}{\beta} \; - \; \frac{\rho+\beta}{\rho+2\beta} \\ &= \; \frac{\beta^2}{(\rho+\beta)(\rho+2\beta)} \; > \; 0 \, . \end{aligned}$$

The function $\phi(\cdot)$ is therefore increasing, and it is positive. $\square$

Appendix C

Auxiliary problems

We gather in this appendix several "auxiliary" problems which provide complementary arguments in different parts of our analysis.

In Section C.1, we develop the finite-horizon approach, which is related with the backwards solution of infinite-horizon problems, and therefore provides an alternate source of results. In particular, we examine in Section C.1.3 the problem of optimizing the consumption while being constrained to stay at the pollution ceiling $Z = \overline{Z}$.

In Section C.2, we discuss the representative agent's optimization problem, and we use the results to argue that there are more solutions to the problem than those identified in Chapter 4.

C.1 Finite-horizon problems

The fact that final phases can be identified opens the way to a finite-horizon approach to determine the optimal trajectories. From initial points located on the boundary $Z = \widetilde{Z}(S)$, the optimal trajectory and the value function are known. There remains to determine the optimal junction point starting from an initial state located in the interior. Neither the "final" state nor the final time are known *a priori*.

We investigate this possibility here. We state in Section C.1.1 a sufficiency theorem related to this situation. We do this essentially for completeness since we do not exploit it. However, we do exploit in Section C.1.2 some of the conditions in order to "guess" the properties of optimal trajectories in various situations.

C.1.1 Sufficient conditions for free finite-horizon problems

As said above, another way to find optimal trajectories of our problem in specific situations, is to use a finite-horizon approach. The following result gives appropriate *sufficient* condition for optimality in this context. The statement is that of Seierstad & Sydsæter (1987, Theorem 13, p. 390) and Seierstad & Sydsæter (1987, Theorem 17, p. 398), without the provision for free initial conditions.

Theorem C.1 (Seierstad & Sydsæter (1987), Theorem 17)**.** *Consider the finite-horizon optimal control problem with free terminal time and scrap value:*

$$\max_{\mathbf{u}(\cdot),t_1} \int_0^{t_1} f_0(\mathbf{x}(t), \mathbf{u}(t), t)\,dt \;+\; S_1(\mathbf{x}(t_1), t_1)$$

where the state vector $\mathbf{x}(\cdot)$ belongs to $\mathbb{R}^n$, $0 \leq T_1 \leq t_1 \leq T_2$, the control vector $\mathbf{u}(\cdot)$ belongs to some fixed convex set $U \subset \mathbb{R}^r$, and $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$. Assume that admissible trajectories must satisfy the vector of s constraints:

$$g_j(\mathbf{x}(t), \mathbf{u}(t), t) \geq 0, j = 1, \ldots, s', \qquad g_j(\mathbf{x}(t), \mathbf{u}(t), t) = \bar{g}_j(\mathbf{x}(t), t) \geq 0, j = s'+1, \ldots, s,$$

as well as the initial condition $\mathbf{x}(0) = \mathbf{x}^0$ *and terminal conditions*

$$R^1_k(x_i(t_1), t_1) = 0, i = 1, \ldots, r'_1, \quad R^1_k(x_i(t_1), t_1) \geq 0, i = r'_1, \ldots, r_1.$$

Assume that:

a) f_0, f *and* g *have derivatives w.r.t.* $\mathbf{x}$ *and* $\mathbf{u}$*, and that these derivatives are continuous;* S_1 *and* $(R^1_1, \ldots, R^1_{r_1})$ *are* C^1 *functions;*

b1) S_1 *is a concave function of* $\mathbf{x}$*;*

b2) $\sum_{k=1}^{r_1} \gamma_k R^1_k$ *is a quasi-concave function of* $\mathbf{x}$*;*

If, for each fixed $T \in [T_1, T_2]$*, there exists an admissible pair* $(\mathbf{x}^T(t), \mathbf{u}^T(t))$*, together with a continuous and piecewise continuously differentiable vector function* $\mathbf{p}^T(t)$*, a piecewise-continuous function* $\mathbf{q}^T(t)$ *and vectors of numbers* $\beta^T = (\beta^T_1, \ldots, \beta^T_s)$ *and* $\gamma^T = (\gamma^T_1, \ldots, \gamma^T_{r_1})$ *such that, defining*

$$\begin{aligned} H(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) &:= f_0(\mathbf{x}, \mathbf{u}, t) + \mathbf{p} \cdot f(\mathbf{x}, \mathbf{u}, t) \\ L(\mathbf{x}, \mathbf{u}, \mathbf{p}, \mathbf{q}, t) &:= H(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) + \mathbf{q} \cdot g(\mathbf{x}, \mathbf{u}, t) \ , \end{aligned}$$

c) for virtually all $t \in [0, T]$*, and all* $\mathbf{u} \in U$*,* $\dfrac{\partial L}{\partial \mathbf{u}}(\mathbf{x}^T(t), \mathbf{u}^T(t), \mathbf{p}^T(t), \mathbf{q}^T(t), t) \cdot (\mathbf{u} - \mathbf{u}^T) \leq 0$*,*

d) for virtually all $t \in [0, T]$*,* $\dot{\mathbf{p}}^T(t) = -\dfrac{\partial L}{\partial \mathbf{x}}(\mathbf{x}^T(t), \mathbf{u}^T(t), \mathbf{p}^T(t), \mathbf{q}^T(t), t)$*,*

e) the Hamiltonian is a concave function of $(\mathbf{x}, \mathbf{u})$*, for all* $t \in [0, T]$*,*

f) $q_j(t) \geq 0$ *and* $= 0$ *if* $g_j(\mathbf{x}^T(t), \mathbf{u}^T(t), t) > 0$*, for all* t *and* $j = 1, \ldots, s$*,*

g) g_j *is a quasi-concave function of* $(\mathbf{x}, \mathbf{u})$*, for all* $t \in [0, T]$ *and* $j = 1, \ldots, s$*,*

h) for each $i = 1, \ldots, n$*,*

$$p_i(T) \ = \ \sum_{j=1}^{s} \beta_j \frac{\partial g_j}{\partial x_i}(\mathbf{x}^T(T), \mathbf{u}^T(T), T) + \frac{\partial S_1}{\partial x_i}(\mathbf{x}^T(T), T) + \sum_{k=1}^{r_1} \gamma_k \frac{\partial R^1_k}{\partial x_i}(\mathbf{x}^T(T), T) \ , \quad \text{(C.1.1)}$$

i) $\beta_j = 0$ *for* $j = 1, \ldots, s'$*,* $\beta_j \geq 0$ *for* $j = s'+1, \ldots, s$ *and* $= 0$ *if* $g_j(\mathbf{x}^T(T), \mathbf{u}^T(T), T) > 0$*,*

j) for $k = r'_1, \ldots, r_1$*,* $\gamma_k \geq 0$ *and* $= 0$ *if* $R^1_k(\mathbf{x}^T(T), T) > 0$*,*

k) $\mathbf{u}^T(t)$ *and* $\mathbf{q}^T(t)$ *take values in fixed, bounded subsets of* $\mathbb{R}^n$ *and* $\mathbb{R}^r$*, respectively,*

l) the function $T \mapsto \mathbf{x}^T(T)$ *is Lipschitz continuous, the functions* $T \mapsto \beta^T$ *and* $T \mapsto \gamma^T$ *are piecewise continuous,*

m) for all T*,* $\mathbf{u}^T(T)$ *belongs to the closure of the set* $\{\mathbf{u} \in U, g_j(\mathbf{x}^T(T), \mathbf{u}^T, T) > 0 \text{ for all } j \leq s'\}$*,*

n) the function

$$F(T) \ = \ H(\mathbf{x}^T, \mathbf{u}^T, \mathbf{q}^T, \mathbf{p}^T) \ + \ \beta^T . \frac{\partial g}{\partial t}(\mathbf{x}^T, \mathbf{u}^T, T) \ + \ \frac{\partial S_1}{\partial t}(\mathbf{x}^T, T) \ + \ \gamma^T . \frac{\partial R^1}{\partial t}(\mathbf{x}^T, \mathbf{u}^T, T)$$

where functions are evaluated at T*, has the property that there exists some* $T^* \in [T_1, T_2]$ *such that* $F(T) \geq 0$ *for* $T < T^*$ *and* $F(T) \leq 0$ *for* $T > T^*$*,*

then the pair $(\mathbf{x}^{T^*}(t), \mathbf{u}^{T^*}(t))$ *and the final time* $t_1 = T^*$ *are optimal.*

In problems, such as ours, where:

$$f_0(\mathbf{x},\mathbf{u},t) \;=\; \bar{f}_0(\mathbf{x},\mathbf{u})e^{-\rho t}, \quad f(\mathbf{x},\mathbf{u},t) \;=\; \bar{f}(\mathbf{x},\mathbf{u}), \quad S_1(\mathbf{x}^T,t) \;=\; V(\mathbf{x}^T)e^{-\rho t}$$

and where constraints do not depend on time, Condition n) of Theorem C.1 takes often the following form in the literature:

$$\bar{f}_0(\mathbf{x}(T^*),\mathbf{u}(T^*)) \;+\; \lambda.\bar{f}(\mathbf{x}(T^*),\mathbf{u}(T^*)) \;=\; \rho V(\mathbf{x}(T^*)) \tag{C.1.2}$$

where $\lambda(t) = e^{-\rho t}p(t)$. We shall use this form in the next section to identify the value of adjoint variables at T^*.

C.1.2 The problem in finite horizon

Since all optimal trajectories eventually end up staying on the boundary $Z = \widetilde{Z}(S)$ of the domain $\mathcal{D}$, a possible approach to the construction of optimal trajectories is to solve a finite-horizon problem with scrap value and free terminal time.

The problem is formulated as:

$$\max_{s(\cdot),x(\cdot),y(\cdot),T} \int_0^T \left[u(x(t)+y(t)) - c_s s(t) - c_x x(t) - c_y y(t)\right] e^{-\rho t}\mathrm{d}t \;+\; e^{-\rho T}V(S) \tag{C.1.3}$$

given the controlled dynamics (2.2.2):

$$\left\{\begin{array}{rcl} \dot{Z} &=& -\alpha Z + \beta S + \zeta x - s \\ \dot{S} &=& -\beta S + s \end{array}\right.$$

with the usual constraints on controls $x(t)$, $s(t)$ and $y(t)$, and the constraint on state variables

$$R(S(t),Z(t)) := \widetilde{Z}(S(t)) - Z(t) \quad \geq \quad 0\;, \quad t \in [0,T]. \tag{C.1.4}$$

Initial and terminal conditions are:

$$S(0) = S^0, \quad Z(0) = Z^0, \quad R(S(T),Z(T)) = 0. \tag{C.1.5}$$

The scrap value V_S is the value function of the problem restricted to the curve $Z = \widetilde{Z}(S)$. Its exact form varies depending on the value of c_s but since in Phase L, $x(t) = s(t) = 0$ and $y(t) = \widetilde{y}$, we always have the relationship:

$$V(S) \;=\; \frac{1}{\rho}\left(u(\widetilde{y}) - c_y\widetilde{y}\right)(1 - e^{-\rho\tau(S)}) + e^{-\rho\tau(S)}V(S_m)\;,$$

where $\tau(S) = -\beta^{-1}\log(S/S_m)$ is the time it takes to go from S to S_m when in Phase L.

