Heterogeneous Beliefs and Prediction Market Accuracy*

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Keywords: Prediction market, heterogeneous beliefs, risk aversion, favorite-longshot bias, complete markets, and asset prices.

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Abstract

We consider a prediction market in which traders have heterogeneous prior beliefs in probabilities. In the two-state case, we derive necessary and sufficient conditions so that the prediction market is accurate in the sense that the equilibrium state price equals the mean probabilities of traders' beliefs. We also provide a necessary and sufficient condition for the well documented favorite-longshot bias. In an extension to many states, we revisit the results of Varian (1985) on the relationship between equilibrium state price and belief heterogeneity.

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1 Introduction

In 1906, Francis Galton attended in Plymouth a prediction contest about the weight of an ox. About 800 people participated, and Galton was surprised by how much they were accurate at predicting the correct weight. He concluded that the result is "*more creditable to the trustworthiness of a democratic judgment that might have been expected*" (Galton 1907, p. 451). Since then, the measurement of popular beliefs has become increasingly common both in academia and in practice of prediction. In particular, prediction markets are now considered as one of the most efficient tool to elicit people's beliefs (Hahn and Tetlock 2006, Surowiecki 2005). They have been repeatedly used to predict the outcome of political elections, like with the Iowa electronic market (www.biz.uiowa.edu/iem/). They are also increasingly used by private companies like Microsoft, Google and Chevron for instance to elicit their employees' beliefs about future sales or industry trends. Given the development of prediction markets, it is thus important to better understand when prediction markets are expected to be accurate at predicting future events.

Technically, prediction markets are simple financial markets in which traders bet on the outcomes of uncertain events. But the main purpose of prediction markets is to predict future events, not to share risks. With that purpose, asset prices in prediction markets are typically interpreted as probabilities. For instance, Arrow et al. (2008) introduce prediction markets as follows: *"Consider a contract that pays \$1 if Candidate X wins the presidential election in 2008. If the market price of an X contract is currently 53 cents, an interpretation is that the market 'believes' X has a 53% chance of winning*" (Arrow et al. 2008: 877).[1]

In this paper, we examine theoretically this interpretation. More precisely, we consider a simple prediction market in which traders have heterogeneous beliefs in probabilities of different states. We then derive conditions so that the prediction market is accurate in the sense that the equilibrium

[1]This interpretation is consistent with the information displayed on one of the leading prediction market, Intrade. Indeed, each prediction market on Intrade is based on whether an event occurs or not (coined "YES/NO proposition"), and for each market the current market price is provided together with the "% chance" that the event occurs. The explanation provided online on Intrade reads as follows: "*The market prices of shares also indicate the probability of the event happening. For example, a market price of \$3.63 indicates a 36.3% probability the event will actually happen, according to the market. In other words, the market is predicting a 36.3% probability*", taken from *www.intrade.com* (April, 12, 2012).

state price equals the mean probabilities of traders' beliefs.[2] Manski (2006) presents a first formal analysis of this problem using a model with risk neutral traders and assuming limited investment budgets. He shows that the equilibrium price can largely differ from the mean beliefs of traders. Wolfers and Zitzewitz (2006) consider a more standard model with risk averse traders and show theoretically that the prediction market is accurate when the utility function of traders is logarithmic. Moreover Wolfers and Zitzewitz explore numerically how the equilibrium price is affected by belief heterogeneity for several utility functions and several beliefs distributions.[3]

Our main contribution in this paper is to derive the exact necessary and sufficient conditions for prediction market accuracy for general utility functions and for general distributions of beliefs. Specifically, we show in the two-state case that the prediction market is accurate i) for all distributions of beliefs *if and only if* the utility function is logarithmic, and ii) for all strictly concave utility functions *if and only if* the distribution of beliefs is symmetric about one half. Moreover, we present several examples in which the (joint) distributions of traders' beliefs, wealth and risk preferences lead to a systematic violation of prediction market accuracy. Nevertheless, we provide indications about the direction of the bias. Most significantly, we exhibit the necessary and sufficient condition for the equilibrium price to be always below/above the mean beliefs for all symmetric beliefs distributions. This condition provides a rationale to the well documented favorite-longshot bias (Ali 1977). We also critically discuss some previous results obtained in a generalized complete market setting with heterogeneous beliefs. In particular, we revisit Varian (1985)'s early results that, first, the equilibrium state price only depends on the distribution of beliefs about that state and, second, that more heterogeneity in beliefs should lead to decrease asset prices when relative risk aversion is high enough. We show with the help of an example that the first result is not correct, and argue that the second result is based on an inappropriate comparative statics analysis.

[2]We realize that the word "accurate" may not be appropriate here. Typically, traders' beliefs may be biased, and inferring the mean of their beliefs may not be enough to predict accurately an uncertain event. We use this word mostly for simplicity, and also because it compactly refers to discussions about the empirical success of prediction markets (see, e.g., Forsythe et al. 1992 and Hanson 2006).

[3]See Gjerstad (2004) for theoretical results under constant relative risk aversion (CRRA) utility functions, and some numerical results. See also Fountain and Harrison (2011) for further numerical results with wealth and beliefs heterogeneity.

We organize the paper as follows. In the next section we introduce a simple binary prediction market model and derive a sufficient condition for the equilibrium state price to be unique. In the next two sections we derive necessary and sufficient conditions for prediction market accuracy. More precisely, Section 3 derives a condition on the utility function, and Section 4 derives a condition on the probability distribution representing the beliefs of traders. Then in Section 5 we examine the conditions leading to the favorite-longshot bias. Finally in Section 6 we study the generalization of previous results to more than two states, and discuss the link with Varian (1985). The last section concludes.

2 The model

We first consider a simple binary prediction market in which risk averse agents can buy and sell a financial asset paying \$1 if a specific event occurs, and nothing otherwise. The main assumption of the model is that the beliefs of the agents about the occurrence of the specific event are heterogeneous. We thus consider a model in which agents "agree to disagree", and therefore have different prior beliefs. Namely, the heterogeneity in beliefs does not come from asymmetric information but rather from intrinsic differences in how agents interpret information.[4] In this section, we derive some properties of the individual asset demand, and then of the equilibrium price in this specific model.

2.1 Individual asset demand

In our model, each agent maximizes his expected utility based on his own beliefs. Formally, when he decides how much to invest in the financial asset paying \$1 if the event occurs, he maximizes over α the following expected utility

$$pu(w + \alpha(1 - \pi)) + (1 - p)u(w - \alpha\pi), \tag{1}$$

in which w is the agent's initial wealth, $p \in (0, 1)$ his subjective probability that the event occurs (i.e., his belief), α his asset demand and π the price

[4] Many models in finance have considered agents with heterogeneous prior beliefs (see, e.g., Varian 1985, Abel 1989, Jouini and Napp 2006 and 2007, Gollier 2007 and Roche 2011). For a justification and implications of these models, see for instance the survey papers by Varian (1989), Scheinkman and Xiong (2004) and Hong and Stein (2007).

of this asset.[5] We assume the utility function $u(\cdot)$ to be strictly increasing, strictly concave and three times differentiable.

The first order condition of this optimization program is given by

$$p(1-\pi)u'(w+\alpha(p,\pi)(1-\pi)) - (1-p)\pi u'(w-\alpha(p,\pi)\pi) = 0, \tag{2}$$

in which $\alpha(p,\pi)$ is the unique solution. Differentiating with respect to p the last equality, we obtain

$$\begin{aligned} 0 = \quad & (1-\pi)u'(w+\alpha(p,\pi)(1-\pi)) + \pi u'(w-\alpha(p,\pi)\pi) \\ & +\alpha_p(p,\pi)\{p(1-\pi)^2 u''(w+\alpha(p,\pi)(1-\pi)) \\ & \qquad\qquad +(1-p)\pi^2 u''(w-\alpha(p,\pi)\pi)\}, \end{aligned} \tag{3}$$

and rearranging we have

$$\alpha_p(p,\pi) = \frac{(1-\pi)u'(w+\alpha(p,\pi)(1-\pi)) + \pi u'(w-\alpha(p,\pi)\pi)}{-p(1-\pi)^2 u''(w+\alpha(p,\pi)(1-\pi)) - (1-p)\pi^2 u''(w-\alpha(p,\pi)\pi)} > 0, \tag{4}$$

that is, the asset demand increases with belief p. Since $\alpha(p,p) = 0$, we conclude that $\alpha(p,\pi) \geq 0$ if and only if $p \geq \pi$. Namely, the agent buys (respectively sells) the asset yielding \$1 when the event occurs if and only if he assigns a probability for this event higher (respectively lower) than the asset price.

