Reprinted from JOURNAL OF ECONOMIC THEORY
All Rights Reserved by Academic Press, New York and London

Vol. 19, No. 2, December 1978
Printed in Belgium

Satisfactory Mechanisms for Environments with Consumption Lower Bounds*

JERRY GREEN[†] AND JEAN-JACQUES LAFFONT[‡]

[†] *Harvard University, Cambridge, Massachusetts 02138; and*
[‡] *École Polytechnique, Paris, France*

Received August 2, 1976; revised July 26, 1977

1. INTRODUCTION

This paper concerns the design of decision-making mechanisms having desirable properties for a certain class of economic environments. The special feature of these environments on which we concentrate is that, although some transfer of resources may be possible among the agents, the extent to which any agent can lose resources is bounded. Such a restriction is well known in general equilibrium theory, where the lower boundedness of consumption sets limits transfers to those that are physically feasible. Another interpretation is that the agents in the model are representatives[1] of segments of the population, and that the limitations on transfers among them arises through the institutional structure.[2]

Without bounds on transfers, Groves and Loeb [6] have given a class of mechanisms that make efficient choices, in a sense to be described below, in an equilibrium in which each player is following a dominant strategy. In this way, much of the strategic or manipulative aspect of the free rider

* This work was supported by National Science Foundation Grant SOC 71-03803 to Harvard University, and by a grant from the NATO Systems Science Program.

[1] This is discussed in detail by Johansen [7]. We believe that the primary practical reason for treating the case of bounded transfers is that these representatives may have means which are much smaller than the intensities of the aggregate willingness-to-pay that are typical of their constituents.

[2] Indeed, in the one example of which we are aware where these methods were applied in a practical situation, it was precisely these considerations that were most important in the design of the decision-making procedures. The case is that of program selection by the stations in the Public Broadcasting System. Each of the 135 station managers who "bid" for programs has a limited programming budget but represents a large population. The differences in aggregate willingnesses-to-pay among alternative possible combinations of programs would surely swamp these budgets. Nevertheless a method of "social" decision-making is necessary which allows the station managers the flexibility necessary to accurately reflect the tastes of their potential viewing audiences.

0022-0531/78/0192-0359$02.00/0

problem has been overcome.[3] Green and Laffont [2] have shown that the class of mechanisms described by Groves and Loeb contains all those with these properties.

In this paper we first extend the use of these mechanisms to systems with transfers bounded from below, maintaining all the other aspects of the model described above. Individuals' strategic options are studied. The main result is that one of the mechanisms of Groves and Loeb, a special one, previously developed by Clarke [1] and Vickrey [8], retains all its desirable properties despite the bounds on transfers. Other members of the original successful class lose this property in the present context. Sufficient conditions for a member of the Groves–Loeb class to remain efficient are presented.

Finally, we offer some remarks on the willingness of individuals to participate in this procedure, on extensions to more complex choice situations and on their unbiasedness.

2. The Model

We concentrate on the case of a decision, d, to be taken between two alternatives which can be denoted 0 ("reject proposed project") and 1 ("accept proposed project").

Let x_i be the amount of the private good consumed by individual i who is assumed to have the utility function, defined for $x_i \geqslant 0$, given by:

$$u_i(x_i, d) = x_i \qquad \text{if} \quad d = 0$$
$$= x_i + v_i \qquad \text{if} \quad d = 1.$$

The initial endowment of the ith individual is denoted $\bar{x}_i$; it is assumed to be non-negative.

Since there are no successful dominant strategy mechanisms always having balanced budgets (see Green and Laffont [3]) we must also consider the net transfer of the decision maker (government) in defining feasibility and efficiency. Thus, a *feasible social state* is defined by:

$$z = \left[d, x_1, \dots, x_n, \sum_i (\bar{x}_i - x_i)\right]$$

such that $x_i \geqslant 0$ for all i.

The last entry represents the net transfer from the individuals to the decision maker. It is natural to suppose that the utility of the decision maker

[3] Of course some problems do remain. Hurwicz [1970] has shown that it is impossible to maintain feasibility in two person environments. This has been extended by Green and Laffont [3], Green, Kohlberg, and Laffont [4], and Walker [9].

is strictly increasing in this quantity, as revenues are usable for other purposes or may be substituted for other taxes being collected.[4]

Without fear of confusion we can write the utility of individual i in social state z as $u_i(z) = u_i(d, x_i)$.

The utility of the decision maker is then:

$$u_0(z) = \sum_i (\bar{x}_i - x_i).$$

A social state z is said to be *efficient relative to* $u_0(z)$, (or "efficient" for brevity) if for any other feasible social state z' such that $u_i(z') \geqslant u_i(z)$, $i = 0, 1, \dots, n$, we have that $u_i(z') = u_i(z)$, $i = 0, 1, \dots, n$. Efficiency is just Pareto optimality for the set of agents expanded to include the decision maker.