Applying Theorem C.1 h), a sufficient condition for an optimal trajectory is the existence of continuous adjoint variables λ_S and λ_Z (written in current value) and real numbers $\beta \geq 0$ and γ, such that, T^* being the final optimal time:

$$\begin{array}{rcl} \lambda_S(T^*) &=& \beta\dfrac{\partial R}{\partial S}(S(T^*),Z(T^*)) \;+\; V'(S(T^*)) \;+\; \gamma\dfrac{\partial R}{\partial S}(S(T^*),Z(T^*)) \\ &=& V'(S(T^*)) \;+\; (\beta+\gamma)\widetilde{Z}'(S(T^*)) \\ \lambda_Z(T^*) &=& \beta\dfrac{\partial R}{\partial Z}(S(T^*),Z(T^*)) \;+\; \gamma\dfrac{\partial R}{\partial Z}(S(T^*),Z(T^*)) \\ &=& -(\beta+\gamma)\;. \end{array}$$

These two conditions are satisfied if

$$\lambda_S(T^*) \;=\; V'(S(T^*)) \;-\; \widetilde{Z}'(S(T^*))\;\lambda_Z(T^*)\;. \tag{C.1.6}$$

Given the definition of $\widetilde{Z}(\cdot)$ in (2.3.27), this last condition is in turn refined into:

$$\lambda_S(T^*) \;=\; \begin{cases} V'(S(T^*)) & \text{if } S(T^*) \le S_m \\ V'(S(T^*)) \;-\; Z'_M(S(T^*))\lambda_Z(T^*) & \text{if } S(T^*) \ge S_m. \end{cases}$$

Moreover, it is easy to show that, on the one hand,

$$Z'_M(S) \;=\; \frac{\alpha Z_M(S) - \beta S}{\beta S},$$

and on the other hand,

$$V'(S) \;=\; \frac{1}{\beta S} e^{-\rho\tau(S)}\left(u(\widetilde{y}) - c_y\widetilde{y} - \rho V(S_m)\right) \;.$$

Combining these with (C.1.6), we get the identity, when $S(T^*) \ge S_m$:

$$\beta S(T^*)\lambda_S(T^*) \;+\; (\alpha Z_M(S(T^*)) - \beta S(T^*))\lambda_Z(T^*) \;=\; e^{-\rho\tau(S(T^*))}\left(u(\widetilde{y}) - c_y\widetilde{y} - \rho V(S_m)\right) \;. \tag{C.1.7}$$

Next, since the terminal time is free, Condition (C.1.2) amounts to requiring:

$$\begin{aligned} & u(x(T^*) + y(T^*)) - c_s s(T^*) - c_x x(T^*) - c_y y(T^*) \\ & +\lambda_S(T^*)(-\beta S(T^*) + s(T^*)) + \lambda_Z(T^*)(-\alpha Z(T^*) + \beta S(T^*) + \zeta x(T^*) - s(T^*)) \;=\; \rho V(S)\;, \end{aligned} \tag{C.1.8}$$

where $Z(T^*) = \widetilde{Z}(S(T^*))$ because of the terminal condition.

C.1.2.1 Junction with $Z = \overline{Z}$

Assume in this section that $Z(T^*) = \overline{Z}$, so that $\widetilde{Z}'(S(T^*)) = 0$. In that case, (C.1.6) determines directly $\lambda_S(T^*)$ as

$$\lambda_S(T^*) \;=\; V'(S(T^*)) \;.$$

C.1.2.2 Junction with $Z = Z_M(S)$

Assume in this section that $S(T^*) \ge S_m$ and $Z(T^*) = Z_M(S(T^*))$: the final point is on the boundary curve $Z = Z_M(S)$. With (C.1.7) and the definition of $V(S)$, (C.1.8) reduces to:

$$\begin{aligned} & u(x(T^*) + y(T^*)) - c_s s(T^*) - c_x x(T^*) - c_y y(T^*) \\ & \qquad +\lambda_S(T^*)s(T^*) + \lambda_Z(T^*)(\zeta x(T^*) - s(T^*)) \\ & \qquad\qquad -e^{-\rho\tau(S)}\left(u(\widetilde{y}) - c_y\widetilde{y} - \rho V(S_m)\right) \;=\; \left(u(\widetilde{y}) - c_y\widetilde{y}\right)(1 - e^{-\rho\tau(S)}) + e^{-\rho\tau(S)}V(S_m) \\ & u(x(T^*) + y(T^*)) - c_s s(T^*) - c_x x(T^*) - c_y y(T^*) \\ & \qquad +\lambda_S(T^*)s(T^*) + \lambda_Z(T^*)(\zeta x(T^*) - s(T^*)) \;=\; u(\widetilde{y}) - c_y\widetilde{y}\;. \end{aligned} \tag{C.1.9}$$

The set of conditions:

$$y(T^*) = 0, \quad x(T^*) = \widetilde{y}, \quad s(T^*) = 0, \quad \lambda_Z(T^*) = \frac{c_y - c_x}{\zeta}$$

turns out to satisfy this equation, independently of the value of $\lambda_S(T^*)$. Assuming the continuity of controls, these conditions correspond to Phase A.

C.1.2.3 Junction at $(S_m, \overline{Z})$ in Phase S

Assume that trajectories are required to stop at time T in state $(S_m, \overline{Z})$ and continue in Phase S. Then the total gain on this trajectory, evaluated from instant T on, is, since the control is $x = \overline{x}$ and $s = \zeta\overline{x}$,

$$V_S \;=\; \int_0^\infty e^{-\rho t}\left(u(\overline{x}) - (c_x + \zeta c_s)\overline{x}\right)\mathrm{d}t \;=\; \frac{1}{\rho}(u(\overline{x}) - (c_x + \zeta c_s)\overline{x})\;. \tag{C.1.10}$$

In that case, Condition (C.1.2) is:

$$\begin{aligned} & u(x(T^*)+y(T^*)) - c_s s(T^*) - c_x x(T^*) - c_y y(T^*) \\ & +\lambda_S(T^*)(-\beta S_m + s(T^*)) + \lambda_Z(T^*)(\zeta x(T^*) - s(T^*)) \;=\; u(\overline{x}) - (c_x + \zeta c_s)\;. \end{aligned} \tag{C.1.11}$$

The set of conditions

$$y(T^*) = 0, \quad x(T^*) = \overline{x}, \quad s(T^*) = \zeta\overline{x},$$

turns out to solve this equation, independently of the values of $\lambda_S(T^*)$ and $\lambda_Z(T^*)$. Assuming the continuity of controls, we see that this set of controls correspond to Phase B since $s = \zeta x$. Inside Phase B, the value of the consumption $x(t)$ is given by: $x(t) = q^d(c_x + \zeta c_s - \zeta\lambda_S(t))$. The continuity of controls is then equivalent to the continuity of $\lambda_S(\cdot)$. The value of $\lambda_Z(T^*)$ remains undetermined, except that it must satisfy some inequality as in Corollary 3.1.

C.1.3 Optimization on the ceiling

We derive here the optimal control when the constraint $Z = \overline{Z}$ is enforced. This analysis is used in Section 4.5.5.3 (page 66) to discuss that such trajectories cannot be optimal when $c_s < \hat{c}_s$.

Since this situation corresponds to what we have called Phase Q, this solution can be obtained from Section 3.5.2, but we quickly re-derive it here.

Imposing $Z = \overline{Z}$ to the problem of this chapter reduces it to the following optimal control problem. Since $\dot{Z} = 0 = -\alpha\overline{Z} + \beta S + \zeta x - s$, we have $\beta S - s = \alpha\overline{Z} - \zeta x = \zeta(\overline{x} - x)$, then $\dot{S} = \zeta(x - \overline{x})$. The scrap value is given by (C.1.10). The reduced problem can therefore be stated as:

$$\max_{x(\cdot),T} \int_0^T \left[u(x(t)) - c_x x(t) - c_s(\beta S(t) + \zeta(x(t) - \overline{x}))\right] e^{-\rho t}\mathrm{d}t \,+\, e^{-\rho T}\frac{u(\overline{x}) - (c_x + \zeta c_s)\overline{x}}{\rho} \tag{C.1.12}$$

given the controlled dynamics $\dot{S} = \zeta(x - \overline{x})$, the constraints on controls $x \geq 0$ and $y \geq 0$, and the terminal condition $S(T) = S_m$. The former constraint on control s, $\zeta x - s \geq 0$, becomes here a constraint on the state: $S \leq S_m$. This constraint is superseded by the terminal constraint for S and is therefore ommited.

Naming μ_S the adjoint variable for state S, the Lagrangian for the problem writes as:

$$L(y, x, Z, S) \;=\; u(x+y) - (c_x + \zeta c_s)x - c_y y + \zeta c_s\overline{x} - \beta c_s S \;+\; \mu_S\zeta(\overline{x} - x) \;+\; \gamma_x x + \gamma_y y\;.$$

The first-order equations are:

$$\begin{aligned} 0 &= u'(x+y) - c_x - \zeta c_s + \zeta\mu_S + \gamma_x \\ 0 &= u'(x+y) - c_y + \gamma_y \\ \dot{\mu}_S &= \rho\mu_S + \beta c_s\;. \end{aligned}$$

In addition, the optimality condition (C.1.2) for the terminal time T is, taking into account the fact that $S(T) = S_m$,

$$u(x(T)+y(T)) - (c_x + \zeta c_s)x(T) - c_y y(T) + \mu_S(T)\zeta(x(T) - \overline{x}) \;=\; u(\overline{x}) - (c_x + \zeta c_s)\;.$$

This equation is clearly satisfied with $x(T) = \overline{x}$ and $y(T) = 0$. We actually expect the solution to be such that $y = 0$ and $x > 0$, hence $\gamma_x = 0$. Solving the equations under this assumption and the terminal condition $x(T) = \overline{x}$, we arrive at:

$$x(t) \;=\; q^d(c_x + \zeta c_s - \zeta\mu_S(t)) \tag{C.1.13}$$

$$\mu_S(t) \;=\; \left(\lambda_S^{(S)} + \frac{\beta c_s}{\rho}\right) e^{\rho(t-T)} \;-\; \frac{\beta c_s}{\rho}\;, \tag{C.1.14}$$

where $\lambda_S^{(S)}$ is the value defined, *e.g.*, on page 66.