2.2 The equilibrium

Let $\widetilde{p}$ be the random variable representing the distribution of beliefs in the population of agents, and let π^* be the equilibrium price. The equilibrium condition is

$$E\alpha(\widetilde{p},\pi^*) = 0, \tag{5}$$

in which E denotes the expectation operator with respect to $\widetilde{p}$. Our main objective in the paper is to compare π^* to $E\widetilde{p}$. In particular, in Sections 3 and 4 we will derive conditions so that there is prediction market accuracy defined

[5] The individual asset demand α can be seen as the net asset demand of one asset in a model with two Arrow-Debreu assets. To see that, let α_s and π_s denote respectively the demand for and the price of Arrow-Debreu assets in state $s = 1,2$. The objective can then be written: $\max_{\alpha_1,\alpha_2}[pu(w+\alpha_1-\pi_1\alpha_1-\pi_2\alpha_2)+(1-p)u(w+\alpha_2-\pi_1\alpha_1-\pi_2\alpha_2)]$. Denoting $\alpha = \alpha_1 - \alpha_2$ and observing that $\pi_1 + \pi_2 = 1$ by arbitrage then leads (with $\pi = \pi_1$) to (1).

by $\pi^* = E\widetilde{p}$. Notice immediately that, when $\widetilde{p}$ is degenerate and is equal to p with probability 1, then $\pi^* = p$ and there is no trade at the equilibrium. This is a trivial case always leading to prediction market accuracy. We rule out this case (until Section 6), and consider nondegenerate $\widetilde{p}$ in the following.

It is easy to see that an equilibrium always exists in such a prediction market. Indeed, when π tends to 0 (respectively tends to 1) $\alpha(p, \pi)$ becomes positive (respectively negative) for all p, so its expectation over $\widetilde{p}$ also becomes positive (respectively negative). Therefore when π increases, the function $E\alpha(\widetilde{p}, \pi)$ must go from a positive to a negative region and thus must cross zero somewhere in between.

We now discuss the uniqueness of the equilibrium, that is we study whether $E\alpha(\widetilde{p}, \pi)$ only crosses the origin once. We know that $\alpha(p, \pi)$ has this single crossing property at $\pi = p$. But that does not guarantee that $E\alpha(\widetilde{p}, \pi)$ also has the single crossing property, as illustrated by the following example.

Example 1 (Multiple equilibria): Consider agents with a quadratic utility function $u(w) = -(1-w)^2$ and initial wealth $w = 1/2$. The optimal asset demand is equal to $\alpha(p, \pi) = \frac{p-\pi}{2(p-2p\pi+\pi^2)}$. In a prediction market with only two agents with respective beliefs denoted $p_1 = 0.1$ and $p_2 = 0.9$, the equilibrium condition is equivalent to $9 - 68\pi + 150\pi^2 - 100\pi^3 = 0$. Solving for this equation, it is found that there are three equilibrium prices in this prediction market: $\pi^* = (0.235, 0.5, 0.764)$.

A sufficient condition for the uniqueness of the equilibrium however is $\alpha_\pi(p, \pi) < 0$ everywhere. Indeed, this implies that the function $E\alpha(\widetilde{p}, \pi)$ is strictly decreasing in π, and therefore crosses zero at most once. Differentiating (2) with respect to π, we have

$$\alpha_\pi(p,\pi) = \frac{-pu'(w+\alpha(p,\pi)(1-\pi)) - (1-p)u'(w-\alpha(p,\pi)\pi)}{-p(1-\pi)^2u''(w+\alpha(p,\pi)(1-\pi)) - (1-p)\pi^2u''(w-\alpha(p,\pi)\pi)} - \alpha(p,\pi)\frac{p(1-\pi)u''(w+\alpha(p,\pi)(1-\pi)) - (1-p)\pi u''(w-\alpha(p,\pi)\pi)}{-p(1-\pi)^2u''(w+\alpha(p,\pi)(1-\pi)) - (1-p)\pi^2u''(w-\alpha(p,\pi)\pi)}. \tag{6}$$

The first term is strictly negative but the second term is of ambiguous sign under risk aversion, so that the demand may increase when the price π increases, as it is the case in Example 1. We now provide a sufficient condition for uniqueness by ensuring that the second term is also negative. We show that this is the case under nonincreasing absolute risk aversion.

Proposition 1 *The equilibrium price π^* is unique if u has nonincreasing absolute risk aversion.*

Proof: We are done if we can show that the second term of the right hand side in (6) is negative. As this is simple to show, we only provide a sketch of the proof. Let $\widetilde{x} = (1-\pi, -\pi; p)$ denote a random variable $\widetilde{x}$ which takes values of $1-\pi$ and $-\pi$ with probabilities p and $1-p$, respectively. Then the first order condition (2) can be written more compactly $E[\widetilde{x}u'(w + \alpha\widetilde{x})] = 0$. We thus are done if we can show that this last equality implies $-\alpha E[\widetilde{x}u''(w+\alpha\widetilde{x})] \leq 0$, that is, the second term of the right hand side in (6) is negative. Then it is direct to see that this implication means that $-u'$ is more risk averse than u, which is equivalent to nonincreasing absolute risk aversion. ■

The intuition is the following. When the price of an asset increases, there are two effects captured by the two terms of the right hand side of equation (6). First, there is a substitution effect that leads to a decrease in its demand, but there is also a wealth effect that may potentially increase its demand. Intuitively, as the terminal wealth distribution deteriorates, the investor's attitude towards risk may change, and this wealth effect might prove sufficiently strong to increase the demand for the risky asset, as initially shown by Fishburn and Porter (1976) in the case of a first-order stochastic dominance (FSD) shift. Under decreasing absolute risk aversion (DARA) however, the negative wealth effect leads the agent to be more risk averse, and therefore further decreases the demand for the risky asset. Under constant absolute risk aversion (CARA), there is no wealth effect, and only the first negative effect is at play. Finally, we note that Example 1 features multiple equilibria because the quadratic utility function has increasing absolute risk aversion.

When there is a unique equilibrium, one can make a simple comment on the effect of a change in the distribution of beliefs on the equilibrium price. Indeed, from the equilibrium condition $E\alpha(\widetilde{p}, \pi^*) = 0$ and $\alpha_p(p, \pi) > 0$, any FSD improvement in the distribution of beliefs must increase the equilibrium price.

3 Which utility functions lead to prediction market accuracy?

Assuming a logarithmic utility function $u(w) = \log w$ (which displays DARA) we can obtain a closed-form solution of the first order condition (2):

$$\alpha(p, \pi) = w\frac{(p-\pi)}{\pi(1-\pi)}.$$

This implies that the equilibrium condition (5) can simply be written $\pi^* = E\widetilde{p}$. This shows that the logarithmic utility function is sufficient for prediction market accuracy (Gjerstad 2004; Wolfers and Zitzewitz 2006). A natural question is whether the utility function must be logarithmic to guarantee prediction market accuracy or whether this is possible for other utility functions, i.e. whether $u(w) = \log w$ is also a necessary condition. We show in the following Proposition that this is indeed the case.

