Dynamic Financial Contracting*

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1 Introduction

Taking stock of Modigliani and Miller's (1958) celebrated result that, with perfect capital markets, financial structure is irrelevant, corporate finance has studied how various market imperfections will make different capital structures more or less attractive. In line with the seminal insights of Jensen and Meckling (1976), a large fraction of the literature has focused on the conflicts of interests arising between investors and managers. When the latter have more information about their firms and their own actions than outside investors, an agency problem arises. Managers can take actions that are in the interest of investors, such as working hard to improve cash flows. Alternatively, they can choose to enjoy private benefits, at the expense of investors. For example, when it comes to hiring staff, managers could prefer friendly but inefficient family members, rather than more competent but potentially threatening outsiders. Or, they could engage in loss-making empire-building and prestige-driven activities. In the financial sector, managers might rely on ratings and brokers' advice, rather than conducting time- and resource-consuming checks on the quality of the assets they consider for their portfolios. When such actions are unobservable by investors and managers have limited liability, a moral-hazard problem arises.

In this context, the first generation of corporate finance models analyzed the equilibrium interaction between managers and investors, for a given type of financial contract, such as, for instance, debt or equity. As noted by Harris and Raviv (1992),

> "A much deeper question, however, is what determines the specific form of the contract (security) under which investors supply funds to the firm. [...] Therefore financial contract design must resolve the problem of allocating the cash flows generated to investors."

The financial contracting literature therefore characterizes optimal contracts designed to mitigate the agency conflicts between investors and managers. Seminal contributions along these lines were offered by Townsend (1979), Gale and Hellwig (1985), Innes (1990), and Bolton and Scharfstein (1990), among many others. This literature identified conditions under which debt contracts are optimal. Tirole (2001, 2006) provides a comprehensive unified treatment of corporate finance within an optimal contracting framework.

Although these approaches have proven fruitful, their applicability is limited by their reliance on models that typically feature one or two periods. In practice, however, financial contracting is inherently dynamic. Correspondingly, finance data sets include time series of cash flows and balance-sheet variables, as well as sequences of corporate events such as security issuances, dividends, or default. Dynamic models are needed to confront theoretical implications to such data. Furthermore, asset-pricing models are also generally dynamic, and often set up in continuous time. Analyses of dynamic financial contracting are therefore needed to bridge the gap between corporate finance and asset pricing, thus providing a unified theory of the design and valuation of securities.

Early contributions to the dynamic analysis of financial contracts were offered by Hart and Moore (1994) and Gromb (1999), in line with Bolton and Scharfstein (1990). More recently, the literature on dynamic financial contracting has made significant progress by relying on the recursive approach to dynamic contracting pioneered by Green (1987), Spear and Srivastava (1987), Thomas and Worall (1990), Phelan and Townsend (1991), and Atkeson and Lucas (1992). This led to the discrete-time models of Clementi and Hopenhayn (2006) and DeMarzo and Fishman (2007a, 2007b), followed by Biais, Mariotti, Plantin, and Rochet (2004, 2007). Further progress was made possible by the martingale methods introduced by Sannikov (2008). This led to the continuous-time models of DeMarzo and Sannikov (2006), Biais, Mariotti, Rochet, and Villeneuve (2010), and DeMarzo, Fishman, He, and Wang (2010). The goal of this paper is to survey some of the main results of this stream of literature in the context of a synthetic model.

In Section 2, we present our basic framework, drawing from Biais, Mariotti, Plantin, and Rochet's (2004) simple model of dynamic moral hazard. A principal and an agent dynamically interact in discrete time. Both are risk neutral. The agent has limited liability and must exert costly unobservable effort to make a project profitable. The principal funds the initial investment as well as the operating costs. Whereas we focus on hidden effort, one could alternatively model the agency problem in terms of cash-flow diversion, as Bolton and Sharfstein (1990) do. Indeed, as long as the agent is always requested to exert effort, the two models are isomorphic. Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007b), and Biais, Mariotti, Plantin, and Rochet (2007) analyze dynamic contracting when managers can divert cash. DeMarzo and Fishman (2007a) offer a general analysis encompassing several contractual imperfections, including hidden effort and cash-flow diversion.