C.2 The representative agent's problem

Following the decentralization principle, the socially optimal trajectory obtained by the regulator can be implemented by imposing taxes on the representative agent. We discuss here this implementation, in the case analyzed in Chapter 4: infinite stock of carbon, absence of limits on sequestered stock S or on renewable energy consumption y.

Accordingly, consider the problem of the representative agent, which faces a unitary tax θ_x on consumption and a unitary tax θ_s on sequestration (this "tax" may actually be negative, resulting in an incentive). Both taxes are possibly depending on time. The representative agent is not constrained by the value of the stocks S or Z. It must therefore solve:

$$\max_{s(\cdot),x(\cdot),y(\cdot)} \int_0^{\infty} [u(x(t)+y(t)) - (c_s+\theta_s(t))s(t) - (c_x+\theta_x(t))x(t) - c_y y(t)]\, e^{-\rho t} \mathrm{d}t \qquad \text{(C.2.1)}$$

given the constraints on controls: $y \geq 0$ and $0 \leq s \leq \zeta x$. There is no state variable nor dynamics to consider in this problem.

Modifying the analysis of Section 2.3.1, we find the first-order conditions:

$$\begin{aligned} c_s + \theta_s &= \delta_s - \delta_{sx} \\ u'(x+y) &= c_x + \theta_x - \zeta\delta_{sx} \\ u'(x+y) &= c_y - \delta_y \ , \end{aligned}$$

where we have used δ_y, δ_s and δ_{sx} as Lagrange multipliers for the constraints on controls. Identifying these with the first-order conditions for the regulator's problem (2.3.2)– (2.3.4), we find the value that should be given to the taxes:

$$\theta_x(t) \ = \ -\zeta\lambda_Z(t) \qquad\qquad \theta_s(t) \ = \ \lambda_Z(t) - \lambda_S(t) \ ,$$

where λ_S and λ_Z are the adjoint variables for the socially optimal trajectory. If these values are used, then the socially optimal control also solves the representative agent's optimization problem, and the respective Lagrange multipliers γ_s, γ_{sx}, $\underline{\gamma}_y$ and δ_s, δ_{sx}, δ_y coincide.

This choice is not the unique possibility however. Assume instead that

$$\theta_x(t) \ = \ -\zeta\tau(t) \qquad\qquad \theta_s(t) \ = \ \tau(t) - \lambda_S(t) \ .$$

Replacing these values in the representative agent's first-order conditions and rearranging, we have:

$$\begin{aligned} \tau + \delta_{sx} &= \lambda_S + \delta_s - c_s \\ u'(x+y) &= c_x - \zeta(\tau+\delta_{sx}) \ = \ c_y - \underline{\gamma}_y \ . \end{aligned}$$

As a consequence, as long as $\tau + \delta_{sx} = \lambda_Z + \gamma_{sx}$, the socially optimal trajectory is still a solution to the agent's problem, using the remaining multipliers $\delta_y = \underline{\gamma}_y$ and $\delta_s = \gamma_s$. Since we must have $\delta_{sx} \geq 0$, the constraint on the function τ is just: $\tau(t) \leq \lambda_Z(t) + \gamma_{sx}(t)$.

Consider now the particular situation where the initial state of the system is $(S_m, \overline{Z})$, in the case where $c_s < \hat{c}_s$. The socially optimal trajectory for this situation is identified in Lemma 4.2 on page 32: this trajectory is stationary. In particular, $\gamma_s = 0$ and

$$\begin{aligned} \lambda_Z \ &= \ \frac{\rho+\beta}{\beta}\left(c_s + \frac{c_x-\overline{p}}{\zeta}\right) & \lambda_S \ &= \ c_s + \frac{c_x-\overline{p}}{\zeta} \\ \gamma_{sx} \ &= \ \frac{\rho+\beta}{\beta}(\hat{c}_s - c_s) & \underline{\gamma}_y \ &= \ c_y - \overline{p} \ . \end{aligned}$$

According to the observation above, any choice of taxes with

$$\theta_x \ \geq \ \overline{p} - c_x \qquad\qquad \theta_s \ = \ -\frac{1}{\zeta}\theta_x - c_s - \frac{c_x-\overline{p}}{\zeta}$$

will result in the representative agent finding this stationary trajectory optimal as well.

Returning to the social planner's problem, this suggests that the solution computed in Lemma 4.2 is not the unique solution to the problem. Any solution with a costate variable λ_Z satisfying

$$\lambda_Z + \gamma_{sx} \;=\; \frac{c_x - \overline{p}}{\zeta}$$

should also work. The value of λ_Z is not uniquely defined in this situation. By extension, when considering a trajectory that starts in a different state (S^0, Z^0) but ends up in state $(S_m, \overline{Z})$, the value of λ_Z when the terminal state has been reached, is not uniquely defined either. We can use as terminal value for λ_Z the value it takes *just before* entering the terminal state, thereby avoiding a jump in this function.

Appendix D

Local analysis of trajectories at junction points

This section contains a local analysis of optimal curves when they connect to the boundary $Z = \widetilde{Z}(S)$; this part is useful to assess the global consistency of the family of optimal curves.

The following analysis gives indications on the several features of the state trajectory and consumption when the system is in Phase A (Section D.1) or Phase B (Section D.2), at particular at junction points. We obtain in particular the direction of variation of $x(t)$, $S(t)$ and $Z(t)$, as well as geometric properties such as tangency of trajectories with the line $Z = \overline{Z}$.

D.1 Phase A

We are interested in the variations of $x(\cdot)$ and in the local expansions of state variables at junction points, when the system is in Phase A. The state and adjoint trajectories are solution to:

$$\left\{ \begin{array}{rcl} \dot{Z} & = & -\alpha Z + \beta S + \zeta x \\ \dot{S} & = & -\beta S \end{array} \right. \qquad \left\{ \begin{array}{rcl} \dot{\lambda}_Z & = & (\rho + \alpha)\lambda_Z \\ \dot{\lambda}_S & = & (\rho + \beta)\lambda_S - \beta\lambda_Z \end{array} \right.$$

and $x(t) = q^d(c_x - \zeta\lambda_Z)$. Obviously, S is always decreasing. It follows from these equations that

$$\ddot{S} = -\beta\dot{S} = \beta^2 S \qquad \dddot{S} = -\beta\ddot{S} = -\beta^3 S \ ,$$

and

$$\begin{array}{rcl} \ddot{Z} & = & -\alpha\dot{Z} + \beta\dot{S} + \zeta\dot{x} \\ & = & \alpha^2 Z - \beta(\alpha + \beta)S - \alpha\zeta x + \zeta\dot{x} \\ \dddot{Z} & = & -\alpha\ddot{Z} + \beta\ddot{S} + \zeta\ddot{x} \\ & = & -\alpha^3 Z + \beta(\alpha^2 + \alpha\beta + \beta^2)S + \alpha^2\zeta x - \alpha\zeta\dot{x} + \zeta\ddot{x} \ . \end{array}$$

Finally, from the specific form of $x(t)$, we have:

$$\begin{array}{rcl} \dot{x} & = & -\zeta\dot{\lambda}_Z \ (q^d)'(c_x - \zeta\lambda_Z) \ = \ -\zeta(\rho + \alpha)\lambda_Z \ (q^d)'(c_x - \zeta\lambda_Z) \\ \ddot{x} & = & -\zeta\ddot{\lambda}_Z \ (q^d)'(c_x - \zeta\lambda_Z) + \zeta(\dot{\lambda}_Z)^2 \ (q^d)''(c_x - \zeta\lambda_Z) \ . \end{array}$$

By assumption, $u'(\cdot)$ and $q^d(\cdot)$ are decreasing: $(q^d)' < 0$. There is no assumption on the sign of $(q^d)''$. The analysis shows that $\lambda_Z < 0$ so that $\ddot{\lambda}_Z < 0$ and $\dot{\lambda}_Z < 0$. Finally, $\dot{x} < 0$ but the sign of $\ddot{x}$ is not determined *a priori*. In the LQ case (see Appendix E), $(q^d)'' = 0$ and $\ddot{x} < 0$.

D.1.1 Junction with $Z = \overline{Z}$.

Assume that the trajectory hits the state $(S, Z) = (\overline{Z}, S^0)$ at time $t = 0$. Then we have the Taylor expansion for Z:

$$Z(t) \;=\; \overline{Z} + t(\beta(S^0 - S_m) + \zeta x(0)) + \frac{t^2}{2}(\alpha\beta S_m - \beta(\alpha+\beta)S^0 - \alpha\zeta x(0) + \zeta\dot{x}(0)) + O(t^3)\;.$$

When the junction occurs in Phase P with continuity of λ_Z, we have from (4.1.1):

$$\lambda_Z(0) \;=\; \frac{1}{\zeta}\left(c_x - u'(\overline{x} - \frac{\beta}{\zeta}S^0)\right)\;,$$

or equivalently, $\zeta x(0) = \zeta q^d(u'(\overline{x} - \frac{\beta}{\zeta}S^0)) = \zeta\overline{x} - \beta S^0$ (see for instance Figure 4.1, top, on page 42). Replacing in the development, we get:

$$\begin{aligned} Z(t) &= \overline{Z} + \frac{t^2}{2}(\alpha\beta S_m - \beta(\alpha+\beta)S^0 - \alpha\zeta\overline{x} + \alpha\beta S^0 + \zeta\dot{x}(0)) + O(t^3) \\ &= \overline{Z} + \frac{t^2}{2}(-\beta^2 S^0 + \zeta\dot{x}(0)) + O(t^3)\;. \end{aligned}$$

We have seen above that $\dot{x} < 0$ in general, so that in fact, $\ddot{Z} < 0$. On the other hand, the development for S is just:

$$S(t) \;=\; S^0 - \beta t S^0 + \frac{t^2}{2}\beta^2 S^0 + O(t^3)\;.$$

The conclusion is: at the junction of phases A/P, the trajectory is tangent to the line $Z = \overline{Z}$, coming from below and from the right.

When the junction occurs in Phase R, we have from (4.2.6),

$$\lambda_Z(0) \;=\; -\frac{c_y - c_x}{\zeta}, \quad \text{or equivalently} \quad x(0) \;=\; \widetilde{y}\;.$$

The development can be expressed as:

$$Z(t) \;=\; \overline{Z} + t(\beta S^0 + \zeta(\widetilde{y} - \overline{x})) + O(t^2) \;=\; \overline{Z} + t\beta(S^0 - S_{\widetilde{y}}) + O(t^2)\;.$$

Then, the trajectory hits the ceiling at an angle of direction $(-S^0, S^0 - S_{\widetilde{y}})$. At the triple point of phases A, R and P, we have $S^0 = S_{\widetilde{y}}$ and this direction is tangent to the line $Z = \overline{Z}$, in accordance with the junction in phase P, see above. At any other point $S_{\widetilde{y}} < S^0 \le S_m$, this angle is sharp.