Proposition 2 *For all $\widetilde{p}$, $\pi^* = E\widetilde{p}$ if and only if $u(w) = \log w$.*

Proof: We just need to prove the necessity. Namely, let $\overline{p} = E\widetilde{p}$, we must show that $E\alpha(\widetilde{p}, \overline{p}) = 0$ for all $\widetilde{p}$ implies $u(w) = \log w$. We are done if we can show that this implication holds for a specific class of probability distribution in $\widetilde{p}$. We consider the class of "small" risk, that is we assume that $\widetilde{p}$ is close enough to $\overline{p}$ in the sense of a second-order approximation: $E\alpha(\widetilde{p}, \pi) = \alpha(\overline{p}, \pi) + 0.5E(\widetilde{p} - \overline{p})^2\alpha_{pp}(\overline{p}, \pi)$. Using this last equality, the necessary condition $E\alpha(\widetilde{p}, \overline{p}) = 0$ implies $\alpha(\overline{p}, \overline{p}) + 0.5E(\widetilde{p} - \overline{p})^2\alpha_{pp}(\overline{p}, \overline{p}) = 0$. Since for all p, we have $\alpha(p, p) = 0$, the necessary condition becomes $\alpha_{pp}(p, p) = 0$. Differentiating again (3) with respect to p to compute $\alpha_{pp}(p, p)$ we obtain

$$\begin{aligned} 0 &= 2\alpha_p(p,\pi)\{(1-\pi)^2u''(w+\alpha(p,\pi)(1-\pi)) - \pi^2u''(w-\alpha(p,\pi)\pi)\} \\ &+ \alpha_{pp}(p,\pi)\{p(1-\pi)^2u''(w+\alpha(p,\pi)(1-\pi)) + (1-p)\pi^2u''(w-\alpha(p,\pi)\pi)\} \\ &+ \alpha_p(p,\pi)^2\{p(1-\pi)^3u'''(w+\alpha(p,\pi)(1-\pi)) - (1-p)\pi^3u'''(w-\alpha(p,\pi)\pi)\}. \end{aligned}$$

Taking $\pi = p$ in the last expression, we have $\alpha_p(p, p) = \frac{1}{p(1-p)} \times \frac{u'(w)}{-u''(w)}$ from (4), then rearranging yields

$$\alpha_{pp}(p,p) = \frac{(1-2p)}{p^2(1-p)^2}\left[\frac{u'(w)}{u''(w)}\right]^2\left[\frac{u'''(w)}{-u''(w)} - 2\frac{-u''(w)}{u'(w)}\right]. \tag{7}$$

Therefore a necessary condition is $\frac{u'''(w)}{-u''(w)} = 2\frac{-u''(w)}{u'(w)}$. Finally, integrating this differential equation gives $u(w) = \log w$. ■

We complement this result with three remarks about its limitation in more general settings.

Remark 1 (Wealth heterogeneity): The result of Proposition 2 cannot be generalized to non-identical wealth, as possible correlation between wealth and beliefs would invalidate the result. Indeed, let $\widetilde{w}$ be the random variable representing wealth heterogeneity. Assuming a logarithmic utility function, we can obtain

$$\pi^* = E\widetilde{p} + \frac{1}{E_{\widetilde{w}}\widetilde{w}} Cov(\widetilde{p}, \widetilde{w}). \tag{8}$$

Therefore there is no utility function that can always ensure prediction market accuracy when beliefs and wealth are potentially correlated. Observe that, despite this impossibility result, the direction of the bias can be inferred if the analyst knows the sign of the correlation between beliefs and wealth. The intuition for equilibrium condition (8) is that richer individuals invest more, and therefore have more influence on the equilibrium price. Thus, if wealth is positively (respectively negatively) correlated with beliefs, the equilibrium price will be higher (respectively lower).

Remark 2 (Stakes): Suppose each agent has a (positive or negative) stake Δ in the event he predicts, so that he now maximizes over α the following expected utility

$$pu(w + \Delta + \alpha(1 - \pi)) + (1 - p)u(w - \alpha\pi).$$

Then it is easy to understand that the result of Proposition 2 is not guaranteed either. Indeed for the logarithmic utility function we have

$$\alpha(p, \pi) = w\frac{(p - \pi)}{\pi(1 - \pi)} - \Delta\frac{\pi(1 - p)}{\pi(1 - \pi)},$$

leading to the equilibrium condition

$$\pi^* = \frac{wE\widetilde{p}}{w + \Delta(1 - E\widetilde{p})}.$$

The intuition is that when there is a positive (respectively negative) stake, the marginal utility decreases (respectively increases) if the event occurs. As a result, the agents want to transfer wealth to the state in which the event does not occur (respectively occurs), and they typically use the prediction market as a hedging scheme to do this. The consequence is that the equilibrium is biased downward (respectively upward). Observe that if the stakes are individual-dependent but uncorrelated with beliefs, and if their mean across individuals is equal to zero, then we retrieve prediction market accuracy under a logarithmic utility function.

Remark 3 (Quantities): Suppose that each agent can now bet on a random quantity payoff $\widetilde{t}$. Formally, he chooses the amount α to maximize

$$E_{\widetilde{t}}u(w+\alpha(\widetilde{t}-\pi)),$$

where $E_{\widetilde{t}}$ denotes his expectation over $\widetilde{t}$. Note that we retrieve the previous prediction market model when $\widetilde{t}=(1,0;p)$, implying that we would retrieve prediction market accuracy for $u(w)=\log w$ in this special case. Nevertheless prediction market accuracy fails when $\widetilde{t}$ is not a binary even for a logarithmic utility, as the following example shows. Suppose that $\widetilde{t}$ can take three values, $1,2$ or 3. Let two agents with beliefs over $\widetilde{t}$ be described respectively by the two following random variables $(1,2,3;1/3,1/3,1/3)$ and $(1,2,3;1/6,1/6,2/3)$. That is the first agent believes equal probabilities for the three values and the second agent believes the probabilities of $1/6,1/6$ and $2/3$ for values of $1,2$ and 3, respectively. This implies that the first agent believes that the mean is 2 and the second agent believes that the mean is 2.5. Then solving for the equilibrium price assuming $u(w)=\log w$ and $w=1$ we find $\pi^*=2.227$, which is different from the mean beliefs of 2.25 across the two agents.

4 Which distributions of beliefs lead to prediction market accuracy?

The previous section provides the conditions on the utility function so that there is market prediction accuracy for all $\widetilde{p}$. In this section, we study the dual problem: which conditions on $\widetilde{p}$ ensure prediction market accuracy for all u? We show that the necessary and sufficient condition is that the probability distribution of beliefs is symmetric about one half.

Proposition 3 *Assume that the equilibrium price π^* is unique. For all u, $\pi^* = E\widetilde{p}$ if and only if $\widetilde{p}$ is symmetric about $1/2$.*

Proof: We first show that if $\widetilde{p}$ is symmetric about $1/2$ then $E\widetilde{p} = \pi^*$ for all u. Observe from the first order condition (2) that $\alpha(p, \pi) = -\alpha(1-p, 1-\pi)$. This implies that the equilibrium condition can be written $E\alpha(\widetilde{p}, \pi^*) = E\alpha(1-\widetilde{p}, 1-\pi^*) = 0$. Observe then that $\widetilde{p}$ symmetric about $1/2$ means that $\widetilde{p}$ is distributed as $1-\widetilde{p}$. Consequently the equilibrium condition implies $E\alpha(\widetilde{p}, \pi^*) = E\alpha(\widetilde{p}, 1-\pi^*)$. Since the equilibrium is assumed to be unique, this last condition implies $\pi^* = 1-\pi^*$, that is $\pi^* = 1/2 = E\widetilde{p}$.

We now demonstrate that if $E\widetilde{p} = \pi^*$ for all u then it must be that the distribution is symmetric about $1/2$. This is proved by contradiction. Consider the following example. Let $u(w) = -e^{-rw}$ with CARA coefficient $r > 0$ and the probability density of beliefs $\tilde{p}$ is given by

$$f(p) = \begin{cases} \frac{2p}{b}, & \text{for } 0 < p \leq b, \\ \frac{2(1-p)}{1-b}, & \text{for } b < p < 1. \end{cases}$$

Obviously, for $b \neq 1/2$, the belief distribution is not symmetric about $1/2$. It can be verified that $E\widetilde{p} > \pi^*$ for $0 < b < 1/2$ and $E\widetilde{p} < \pi^*$ for $1/2 < b < 1$. ■

The intuition for Proposition 3 is simple. When $\widetilde{p}$ is symmetric about one half, the two states are formally indistinguishable, and therefore it cannot be that the price of an asset yielding one dollar in one state is different from that of an asset yielding one dollar in the other state, implying $\pi^* = 1/2$. We note, however, that if heterogeneity in individual utility functions is introduced, prediction market accuracy may not hold anymore even under $\widetilde{p}$ symmetric about $1/2$. The intuition is essentially the same as the one presented in Remark 1. This is illustrated by the following example which considers heterogeneity over (constant absolute) risk aversion.