Within this framework, in Sections 2 and 3, we revisit some of the results of Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007b), and Biais, Mariotti, Plantin, and Rochet (2004, 2007). The optimal contract relies on two state variables: the continuation utility of the agent, and the size of the project. Incentive compatibility requires that the continuation utility of the agent be raised after success and reduced after failure. Thus this state variable reflects the cumulative performance of the project. That the optimal contract depends on cumulative performance illustrates the memory property first identified by Rogerson (1985).[1] When performance reaches a milestone, the agent gets paid. By contrast, when performance is poor, the project can be downsized or even liquidated. Then, in Section 4, we consider two continuous-time limits of this model, namely, a Brownian limit, as in DeMarzo and Sannikov (2006) and Biais, Mariotti, Plantin, and Rochet (2007), and a Poisson limit, as in Biais, Mariotti, Rochet, and Villeneuve (2010).

Throughout the analysis, our main emphasis is on the implementation of the abstract optimal contract with financial instruments, and the empirical implications to which it leads. The project is undertaken within a firm. On the asset side of its balance sheet are its cash reserves and the value of its physical capital, as well as intangible assets corresponding to the net present value of the investment project. On the liability side of the balance sheet are the securities issued by the firm, namely, debt, held by the investors, and stocks, held in part by the investors and in part by the manager. The latter cannot sell his shares, lest this should jeopardize his incentives to exert effort.

This implementation of the optimal contract exhibits several features that are in line with stylized facts and empirical evidence. The greater the severity of the moral-hazard problem, the greater the fraction of the shares held by the manager. Unlike the optimal securities obtained in one-period models, which are defined as functions of a single cash flow, the securities implementing the optimal contract are defined as functions of streams of cash flows. Whereas debt pays a steady stream of coupons, stocks pay dividends only when performance milestones are reached, in line with the empirical findings of Kaplan and Strömberg (2003). When its liquidity ratio falls below a threshold, the firm must be downsized. This endogenizes why firms are at least partially liquidated when they run out of cash, even if their projects still have positive net present value.

In the Brownian limit, the dynamics of the stock price are given by a stochastic differential equation similar to that posited in Black and Scholes (1973) and Merton (1973). Two major differences are that the stock price can drop to zero, corresponding to the liquidation of the firm, and that its volatility is decreasing in the level of the stock price, in line with the leverage effect pointed out by Black (1976), Christie (1982), and Nelson (1991). Furthermore, the stock price decreases, and the credit yield spread increases, when the moral-hazard problem becomes more severe. Also, the payment scheme to the manager in the optimal contract has the same structure as the *high-water-mark* compensation schemes used in the hedge-fund industry: for the manager to receive a payment, it must be that cumulative performance

[1]See Laffont and Martimort (2002, Chapter 8) for an extensive discussion of this point.

exceeds its previous maximum.[2] In the Poisson limit, we also characterize the endogenous dynamics of firm size. We show that if the maximum feasible rate of investment in the project is low, then firm size shrinks to zero in the long run. By contrast, if the investment rate can be large, the firm grows without bounds in the long run, despite the agency problem.

2 The Discrete-Time Model when the Principal and the Agent Are Equally Impatient

2.1 The Model

There is a principal (referred to as "she") and an agent (referred to as "he"). Both are risk neutral and discount future payments at rate r. The principal has deep pockets, whereas the agent has limited liability and initial wealth A. They have access to an investment opportunity for which the managerial skills of the agent, as well as the funds of the principal, are needed.[3] Together they form a firm to operate this project.

The firm is started at initial size $X_0 \leq 1$. The unit cost of investment is c, so that the initial investment is $X_0 c$. The firm can operate in periods $n = 1, \ldots, N$, with N possibly infinite. Its size in period n is X_n. There are constant returns to scale. Thus, in period n, the project generates a cash flow equal to $X_n R_n$, where the size-adjusted cash flow R_n is independent of X_n. The firm can be successful, and generate revenues greater than its costs, resulting in a size-adjusted benefit $R_n = R^+ > 0$. Or it can fail, and generate revenues lower than its costs, resulting in a size-adjusted loss $R_n = R^- < 0$. Denote $\Delta R \equiv R^+ - R^-$.