It is easy to see that there may be some efficient states for which $d = 0$ and $\sum v_i \geqslant 0$, or $d = 1$ and $\sum v_i < 0$. For example, consider the two-person economy defined by:

$$\bar{x}_1 = 0, \qquad v_1 = 5$$
$$\bar{x}_2 = 10, \qquad v_2 = -1.$$

The social state $z = (0, 0, 10, 0)$ which produces utilities

$$u_0 = 0, \qquad u_1 = 0, \qquad u_2 = 10$$

is Pareto optimal. This can be seen because if $u_0 \geqslant 0$, then $x_1 + x_2 \leqslant 10$ is required. If $u_2 \geqslant 10$, then $d = 0$ requires $x_2 \geqslant 10$ and $d = 1$ requires $x_2 \geqslant 11$. In the former instance $x_1 \leqslant 0$ and z is the only feasible social state; in the latter, there are no feasible social states since $x_1 \leqslant -1$ would be implied.

Let $w_i \in R$ be the strategy of agent i, $i = 1, \dots, n$.

We define a *mechanism* as a mapping, f, from R^n to social states.

$$w = (w_1, \dots, w_n) \to f(w) = z.$$

We use the notation:

$$w_{-i} = (w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n),$$
$$(w_{-i}, \xi) = (w_1, \dots, w_{i-1}, \xi, w_{i+1}, \dots, w_n).$$

[4] Of course, if individuals were to recognize this dependence, the favorable incentive properties of the mechanisms to be studied would be destroyed (see, however, Green, Kohlberg, and Laffont [4]).

A strategy w_i is called *admissible* if, for every w_{-i},

$$f(w_{-i}, w_i) = \left[d, x_1, \ldots, x_n, \sum (\bar{x}_i - x_i)\right]$$

is such that $x_i \geqslant 0$.

An admissible strategy w_i, is called *admissible dominant* (or just *dominant* for brevity) if, for every other admissible strategy, w_i'

$$u_i[f(w_{-i}, w_i)] \geqslant u_i[f(w_{-i}, w_i')]$$

for all w_{-i}.

If $w_i = v_i$ is a *dominant* strategy for every i, then the mechanism will be said to be *strongly individually incentive compatible*. Note, however, that v_i may not be admissible for some mechanisms and some individuals. There may nevertheless exist other dominant strategies.

A mechanism will be called *satisfactory* if

(i) for every i there exists an admissible dominant strategy, w_i^*, and

(ii) $f(w^*) = f(w_1^*, \ldots, w_n^*)$ is an efficient state.

In Green and Laffont [2] it was shown that, for economies without bounds on consumption of the private good, the class of all mechanisms that are strongly individually incentive compatible and satisfactory is given by:

$$f(w) = \left[d, x_1, \ldots, x_n, \sum_i (\bar{x}_i - x_i)\right] \tag{2.1}$$

where

$$\begin{aligned} d &= 0 && \text{iff } \textstyle\sum w_i < 0 \\ x_i &= \bar{x}_i + \sum_{j \neq i} w_j + h_i(w_{-i}) && \text{if } d = 1 \\ x_i &= \bar{x}_i + h_i(w_{-i}) && \text{if } d = 0 \end{aligned}$$

where the $h_i(\cdot)$ are an arbitrary set of real-valued functions defined on R^{n-1}.

3. The Pivotal Mechanism

In this section we consider the special case in which, for all i.

$$h_i(w_{-i}) = \min\left(-\sum_{j \neq i} w_j, 0\right). \tag{3.1}$$

This defines the *pivotal mechanism* (see Green and Laffont [3], Clarke [1] and Vickrey [8]). Under this mechanism we always have $x_i \leqslant \bar{x}_i$. The strategy w_i is pivotal when $w_i \cdot \sum_{j \neq i} w_j < 0$ and $| w_i | > | \sum_{j \neq i} w_j |$, because the presence of individual i changes the social decision. In such a case $x_i \neq \bar{x}_i$.

Lemma 1. *For the mechanism defined by* (3.1), *the strategies*

$$\begin{aligned} \hat{w}_i(\bar{x}_i, v_i) &= v_i - \max(v_i - \bar{x}_i, 0) && v_i \geqslant 0 \\ &= v_i - \min(v_i + \bar{x}_i, 0) && v_i < 0 \end{aligned}$$

are the unique and dominant strategies.

Proof. Consider first the case in which $v_i \geqslant 0$. Note that $u_i(w_{-i}, w_i)$ is defined for all w_{-i}.

However, for $| \tilde{w}_i | > \bar{x}_i$, $u_i(w_{-i}, \tilde{w}_i)$ is not defined whenever w_{-i} is such that

$$| \tilde{w}_i | > \left| \sum_{j \neq i} w_j \right| > \bar{x}_i$$

and

$$\tilde{w}_i \cdot \sum_{j \neq i} w_j < 0$$

because then

$$x_i = \bar{x}_i - \left| \sum_{j \neq i} w_j \right| < 0.$$

For $| \tilde{w}_i | \leqslant \bar{x}_i$, $u_i(w_{-i}, \tilde{w}_i)$ is defined for all w_{-i}, and hence these values of $\tilde{w}_i$ are admissible strategies.