When junction occurs in Phase Q, then according to (4.2.3) we have: $s(0) = \zeta x(0) - \beta(S_m - S^0) = \zeta(x(0) - \overline{x}) + \beta S^0$. Replacing in the development of Z, we get:

$$Z(t) \;=\; \overline{Z} + ts(0) + O(t^2)\;,$$

and again, the trajectory hits the line $Z = \overline{Z}$ with an angle of direction $(-\beta S^0, s(0))$. As the junction point S^0 moves from S^{QP} to S^{QR}, this angle moves continuously between the tangent to $Z = \overline{Z}$ to the same angle as in Phase R.

D.1.2 Junction on the curve $Z = Z_M(S)$.

When an optimal trajectory joins the boundary curve at some point $(S, Z_M(S))$, its tangent vector is $(-\beta S, -\alpha Z + \beta S + \zeta x)$. The tangent vector to the boundary itself is, since the curve is a "free" trajectory: $(-\beta S, -\alpha Z + \beta S)$. The tangent vector of the optimal trajectory is therefore pointing "outwards" as required.

When the junction point is close to $S = S_m$, the tangent vector tends to $(-\beta S_m, \zeta\widetilde{y})$, This is the same limit as in Phase R: according to what was said above, the tangent vector in Phase R close to $S = S_m$ has the direction: $(-\beta S_m, \beta(S_m - S_{\widetilde{y}})) = (-\beta S_m, \zeta\widetilde{y})$ (see page 29). There is therefore continuity of directions at that point.

D.2 Phase B

We are interested here in the sign of $\dot{x}$ and in asymptotic expansions of $S(t)$ and $Z(t)$ when in Phase B. We start with generic formulas, then specialize them when a junction with line $Z = \overline{Z}$ takes place. In Section D.2.2, we perform a detailed local analysis of the state close to point $(S_m, \overline{Z})$.

When in Phase B, the state and adjoint trajectories are solution to:

$$\begin{cases} \dot{Z} &= -\alpha Z + \beta S \\ \dot{S} &= -\beta S + \zeta x \end{cases} \qquad \begin{cases} \dot{\lambda}_Z &= (\rho+\alpha)\lambda_Z \\ \dot{\lambda}_S &= (\rho+\beta)\lambda_S - \beta\lambda_Z \end{cases}$$

and $x(t) = q^d(c_s + \zeta c_s - \zeta\lambda_S)$. It follows that:

$$\begin{aligned} \ddot{Z} &= -\alpha\dot{Z} + \beta\dot{S} \\ &= \alpha^2 Z - \beta(\alpha+\beta)S + \beta\zeta x \\ \ddot{S} &= -\beta\dot{S} + \zeta\dot{x} \\ &= \beta^2 S - \beta\zeta x + \zeta\dot{x} \ , \end{aligned}$$

and

$$\begin{aligned} \dddot{Z} &= -\alpha\ddot{Z} + \beta\ddot{S} \\ &= -\alpha^3 Z + \beta(\alpha^2 + \alpha\beta + \beta^2)S - \beta(\alpha+\beta)\zeta x + \beta\zeta\dot{x} \\ \dddot{S} &= -\beta\ddot{S} + \zeta\ddot{x} \\ &= -\beta^3 S + \beta^2\zeta x - \beta\zeta\dot{x} + \zeta\ddot{x} \ . \end{aligned}$$

Finally, from the specific form of $x(t)$, we have:

$$\begin{aligned} \dot{x} &= -\zeta\dot{\lambda}_S \ (q^d)'(c_x + \zeta c_s - \zeta\lambda_S) \\ \ddot{x} &= -\zeta\ddot{\lambda}_S \ (q^d)'(c_x + \zeta c_s - \zeta\lambda_S) + \zeta(\dot{\lambda}_S)^2 \ (q^d)''(c_x + \zeta c_s - \zeta\lambda_S) \ . \end{aligned}$$

We conclude that the sign of $\dot{x}$ is the same as the sign of λ_S, but the latter can be + or - in Phase B. A more precise analysis in function of c_s is necessary.

D.2.1 Junction with $Z = \overline{Z}$.

The analysis which follows suggests that only two possibilities occur for a junction in phase B: 1) either $c_s < \hat{c}_s$ and the trajectory may actually *leave* the line $Z = \overline{Z}$ to enter phase B; 2) the trajectory hits $(S_m, \overline{Z})$ in phase B.

When the trajectory hits the point $(S^0, \overline{Z})$, the Taylor developments of the state variables are generally:

$$Z(t) = \overline{Z} + t\beta(S^0 - S_m) + \frac{t^2}{2}(\alpha\beta S_m - \beta(\alpha+\beta)S^0 + \beta\zeta x(0)) + O(t^3) \qquad \text{(D.2.1)}$$

$$S(t) = S^0 + t(\zeta x(0) - \beta S^0) + \frac{t^2}{2}(\beta^2 S^0 - \beta\zeta x(0) + \zeta\dot{x}(0)) + O(t^3) \ . \qquad \text{(D.2.2)}$$

Assume first that $S^0 < S_m$. Then clearly $\dot{Z}(0) < 0$ and the trajectory *cannot arrive at* the line $Z = \overline{Z}$: it must be leaving. Its direction is $(\zeta x(0) - \beta S^0, S^0 - S_m)$.

Assume next that $S^0 = S_m$. Then the development is simplified into:

$$\begin{aligned} Z(t) &= \overline{Z} + \frac{t^2}{2}(\alpha\beta S_m - \beta(\alpha+\beta)S_m + \beta\zeta x(0)) + O(t^3) \\ &= \overline{Z} + \frac{t^2}{2}\beta\zeta(x(0) - \overline{x}) + O(t^3) \qquad \text{(D.2.3)} \end{aligned}$$

$$S(t) \quad = \quad S_m + t\zeta(x(0) - \overline{x}) + \frac{t^2}{2}(\beta\zeta\overline{x} - \beta\zeta x(0) + \zeta\dot{x}(0)) + O(t^3) \; . \tag{D.2.4}$$

If $x(0) \neq \overline{x}$, by elimination of the time variable, one gets that

$$t \;\sim\; \frac{S(t) - S_m}{\zeta(x(0) - \overline{x})}$$

so that the trajectory is, asymptotically,

$$\begin{aligned} Z &= \overline{Z} + \frac{1}{2}\beta\zeta(x(0) - \overline{x})\left(\frac{S - S_m}{\zeta(x(0) - \overline{x})}\right)^2 + o((S - S_m)^2) \\ &= \overline{Z} + \frac{1}{2}\frac{\beta}{\zeta}\frac{(S - S_m)^2}{x(0) - \overline{x}} + o((S - S_m)^2) \; . \end{aligned}$$

On the condition that $x(0) < \overline{x}$, this trajectory is tangent to the line $Z = \overline{Z}$ and arrives from below and from the right. If $x(0) > \overline{x}$, the trajectory arrives from above, which is not consistent.

However, if $x(0) = \overline{x}$, then we have $\dot{S} = \dot{Z} = \ddot{Z} = 0$, and the development of $Z(t)$ has to be refined to get, using the formula $\dddot{Z} = -\alpha\ddot{Z} + \beta\ddot{S}$:

$$Z(t) \quad = \quad \overline{Z} \; + \; \frac{t^3}{6}\beta\zeta\dot{x}(0) \; + \; O(t^4) \tag{D.2.5}$$

$$S(t) \quad = \quad S_m \; + \; \frac{t^2}{2}\zeta\dot{x}(0) \; + \; O(t^3) \; . \tag{D.2.6}$$

If $\dot{x}(0) > 0$, which happens when $\dot{\lambda}_S > 0$, then the trajectory is tangent to the line $Z = \overline{Z}$ and approaches it from below and from the right. In the case $\dot{x}(0) < 0$, it approaches it from above, and this is not consistent. In the first case, eliminating the time variable gives (remembering that $t \leq 0$):

$$t \;\sim\; -\left(\frac{2(S(t) - S_m)}{\zeta\dot{x}(0)}\right)^{1/2}$$

so that the trajectory is, asymptotically,

$$\begin{aligned} Z &= \overline{Z} - \frac{1}{6}\zeta\dot{x}(0)\left(\frac{2(S - S_m)}{\zeta\dot{x}(0)}\right)^{3/2} + o((S - S_m)^{3/2}) \\ &= \overline{Z} - \frac{2^{3/2}}{6}\frac{(S - S_m)^{3/2}}{(\zeta\dot{x}(0))^{1/2}} + o((S - S_m)^{3/2}) \; . \end{aligned}$$

D.2.2 Local analysis around $(S_m, \overline{Z})$

This section is devoted to a proof that the general scheme of Figure 4.4 page 51 is correct, at least for a set of trajectories "close" to the point $(S_m, \overline{Z})$. The result is stated as Lemma D.1 next. As a corollary, we state in Lemma 4.14 (page 50) that some optimal trajectories consist in a Phase B followed by the Phase S.

This lemma describes a property of the dynamical system of Phase B around particular initial values. It does not depend on costs. The fact that it describes *optimal* trajectories holds however only for $c_s < \hat{c}_s$. The critical value $\lambda_S^{(S)} = c_s + (c_x - c_y)/\zeta$ is central in the analysis.