In any period, the probability of a high or a low cash-flow realization depends only on the current effort exerted by the agent, so that, for a given effort profile, cash flows are independent over time. Moreover, the technology that generates these cash flows is the same in all periods. If the agent exerts effort in period n, which we denote by $e_n = 1$, the probability of a high cash-flow realization is $p > 0$. If the agent exerts no effort in period n, which we denote by $e_n = 0$, the probability of a high cash-flow realization is $p - \Delta p < p$, where $0 < \Delta p < p$. Thus shirking reduces the profitability of the project.

If the agent exerts no effort in period n, he enjoys private benefits $X_n B$ from shirking.[4]

[2] Goetzmann, Ingersoll, and Ross (2003) discuss the features of high-water-mark contracts and offer an analysis relying on asset-pricing tools. Whereas they take high-water marks as given, we show how they emerge as part of an optimal contract.

[3] Empirically, the assumption that the agent has unique necessary skills to run the project is particularly relevant in the case of small businesses, where the entrepreneur-manager is often indispensable for operating the firm efficiently (Sraer and Thesmar (2007)).

[4] We follow Holmström and Tirole (1997) and Tirole (2006) in reasoning in terms of a private benefit from shirking. One could equivalently cast the model in terms of a cost of exerting effort.

Note that we impose a constant-returns-to-scale assumption to private benefits as well as to cash flows. It is natural to assume that private benefits and cash flows are increasing in firm size. What is less obvious is why they should be proportional to firm size. We make this assumption because it allows us to reduce the number of relevant state variables and to derive fairly explicit results.[5]

At the beginning of period n, the firm can be irreversibly downsized from X_{n-1} to $X_n = x_n X_{n-1}$, where $x_n \in [0, 1]$.[6] Thus the size of the firm in period n is

$$X_n = X_0 \prod_{k=1}^{n} x_k.$$

For simplicity, the proceeds from liquidating a fraction of the firm are normalized to zero.[7] We assume that the project has positive net present value if the agent always exerts effort,

$$\sum_{k=1}^{N} \frac{pR^+ + (1-p)R^-}{(1+r)^{k-1}} > c, \tag{1}$$

and that it is inefficient to operate the project if the agent shirks,

$$(p - \Delta p)R^+ + (1 - p + \Delta p)R^- + B < 0. \tag{2}$$

Assumptions (1)–(2) imply that exerting effort is efficient,

$$\Delta p \Delta R > B.$$

A dynamic contract maps in any period the history of the firm up to that period into downsizing and compensation decisions. The first-best case arises when the agent's actions are observable and contractible. It follows from (1)–(2) that, in this first-best benchmark, it is optimal to undertake the project at full scale, and to require the agent to always exert effort; compensation decisions are irrelevant as long as they provide the agent with his required intertemporal utility. Under unobservable action choices and limited liability, the first-best

[5] It would be interesting, in future research, to investigate to which extent these results are robust when the constant-returns-to-scale assumption is relaxed. Biais, Mariotti, Rochet, and Villeneuve (2010) offer a heuristic analysis of a model in which private benefits given firm size X are given by $B^\varepsilon(X) \equiv XB + \varepsilon X \beta(X)$, for some small number ε and some bounded function β. Their analysis suggests that, under regularity conditions, the qualitative properties of the optimal contract can reasonably expected to be upheld for such a small perturbation. One can similarly conjecture that the results are robust to a small perturbation in the specification of cash flows.

[6] The possibility of increasing the size of the firm through investment will be considered in Section 3.3.

[7] One could easily allow for a positive resale value. Setting the resale value below c captures the notion that distressed firms' assets often trade at unfavorable prices. Even if the resale price were equal to c, downsizing would generate inefficiencies, to the extent that it would imply an irreversible reduction in the scale of operation of a positive net present value project.

outcome may not be attainable. The second-best contract maximizes expected surplus, subject to feasibility, participation, limited-liability, and incentive-compatibility constraints. To focus on a single imperfection, namely, managerial moral hazard, we assume that the principal and the agent can fully commit to a long-term contract.