If $v_i < \bar{x}_i$, then $\hat{w}_i = v_i$ and the proof in Green and Laffont [2, Theorem 3] establishes the dominance of $\hat{w}_i$.

If $v_i > \bar{x}_i$, so that $\hat{w}_i = \bar{x}_i$, and $\tilde{w}_i$ is any other admissible strategy, we have the following cases: the value of w_{-i} is such that

(i) $\hat{w}_i$ is pivotal but $\tilde{w}_i$ is not,

(ii) $\hat{w}_i$ and $\tilde{w}_i$ are both pivotal,

(iii) neither $\hat{w}_i$ nor $\tilde{w}_i$ are pivotal.

Case (*i*). Since $\hat{w}_i = \bar{x}_i \geqslant 0$ is pivotal, it must be that $\sum_{j \neq i} w_j < 0$, and $\bar{x}_i > -\sum_{j \neq i} w_j$. Thus:

$$u_i[f(w_{-i}, \hat{w}_i)] = v_i + \bar{x}_i + \sum_{j \neq i} w_j$$

and

$$u_i[f(w_{-i}, \tilde{w}_i)] = \bar{x}_i.$$

Using $v_i > \bar{x}_i > -\sum_{j \neq i} w_j$, the former outcome dominates the latter.

Case (ii). We must have $\hat{w}_i \geqslant \tilde{w}_i > 0$, $\sum_{j\neq i} w_j < 0$, $\tilde{w}_i > -\sum_{j\neq i} w_j$. Therefore the outcome associated with $(w_{-i}, \hat{w}_i)$ is the same as that associated with $(w_{-i}, \tilde{w}_i)$.

Case (iii). Similarly, the two outcomes are identical.

The case of $v_i < 0$ is analogous and therefore $\hat{w}_i$ is the unique dominant strategy. Q.E.D.

Our definition of the pivotal mechanism includes a rule by which "ties" are broken, namely, if $\sum_i w_i = 0$ then $d = 1$ is the decision chosen. In unconstrained environments, the choice of tie breaking rules is clearly irrelevant since $v_i = w_i$ and thus either $d = 0$ or $d = 1$ results in an efficient state. The present context differs from this one, however, in that inefficiency may result if the tie is broken in the "wrong" way. Consider the following example.

$$\bar{x}_1 = 5 \qquad v_1 = -6$$
$$\bar{x}_2 = 10 \qquad v_2 = +5.$$

By virtue of Lemma 1, $w_1 = -5$, $w_2 = +5$ are the strategies played. Hence, under the pivotal mechanism for instance,

$$z = (1, 5, 5, 5)$$

is the resulting state. The associated utilities are -1, 10, 5 for the two agents and the decision maker, respectively. However, the state

$$z' = (0, 0, 10, 5)$$

gives rise to the utilities 0, 10, 5, which obviously dominates z (relative to $u_0 = 5$). In some sense, however, the likelihood of such cases of exact ties is negligible. This motivates the following definition:

A mechanism is said to be *essentially satisfactory* if

(i) it is satisfactory, or

(ii) it fails to be satisfactory only at environments for which $\sum w_i$ is zero, where w is the vector of dominant strategies.

THEOREM 1. *The pivotal mechanism is essentially satisfactory.*

Proof. By virtue of the lemma it suffices to prove that $f(\hat{w}_1, ..., \hat{w}_n)$ is efficient, or else $\sum \hat{w}_i = 0$.

Let

$$f(\hat{w}_1, ..., \hat{w}_n) = \left[d, x_1, ..., x_n, \sum (\bar{x}_i - x_i)\right].$$

Consider any feasible social state $z' = [d', x_1', ..., x_n', \sum (\bar{x}_i - x_i)]$. If $d' = d$ then it is clear that we cannot have $x_i' \geqslant x_i$ and $u_0' \geqslant u_0$ without havin $x_i' = x_i$ for all i. Thus if $f(\hat{w}_1, ..., \hat{w}_n)$ can be dominated, it must be by social state such that $d' \neq d$.

Consider first the case of $d = 0$:

Let

$$S_+ = \{i \mid 0 \leqslant \hat{w}_i = v_i \leqslant \bar{x}_i\}$$
$$S_- = \{i \mid 0 \geqslant \hat{w}_i = v_i \text{ and } |v_i| \leqslant \bar{x}_i\}$$
$$T_+ = \{i \mid 0 \leqslant \hat{w}_i = \bar{x}_i < v_i\}$$
$$T_- = \{i \mid 0 \geqslant \hat{w}_i = -\bar{x}_i < v_i\}.$$

Let

$$t_i = \bar{x}_i - x_i, \qquad i = 1, ..., n$$

be the transfer under the pivotal mechanism.