Lemma D.1. *Consider the dynamical system characteristic of Phase B, under Assumption 1 and assuming that $u(\cdot)$ has a bounded third derivative. There exists a constant $\overline{\ell}$ such that, for all $\ell \in (0, \overline{\ell}]$, the trajectories which terminate at $S(T) = S_m$, $Z(T) = \overline{Z}$, $\lambda_S(T) = \lambda_S^{(S)}$ and $\lambda_Z(T) = \lambda_Z^{(S)} - \ell$, have the following property: there exist $\tau_1 < \tau_2 < \tau_3 < \tau_4 < \tau_5 < \tau_6 < T$ such that the table of variation in Table D.1 holds.*

	τ_1		τ_2		τ_3		τ_4		τ_5		τ_6		T	
					$-$						0	$+$		$\zeta\dot{x}$
				$\searrow$					$\zeta\overline{x}$	$\searrow$		$\nearrow$	$\zeta\overline{x}$	ζx
			$+$				0			$-$			0	$\beta\dot{S}$
		$\nearrow$			βS_m	$\nearrow$				$\searrow$			βS_m	βS
	$-$		0					$-$					0	$\alpha\dot{Z}$
$\searrow$	$\alpha\overline{Z}$	$\searrow$						$\nearrow$					$\alpha\overline{Z}$	αZ

Table D.1: Table of variation of trajectories in Phase B

The time instants τ_i are illustrated in Figure 4.4 on page 51. The proof uses in part the following intermediate result:

Lemma D.2. *Consider the dynamical system characteristic of Phase B, under Assumption 1, assuming that $u'(\cdot)$ has a bounded third derivative. There exist constants, C_1, C_2, C_3 and $\overline{\ell}$ such that, for all $\ell \in (0, \overline{\ell}]$, the trajectories which terminate at $S(T) = S_m$, $Z(T) = \overline{Z}$, $\lambda_S(T) = \lambda_S^{(S)}$ and $\lambda_Z(T) = \lambda_Z^{(S)} - \ell$, are such that:*

$$\begin{aligned} \zeta x(T - C_1\ell) &> \beta S(T - C_1\ell) && \text{(D.2.7)} \\ \alpha Z(T - C_2\ell) &> \beta S(T - C_2\ell) && \text{(D.2.8)} \\ \alpha Z(T - C_3\ell) &> \alpha\overline{Z}\ . && \text{(D.2.9)} \end{aligned}$$

Proof. The proof consists in computing Taylor expansions of the three different functions $\zeta x(t)$, $\beta S(t)$ and $\alpha Z(t)$ around $t = T$, while at the same time considering $\lambda_Z(T) = \lambda_Z^{(S)} - \ell$. In a second phase, the value of ℓ is linked appropriately to the time parameter in the expansion.

We start with $\lambda_S(t)$, the formula of which is given in (3.2.5). Using the boundary conditions, and the fact that $\beta\lambda_Z^{(S)} = (\rho + \beta)\lambda_S^{(S)}$, we have:

$$\begin{aligned} \lambda_S(T + u) &= \lambda_S^{(S)} e^{(\rho+\beta)u} \; - \; \frac{1}{\alpha - \beta}\left((\rho + \beta)\lambda_S^{(S)} - \beta\ell\right)\left(e^{(\rho+\alpha)u} - e^{(\rho+\beta)u}\right) \\ &= \lambda_S^{(S)} \; + \; u\beta\ell \; + \; \frac{1}{2}Au^2 \; + \; O(u^3)\ , \end{aligned}$$

where we have used the shorthand notation $A := \beta\ell(2\rho+\alpha+\beta) - (\rho+\alpha)(\rho+\beta)\lambda_S^{(S)}$. The function $O(u^3)$ in this expansion is bounded by Mu^3, for some constant M, uniformly for ℓ in any compact containing 0. Next, consider the expansion of $x(t)$:

$$\begin{aligned} x(T + u) &= q^d(c_x + \zeta c_s - \zeta\lambda_S(T + u)) \; = \; q^d(\overline{p} + \zeta(\lambda_S^{(S)} - \lambda_S(T + u))) \\ &= \overline{x} \; + \; (q^d)'(\overline{p})\zeta(\lambda_S^{(S)} - \lambda_S(T + u)) \\ &\quad + \; \frac{1}{2}(q^d)''(\overline{p})\zeta^2(\lambda_S^{(S)} - \lambda_S(T + u))^2 \; + \; O(|\lambda_S^{(S)} - \lambda_S(T + u)|^3) \\ &= \overline{x} \; - \; (q^d)'(\overline{p})\zeta\left(\beta\ell \; + \; \frac{1}{2}uA\right)u \; + \; \frac{1}{2}(q^d)''(\overline{p})\zeta^2\beta^2\ell^2u^2 \; + \; O(u^3). \end{aligned}$$

Again, the "$O(u^3)$" term is uniform for ℓ in a compact, assuming that q^d admits a bounded third derivative. The expansion for $S(\cdot)$ is derived from that of x, through the integral formula (3.4.8). After a change of variables:

$$\begin{aligned} S(T + u) &= S_m e^{-\beta u} \; + \; \zeta e^{-\beta u}\int_0^u e^{\beta w} x(T + w)\mathrm{d}w \\ &= e^{-\beta u}\Bigg(S_m \; + \; \zeta\overline{x}\,\frac{e^{\beta u} - 1}{\beta} \; - \; \zeta^2\beta\ell(q^d)'(\overline{p})\int_0^u we^{\beta w}\mathrm{d}w \end{aligned}$$

$$
\begin{aligned}
& - \frac{1}{2}\zeta^2 \left((q^d)'(\overline{p})A - (q^d)''(\overline{p})\zeta\beta^2\ell^2\right) \int_0^u w^2 e^{\beta w}\mathrm{d}w \; + \; \int_0^u O(w^3)e^{\beta w}\mathrm{d}w \Big) \\
= \;& e^{-\beta u}\Big(S_m e^{\beta u} \; - \; \frac{1}{2}(q^d)'(\overline{p})\zeta^2\beta\ell u^2 \; - \; \frac{1}{3}(q^d)'(\overline{p})\zeta^2\beta^2\ell u^3 \\
& - \frac{1}{6}(q^d)'(\overline{p})\zeta^2 A u^3 \; + \; \frac{1}{6}(q^d)''(\overline{p})\zeta^3\beta^2\ell^2 u^3 \Big) \; + \; O(u^4) \; .
\end{aligned}
$$

Finally, the expansion for $Z(\cdot)$ follows from (3.4.7):

$$
\begin{aligned}
Z(T+u) \;&=\; \overline{Z}e^{-\alpha u} \; + \; \beta e^{-\alpha u} \int_0^u e^{\alpha w} S(T+w)\mathrm{d}w \\
&=\; \overline{Z}e^{-\alpha u} \; + \; \beta e^{-\alpha u} \int_0^u e^{\alpha w} S_m \mathrm{d}w \\
&\quad - \; e^{-\alpha u} \int_0^u e^{(\alpha-\beta)w}(q^d)'(\overline{p})\zeta^2 w^2 \left(\frac{1}{2}\beta\ell + \frac{1}{6}A + lO(w) + O(w^2)\right)\mathrm{d}w \\
&=\; \overline{Z} \; - \; (q^d)'(\overline{p})\zeta^2\beta\left(\frac{1}{6}\beta\ell - \frac{1}{24}(\rho+\alpha)(\rho+\beta)\lambda_S^{(S)}u\right)u^3 \; + \; \ell O(u^4) + O(u^5) \; .
\end{aligned}
$$

If we choose now to set $u = -C\ell$ for some positive constant C, we get the expansions:

$$
\begin{aligned}
x(T-C\ell) \;&=\; \overline{x} \; + \; (q^d)'(\overline{p})\zeta C\left(\beta \; + \; \frac{1}{2}C(\rho+\alpha)(\rho+\beta)\lambda_S^{(S)}\right)\ell^2 \; + \; O(\ell^3) \\
S(T-C\ell) \;&=\; S_m \; - \; (q^d)'(\overline{p})\zeta^2 C^2\left(\frac{1}{2}\beta \; + \; \frac{1}{6}C(\rho+\alpha)(\rho+\beta)\lambda_S^{(S)}\right)\ell^3 \; + \; O(\ell^4) \\
Z(T-C\ell) \;&=\; \overline{Z} \; - \; (q^d)'(\overline{p})\zeta^2 C^3\beta\left(\frac{1}{6}\beta + \frac{1}{24}C(\rho+\alpha)(\rho+\beta)\lambda_S^{(S)}\right)\ell^4 \; + \; O(\ell^5) \; .
\end{aligned}
$$

By assumption, $(q^d)' < 0$. If the constants C_1, C_2 and C_3 are chosen such that

$$
C_1 \; > \; 2C_0, \quad C_2 \; > \; 3C_0, \quad C_3 \; > \; 4C_0, \qquad C_0 := -\frac{\beta}{(\rho+\alpha)(\rho+\beta)}\,\frac{1}{\lambda_S^{(S)}},
$$

then the different orders of the expansions allow to conclude that for ℓ sufficiently close to 0, $\zeta x(T-C_1\ell) > \beta S(T-C_1\ell)$, $\beta S(T-C_2\ell) > \alpha Z(T-C_2\ell)$ and $\alpha Z(T-C_3\ell) > \alpha\overline{Z}$. $\square$

Proof of Lemma D.1. We begin with $x(t)$ and its related function $\lambda_S(t)$, since $x(t) = q^d(c_x + \zeta c_s - \zeta\lambda_S(t))$. Using the results of Section 3.2, it is straightforward to show that there exists $\tau_6 < T$ such that $\dot{\lambda}_S(\tau_6) = 0$. Indeed, $\dot{\lambda}_S(t) = 0$ iff $(\rho+\beta)\lambda_S(t) = \beta\lambda_Z(t)$, and from the observations in Section 3.2.2, the ratio $r_\lambda = \lambda_S/\lambda_Z$ is decreasing on the interval $t \in (-\infty, T]$. There exists therefore a unique τ_6 where $\lambda_S(\tau_6)$ is minimal: $\lambda_S(t)$ is decreasing up to τ_6, then increasing.

Next, we have

$$
\dot{x}(t) \;=\; -\,\zeta\dot{\lambda}_S \; (q^d)'(c_x + \zeta c_s - \zeta\lambda_S)
$$

and since $(q^d)' < 0$ under Assumption 1, the variation of $\zeta\dot{x}$ is as in Table D.1. When $t \to -\infty$, $\lambda_S(t) \to 0$ so that $x(t) \to q^d(c_x + \zeta c_s)$. Under the assumption that $c_s < \hat{c}_s$, we find that $q^d(c_x + \zeta c_s) > \overline{x}$. This implies the existence of $\tau_5 < \tau_6$ such that $x(\tau_5) = 0$. The variation of $\zeta x(t)$ is therefore as claimed in Table D.1.

Consider now the function $\beta S(t)$. According to the development close to $t = T$ computed in the proof of Lemma D.2 (see also Section D), $\beta S(t) > \beta S_m = \zeta\overline{x} > \zeta x(t)$ for t sufficiently close to T. On the other hand, from Lemma D.2, there exists $\bar{\ell}$ such that for all $\ell \in (0, \bar{\ell}]$, there is a time τ such that $\beta S(\tau) < \zeta x(\tau)$. By continuity, this implies the existence of at least one t such that $\beta S(t) = \zeta x(t)$. Let τ_4 be the largest of them. Necessarily, $x(\tau_4) > \overline{x}$ because $\dot{S}(\tau_4) = -\beta S(\tau_4) + \zeta x(\tau_4) = \beta(S_m - S(\tau_4)) + \zeta(x(\tau_4) - \overline{x}) = 0$, and because $\dot{S}(t) < 0$ for $t \in (\tau_4, T)$ implies $S(\tau_4) > S_m$. From the variation of $x(t)$, this implies in turn that $\tau_4 < \tau_5$.