At the beginning of the contractual relationship, the contract is designed and $X_0 c$ is invested in the firm. Then, in any period n, the timing of events is the following:

- First, there may be downsizing, $x_n < 1$, or no change in firm size, $x_n = 1$.
- Second, the agent may decide to exert effort, $e_n = 1$, or to shirk, $e_n = 0$.
- Third, the project may be successful, $R_n = R^+$, or unsuccessful, $R_n = R^-$.
- Fourth, the agent receives a compensation $X_n u_n \geq 0$, and the principal receives the remaining cash flow $X_n(R_n - u_n)$.

Let $\mathcal{H}_n$ be the publicly available information at the beginning of period n, that is, the history of realized cash flows and investment and downsizing decisions until period n. Then X_n is measurable with respect to $\mathcal{H}_n$, whereas u_n also depends on the realization of the period n cash flow R_n.

2.2 The One-Period Case

To build intuition, consider the one-period case, $N = 1$, which corresponds to the baseline corporate finance model analyzed in Holmström and Tirole (1997) and further developed in Tirole (2006). The size-adjusted compensation of the agent is u^+ if the project is successful, and u^- otherwise. The incentive-compatibility constraint of the agent is

$$pu^+ + (1-p)u^- \geq (p - \Delta p)u^+ + (1 - p + \Delta p)u^- + B,$$

where the left-hand side is the size-adjusted expected utility of the agent when he exerts effort, and the right-hand side its counterpart when he shirks. This simplifies to

$$\Delta p(u^+ - u^-) \geq B. \tag{3}$$

To relax the incentive-compatibility constraint, it is optimal to set $u^- \equiv 0$, yielding

$$u^+ \geq \frac{B}{\Delta p}. \tag{4}$$

The ratio $\frac{B}{\Delta p}$ is the minimum rent that must be promised to the agent in case of success to induce him to exert effort, and measures the severity of the moral-hazard problem. As B

increases, it becomes more tempting for the agent to shirk, and as Δp decreases, it becomes more difficult for the principal to detect shirking. Inequality (4) implies that, in size-adjusted terms, the one-period pledgeable income, that is, the maximum expected revenue that can be promised to the principal without jeopardizing the incentives of the agent, is

$$pR^{+} + (1-p)R^{-} - p\,\frac{B}{\Delta p}.$$

The participation constraint of the principal is

$$X_0[p(R^{+} - u^{+}) + (1-p)(R^{-} - u^{-})] \geq X_0 c - I_A,$$

where the left-hand side is the expected net cash flow received by the principal, whereas the right-hand side is the initial funding she provides, with $I_A \leq A$ denoting the initial investment of the agent. Setting $u^{-} \equiv 0$, the participation constraint of the principal is consistent with the incentive-compatibility constraint of the agent if and only if

$$X_0\left[pR^{+} + (1-p)R^{-} - p\,\frac{B}{\Delta p} - c\right] \geq -I_A. \tag{5}$$

If the one-period pledgeable income is not less than the investment cost, so that the bracketed term in (5) is nonnegative, the venture can be undertaken at full scale with no funding from the agent, that is, with $X_0 = 1$ and $I_A = 0$. By contrast, if

$$pR^{+} + (1-p)R^{-} - p\,\frac{B}{\Delta p} < c,$$

funding by the agent is requested. Binding the incentive-compatibility and participation constraints then yields the largest possible initial scale of operation for the firm,

$$X_0 = \min\left\{\frac{A}{c + p\,\frac{B}{\Delta p} - pR^{+} - (1-p)R^{-}}\,, 1\right\}. \tag{6}$$

This foreshadows a result we will obtain in the dynamic model: when the moral-hazard problem is severe, that is, when $\frac{B}{\Delta p}$ is high, and the wealth A of the agent is low, it is necessary to operate the project at a smaller scale than what would be efficient in the first-best benchmark. In the following, we maintain the assumption that

$$pR^{+} + (1-p)R^{-} < p\,\frac{B}{\Delta p}.$$

Hence the moral-hazard problem is so severe that the one-period pledgeable income is negative. Yet we will see that, for some parameter values, when the principal and the agent interact over several periods, that is, $N > 1$, the project can be undertaken even if the agent has no initial wealth, that is, $A = 0$. Thus dynamic contracting alleviates the moral-hazard problem.