In order that z' dominate $f(\hat{w}_1, ..., \hat{w}_n)$, we require that for all $i = 1, ..., n$

$$v_i + x_i' \geqslant x_i$$

or

$$x_i' = x_i - v_i = \bar{x}_i - t_i - v_i.$$

For the four groups of individuals above this means

$$i \in S_+ \rightarrow x_i' \geqslant \bar{x}_i - t_i - \hat{w}_i$$

$$i \in S_- \rightarrow x_i' \geqslant \bar{x}_i - t_i - \hat{w}_i$$

$$i \in T_+ \rightarrow x_i' \geqslant \bar{x}_i - t_i - v_i$$

$$i \in T_- \rightarrow x_i' \geqslant \bar{x}_i - t_i - v_i \geqslant \bar{x}_i - t_i - \hat{w}_i.$$

Note that for $i \in T_+$, the relevant constraint is really $x_i' \geqslant 0$ since $\bar{x}_i < v$ and $-t_i \leqslant 0$, so that the right hand side of the inequality above is negative

Therefore the minimal amount of the private good necessary to sustain Pareto superior point is

$$P = \sum_{S_+} (\bar{x}_i - t_i - \hat{w}_i) + \sum_{S_-} (\bar{x}_i - t_i - \hat{w}_i)$$
$$+ \sum_{T_+} 0 + \sum_{T_-} (\bar{x}_i - t_i - \hat{w}_i) + \sum_i t_i$$
$$= \sum_{S_+} \bar{x}_i + \sum_{S_-} \bar{x}_i + \sum_{T_-} \bar{x}_i + \sum_{T_+} t_i - \sum_{S_+ \cup S_- \cup T_-} \hat{w}_i.$$

Now, since $d = 0$ was the decision with strategies $\hat{w}_i$, $i = 1,..., n$, we have

$$0 > \sum_i \hat{w}_i = \sum_{S_+ \cup S_- \cup T_-} \hat{w}_i + \sum_{T_+} \hat{w}_i$$

or

$$- \sum_{S_+ \cup S_- \cup T_-} \hat{w}_i > \sum_{T_+} \hat{w}_i = \sum_{T_+} \bar{x}_i .$$

Substituting above

$$P > \sum_i \bar{x}_i + \sum_{T_+} t_i \geqslant \sum_i \bar{x}_i$$

since $t_i \geqslant 0$ for all i, under the pivotal mechanism.

This inequality establishes the infeasibility of a Pareto superior allocation with $d = 0$.

With $d = 1$ and $\sum_i \hat{w}_i > 0$, the proof is analogous. When $\sum_i \hat{w}_i = 0$, the second part of the definition of essential satisfactoriness applies. Q.E.D.

4. The General Case

In order to discuss the results above in the context of mechanisms other than the pivotal mechanism, we will proceed in two stages. First we ascertain the class of mechanisms for which dominant strategies exist. Then, we study those which produce Pareto optimal results.

For simplicity we concentrate on the symmetric case, where $h_i(\cdot)$ is the same function for all i and is a symmetric function of w_{-i}.[5] We denote this common function by $h(\cdot)$. In order that dominant strategies exist for the mechanism defined by h, it is first necessary that it have a non-empty set of admissible strategies for all possible levels of the endowment.

Writing

$$-t_i(w_{-i}, w_i) = \sum_{j \neq i} w_j + h(w_{-i}) \qquad \text{if } \sum w_i \geqslant 0$$

$$= h(w_{-i}) \qquad \text{if } \sum w_i < 0$$

we can define

$$m(h, w_i) = \min_{w_{-i}} -t_i(w_{-i}, w_i)$$

Since $\bar{x}_i \geqslant 0$, it is required that

$$\max_{w_i} m(h, w_i) \geqslant 0$$

[5] None of our results depend on this restriction used for simplicity of notation.

or that

$$\min_{w_{-i}} -t_i(w_{-i}, w_i) \geqslant 0 \qquad \text{for some choice of } w_i .$$

Lemma 2. *If for some w_{-i}, $h(w_{-i}) < \min(-\sum_{j \neq i} w_j, 0)$ then the mechanism defined by $h(\cdot)$ has no admissible strategies for $\bar{x}_i$ sufficiently small.*

Proof. The proof is immediate for if $h(\bar{w}_i) < \min(-\sum_{j \neq i} \bar{w}_j, 0)$ then $t_i(w_i, \bar{w}_i)$ takes on one of the two values $-\sum_{j \neq i} w_j - h(\bar{w}_{-i})$ or $-h(\bar{w}_{-i})$ according to the sign of $\sum w_i$, and each of these is strictly positive.

For $h(\cdot)$ satisfying

$$h(w_{-i}) \geqslant \min\left(- \sum_{j \neq i} w_j , 0\right)$$

admissible strategies exist. Before we analyze the dominance of one strategy within this set, we characterize the admissible set according to the following:

Lemma 3. *For any mechanism defined by a function $h(\cdot)$, the set of admissible strategies for each $\bar{x}_i \geqslant 0$ is an interval. Moreover,*

(i) *it is a closed interval if $h(\cdot)$ is continuous,*

(ii) *it contains zero if $h(\cdot) \geqslant \min(-\sum_{j \neq i} w_j, 0)$.*

Proof. Suppose w and w' are admissible strategies and $w > w'$. The following inequalities must then hold:

(i) $h(w_{-i}) \geqslant -\bar{x}_i$ for $\sum_{j \neq i} w_j < -w$

(ii) $h(w_{-i}) \geqslant -\sum_{j \neq i} w_j - \bar{x}_i$ for $\sum_{j \neq i} w_j \geqslant -w$

(iii) $h(w_{-i}) \geqslant -\bar{x}_i$ for $\sum_{j \neq i} w_j < -w'$

(iv) $h(w_{-i}) \geqslant -\sum_{j \neq i} w_j - \bar{x}_i$ for $\sum_{j \neq i} w_j \geqslant -w'$.