Example 2 (Heterogeneous CARA): Let $u_i(w) = -e^{-r_i w}$ in which $r_i > 0$ represents the CARA coefficient of agent $i = 1, 2$ with respective beliefs $p_1 = 0.1$ and $p_2 = 0.9$. Under positive correlation between beliefs and risk aversion $(r_1, r_2) = (1, 3)$, we have $\pi^* = 1/4 < 1/2 = E\widetilde{p}$, while under negative correlation $(r_1, r_2) = (3, 1)$, we have $\pi^* = 3/4 > 1/2 = E\widetilde{p}$.

We have characterized the necessary and sufficient conditions for prediction market accuracy, for all $\widetilde{p}$ in Proposition 2, and then for all u in

Proposition 3. These conditions are rather stringent. However, one can relax these conditions in the sense that it is possible to find well-chosen pairs $(u, \widetilde{p})$ also yielding prediction market accuracy. This is shown in the following example which uses a specific constant relative risk aversion (CRRA) utility function and a specific nonsymmetric distribution of beliefs.

Example 3 (Prediction market accuracy under CRRA and nonsymmetric beliefs). Consider agents with utility function $u(w) = -1/w$. Two groups of agents participate in the prediction market: one group has beliefs $p_1 = p$, and the other group has beliefs $p_2 = 1 - p$. Denoting a the proportion of agents in the first group, we have $E\widetilde{p} = ap + (1-a)(1-p)$. One may then easily obtain that $E\alpha(\tilde{p}, \pi) = 0$ implies $\sqrt{\pi(1-\pi)}\{ap + (1-a)(1-p) - \pi\} = 0$ leading to $\pi^* = E\widetilde{p}$.

Examples 2 and 3 indicate that symmetric beliefs about 1/2 is neither necessary nor sufficient for prediction market accuracy for general utility function.

5 A necessary and sufficient condition for the favorite-longshot bias

In the previous analysis, we have examined under which conditions the prediction market is accurate in the sense that the equilibrium price π^* is equal to mean belief $E\widetilde{p}$. In this section, we derive necessary and sufficient conditions for π^* to be systematically above or below $E\widetilde{p}$.

The analysis developed in this section may provide a rationale for the favorite-longshot bias, namely for the empirical observation that longshots tend to be over-valued and that favorites tend to be under-valued (Ali 1977; Thaler and Ziemba 1988). More explicitly, consider a horse race with only two horses, and call the first horse the favorite (resp. longshot) if the mean beliefs that this horse wins are such that $E\widetilde{p} \geq 1/2$ (resp. $E\widetilde{p} \leq 1/2$). As we will see, the necessary and sufficient condition so that this horse is under-valued, i.e. $\pi^* \leq E\widetilde{p}$, critically depends on whether it is a favorite or a longshot. This result is presented in the following Proposition 4 in which $P(w) = -u'''(w)/u''(w)$ denotes the coefficient of absolute prudence (Kimball 1990) and $A(w) = -u''(w)/u'(w)$ denotes the coefficient of absolute risk aversion.

Proposition 4 *Assume that the equilibrium price π^* is unique. Then for all symmetric $\widetilde{p}$, $\pi^* \geq E\widetilde{p}$ if and only if $(1/2 - E\widetilde{p})(P(w) - 2A(w)) \geq 0$ for all w.*

Proof: See Appendix A. ■

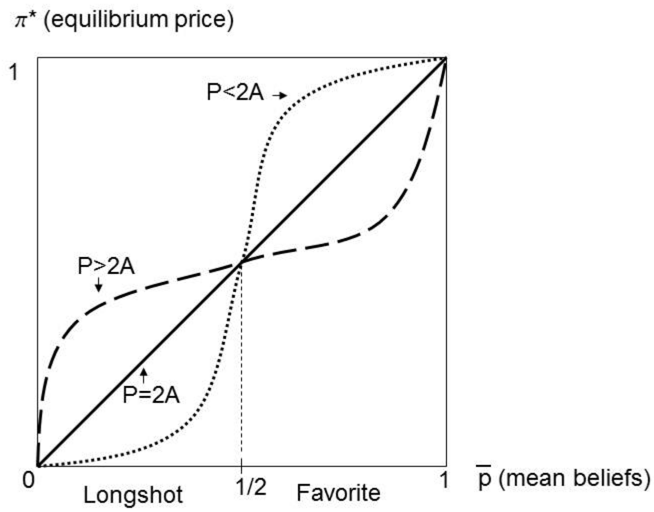
The difference between the mean belief and the equilibrium price therefore depends on whether the mean belief is less than 1/2, and on whether absolute prudence is greater than twice absolute risk aversion. The sign of $P - 2A$ is a familiar condition on utility functions derived from comparative statics analysis within expected utility models (Gollier 2001). Under CRRA utility functions, $P \leq 2A$ is equivalent to a parameter of constant relative risk aversion greater than 1. Notice also that DARA is equivalent to that P is larger than A.

The result presented in Proposition 4 is illustrated in Figure 1. The horizontal axis represents the mean belief and the vertical axis represents the equilibrium price. The diagonal therefore represents prediction market accuracy, which holds everywhere if and only if $P = 2A$ (i.e., u is logarithmic). The result therefore shows that there is a favorite-longshot bias if and only if the utility function displays $P > 2A$.

Observe that the result in Proposition 4 is consistent with Propositions 2 and 3. Indeed, this result shows that there is prediction market accuracy under two extreme and separate conditions on the utility functions and the distribution of beliefs: either as in Proposition 2 when the utility is logarithmic ($P = 2A$) or as in Proposition 3 when mean beliefs equal one half and are symmetric. This result also generalizes the theoretical results of Gjerstad (2004) obtained for CRRA utility functions and symmetric beliefs, and provides a theoretical foundation for the numerical simulations presented in Wolfers and Zitzewitz (2006) for various utility functions and distributions of beliefs.

One may wonder whether the condition $(1/2-E\widetilde{p})(P(w)-2A(w)) \geq 0$ is also necessary and sufficient for all distributions $\widetilde{p}$, not only symmetric ones. To see this, note first that $\pi^* \geq E\widetilde{p}$ is equivalent to $E\alpha(\widetilde{p}, \overline{p}) \geq 0$, and since $\alpha(\overline{p}, \overline{p}) = 0$, by Jensen's inequality the necessary and sufficient condition for all $\widetilde{p}$ is simply given by $\alpha_{pp}(p, \overline{p}) \geq 0$ for all p and $\overline{p}$. The computation of $\alpha_{pp}(p, p)$ in (7) shows that the condition $(1/2 - E\widetilde{p})(P(w) - 2A(w)) \geq 0$ is

Figure 1: This figure plots the equilibrium price π^* as a function of mean beliefs $\overline{p}$. Under symmetric beliefs, there is a "favorite-longshot bias" for the class of utility functions u for which absolute prudence $P(w)$ is greater than twice absolute risk aversion $A(w)$.



indeed necessary for the favorite-longshot bias. However this condition is not sufficient, as the following example shows.

Example 4 (Nonsymmetric beliefs): Consider two groups of agents with $u(w) = \sqrt{w}$ (i.e., $P > 2A$) and heterogeneous beliefs $p_1 = 0.1$ and $p_2 = 0.9$. When the proportion of agents with beliefs $p_1 = 0.1$ is 75% then $\pi^* = 0.272 < E\widetilde{p} = 0.3$ (i.e., the longshot is undervalued), and when the proportion of agents with beliefs $p_1 = 0.1$ is 25% then $\pi^* = 0.727 > E\widetilde{p} = 0.7$ (i.e., the favorite is overvalued).