2.3 The Infinite-Horizon Case

Now, consider the case where $N = \infty$. The analysis below is then in line with the infinite-horizon analyses of Clementi and Hopenhayn (2006) and Biais, Mariotti, Plantin, and Rochet (2004, 2007), and is the stationary counterpart of the finite-horizon analysis of DeMarzo and Fishmann (2007b).[8] Notice that Clementi and Hopenhayn (2006) also allow for short-lived investment in working capital, which raises the scale of operations in the current period. In that respect, their framework is richer than the present one. For simplicity, we restrict the analysis to the case where the agent must always exert effort, that is, $e_n = 1$ for all $n \geq 1$. Biais, Mariotti, Plantin, and Rochet (2004, Proposition 13) show that when the adverse consequences of shirking are severe, it is indeed optimal to always request high effort from the agent.

2.3.1 Continuation Utilities

On the equilibrium path, where effort is exerted, the expected continuation utility of the agent at the beginning of period n is

$$W_n \equiv \mathbf{E}\left[\sum_{k=0}^{\infty} \frac{X_{n+k} u_{n+k}}{(1+r)^k} \,|\, \mathcal{H}_n\right],$$

and the expected discounted future profit of the principal at the beginning of period n is

$$F_n \equiv \mathbf{E}\left[\sum_{k=0}^{\infty} \frac{X_{n+k}(R_{n+k} - u_{n+k})}{(1+r)^k} \,|\, \mathcal{H}_n\right],$$

where $\mathbf{E}[\,\cdot\,|\,\mathcal{H}_n]$ is the expectation operator conditional on $\mathcal{H}_n$ and the agent always exerting effort, and the controls are X_{n+k} and u_{n+k}. The size-adjusted continuation utility of the agent and the size-adjusted value of the principal are

$$w_n \equiv \frac{W_n}{X_{n-1}},$$
$$f_n \equiv \frac{F_n}{X_{n-1}}.$$

Because x_n is measurable with respect to $\mathcal{H}_n$, these can be recursively expressed as

$$w_n = x_n \mathbf{E}\left[u_n + \frac{w_{n+1}}{1+r} \,|\, \mathcal{H}_n\right], \tag{7}$$

$$f_n = x_n \mathbf{E}\left[R_n - u_n + \frac{f_{n+1}}{1+r} \,|\, \mathcal{H}_n\right]. \tag{8}$$

The equality (7) is often referred to as the *promise-keeping constraint.*

[8] One of the advantages of the finite-horizon analysis of DeMarzo and Fishman (2007b) is that it clarifies the recursive nature of the problem, operating backward from a finite value of N. On the other hand, one advantage of the stationary setting we focus on is that it simplifies the computations, and hence enables one to obtain more explicit results.

2.3.2 The Optimal Contract

It can be shown that the optimal dynamic contract is contingent on two state variables only: the size-adjusted continuation utility of the agent, w, and the size of the project, X. Furthermore, because of the constant-returns-to-scale assumption, the downsizing factor, x, the size-adjusted compensation of the agent after a success, u^+, or after a failure, u^-, and the size-adjusted continuation utility of the agent after a success, w^+, or after a failure, w^-, only depend on w. Thus the promise-keeping constraint (7) rewrites as

$$w = x\left[pu^+ + (1-p)u^- + \frac{pw^+ + (1-p)w^-}{1+r}\right], \tag{9}$$

whereas (8) leads to the following Bellman equation:

$$f(w) = \max\left\{x\left[p(R^+ - u^+) + (1-p)(R^- - u^-) + \frac{pf(w^+) + (1-p)f(w^-)}{1+r}\right]\right\}, \tag{10}$$

where the maximization in (10) is over the controls x, u^+, u^-, w^+, and w^-, subject to the feasibility and limited-liability constraints

$$(x, u^+, u^-, w^+, w^-) \in [0,1] \times [0,\infty)^4, \tag{11}$$

the promise-keeping constraint (9), and the incentive-compatibility constraint

$$\begin{aligned} pu^+ + (1-p)u^- &+ \frac{pw^+ + (1-p)w^-}{1+r} \\ &\geq B + (p-\Delta p)u^+ + (1-p+\Delta p)u^- + \frac{(p-\Delta p)w^+ + (1-p+\Delta p)w^-}{1+r}. \end{aligned}$$