Since (iii) implies (i) and (ii) implies (iv), then for w'' such that $w' < w'' < w$ we will have that

$$h(w_{-i}) \geqslant - \sum_{j \neq i} w_j - \bar{x}_i \qquad \text{for } \sum_{j \neq i} w_j \geqslant -w$$

and therefore for

$$\sum_{j \neq i} w_j \geqslant w''$$

and

$$h(w_{-i}) \geqslant -\bar{x}_i \qquad \text{for } \sum_{j \neq i} w_j < -w'$$

and therefore for

$$\sum_{j \neq i} w_j < -w''.$$

The closedness of the interval of admissible strategies follows from the closedness of the requirement $x_i \geqslant 0$ and the continuity of $h(\cdot)$.

The admissibility of zero for the function $h(\cdot) = \min(-\sum_{j \neq i} w_j, 0)$ follows from the definition.

This characterizes the admissible strategies for every endowment. Because of this one can find the dominant admissible strategy directly. It will be the closest point to the true willingness-to-pay within the admissible set. From now on, we will assume that $h(\cdot)$ is continuous.

THEOREM 2. *Let $[-\alpha, \beta]$ be the interval of admissible strategies for an individual with endowment $\bar{x}$ and preferences v, when the mechanism is defined by $h(\cdot)$.*

Let w^* solve $\min_{w \in [-\alpha, \beta]} | v - w |$. Then w^* is the dominant strategy.

Proof. We consider individual i but drop the index when there is no risk of confusion.

Take the case $v < -\alpha \leqslant \beta$ and suppose $-\alpha < w \leqslant \beta$. Consider

$$u[f(-\alpha, w_{-i})] - u[f(w, w_{-i})] = 0 \qquad \text{if} \quad \sum_{j \neq i} w_j \notin [-w, \alpha]$$

$$= \bar{x} - t(-\alpha, w_{-i}) - v - \bar{x} + t(w, w_{-i})$$

or

$$= -v - \sum_{j \neq i} w_j \qquad \text{if} \quad \sum_{j \neq i} w_j \in [-w, \alpha]$$

but $v < -\alpha$ and $\sum_{j \neq i} w_j < \alpha$ implies $-v - \sum_{j \neq i} w_j > 0$ and hence setting $w = -\alpha$ dominates any other admissible strategy. Other cases can be treated analogously.

We have seen that we must place some restrictions on the arbitrary h—it must be continuous and bounded below by the pivotal mechanism—in order to insure the existence of a dominant admissible strategy for all agents. It is nevertheless still not true that the class of all mechanisms defined by functions h with these characteristics are satisfactory. That is, there are some dominant strategy mechanisms which may, in some cases, produce inefficient outcomes in an essential way. An example is the following.

There are two individuals. For each i, the function $h(w_j)$ for $j \neq i$ is defined by

$$h(w_j) = -2(w_j + 1) \qquad w_j < -1$$
$$= 0 \qquad w_j \geqslant -1.$$

This function clearly satisfies our requirements. One can verify that the

set A of admissible strategies for each agent, as a function of his endowment, $\bar{x}_i$, is given by

$$A(\bar{x}_i) = [-\infty, +\bar{x}_i] \qquad \text{if} \quad \bar{x}_i < 1$$
$$= [-\infty, +\infty] \qquad \text{if} \quad \bar{x}_i \geqslant 1.$$

Let

$$\bar{x}_1 = 0, \qquad v_1 = 5$$
$$x_2 = 10, \qquad v_2 = -3.$$

Thus

$$w_1 = 0$$
$$w_2 = -3$$

will be the dominant admissible strategies of the agents. The project will be rejected and a transfer of $+4$ will be given to agent 1. The social state attained is therefore

$$z = (0, 4, 10, -4)$$

and the utilities attained are just the indicated consumption levels.

However, consider the alternative social state

$$z = (1, 0, 14, -4).$$

The utilities are now 5, 11 and -4 for the two agents and the government respectively. This clearly dominates the equilibrium attained by the mechanism.

It is therefore clear that only some of the satisfactory mechanisms for economies with unrestricted transfers of the private good continue to have this property when the non-negativity of consumption is required. We can then ask, naturally, for the class of mechanisms that are satisfactory in this case.

Conditions sufficient to insure the successfulness of a dominant-strategy inducing mechanism are not hard to derive. It does not seem possible, however, to state a necessary and sufficient condition on the function h, in a readily interpretable form.

By virtue of Lemmas 2 and 3 the endpoints of the interval defining the admissible strategies are both at least equal to the level of endowment in absolute value. If h is above the pivotal h function, then strategies that exceed $\bar{x}$ in absolute value are allowable. A sufficient condition that a mechanism be satisfactory when consumption is bounded below by zero is that it never provide any individual a subsidy that is greater than the absolute value of the difference between his strategy and his endowment. It may be seen that the pivotal mechanism, which never provides any subsidies, satisfies this criterion.