We argue now that $\dot{S}(t) > 0$ for all $t < \tau_4$, so that the variation of $\dot{S}$ is as claimed in Table D.1. Assume by contradiction that $\dot{S}(\tau) = 0$ for some $\tau < \tau_4$, and consider the largest of such values. Then $\dot{S}(t) > 0$ for all t in the interval (τ, τ_4). Then, since $\ddot{S} = -\beta\dot{S} + \zeta\dot{x}$, and since $\dot{x}(t) < 0$ on the interval, according to the variation of $\dot{x}$, we conclude that $\ddot{S}(t) < 0$ over the interval. We reach a contradiction with the fact that $\dot{S} = 0$ at both extremities.

Finally, according again to Lemma D.2, there exists a τ such that $\beta S(\tau) < \alpha Z(\tau)$. Similarly as above, this implies the existence of a unique τ_2 such that $\beta S(\tau_2) = \alpha Z(\tau_2)$. Clearly, Z is increasing on the interval $[\tau_2, T]$ so that $\beta S(\tau_2) = \alpha Z(\tau_2) < \alpha Z(T) = \alpha\overline{Z} = \beta S_m$. This implies in turn: on the one hand that $\tau_2 < \tau_4$, and on the other hand that there exists τ_3 such that $S(\tau_3) = S_m$ and $\tau_2 < \tau_3 < \tau_4$. This concludes the proof that the variation of S is as in Table D.1.

There remains to complete the analysis of $Z(t)$. By the same convexity argument, $\alpha Z(t)$ cannot cross twice $\beta S(t)$ because $\ddot{Z} = -\alpha\dot{Z} + \beta\dot{S}$ is positive on any interval ending at τ_2. Therefore, $\dot{Z}$ cannot vanish on interval $(-\infty, \tau_2)$ and the variation of $\dot{Z}$ is as shown in Table D.1.

Using Lemma D.2 a last time, we conclude that there exists a value τ_1 such that $Z(\tau_1) = \overline{Z}$. The function $Z(\cdot)$ therefore evolves as described in Table D.1. This concludes the proof. $\square$

Appendix E

The Linear-Quadratic case

In this section, we develop explicit formulas for the case where $u(\cdot)$ is quadratic, in the situation where X is infinite. The relationship between critical parameter values becomes clear in this case. A numerical example is developped using these formulas.

E.1 Relationships between parameters

In that case, $u'(\cdot)$ is linear. Let $-W$ denote its slope, with $W > 0$. Let us choose the form:

$$u'(x) = \overline{p} - W(x - \overline{x}) \tag{E.1.1}$$

$$u(x) = u(\overline{x}) + \overline{p}(x - \overline{x}) - \frac{1}{2}W(x - \overline{x})^2 \tag{E.1.2}$$

$$q^d(p) = \overline{x} - \frac{1}{W}(p - \overline{p}) \; . \tag{E.1.3}$$

Since $c_y = u'(\widetilde{y})$, and $c_x = u'(\widetilde{x})$, we have the alternate forms for W:

$$W = \frac{\overline{p} - c_y}{\widetilde{y} - \overline{x}} = \frac{\overline{p} - c_x}{\widetilde{x} - \overline{x}} = \frac{c_y - c_x}{\widetilde{x} - \widetilde{y}} = \frac{c_y - \overline{p}}{S_{\widetilde{y}}} \frac{\zeta}{\beta} \; . \tag{E.1.4}$$

Other formulas linking W and previously introduced quantities are:

$$\hat{c}_s = \frac{\rho}{\rho + \beta} \frac{\widetilde{x} - \overline{x}}{\zeta W} \tag{E.1.5}$$

$$\bar{c}_s = \frac{\rho}{\rho + \beta} \frac{\widetilde{x} - \widetilde{y}}{\zeta W} \; . \tag{E.1.6}$$

$$\hat{c}_s - \bar{c}_s = \frac{\rho}{\rho + \beta} \frac{\widetilde{y} - \overline{x}}{\zeta W} \; .$$

E.2 Phase P

The functions $M(\cdot)$ and $L(\cdot)$ are respectively given by:

$$M(S) = -\frac{W}{\zeta} \frac{\beta^2 S}{\rho + 2\beta} \tag{E.2.1}$$

$$L(S) = \frac{\beta}{\rho + \beta}(c_x - \overline{p}) - \frac{W}{\zeta} \frac{\beta^2 S}{\rho + 2\beta} \; . \tag{E.2.2}$$

The value S^{QP} solves equation (4.3.3) or (4.3.4), which gives:

$$\zeta(c_s - \hat{c}_s) - \frac{W\beta}{\zeta} S^{QP} = -\frac{W}{\zeta} \frac{\beta^2 S^{QP}}{\rho + 2\beta}$$

$$S^{QP} = (c_s - \hat{c}_s)\frac{\zeta^2}{W}\frac{\rho+2\beta}{\beta(\rho+\beta)} \ . \tag{E.2.3}$$

One checks directly that $S^{QP} < S_{\widetilde{y}}$ when $c_s < \bar{c}_s$. Indeed, we have:

$$\begin{aligned}
S^{QP} < S_{\widetilde{y}} &\iff (c_s - \hat{c}_s)\frac{\zeta^2}{W}\frac{\rho+2\beta}{\beta(\rho+\beta)} \leq (c_y - \overline{p})\frac{\zeta}{\beta W} \\
&\iff c_s - \hat{c}_s \leq \frac{\rho+\beta}{\rho+2\beta}\frac{c_y-\overline{p}}{\zeta} = \frac{\rho+\beta}{\rho+2\beta}(\bar{c}_s - \hat{c}_s) \\
&\iff c_s \leq \frac{\beta}{\rho+2\beta}\hat{c}_s + \frac{\rho+\beta}{\rho+2\beta}\bar{c}_s \ .
\end{aligned}$$

The right-hand side is a convex combination of $\hat{c}_s$ and $\bar{c}_s$, and since $\hat{c}_s < \bar{c}_s$, it lies between these two values.

The adjoint variables in phase P are given by (4.1.1) and $\lambda_S^{(P)}(t) = L(S(t))/\zeta$. Therefore we have the formulas expressed as a state feedback:

$$\begin{aligned}
\lambda_Z &= \frac{1}{\zeta}\left(c_x - u'\left(\overline{x} - \frac{\beta}{\zeta}S\right)\right) \\
&= \frac{c_x - \overline{p}}{\zeta} + \frac{W}{\zeta}\left(\overline{x} - \frac{\beta}{\zeta}S - \overline{x}\right) \\
&= \frac{c_x - \overline{p}}{\zeta} - \frac{W\beta}{\zeta^2}S & \text{(E.2.4)} \\
\lambda_S &= \frac{\beta}{\rho+\beta}\frac{c_x-\overline{p}}{\zeta} - \frac{W}{\zeta^2}\frac{\beta^2 S}{\rho+2\beta} & \text{(E.2.5)} \\
&= \frac{\beta}{\rho+\beta}\frac{c_x-\overline{p}}{\zeta} - \frac{\beta}{\rho+2\beta}\left(\frac{c_x-\overline{p}}{\zeta} - \lambda_Z\right) \\
&= \frac{c_x-\overline{p}}{\zeta}\frac{\beta^2}{(\rho+\beta)(\rho+2\beta)} + \frac{\beta}{\rho+2\beta}\lambda_Z \ . & \text{(E.2.6)}
\end{aligned}$$

According to this last formula, the trajectory of $(\lambda_Z(t), \lambda_S(t))$ in the $\lambda_Z - \lambda_S$ plane is a straight line with a slope that does not depend on W.

When $S \to 0$, the point tends to the point P_∞ defined in (4.1.9) on page 31. When $S \to S_{\widetilde{y}}$, it tends to:

$$\begin{aligned}
(\lambda_Z(S_{\widetilde{y}}), \lambda_S(S_{\widetilde{y}})) &= \left(\frac{c_x-c_y}{\zeta}, \frac{\beta}{\rho+\beta}\frac{c_x-\overline{p}}{\zeta} - \frac{\beta}{\rho+2\beta}\frac{c_y-\overline{p}}{\zeta}\right) \\
&= \left(\frac{c_x-c_y}{\zeta}, \frac{\beta}{\rho+\beta}\frac{c_x-c_y}{\zeta} + \frac{\beta^2}{(\rho+\beta)(\rho+2\beta)}\frac{c_y-\overline{p}}{\zeta}\right) \ . & \text{(E.2.7)}
\end{aligned}$$

Value of c_{sm}. By definition of c_{sm}, the point given by (E.2.7) is on the line $\lambda_S = \lambda_Z + c_{sm}$, because Phase Q occurs just at $S = S_{\widetilde{y}}$. Therefore, it follows that:

$$\begin{aligned}
c_{sm} &= \frac{\rho}{\rho+\beta}\frac{c_y-c_x}{\zeta} + \frac{\beta^2}{(\rho+\beta)(\rho+2\beta)}\frac{c_y-\overline{p}}{\zeta} & \text{(E.2.8)} \\
&= \bar{c}_s + \frac{\beta^2}{(\rho+\beta)(\rho+2\beta)}\frac{c_y-\overline{p}}{\zeta} \ .
\end{aligned}$$

As expected, it follows from the last line that $c_{sm} > \bar{c}_s$.

Alternately, when $c_s = c_{sm}$, we must have $S^{QP} = S_{\widetilde{y}}$. Accordingly, using (E.2.3) and (E.1.4) and simplifying, we get the second identity:

$$c_{sm} = \hat{c}_s + \frac{\rho+\beta}{\rho+2\beta}\frac{c_y-\overline{p}}{\zeta} \ . \tag{E.2.9}$$

Value function. Finally, the value function $V_P(S)$ is computed directly from its definition as:

$$\begin{aligned}
V_P(S) &= \int_0^\infty e^{-\rho v}\left(u(\overline{x}-\frac{\beta}{\zeta}Se^{-\beta v})-c_x\left(\overline{x}-\frac{\beta}{\zeta}Se^{-\beta v}\right)\right)\mathrm{d}v \\
&= \int_0^\infty e^{-\rho v}\left(u(\overline{x})+\overline{p}\left(-\frac{\beta}{\zeta}Se^{-\beta v}\right)-\frac{W}{2}\left(-\frac{\beta}{\zeta}Se^{-\beta v}\right)^2\right)\mathrm{d}v-\frac{c_x\overline{x}}{\rho}+\frac{c_x\beta S}{\zeta}\frac{1}{\rho+\beta} \\
&= \frac{u(\overline{x})-c_x\overline{x}}{\rho}+\frac{\beta S}{\rho+\beta}\frac{c_x-\overline{p}}{\zeta}-\frac{W}{2}\frac{\beta^2S^2}{\zeta^2}\frac{1}{\rho+2\beta}\ . \qquad \text{(E.2.10)}
\end{aligned}$$

It is possible to check the Hamilton-Jacobi-Bellman identity from (E.2.10), as well as the identity $V_P' = \lambda_S$ from (E.2.10) and (E.2.5). Also, it is verified that $H(S^{(P)}(t^0), \overline{Z}, \lambda_S^{(P)}(t^0), \lambda_Z^{(P)}(t^0)) = \rho V_P(S^{(P)}(t^0))$.