We believe that the general results and selected numerical examples presented so far provide a fairly complete view of the properties of equilibrium prices of a prediction market in the two-state case. This is the case that has been usually considered in the handful of theoretical papers on prediction markets that we have found in the literature. We next extend our discussion to a prediction market with more than two states.

6 The $S-$state case

Until now, we have considered a prediction market with only 2 states. In this section we discuss prediction market accuracy for any finite number S of states with $S > 2$. We also discuss the effect of more heterogeneity in agents' beliefs over one state compared to another state. Note that this effect cannot be studied in the 2-state case. Consistent with previous notations, we denote by the vector $\mathbf{p}_i = (p_{i1}, ..., p_{iS})$ the agent i's beliefs over states $s = 1, ..., S$, and by π_s the equilibrium price of state s. Prediction market accuracy for state s therefore means $\pi_s = \frac{1}{N}\sum_{i=1}^{N} p_{is}$.

6.1 Generalization of previous results

As indicated in Section 3, a well-known result in the literature is that under a logarithmic utility function $u(w) = \log w$ there is prediction market accuracy in the $2-$state case (Gjerstad 2004; Wolfers and Zitzewitz 2006). It turns out that this sufficiency result can be generalized in the sense that all S state prices equal the mean of agents' beliefs for each state under $u(w) = \log w$.[6] Since we have shown in Section 3 that $u(w) = \log w$ is also necessary for prediction market accuracy in the $2-$state case, we can therefore state the following result[7].

Proposition 5 *(Generalization of Proposition 2) There is prediction market accuracy for all distributions of beliefs in the general $S-$state case if and only if $u(w) = \log w$.*

This result should not suggest, however, that previous results can be directly generalized to the $S-$state case. Indeed, we first show that a basic result derived in the $2-$state case is no longer valid in the general case. More precisely, we next show with the help of an example that, even if all agents

[6] This result is not inconsistent with the example in Remark 3 as the framework differs. Specifically, in our S-state prediction markets, markets are complete.

[7] All Propositions and statements in this section are proved in Appendix B.

have the homogeneous belief over one state, the equilibrium price of that state could nevertheless be different from its homogeneous belief.

Example 5 (Failure of prediction market accuracy under homogeneous beliefs): Let $N = 2$ and $S = 3$, and assume the following agents' beliefs: $\mathbf{p}_1 = (1 - 2p, p - \epsilon, p + \epsilon)$ and $\mathbf{p}_2 = (1 - 2p, p + \epsilon, p - \epsilon)$. Observe that the two agents have homogeneous beliefs $1 - 2p$ over state 1. Under CARA $u(w) = -e^{-rw}$ with $r > 0$, we have however $\pi_1 = \frac{1-2p}{1-2p+2\sqrt{p^2-\epsilon^2}} > 1 - 2p$ for any $\epsilon \neq 0$.

We now make another observation using the previous example. Start from the same numerical values for individual beliefs and utilities, but consider an alternative prediction market. Assume that there are only two Arrow-Debreu assets: an asset that pays \$1 if state 1 occurs, and another asset that pays \$1 if either state 2 or state 3 occurs. We have therefore a binary prediction market. But since the agents' beliefs over the two states $(1 - 2p, 2p)$ are now homogeneous, we obviously retrieve prediction market accuracy. Therefore, this simple observation shows that the equilibrium state price varies depending on the number of states on which it is possible to bet. In other words, this means that the "design" of the prediction market matters for equilibrium state prices.[8] This makes sense since the design may affect trading opportunities under heterogeneous beliefs. Following this observation, one may ask: when is an equilibrium state price always independent from the design of the prediction market? It can be easily shown that this is the case when all agents believe that all other $S - 1$ states are equally likely (see Proposition 6 below).

Example 5 also shows that the equilibrium price of one state depends on the distribution of beliefs in other states (through the parameter ϵ). Note that this is in contradiction with "FACT 1" in Varian (1985)'s early paper on the topic.[9] One may therefore ask: When does the equilibrium price of one

[8] To illustrate this interpretation, consider the horse race illustration. Example 5 and the last observation indicate that in a race with $S > 2$ horses, the equilibrium state price that one specific horse wins the race depends on whether it is possible to place bets separately on each of the other horses participating in the race.

[9] We explain here why this is a contradiction. Varian's result is obtained in a $S-$state complete market setting with heterogeneous beliefs and heterogeneous initial wealth. Our prediction market model in this section is thus a particular case of Varian's model since we assume a common initial wealth (thus removing any initial risk-sharing motivations

state depends only on the beliefs about that state? Interestingly the answer to this question is exactly the same as the one to the question asked above about the prediction market design. Indeed, if the equilibrium price of one state in a $S-$state prediction market is always equal to the equilibrium price of that state in a binary prediction market, this precisely means that the distribution of beliefs in all the other states is irrelevant for that equilibrium state price. We state this result in the following proposition.

Proposition 6 *For all u, the equilibrium price of one state, say $s = 1$, in a $S-$state prediction market only depends on the beliefs about that state if and only if $p_{is} = p_i$ for all $s = 2, ..., S$, and for all $i = 1, ..., N$ (i.e. if and only if states 2 to S are judged as equally likely by all agents).*

The previous Example 5 has shown that prediction market accuracy in a state may fail despite homogeneous beliefs in that state. The next example shows the dual result that prediction market accuracy may hold despite heterogeneous beliefs. Specifically, Example 6 identifies a case with symmetric beliefs where there is prediction market accuracy for *all* constant relative risk aversion (CRRA) utility functions with CRRA parameter $\gamma > 0$.

Example 6 (Prediction market accuracy under heterogeneous beliefs): Let $N = 2$ and $S = 3$, and assume the following agents' beliefs: $\mathbf{p}_1 = (1/2 + 1/4, 1/3 - 1/6, 1/6 - 1/12)$ and $\mathbf{p}_2 = (1/2 - 1/4, 1/3 + 1/6, 1/6 + 1/12)$. Then under $u(w) = w^{1-\gamma}/(1-\gamma)$ with $\gamma > 0$, we have prediction market accuracy for all states, i.e., $\pi_1 = 1/2, \pi_2 = 1/3$ and $\pi_3 = 1/6$.

In contrast, remember that Proposition 4 implies for symmetric beliefs of two-state and under CRRA utility functions that there is underpricing if and only if $\gamma > 1$, and thus that there is prediction market accuracy only in the knife-edge logarithmic case, i.e. $\gamma = 1$. This example illustrates that the prediction market accuracy in many state markets can be very different from 2-state markets.

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for trade). Therefore Varian's result should also hold in our simpler setting. Note that Ingersoll (1987, p. 214) also presents a similar result as that of Varian: "*The Arrow-Debreu price for an insurable state depends only on agregate wealth in that state and the pattern of beliefs about that state*". Technically, the problem arises because both Varian and Ingersoll "fix" the Lagrangian multiplier in their equilibrium conditions. This is misleading since that multiplier may also depend on the distribution of beliefs about other states. We will come back to a related problem in the next subsection.

6.2 The effect of more heterogeneity in beliefs

We finally discuss the effect of more heterogeneity in beliefs on equilibrium state prices. This question has been initially studied by Varian (1985) who identifies conditions under which the equilibrium prices of Arrow-Debreu assets in a complete market setting decrease when beliefs are more dispersed. Interestingly, Varian shows that the necessary and sufficient condition under a common utility function u is $A' \geq -A^2$, or equivalently $P \leq 2A$ using our previous notations. As we have indicated above, this condition always holds under CARA utility functions (which is equivalent to $P = A$). Varian considers this condition as plausible, which leads him to conclude that "*equilibrium asset prices should generally decrease with an increase in diversity of opinion*" (Varian 1985, p. 316). This result may be seen as intuitive since the heterogeneity in beliefs about an asset's payoff makes the asset appear more risky, and therefore should decrease its market value and increase the expected return for compensating that risk.