Using the notation paralleling that of Theorem 1, let

$$S_+ = \{i \mid 0 \leqslant w_i = v_i\}$$
$$S_- = \{i \mid 0 \geqslant w_i = v_i\}$$
$$T_+ = \{i \mid 0 \leqslant w_i < v_i\}$$
$$T_- = \{i \mid 0 \geqslant w_i > v_i\}.$$

Suppose the strategies $(w_1^*,..., w_n^*)$ are played and the mechanism rejects the project, resulting in the social state

$$z = \left(0, x_1^*,..., x_n^*, \sum \bar{x}_i - \sum x_i^*\right)$$

and that

$$z' = \left(1, x_1',..., x_n', \sum \bar{x}_i - \sum x_i'\right)$$

dominates z. This would require:

$$x_i' \leqslant \sum x_i^*$$

$$x_i' \geqslant x_i^* - v_i$$

$$x_i' \geqslant 0$$

for each i. Hence the amount of the private good necessary to sustain z', P, would have to satisfy

$$P = \sum x_i' + \sum \bar{x} - \sum x_i^*$$

$$\geqslant \sum_{S_+ \cup S_- \cup T_-} (x_i^* - w_i^*) + \sum \bar{x}_i - \sum x_i^* + \sum_{T_+} x_i'$$

$$\geqslant -\sum_{T_+} x_i^* - \sum w_i^* + \sum_{T_+} w_i^* + \sum \bar{x}_i$$

$$\geqslant \sum \bar{x}_i + \sum_{T_+} [(w_i^* - \bar{x}_i) - (x_i^* - \bar{x}_i)]$$

since $\sum w_i^* < 0$.

Thus, if for every individual whose strategy is constrained from above, and who therefore responds with the upper endpoint of the set of admissible strategies, the maximum possible subsidy is less than the difference between this strategy and the endowment, the last sum will be positive and a contradiction to the non-optimality of z is derived.

By a parallel argument for the case of $\sum w_i^* > 0$, we arrive at the condition

$$P \geqslant \sum \bar{x}_i + \sum_{T_-} [(-w_i - \bar{x}_i) - (x_i^* - \bar{x}_i)]$$

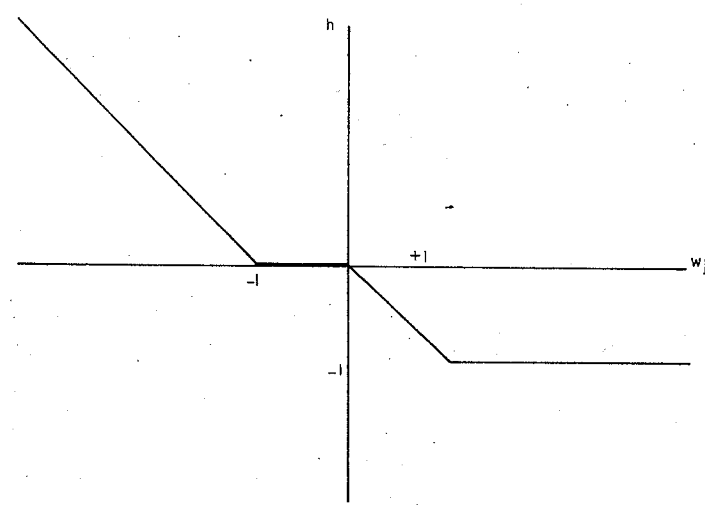
which has the interpretation given above.

Since this is a sufficient condition, one can extend the class of satisfactory mechanisms by checking that they satisfy it. For example, the pivotal mechanism plus any constant transfer K will be successful since the maximum subsidy is K and the admissible strategies are $[-\bar{x}_i - K, \bar{x}_i + K]$, so that members of T_+ will choose $\bar{x}_i + K$ and members of T_- will choose $-\bar{x}_i - \bar{K}$.

It is not the case, however, that any mechanism admitting the possibility of a subsidy greater than $| w_i | - \bar{x}_i$, will fail to be a member of the satisfactory class. To see this, we consider an example closely related to that used above.

Let

$$\begin{aligned} h &= -w_j - 1 & w_j &< 1 \\ &= 0 & -1 \leqslant w_j &< 0 \\ &= -w_j & 0 \leqslant w_j &< 1 \\ &= -1 & 1 \leqslant w_j&. \end{aligned}$$



The mechanism will have dominant admissible strategies by virtue of Lemma 3 and Theorem 2. The admissible strategy correspondence is given by

$$\begin{aligned} A(\bar{x}_i) &= [-\bar{x}_i, +\bar{x}_i] & \text{if} \quad | \bar{x}_i | < 1 \\ &= [-\infty, +\infty] & \text{if} \quad | \bar{x}_i | \geqslant 1. \end{aligned}$$

We will now show that this mechanism is essentially satisfactory for all two-person economies even though when

$$\bar{x}_1 = 0, \qquad v_1 = 5$$
$$\bar{x}_2 = 10, \qquad v_2 = -3$$

we have that the project is rejected, since $w_1 = 0$, $w_2 = -3$, and agent 1 is in T_+ with $h(w_2) = 2 > 0 = w_1 - \bar{x}_1$. This will demonstrate that the nonexistence of such situations is not a necessary condition for satisfactoriness.