E.3 Phase Q

The value of λ_Z is expressed from (4.3.7) and the value of S^{QP} in (E.2.3) as:

$$\begin{aligned}
\lambda_Z(t) &= e^{\rho(t-t^{QP})}\left[\frac{\rho+\beta}{\rho}(c_s-\hat{c}_s)+\frac{W}{\zeta}\left(-\frac{\beta}{\zeta}S^{QP}\right)\right]-\frac{\rho+\beta}{\rho}c_s \\
&= e^{\rho(t-t^{QP})}\left[\frac{\rho+\beta}{\rho}(c_s-\hat{c}_s)-\frac{W\beta}{\zeta^2}(c_s-\hat{c}_s)\frac{\zeta^2}{W}\frac{\rho+2\beta}{\beta(\rho+\beta)}\right]-\frac{\rho+\beta}{\rho}c_s \\
&= e^{\rho(t-t^{QP})}(c_s-\hat{c}_s)\left[\frac{\rho+\beta}{\rho}-\frac{\rho+2\beta}{\rho+\beta}\right]-\frac{\rho+\beta}{\rho}c_s \\
&= e^{\rho(t-t^{QP})}(c_s-\hat{c}_s)\frac{\beta^2}{\rho(\rho+\beta)}-\frac{\rho+\beta}{\rho}c_s\ . \qquad \text{(E.3.1)}
\end{aligned}$$

Next, the value of $x^{(Q)} = q^d(c_x - \zeta\lambda_Z)$ is, using (E.1.3),

$$\begin{aligned}
x^{(Q)}(t) &= \overline{x}-\frac{1}{W}\left(c_x-\zeta e^{\rho(t-t^{QP})}(c_s-\hat{c}_s)\frac{\beta^2}{\rho(\rho+\beta)}+\zeta\frac{\rho+\beta}{\rho}c_s-\overline{p}\right) \\
&= \overline{x}-\frac{1}{W}\left(\zeta\frac{\rho+\beta}{\rho}\left(c_s+\frac{\rho}{\rho+\beta}\frac{c_x-\overline{p}}{\zeta}\right)-\zeta e^{\rho(t-t^{QP})}(c_s-\hat{c}_s)\frac{\beta^2}{\rho(\rho+\beta)}\right) \\
&= \overline{x}-\frac{\zeta}{\rho W}(c_s-\hat{c}_s)\left(\rho+\beta-e^{\rho(t-t^{QP})}\frac{\beta^2}{\rho+\beta}\right)\ . \qquad \text{(E.3.2)}
\end{aligned}$$

As a particular value, we can evaluate $x^{(Q)}(t^{QP})$, see Figure 4.1. We have:

$$\begin{aligned}
x^{(Q)}(t^{QP}) &= \overline{x}-\frac{\zeta}{\rho W}(c_s-\hat{c}_s)\left(\rho+\beta-\frac{\beta^2}{\rho+\beta}\right) = \overline{x}-\frac{\zeta}{\rho W}(c_s-\hat{c}_s)\frac{\rho(\rho+2\beta)}{\rho+\beta} \\
&= \overline{x}-\frac{\beta}{\zeta}S^{QP}\ ,
\end{aligned}$$

where we have used the value of S^{QP} obtained in (E.2.3). This is of course consistent with the general relationship which prevails in Phase P: $x = \overline{x} - \beta S/\zeta$. Next, the dynamics of $S(t)$ are integrated with (4.3.1) as:

$$\begin{aligned}
S^{(Q)}(t) &= S^{QP}-\zeta\int_t^{t^{QP}}(x^{(Q)}(t)-\overline{x})\ \mathrm{d}t \\
&= S^{QP}+\frac{\zeta^2}{\rho W}(c_s-\hat{c}_s)\int_t^{t^{QP}}\left(\rho+\beta-e^{\rho(t-t^{QP})}\frac{\beta^2}{\rho+\beta}\right)\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= S^{QP} + \frac{\zeta^2}{W}(c_s - \hat{c}_s)\frac{\rho+\beta}{\rho}(t^{QP} - t) - \frac{\beta^2\zeta^2}{\rho(\rho+\beta)W}(c_s - \hat{c}_s)\int_t^{t^{QP}} e^{\rho(u-t^{QP})}\,\mathrm{d}u \\
&= S^{QP} + \frac{\zeta^2}{W}(c_s - \hat{c}_s)\frac{\rho+\beta}{\rho}(t^{QP} - t) - \frac{\beta^2\zeta^2}{W}(c_s - \hat{c}_s)\frac{1 - e^{\rho(t-t^{QP})}}{\rho^2(\rho+\beta)} \,. \qquad \text{(E.3.3)}
\end{aligned}
$$

Comparing Equation (E.3.2) for x^Q and Equation (E.3.3) for S^Q, we see that the optimal control is not an affine function of the state.

The value t^{RQ} satisfies $\lambda_Z^{(Q)}(t^{RQ}) = (c_x - c_y)/\zeta$. Accordingly, from (E.3.1) (see also (4.3.13)), we have:

$$
\begin{aligned}
t^{RQ} - t^{QP} &= \frac{1}{\rho}\log\left[\frac{\frac{\rho+\beta}{\rho}c_s + \frac{c_x - c_y}{\zeta}}{(c_s - \hat{c}_s)\frac{\beta^2}{\rho(\rho+\beta)}}\right] = \frac{1}{\rho}\log\left[\frac{\frac{\rho+\beta}{\rho}(c_s - \bar{c}_s)}{(c_s - \hat{c}_s)\frac{\beta^2}{\rho(\rho+\beta)}}\right] \\
&= \frac{1}{\rho}\log\left[\left(\frac{\rho+\beta}{\beta}\right)^2 \frac{c_s - \bar{c}_s}{c_s - \hat{c}_s}\right] \,. \qquad \text{(E.3.4)}
\end{aligned}
$$

It is easy to check with identities (E.2.8) and (E.2.9) that when $c_s = c_{sm}$, this quantity reduces to 0. This is of course consistent with the fact that Phase Q vanishes in that situation.

When Equation (E.3.3) is evaluated at $t = t^{RQ}$, Equations (E.3.4) for $t^{RQ} - t^{PQ}$ and (E.2.3) for S^{QP} allow to obtain the value of $S^{RQ} = S(t^{RQ})$:

$$
S^{RQ} = \frac{\zeta^2}{W}(c_s - \hat{c}_s)\frac{\rho+\beta}{\rho^2}\log\left[\left(\frac{\rho+\beta}{\beta}\right)^2 \frac{c_s - \bar{c}_s}{c_s - \hat{c}_s}\right] + \frac{\zeta^2}{W\rho}\left[\frac{\overline{p} - c_y}{\zeta} + \frac{\rho+2\beta}{\beta}(c_s - \hat{c}_s)\right] \,. \qquad \text{(E.3.5)}
$$

Again, it can be checked that when $c_s = c_{sm}$, this formula reduces to $S_{\widetilde{y}}$. The value of c_{sQ} is obtained when solving $S^{RQ} = S_m$.

E.4 Phase A

Assuming that the system is in state $S^0 = S(t^0)$ at some arbitrary time instant t^0, we have: $\lambda_Z^{(A)}(t) = \lambda_Z^0 e^{(\rho+\alpha)(t-t^0)}$ and consequently, since $x^{(A)}(t) = q^d(c_x - \zeta\lambda_Z)$,

$$
x^{(A)}(t) = \overline{x} - \frac{c_x - \overline{p}}{W} + \frac{\zeta}{W}\lambda_Z^0 e^{(\rho+\alpha)(t-t^0)} = \widetilde{x} + \frac{\zeta}{W}\lambda_Z^0 e^{(\rho+\alpha)(t-t^0)} \,, \qquad \text{(E.4.1)}
$$

where we have used, from (E.1.4): $(c_x - \overline{p})/W = \overline{x} - \widetilde{x}$. Next, according to (3.4.3),

$$
\begin{aligned}
Z(t) &= Z^0 e^{-\alpha(t-t^0)} + S^0\frac{\beta}{\alpha-\beta}\left(e^{-\beta(t-t^0)} - e^{-\alpha(t-t^0)}\right) + \zeta\int_{t^0}^{t} e^{-\alpha(t-u)}x^{(A)}(u)\,\mathrm{d}u \\
&= Z^0 e^{-\alpha(t-t^0)} + S^0\frac{\beta}{\alpha-\beta}\left(e^{-\beta(t-t^0)} - e^{-\alpha(t-t^0)}\right) \\
&\quad + \zeta\widetilde{x}\frac{1 - e^{-\alpha(t-t^0)}}{\alpha} + \frac{\zeta^2}{W}\lambda_Z^0\frac{e^{(\rho+\alpha)(t-t^0)} - e^{-\alpha(t-t^0)}}{\rho+2\alpha} \,. \qquad \text{(E.4.2)}
\end{aligned}
$$

Using the dynamics of S: $S^{(A)}(t) = S^0 e^{-\beta(t-t^0)}$, it is possible to eliminate the time variable so as to obtain the equation of the trajectory in the (S, Z) space:

$$
Z = Z_M(S) + \frac{\zeta\widetilde{x}}{\alpha}\left(1 - \left(\frac{S}{S^0}\right)^{\alpha/\beta}\right) + \frac{\zeta^2}{W}\frac{\lambda_Z^0}{\rho+2\alpha}\left(\left(\frac{S}{S^0}\right)^{-(\rho+\alpha)/\beta} - \left(\frac{S}{S^0}\right)^{\alpha/\beta}\right) \,. \qquad \text{(E.4.3)}
$$

E.5 Phase B

Assume that the system is in Phase B at time t^0, with corresponding values λ_Z^0 and λ_S^0 for the costate variables. Since $x^{(B)} = q^d(c_x + \zeta c_s - \zeta\lambda_S)$, we get:

$$x^{(B)}(t) \;=\; \tilde{x} - \frac{\zeta c_s}{W} + \frac{\zeta}{W}\left(\lambda_S^0 e^{(\rho+\beta)(t-t^0)} \;-\; \frac{\beta}{\alpha-\beta}\lambda_Z^0\left(e^{(\rho+\alpha)(t-t^0)} - e^{(\rho+\beta)(t-t^0)}\right)\right) .$$