In the rest of the paper, we revisit this result by Varian. We first note that, as recognized by Varian himself, this result is not based on the comparison of two different equilibria. Indeed Varian compares the state prices within the *same* equilibrium.[10] In the following, we also consider this within-equilibrium comparison, but we are nevertheless concerned by the specific nature of the comparative statics analysis of more heterogeneity considered by Varian. Indeed Varian examines the effect of a mean-preserving spread (MPS) of "*weighted probabilities*" (Varian 1985, p. 314), i.e. a MPS of individual probabilities divided by individual marginal utilities. Although this specific notion of beliefs dispersion drastically simplifies the theoretical analysis,[11] we wonder about the economic meaning of this comparison. In particular, Varian's notion of more heterogenous beliefs is not solely based

[10] To illustrate this comparison, we use again the horse race illustration. Varian considers a race with S horses and compares the equilibrium price of two horses s and t within that race, given that the mean beliefs that horses s and t win are identical but the beliefs over horse s are more dispersed than those over horse t. Observe that such a comparison requires $S > 2$.

[11] Technically, this leads to "fixing" the Lagrangian multiplier in the equilibrium conditions. Several authors have used a similar approach (see, e.g., Ingersoll 1987, Gollier 2007 and Roche 2011).

on the beliefs of agents since it also depends on marginal utilities (which are determined endogenously). We thus argue that Varian's definition of more heterogeneity in beliefs is inappropriate from a conceptual point of view when one is concerned by the pure effect of the heterogeneity in beliefs.

The following final proposition revisits Varian's analysis in order to study such a pure effect of beliefs heterogeneity, that is, the effect of a MPS of individual "unweighted probabilities". We can demonstrate that Varian's condition on the utility function still holds if one restricts the analysis to the common CARA utility functions.

Proposition 7 *Under CARA $u(w) = -e^{-rw}$ with $r > 0$, equilibrium state prices decrease with a mean-preserving spread in agents' beliefs.*

7 Conclusion

In the last decades, academics as well as private-sector operators have increasingly used financial prediction markets with the primary objective to better predict future uncertain events. But under which conditions should prediction markets be accurate? This paper has derived generic theoretical conditions so that equilibrium state prices in prediction markets reflect the mean of the beliefs held by the participants in the market. The bad news is that these conditions are very stringent. They require the utility function of all participants to be logarithmic or their beliefs to be distributed symmetrically. Moreover, several examples have illustrated that no general conditions can be found under heterogeneity over participants' characteristics (e.g. wealth, risk preferences). Consistent with the early study by Manski (2006), the general message from this theoretical analysis is that we cannot realistically expect that equilibrium prices in prediction markets only reflect the mean of participants' beliefs. Typically, they should also reflect the distribution of beliefs, as well as the individual characteristics of the participants. The good news is that the paper has also provided a set of conditions that are informative about how prices vary with the heterogeneity in beliefs and the risk aversion of participants.

Appendix A: Proof of Proposition 4

Recall that, when the equilibrium is unique, $\pi^* \geq \overline{p}$ if and only if $E\alpha(\widetilde{p}, \overline{p}) \geq 0$. For symmetric distributions, this holds true if and only if for all $\overline{p}$ (hereafter denoted p) we have

$$g(\delta) = \alpha(p+\delta, p) + \alpha(p-\delta, p) \geq 0, \tag{A.1}$$

in which $\alpha(p+\delta, p)$ is the unique solution of

$$(p+\delta)(1-p)u'(w+\alpha(p+\delta,p)(1-p)) - (1-p-\delta)pu'(w-\alpha(p+\delta,p)p) = 0 \tag{A.2}$$

and $\alpha(p-\delta, p)$ is the unique solution of

$$(p-\delta)(1-p)u'(w+\alpha(p-\delta,p)(1-p)) - (1-p+\delta)pu'(w-\alpha(p-\delta,p)p) = 0 \tag{A.3}$$

for $\delta \in [0, \min\{p, 1-p\}]$.

Observe that $g(0) = 0$ and $g'(0) = 0$. Moreover, we have $g''(0) = 2\alpha_{pp}(p,p)$. Then, taking $\alpha_{pp}(p,p)$ from (7), we can see that $g''(0) \geq 0$ is equivalent to $(1/2 - p)(P(w) - 2A(w)) \geq 0$ for all w. This provides the necessity part of the Proposition.

We now prove the sufficiency. From (A.3), condition (A.1) is equivalent to

$$(p-\delta)(1-p)u'(w-\alpha(p+\delta,p)(1-p)) - (1-p+\delta)pu'(w+\alpha(p+\delta,p)p) \geq 0. \tag{A.4}$$

Denoting $\phi(x) = 1/u'(x)$ and $\alpha = \alpha(p+\delta, p) \geq 0$, $\pi^* \geq p$ is therefore satisfied if

$$(p+\delta)(1-p)\phi(w-\alpha p) - (1-p-\delta)p\phi(w+\alpha(1-p)) = 0 \tag{A.5}$$

implies

$$(p-\delta)(1-p)\phi(w+\alpha p) - (1-p+\delta)p\phi(w-\alpha(1-p)) \geq 0. \tag{A.6}$$

We now introduce two random variables:

$$\tilde{x} = \begin{cases} w+\alpha p, & \frac{p-\delta}{2p} \\ w-\alpha p, & \frac{p+\delta}{2p} \end{cases}, \qquad \tilde{y} = \begin{cases} w+\alpha(1-p), & \frac{1-p-\delta}{2(1-p)} \\ w-\alpha(1-p), & \frac{1-p+\delta}{2(1-p)} \end{cases}.$$

Then it can be verified that $E\tilde{x} = E\tilde{y} = w - \alpha\delta$ and $\tilde{x}$ is a mean-preserving spread of $\tilde{y}$ if and only if $p \geq 1/2$. Note that $\phi''(x) \geq 0$ if and only if $P \leq 2A$. Therefore, when $p \geq 1/2$ and $P \leq 2A$, we have

$$E\phi(\tilde{x}) \geq E\phi(\tilde{y}), \tag{A.7}$$

which is equivalent to

$$\begin{aligned}
&\frac{1}{2p}\Big[(p-\delta)\phi(w+\alpha p) + (p+\delta)\phi(w-\alpha p)\Big] \\
\geq &\frac{1}{2(1-p)}\Big[(1-p-\delta)\phi(w+\alpha(1-p)) + (1-p+\delta)\phi(w-\alpha(1-p))\Big].
\end{aligned}$$

This last inequality then leads to

$$\begin{aligned}
&(1-p)(p-\delta)\phi(w+\alpha p) - p(1-p+\delta)\phi(w-\alpha(1-p)) \\
\geq -&\Big[(1-p)(p+\delta)\phi(w-\alpha p) - p(1-p-\delta)\phi(w+\alpha(1-p))\Big] = 0,
\end{aligned}$$

where the last equality is given by (A.5). This shows that the condition (A.6) is satisfied. Hence $\pi^* \geq p$ when $p \geq 1/2$ and $P \leq 2A$. Moreover, when $p \leq 1/2$, $\tilde{y}$ is a mean-preserving spread of $\tilde{x}$, and $\phi''(x) \leq 0$ is equivalent to (A.7), leading to $\pi^* \geq p$. The case $\pi^* \leq p$ under $(1/2 - p)(P - 2A) \leq 0$ can be demonstrated in an analogous fashion. This concludes the proof. ■

Appendix B [*Not for Publication*]: Proof of Results in Section 6

In this appendix, we setup a prediction market of S-state, derive the equilibrium state prices, and provide the proofs of the propositions and details of the examples in Section 6.

B.1 The model of S-state

Consider a prediction market with N agents, indexed by $i = 1, \cdots, N$, and S states, indexed by $s = 1, \cdots, S$. Agents have the same utility function $u(\cdot)$, however they have heterogeneous beliefs in the probability distribution over the states of nature, denoted by $\mathbf{p}_i = (p_{i1}, p_{i2}, \cdots, p_{iS})$ for agent i. Let π_s be the price of the Arrow-Debreu asset s that delivers \$1 in state s and \$0 in other states for $s = 1, \cdots, S$. Agent i chooses a portfolio $\alpha_i = (\alpha_{i1}, \cdots, \alpha_{iS})$ of the Arrow-Debreu assets to maximize his expected utility of portfolio wealth based on his belief $\mathbf{p}_i$. This leads to the standard first-order condition (FOC):

$$p_{is} u_i'(w_{is}) = \lambda_i \pi_s, \tag{B.1}$$

where λ_i is the Lagrange multiplier,

$$w_{is} = w_o + \alpha_{is} - \sum_{j=1}^{S} \pi_j \alpha_{ij}, \qquad i = 1, \cdots, N; \qquad s = 1, \cdots, S$$

is the portfolio wealth of agent i in state s, and w_o is the initial wealth.