In the demonstration of satisfactoriness to follow we use two conditions which are each sufficient to insure that a particular social state is efficient. They have both been proven and discussed above.

Let

$$z = \left(d, x_1, \dots, x_n, \sum \bar{x}_i - \sum x_i\right).$$

If

(*) $d = 1$ and $\sum v_i \geqslant 0$, or $d = 0$ and $\sum v_i < 0$, then z is efficient relative to $\sum \bar{x}_i - x_i$.

(**) If z is such that for each i in T_+, T_-, $x_i - \bar{x}_i = w_i - \bar{x}_i$, then z is efficient relative to $\sum \bar{x}_i - x_i$.

We consider three cases, which, by symmetry, are exhaustive:

(I) $\bar{x}_1 \geqslant 1, \bar{x}_2 \geqslant 1$
(II) $\bar{x}_1 > 1, \bar{x}_2 \geqslant 1$
(III) $\bar{x}_1 > 1, \bar{x}_2 < 1$.

Case (I). Here, $w_1 = v_1$ and $w_2 = v_2$ are both admissible. Thus the resulting social state will satisfy the hypothesis in condition (*) and will be efficient.

Case (II). We will have $w_1 \leqslant \bar{x}_1$, and $w_2 = v_2$. Only individual 1 can be in T_+ or T_-. Take the case in which he is in T_+, so that $v_1 > \bar{x}_1$, and $w_1 = \bar{x}_1$ is his dominant admissible strategy. If $w_2 \geqslant -\bar{x}_1$, condition (*) applies and the result is Pareto optimal. If $w_1 < \bar{x}_2$, the project will be rejected even though $v_1 + v_2$ may be positive. However, $w_2 < 0$ and rejection of the project implies that the subsidy received by agent 1 is exactly $h(w_2)$. If there exists a feasible superior situation, it means that the quantity of the private good consumed by agent 1, $h(w_1) + \bar{x}_2$ exceeds the amount of compensation it is necessary to give agent 2, $-v_2$, for the change in the decision towards acceptance of the project. Since in this case $w_2 = v_2$, the nonoptimality of the social state selected means

$$h(w_2) + \bar{x}_1 \geqslant -w_2.$$

But the left hand side of this inequality is either $-w_2 - 1$ or zero, according to whether w_2 is above or below -1.

Consider first the situation where $w_2 < -1$. Hence we have

$$-w_2 - 1 + \bar{x}_1 \geqslant -w_2$$

or

$$\bar{x}_1 \geqslant 1$$

as the condition for inefficiency—but this contradicts the hypothesis of case (II).

If $w_2 \geqslant -1$, then we need $\bar{x}_1 \geqslant -w_2$ for nonoptimality. But since $w_1 = \bar{x}_1$ $(1 \in T_+)$ and $w_1 + w_2 < 0$ $(d = 0)$, this condition cannot be satisfied. Therefore a superior social state cannot exist in this case.

Case (III). Here $|w_1| \leqslant \bar{x}_1 \leqslant 1$ and $|w_2| \leqslant \bar{x}_2 \leqslant 1$. But for this strategy, the h function coincides with the pivotal mechanism. Therefore no positive subsidies can ever arise and condition (**) applies to insure optimality of the resulting social state.

In summary, the nonexistence of potential situations in which subsidies in excess of $|w_i| - \bar{x}_i$ are paid is a sufficient, but not a necessary condition for the satisfactoriness of social decision mechanisms with bounded consumption sets. A more precise delineation of the class of satisfactory mechanisms is at present an open problem.

5. Remarks

The pivotal mechanism has been shown to be quite useful even in environments with bounded consumption in that it can elicit dominant strategies that produce efficient outcomes. It has, in addition, two further properties which are highly desirable. We will comment on these below.

Our definition of efficiency allowed for negative transfers to the decision maker, even though we restricted individual agents to have positive net holdings of the transferable resource. This was justified by the prospect that deficits might be covered out of "general revenues." If this were impossible, or irrelevant, one would want to insure a non-negative level of revenue generation. It is clear that this property is satisfied by the pivotal mechanism, which never gives a positive transfer to any agent. Further, any h-function which is greater than the pivotal h-function at any point runs the risk of subsidizing someone. By virtue of Lemma 2, therefore, the pivotal mechanism is the only one in the satisfactory class which can insure that the decision-makers transfer is never negative.

Another property of social decision mechanisms that is often required is "individual rationality"—that no agent can ever suffer a decrease in utility vis à vis his initial position. Clearly, no member of the satisfactory class has this property: if v_i is negative and $\sum_{j \neq i} w_j$ is larger in absolute value, but positive, then agent i will be hurt, and uncompensated, by the project's acceptance under the pivotal mechanism. Even if the transfer to such an agent were positive in another mechanism, he would lose if $|v_i|$ were sufficiently large.