The latter constraint simplifies to

$$\Delta p\left[\left(u^+ + \frac{w^+}{1+r}\right) - \left(u^- + \frac{w^-}{1+r}\right)\right] \geq B. \tag{12}$$

Constraint (12) is similar to its one-period counterpart (3). In both, the left-hand side is the effect of effort on the expected size-adjusted compensation of the agent, whereas the right-hand side is the size-adjusted private benefit from shirking. The difference between (12) and (3) lies in the continuation utilities w^+ and w^-, which reflect the effect of current effort on expected future compensation. Relying on such deferred payments helps relaxing the incentive-compatibility constraint.

Using standard techniques (see, for instance, Stokey and Lucas, with Prescott (1989)), one can show that there exists a unique solution f to (10) subject to (9), (11), and (12) (Biais, Mariotti, Plantin, and Rochet (2004, Proposition 1)). The value function f is continuous, concave, and vanishes at $w = 0$. To spell out the optimal contract, define the two following thresholds for the agent's size-adjusted continuation utility:

$$w^l \equiv \frac{pB}{\Delta p}, \tag{13}$$

$$w^p \equiv \frac{1+r}{r}\frac{pB}{\Delta p}. \tag{14}$$

The following result, a proof of which can be found in Biais, Mariotti, Plantin, and Rochet (2004, Proposition 3), characterizes an optimal contract.[9]

Proposition 1 *Suppose that $p > r$. Then, in any optimal contract, there is no downsizing as long as $w \geq w^l$, and the agent receives no payment in case of failure as long as $w \leq w^p$. The following describes an optimal contract:*

(i) *w^p is an absorbing boundary for the continuation utility of the agent. Once it is reached, the project is operated with certainty forever, and in any subsequent period, the agent is paid $u^+ = \frac{B}{\Delta p}$ in case of success, whereas in case of failure he is not paid, $u^- = 0$.*

(ii) *When $w \in [w^l, w^p)$, the agent is paid $u^+ = \max\{w - (w^p - \frac{B}{\Delta p}), 0\}$ in case of success and his continuation utility then moves up to $w^+ = \min\{(1+r)[w + (1-p)\frac{B}{\Delta p}], w^p\}$, whereas in case of failure he is not paid, $u^- = 0$, and his continuation utility moves down to $w^- = (1+r)(w - p\frac{B}{\Delta p})$.*

(iii) *When $w \in (0, w^l)$, the firm is downsized, $x = \frac{w}{w^l}$, and there are no immediate payments, $u^+ = u^- = 0$. The continuation utility of the agent moves up to $w^+ = \min\{(1+r)[w^l + (1-p)\frac{B}{\Delta p}], w^p\}$ in case of success, whereas in case of failure it moves down to $w^- = 0$ and the firm is liquidated.*

The value function f corresponding to the optimal contract stated in Proposition 1 is linear over $[0, w^l]$, and affine with slope -1 over $[w^p, \infty)$. As mentioned in Proposition 1, when $w \leq w^p$ the agent gets no current compensation after a failure. This feature of the optimal contract arises, like in the one-period case, in order to relax the incentive-compatibility constraint. Once the agent's size-adjusted continuation utility reaches w^p, the firm is insulated from the risk of liquidation, and financial constraints cease to bind. Because the principal and the agent are risk neutral and discount future payments at the same rate, there are actually many ways to induce the agent to exert effort from that point

[9] All quantities in the statement of Propositions 1 and 3 are in size-adjusted terms. This is not mentioned systematically in order to make the exposition more concise.

on. Proposition 1 focuses on the case where incentives are provided by current size-adjusted payments of $\frac{B}{\Delta p}$ in case of success; consistent with this, w^p as given by (14) is the value of a perpetual annuity paying $\frac{B}{\Delta p}$ with probability p in every period. Note that, although the firm faces no risk of liquidation once the agent's size-adjusted continuation utility has reached w^p, say, in period n, this does not mean that the first-best outcome is always attained in such case. Indeed, the firm may have been downsized prior to period n, resulting in $X_n < 1$. Then, the project is not operated at full scale, unlike in the first-best benchmark.