Integrating Equations (3.4.8) then (3.4.7), *with terminal conditions* $S(0) = S_m$ and $Z(0) = \overline{Z}$, we get:

$$\begin{aligned}
S^{(B)}(t) \;&=\; S_m e^{-\beta t} + \frac{\zeta}{\beta}(\widetilde{x} - \zeta c_s/W)(1 - e^{-\beta t}) \\
&\quad + \frac{\zeta^2}{W}(\lambda_S^0 + \lambda_Z^0\frac{\beta}{\alpha-\beta})\frac{1}{\rho+2\beta}(e^{(\rho+\beta)t} - e^{-\beta t}) \\
&\quad - \frac{\zeta^2}{W}\lambda_Z^0\frac{\beta}{\alpha-\beta}\frac{1}{\rho+\alpha+\beta}(e^{(\rho+\alpha)t} - e^{-\beta t}) \qquad \text{(E.5.1)}\\
Z^{(B)}(t) \;&=\; \zeta(\widetilde{x} - \zeta c_s/W)/\alpha \\
&\quad + e^{-\alpha t}\Bigg(\overline{Z} - \frac{\beta}{\alpha-\beta}S_m + \beta\zeta(\widetilde{x} - \zeta c_s/W)/\alpha/(\alpha-\beta) \\
&\qquad + \frac{\zeta^2}{W}(\lambda_S^0 + \lambda_Z^0\frac{\beta}{\alpha-\beta})\frac{\beta}{(\rho+\alpha+\beta)(\alpha-\beta)} \\
&\qquad - \frac{\zeta^2}{W}\lambda_Z^0\frac{\beta^2}{(\alpha-\beta)^2}\frac{1}{\rho+2\alpha}\Bigg) \\
&\quad + \frac{e^{-\beta t}}{\alpha-\beta}\Bigg(\beta S_m - \zeta(\widetilde{x} - \zeta c_s/W) \\
&\qquad - \frac{\zeta^2}{W}(\lambda_S^0 + \lambda_Z^0\frac{\beta}{\alpha-\beta})\frac{\beta}{\rho+2\beta} \\
&\qquad + \frac{\zeta^2}{W}\lambda_Z^0\frac{\beta}{\alpha-\beta}\frac{\beta}{\rho+\alpha+\beta}\Bigg) \\
&\quad + \frac{\zeta^2}{W}(\lambda_S^0 + \lambda_Z^0\frac{\beta}{\alpha-\beta})\frac{\beta}{(\rho+2\beta)(\rho+\alpha+\beta)}e^{(\rho+\beta)t} \\
&\quad - \frac{\zeta^2}{W}\lambda_Z^0\frac{\beta}{\alpha-\beta}\frac{\beta}{(\rho+\alpha+\beta)(\rho+2\alpha)}e^{(\rho+\alpha)t} . \qquad \text{(E.5.2)}\\
&=\; \overline{Z}e^{-\alpha t} + \frac{\beta S_m}{\alpha-\beta}\left(e^{-\beta t} - e^{-\alpha t}\right) + \frac{\zeta}{\alpha}(\widetilde{x} - \frac{\zeta c_s}{W})\left(1 + \frac{\beta}{\alpha-\beta}e^{-\alpha t} - \frac{\alpha}{\alpha-\beta}e^{-\beta t}\right) \\
&\quad + \frac{\zeta^2}{W}\lambda_S^0\frac{\beta}{\alpha-\beta}\frac{1}{\rho+\alpha+\beta}\left(e^{-\alpha t} - \frac{\rho+\alpha+\beta}{\rho+2\beta}e^{-\beta t} + \frac{\alpha-\beta}{\rho+2\beta}e^{(\rho+\beta)t}\right) \\
&\quad + \frac{\zeta^2}{W}\lambda_Z^0\frac{\beta^2}{(\alpha-\beta)^2}\frac{1}{\rho+\alpha+\beta}\Bigg(e^{-\alpha t} - \frac{\rho+\alpha+\beta}{\rho+2\alpha}e^{-\alpha t} + e^{-\beta t} - \frac{\rho+\alpha+\beta}{\rho+2\beta}e^{-\beta t} \\
&\qquad - \frac{\alpha-\beta}{\rho+2\alpha}e^{(\rho+\alpha)t} + \frac{\alpha-\beta}{\rho+2\beta}e^{(\rho+\beta)t}\Bigg) . \qquad \text{(E.5.3)}
\end{aligned}$$

E.6 Numerical Example

Figure E.1 represents the value function (left) and the optimal consumption x (right) in a parameter configuration "c_s small". The origin value $(S, Z) = (0, 0)$ is placed at the back of the figure, so that the behavior of the function at the boundary $Z = \widetilde{Z}(S)$ becomes more visible.

The values given to the model parameters are as follows:

$$\overline{Z} = 1, \quad \alpha = 1, \quad \beta = 1/2, \quad \rho = 5, \quad \xi = 1/2, \quad c_x = 1, \quad c_y = 9, \quad c_s = 1. \tag{E.6.1}$$

The utility function $u(\cdot)$ is given as in (E.1.2) with $\overline{p} = 8$ and $W = 1$. For these values, we have $\hat{c}_s \simeq 12.73$. We are therefore indeed in the "c_s small situation". Other special values are: $S_m = 2$, $S_M = 4$, $\overline{x} = 2$ and $\widetilde{y} = 1$.

Trajectories starting with a large value of S run from left to right, then (for some of them) experience the change of phase, from Phase A to Phase B. At this point, a sharp decrease occurs, both for the value and for the consumption. The trajectory eventually approaches the limit of the domain $Z = \widetilde{Z}(S)$. There, consumption drops to 0. The optimal trajectory then stays close to the boundary until it reaches the terminal state $(S_m, \overline{Z}) = (2, 1)$.

Trajectories starting with a small initial S stay in Phase A with a relatively constant consumption until the ceiling $Z = \overline{Z}$ is hit. They then follow the ceiling, but eventually leave it (the location is approximately $(S, Z) = (1.1577, 1)$ to enter the loop described in Section 4.4.3.2. Along this loop, the value and the consumption first sharply decrease, then increase again.

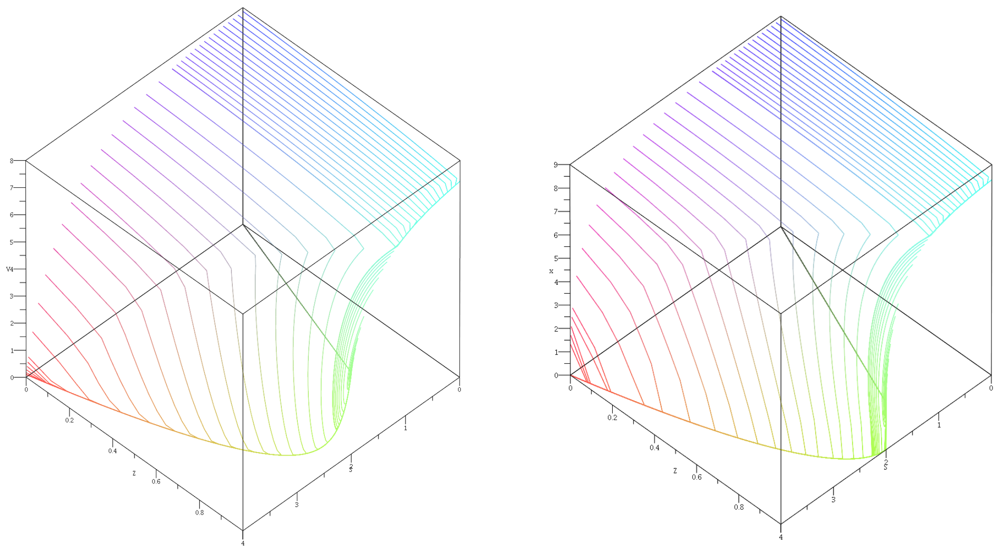

Figure E.1: Value function (left) and (right) for parameter values as in (E.6.1)

Bibliography

Amigues, J., Moreaux, M. & Schubert, K. (2011), 'Optimal use of a polluting non renewable resource generating both manageble and catastrophic damages', *Annals of Economics and Statistics* **103-104**, 107–141.

Augeraud-Veron, E. & Leandri, M. (2013), Optimal pollution control with distribued delays, Technical Report 2013-03, Cahier du GRETha.

Chakravorty, U., Magné, B. & Moreaux, M. (2006), 'A Hotelling model with a ceiling on the stock of pollution', *Journal of Economic Dynamics and Control* **30**, 2875–2904.

Ha-Duong, M. & Keith, D. W. (2003), 'Carbon storage: the economic efficiency of storing CO_2 in leaky reservoirs', *Clean. Techn. Environ. Policy* **5**, 181–189.

Lafforgue, G., Magné, B. & Moreaux, M. (2008*a*), 'Energy substitutions, climate change and carbon sinks', *Ecol. Econ.* **67**, 589–597.

Lafforgue, G., Magné, B. & Moreaux, M. (2008*b*), The optimal sequestration policy with a ceiling on the stock of carbon in the atmosphere, *in* R. Guesnerie & H. Tulkens, eds, 'The Design of Climate Policy', The MIT Press, Boston, pp. 273–304.

Livernois, J. & Martin, P. (2001), 'Price, Scarcity Rent, and a Modified r-Percent Rule for Nonrenewable Resources', *Can. J. Economics* **34**(3), 827–845.

Lontzek, T. S. & Rickels, W. (2008), Carbon Capture and Storage & the Optimal Path of the Carbon Tax, Technical Report 1475, Kiel Institute for the World Economy.

Rickels, W. & Lontzek, T. (2012), 'Optimal global carbon management with ocean sequestration', *Oxford Economic Papers* **64**(3), 323–349.

Seierstad, A. & Sydsæter, K. (1977), 'Sufficient conditions in optimal control theory', *International Economic Review* **18**(2), 367–391.

Seierstad, A. & Sydsæter, K. (1987), *Optimal Control Theory with Economic Applications*, North Holland.

Sethi, S. P. & Thompson, G. L. (2000), *Optimal Control Theory*, second edn, Springer.

Soner, H. M. (1986), 'Optimal control with state-space constraints I', *SIAM J. Control and Optimization* **24**(3), 552–561.

Van der Zwaan, B. (2005), 'Will coal depart or will it continue to dominate global power production during the 21st century?', *Climate Policy* **5**, 445–453.

Van der Zwaan, B. & Gerlagh, R. (2009), 'Economics of geological CO_2 storage and leakage', *Climate Change* **93**, 285–309.

Van der Zwaan, B. & Smekens, K. (2009), 'CO_2 capture and storage with leakage in an energy-climate model', *Environ. Model Assess.* **24**, 135–148.

Contents