There is another concept which is related in spirit to "individual rationality" but distinct from it. There are many situations, especially those involving public goods, where the appropriate benchmark from which to measure utility change is ill-defined. For instance, "$d = 1$" may represent the cancellation of an ongoing project. In a non-coercive society, however, one option open to any agent is to decline participation in the mechanism. He should be able to avoid all transfers in this way, and the decision will reflect the strategies of the remaining agents.

The pivotal mechanism has the property that no agent would decline participation in this way. To see this, observe that the strategy of non-participation in this mechanism is precisely equivalent to setting $w_i = 0$. If this is dominated by any other value of w_i, non-participation is dominated as well. (In cases of indifference, no bias in the decision results in any event.) Other members of the successful class possess this property as well, but since the pivotal mechanism has all the attributes we are looking for, we have not attempted to characterize the set of those possessing this feature in more detail.

Finally, and unfortunately, we must point out the failure of these satisfactory mechanisms in situations with three or more possible public decisions.

Let $K = \{0, 1, 2\}$ and consider an agent, say 1, for whom $v_1 = (0, 4, 6)$ describes his willingness-to-pay for each of the three public projects in K, and $\bar{x}_1 = 5$. For the pivotal mechanism generalized to the case of vector-valued responses representing evaluations of $k \in K$, one can verify straightforwardly that the admissible strategy space is

$$\{w_i(\cdot) \mid \max_K w_i(k) - \min_K w_i(k) \leqslant \bar{x}_i\}.$$

The problem with using the pivotal mechanism is not that an inefficient outcome might be selected, but rather that dominant strategies might fail to exist. In the present instance, consider strategies of the form

$$w_1 = (0, w, 5)$$

where $0 < w < 5$, which are the obvious candidates for dominant strategies.

To prove that no strategy of this form is dominant, consider three cases:

I. $w < 3$

II. $4 \geqslant w \geqslant 3$

III. $w > 4$

and let

$$\left(0, \sum_{j \neq 1} w_j(1), \sum_{j \neq 1} w_j(2)\right)$$

be the (normalized) response of all the other agents.

Strategies in case I fail to be dominant when, for example, $\sum_{j \neq 1} w_j(1) = -3$ and $\sum_{j \neq 1} w_j(2) = -10$. The project $K = 0$ would be adopted, yielding a payoff of zero, whereas $\tilde{w} = (0, 3 + \epsilon, 5)$ would produce a payoff of one. Case III can be eliminated in a similar fashion.

In case II, the matter is more delicate: when $w < 4$ a counterexample similar to that in case I can be arranged by taking $-4 < \sum_{j \neq i} w_j(1) < -w$. The payoff to the strategy $w = (0, 3 + \epsilon, 5)$ would be superior.

When $w = 4$, let $\sum_{j \neq 1} w_j(1) = -3\frac{1}{2}$ and $\sum_{j \neq 1} w_j(2) = -4\frac{3}{4}$. Thus,

$$\sum_i w_i(1) = \tfrac{1}{2} \quad \text{and} \quad \sum_i w_i(2) = \tfrac{1}{4}$$

so that $d = 1$ and the net payoff to agent 1 is $+\frac{1}{2}$. By playing $w = (0, 3, 5)$, the decision $d = 2$ would be taken and a net payoff of $1\frac{1}{4}$ would be obtained.

Thus, the lack of a dominant strategy within the admissible strategy space—even though there would be a dominant strategy for a consumer with a large enough endowment—precluded the existence of satisfactory mechanisms in these environments.

References

1. E. H. Clarke, Multipart pricing of public goods, *Public Choice* (1971), 19–33.
2. J. Green and J. J. Laffont, Characterization of satisfactory mechanisms for the revelation of preferences for public goods, *Econometrica* **45** (1977), 427–438.
3. J. Green and J. J. Laffont, "Révélation des Préférences pour les Biens Publics: Première partie, Caractérisation des Mécanismes Satisfaisants" (E. Malvinaud, Ed.), pp. 83–103, Cahiers du Seminaire d'Économetrie No. 19, Centre National de la Recherche Scientifique, Paris, 1977.
4. J. Green, E. Kohlberg and J. J. Laffont, Partial equilibrium approach to the free-rider problem, *J. Public Econ.* **6** (1976), 375–394.
5. T. Groves, Incentives in teams, *Econometrica* **41** (1973), 617–631.
6. T. Groves and M. Loeb, Incentives and public inputs, *J. Public Econ.* **4** (1975), 211–226.
7. L. Johansen, The theory of public goods: Misplaced emphasis, *J. Public Econ.* **7** (1977), 147–152.
8. W. Vickrey, Counterspeculation, auctions and competitive sealed tenders, *J. Finance* **16** (1961), 1–17.
9. M. Walker, "On the Impossibility of a Dominant Strategy Mechanism to Optimally Decide Public Goods Questions," Department of Economics Working Paper No. 160, S.U.N.Y. at Stony Brook, Stony Brook, New York, 1976.

Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.