B.2 The equilibrium state prices

Based on the setup in subsection B.1, the equilibrium state prices $\{\pi_s\}$ are determined by the market clear condition

$$\sum_{i=1}^{N} \alpha_{is} = 0, \qquad s = 1, \cdots, S.$$

To derive the equilibrium state prices, we consider three types of utility functions and the results are summarized in three Lemmas.

Lemma 1. For $u(x) = \log(x)$, the equilibrium state prices are given by

$$\pi_s = \frac{1}{N} \sum_{i=1}^{N} p_{is}, \qquad s = 1, \cdots, S, \tag{B.2}$$

that is, the state price is the mean probability belief of agents in the state.

Proof: With $u(x) = \log(x)$, the FOC (B.1) becomes

$$w_o + \alpha_{is} - \sum_{k=1}^{S} \pi_k \alpha_{ik} = \frac{1}{\lambda_i} \frac{p_{is}}{\pi_s}, \qquad i = 1, \cdots, N; s = 1, \cdots, S, \tag{B.3}$$

which leads to

$$\alpha_{is} = \alpha_{iS} + \frac{1}{\lambda_i}\left(\frac{p_{is}}{\pi_s} - \frac{p_{iS}}{\pi_S}\right), \qquad s = 1, \cdots, S.$$

Substituting the above expressions into (B.3) for $s = S$, we obtain that $\lambda_i = w_o$. This implies that the Lagrange multiplier λ_i is a constant in this case. Applying the market clearing condition to (B.3) then leads to the equilibrium state prices (B.2). ■

Lemma 2. For CARA utility $u(x) = -e^{-rx}/r$ with $r > 0$, the equilibrium state prices are given by

$$\pi_s = \frac{p_s^*}{\sum_{k=1}^S p_k^*} \qquad \text{with } p_s^* = \left(\prod_{i=1}^N p_{is}\right)^{1/N} \qquad \text{for } s = 1, \cdots, S. \tag{B.4}$$

Proof: With $u(x) = -e^{-rx}/r$, the FOC (B.1) becomes

$$\frac{p_{is}}{\pi_s} e^{-rw_{is}} = \lambda_i, \qquad i = 1, \cdots, N, \qquad s = 1, \cdots, S.$$

This leads to

$$\alpha_{is} - \sum_{k=1}^S \pi_k \alpha_{ik} = -w_o + \frac{1}{r}\left[\log\left(\frac{p_{is}}{\pi_s}\right) - \log(\lambda_i)\right], \qquad s = 1, \cdots, S. \tag{B.5}$$

Applying the market clearing conditions to (B.5), we obtain

$$rw_o + \frac{1}{N}\sum_{i=1}^N \log(\lambda_i) = \log\left(\frac{p_s^*}{\pi_s}\right), \qquad s = 1, \cdots, S, \tag{B.6}$$

where p_s^* is defined by $\log(p_s^*) = \frac{1}{N}\sum_{i=1}^N \log(p_{is})$ for $s = 1, \cdots, S$. Also, from (B.5),

$$\alpha_{is} = \alpha_{iS} + \frac{1}{r}\Big[\log\left(\frac{p_{is}}{\pi_s}\right) - \log\left(\frac{p_{iS}}{\pi_S}\right)\Big], \tag{B.7}$$

for $s = 1, \cdots, S$. Substituting (B.7) into (B.5) for $s = S$, we have

$$\sum_{s=1}^S \pi_s \log\left(\frac{p_{is}}{\pi_s}\right) = rw_o + \log(\lambda_i), \qquad i = 1, \cdots, N. \tag{B.8}$$

Aggregating (B.8) over i then leads to

$$rw_o + \frac{1}{N}\sum_{i=1}^{N}\log(\lambda_i) = \sum_{s=1}^{S}\pi_s \log\left(\frac{p_s^*}{\pi_s}\right). \tag{B.9}$$

Substituting (B.9) into (B.6), we then have

$$\log\left(\frac{p_s^*}{\pi_s}\right) = \sum_{k=1}^{S}\pi_k \log\left(\frac{p_k^*}{\pi_k}\right), \qquad s = 1, \cdots, S.$$

Therefore $p_s^*/\pi_s = \beta$ is a constant, independent of the state. Then (B.4) follows from $\sum_{s=1}^{S} p_s^* = \sum_{s=1}^{S}\pi_s\beta = \beta$. ■

Lemma 3. For CRRA utility $u(x) = x^{1-\gamma}/(1-\gamma)$ with $\gamma \neq 1$. the equilibrium state prices π_s satisfy

$$\pi_s^{1/\gamma} = \frac{1}{N}\sum_{i=1}^{N}\frac{p_{is}^{1/\gamma}}{\sum_{k=1}^{S}\pi_k\left(\frac{p_{ik}}{\pi_k}\right)^{1/\gamma}}, \qquad s = 1, \cdots, S. \tag{B.10}$$

Proof: With the CRRA utility function, the FOC (B.1) becomes $u'(w_{is}) = w_{is}^{-\gamma} = \lambda_i\pi_s/p_{is}$. Hence, with $g = -1/\gamma$, $w_{is} = \left(\frac{\lambda_i}{p_{is}}\pi_s\right)^g$. This, together with $w_{is} = w_o + \alpha_{is} - \sum_{k=1}^{S}\pi_k\alpha_{ik}$, leads to

$$\alpha_{is} = -w_o + \sum_{k=1}^{S}\pi_k\alpha_{ik} + \left(\frac{\lambda_i}{p_{is}}\pi_s\right)^g. \tag{B.11}$$

Equation (B.11) implies that, for $s = 1, \cdots, S$,

$$\alpha_{is} = \alpha_{iS} + \left[\left(\frac{\lambda_i}{p_{is}}\pi_s\right)^g - \left(\frac{\lambda_i}{p_{iS}}\pi_S\right)^g\right]. \tag{B.12}$$

Substituting (B.12) into (B.11) for $s = S$ and using the market clear condition, we obtain

$$w_o = \sum_{s=1}^{S}\pi_s\left(\frac{\lambda_i}{p_{is}}\pi_s\right)^g, \qquad i = 1, \cdots, N \tag{B.13}$$

Also, applying the market clearing condition to (B.11), we have

$$w_o = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\lambda_i}{p_{is}}\pi_s\right)^g, \qquad s = 1, \cdots, S. \tag{B.14}$$

Combining (B.13) with (B.14) leads to the state prices π_s in (B.10). ■

B.3 Proofs of Propositions 5 and 6

The proof of Proposition 5 follows easily from the equilibrium state prices (B.2) in Lemma 1 for $u(x) = \log(x)$. To prove Proposition 6, we first show that $p_{is} = p_i$ for $s = 2, ..., S$ and $i = 1, ..., N$ implies $\pi_{2,S} = \sum_{s=2}^{S} \pi_s$, where $\pi_{2,S}$ denote the state price of an Arrow-Debreu security that delivers \$1 if either state $j = 2, ..., S$ occurs and \$0 if state 1 occurs. If $p_{is} = p_i$ for $s = 2, ..., S$, we have from the FOC (B.1) that

$$\frac{u'(w_{ij})}{u'(w_{ik})} = \frac{\pi_j}{\pi_k}, \text{ for all } i = 1, ..., N \text{ and } j, k = 2, ..., S.$$

Assume then by contradiction that $\pi_j > \pi_k$ for some $j, k = 2, ..., S$ and $j \neq k$. This implies $w_{ij} < w_{ik}$, and hence $\alpha_{ij} < \alpha_{ik}$ for all $i = 1, ..., N$. The last inequality cannot hold at the equilibrium due to the market clearing condition. As a result we must have $\pi_j = \pi_k \equiv \pi$ for all $j = 2, ..., S$. It is immediate that each agent i must demand the same amount, say α_i, for $j, k = 2, ..., S$. The problem of each agent i is then to select α_{i1} and α_i to maximize

$$p_{i1}u(w + \alpha_{i1} - \alpha_{i1}\pi_1 - \alpha_i(S-1)\pi) + (1 - p_{i1})u(w + \alpha_i - \alpha_{i1}\pi_1 - \alpha_i(S-1)\pi).$$

This is equivalent to a binary-prediction market in which $(S-1)\pi$ denotes the equilibrium price of an Arrow-Debreu security that delivers \$1 if either state $j = 2, ..., S$ occurs. Therefore we have $\sum_{j=2}^{S} \pi_j = (S-1)\pi = \pi_{2,S}$. This implies the S-state problem is equivalent to a reduced two-state problem by combining states 2 to S into one state. Thus the equilibrium price of state 1 only depends on the beliefs of the state.

We now show that if there exists an individual i who assigns different probabilities for two states $j, k = 2, ..., S$ then we may always have $\pi_{2,S} \neq \sum_{j=2}^{S} \pi_j$. Consider a simple example with 3 states and 2 agents, with the following structure of beliefs: $\mathbf{p}_1 = (1-2p_1, p_1, p_1)$ and $\mathbf{p}_2 = (1-2p_2, p_2 + e, p_2 - e)$. Namely agent 1 judges states 2 and 3 as equally likely, while agent 2 judges states 2 and 3 as equally likely if and only if $e = 0$. With CARA preferences, in the three-state prediction market we use the equilibrium state prices in (B.4) of Lemma 2 and obtain $\pi_i = p_i^*/(p_1^* + p_2^* + p_3^*)$ for $i = 1, 2, 3$ with $p_1^* = \sqrt{(1-2p_1)(1-2p_2)}$, $p_2^* = \sqrt{p_1(p_2+e)}$ and $p_3^* = \sqrt{p_1(p_2-e)}$. However, in the two-state prediction market where states 2 and 3 are combined into one state, the beliefs $\mathbf{p}_1$ and $\mathbf{p}_2$ become $\bar{\mathbf{p}}_1 = (1-2p_1, 2p_1)$ and

$\bar{\mathbf{p}}_2 = (1-2p_2, 2p_2)$, respectively. Hence the corresponding state prices become (using obvious notations) $\bar{\pi}_1 = \bar{p}_i^*/(\bar{p}_1^* + \bar{p}_{2,3}^*)$ and $\bar{\pi}_{2,3} = \bar{p}_{2,3}^*/(\bar{p}_1^* + \bar{p}_{2,3}^*)$ with $\bar{p}_1^* = \sqrt{(1-2p_1)(1-2p_2)}$ and $\bar{p}_{2,3}^* = 2\sqrt{p_1 p_2}$. Therefore $\bar{\pi}_{2,3} = \pi_2 + \pi_3$ for all p_1 and p_2 if and only if $e = 0$. A similar example can be generated for any arbitrary number of states. This completes the proof of Proposition 6.

B.4 Proofs of the results in Examples 5 and 6

In Example 5 with CARA utility function, we apply the equilibrium state price (B.4) in Lemma 2 to agents' beliefs and obtain that $p_1^* = 1 - 2p$ and $p_2^* = p_3^* = \sqrt{p^2 - \epsilon^2}$, leading to the state price π_1 in the example.

To show the result in Example 6, we apply the equilibrium state prices (B.10) in Lemma 3 for CRRA utility function. With $g = -1/\gamma$, the state prices π_s for $s = 1, 2, 3$ in this case satisfy

$$2 = (\pi_1)^g \left[p_{11}^{-g}/\Delta_1 + p_{21}^{-g}/\Delta_2\right], \tag{B.15}$$
$$2 = (\pi_2)^g \left[p_{12}^{-g}/\Delta_1 + p_{22}^{-g}/\Delta_2\right], \tag{B.16}$$
$$2 = (\pi_3)^g \left[p_{13}^{-g}/\Delta_1 + p_{23}^{-g}/\Delta_2\right], \tag{B.17}$$

where

$$\Delta_1 = \pi_1(\pi_1/p_{11})^g + \pi_2(\pi_2/p_{12})^g + \pi_3(\pi_3/p_{13})^g,$$
$$\Delta_2 = \pi_1(\pi_1/p_{21})^g + \pi_2(\pi_2/p_{22})^g + \pi_3(\pi_3/p_{23})^g.$$

With the specified heterogeneous probabilities, $\Delta_1 = (12)^g \delta_1$ and $\Delta_2 = (12)^g \delta_2$, where

$$\delta_1 = \pi_1(\pi_1/9)^g + \pi_2(\pi_2/2)^g + \pi_3(\pi_3)^g,$$
$$\delta_2 = \pi_1(\pi_1/3)^g + \pi_2(\pi_2/6)^g + \pi_3(\pi_3/3)^g.$$

Correspondingly, equations (B.15)-(B.17) lead to

$$2(\pi_1)^{-g} = 9^{-g}/\delta_1 + 3^{-g}/\delta_2, \tag{B.18}$$
$$2(\pi_2)^{-g} = 2^{-g}/\delta_1 + 6^{-g}/\delta_2, \tag{B.19}$$
$$2(\pi_3)^{-g} = 1/\delta_1 + 3^{-g}/\delta_2, \tag{B.20}$$

From (B.19) and (B.20), we obtain $\pi_3 = \pi_2/2$. Hence

$$\delta_1 = \pi_1(\pi_1/9)^g + (3/2)\pi_2(\pi_2/2)^g, \tag{B.21}$$
$$\delta_2 = \pi_1(\pi_1/3)^g + (3/2)\pi_2(\pi_2/6)^g. \tag{B.22}$$

Also, from (B.18) and (B.19),

$$3^g[\delta_1 + 3^g\delta_2]\pi_2^g = 2^g[3^g\delta_1 + \delta_2]\pi_1^g. \tag{B.23}$$

Substituting (B.21) and (B.22) into (B.23), we obtain

$$[3^g + 3^{-g}][\pi_1 - (3/2)\pi_2]\pi_1^g\pi_2^g = 3(2/3)^{1+g}\pi_2^{1+2g}[(\pi_1/\pi_2)^{1+2g} - (3/2)^{1+2g}],$$

leading to $\pi_1 = (3/2)\pi_2$.

B.5 Proof of Proposition 7

To examine the effect of heterogeneous beliefs, Varian introduces the *weighted probabilities* $q_{is} = p_{is}/\lambda_i$ for $i = 1, \cdots, N$ and $s = 1, \cdots, S$. By defining $f(x)$ as the inverse function of strictly decreasing function $u'(x)$, Varian shows that, when $P < 2A$, increasing the dispersion of heterogeneous beliefs decreases the state prices. In fact, for the S state market, the FOC (B.1) is given by

$$w_{is} = w_o + \alpha_{is} - \sum_{k=1}^{S} \pi_k\alpha_{ik} = f(\pi_s/q_{is}) \tag{B.24}$$

for $i = 1, \cdots, N$ and $s = 1, \cdots, S$. Applying the market clearing condition to (B.24), we have

$$Nw_o = \sum_{i=1}^{N} f(\pi_s/q_{is}), \qquad s = 1, \cdots, S. \tag{B.25}$$

Note the facts that $f(\pi_s/q_{is})$ is an increasing function of q_{is} and that it is a concave function of q_{is} if $P < 2A$. Based on (B.25), Varian shows that a mean-preserving spread in the weighted probabilities q_{is} must decrease the state price π_s when $f(\pi_s/q_{is})$ is an increasing concave function of q_{is}. Proposition 7 claims that this result also holds for the *unweighted probabilities* p_{is} for CARA utility functions. In fact, for CARA utility functions, $P = A$. Note that $\log(p_s^*) = (1/N)\sum_{i=1}^{N} \log(p_{is})$ is a concave function of agents' heterogeneous probabilities p_{is}. Then Proposition 7 follows from the equilibrium state price (B.4).

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