According to Proposition 1, the agent is not paid until his size-adjusted continuation utility exceeds $w^p - \frac{B}{\Delta p}$ and the project is successful, which finally brings his size-adjusted continuation utility to the absorbing boundary w^p. Why not reward the agent earlier, that is, for lower values of w? The reason is that this would be dominated by promising the same expected discounted payment, contingent on his size-adjusted continuation utility reaching this upper bound, which would provide stronger incentives to the agent, at the same cost for the principal. The result that it is optimal to defer payments to the agent is in line with Becker and Stigler (1974).

Over the range $\left[w^l, w^p - \frac{B}{\Delta p}\right)$, the agent is solely motivated by promises: his size-adjusted continuation utility evolves as a function of the performance of the project, increasing after successes and decreasing after failures, so that

$$w_{n+1} = (1+r)(w_n + k\varepsilon_n), \tag{15}$$

where $\varepsilon_n \equiv R_n - \mathbf{E}[R_n]$ is the innovation in the cumulative cash flow under effort, and

$$k \equiv \frac{B}{\Delta p \Delta R} < 1 \tag{16}$$

measures the sensitivity of the agent's reward to the performance of the project, which increases in the severity of the moral-hazard problem, as measured by $\frac{B}{\Delta p}$. Over this range, the size-adjusted continuation utility of the agent is an r–discounted martingale,

$$\mathbf{E}[w_{n+1} \,|\, \mathcal{H}_n] = (1+r)w_n.$$

Deviations around this deterministic trend reflect the performance of the project.

Consider finally the range $[0, w^l)$. To motivate the agent, his continuation utility must be reduced after a failure. However, for $w < p\frac{B}{\Delta p}$, such a reduction conflicts with the limited-liability constraint. To restore the consistency between limited liability and incentive compatibility, downsizing is then necessary. Downsizing reduces the private benefit from shirking, thus making it commensurate with the reduction in the agent's continuation utility that can be implemented without violating the limited-liability constraint.

2.3.3 Initialization

If, at the outset of the contractual relationship, the size-adjusted continuation utility of the agent is set at a level greater than or equal to w^p, downsizing is completely avoided. This is feasible if, for the principal, the corresponding size-adjusted value $f(w^p)$ is greater than the investment cost, that is,

$$\frac{1+r}{r}\left[pR^+ + (1-p)R^-\right] - w^p \geq c - A. \tag{17}$$

The first term on the left-hand side of (17) is the present value of the cash flows generated by the firm, if it is never downsized and the agent always exerts effort. The second one is the present value of the payments to the agent. If condition (17) holds, the firm can be indefinitely operated at full scale, and the first-best outcome is achieved.

When condition (17) does not hold, this outcome cannot be immediately attained. But can the project be operated, and at which scale? The maximum size-adjusted expected income that can be pledged to the principal is

$$\max_{w\in[0,w^p]} \{f(w)\}.$$

Denote by $w^* < w^p$ the value of w for which $f(w)$ is maximal. The participation constraint of the principal is

$$X_0[f(w^*) - c] \geq -I_A,$$

which is the dynamic counterpart of (5). If the left-hand side of this inequality is nonnegative, the project can be initially operated at full scale, no matter the initial wealth A of the agent. Otherwise, the maximum possible initial size of the project is

$$X_0 = \min\left\{\frac{A}{c - f(w^*)}, 1\right\}.$$

It is obtained when $I_A = A$, and is the dynamic counterpart of (6).

2.4 Implementing the Optimal Contract with Cash and Securities

We now study which financial instruments can be used to implement the optimal contract characterized in Proposition 1. In general, several implementations are possible. Indeed, as long as feasibility, participation, limited-liability, and incentive-compatibility constraints are satisfied, the Modigliani and Miller (1958) logic applies: slicing and dicing of cash flows is irrelevant. To narrow down the set of implementations, we shall therefore impose two restrictions.

First, we impose that the stake of the principal be implemented with securities, that is, claims with limited liability, excluding negative payoffs. These claims can be held by a diffuse investor basis.[10] Now, cash flows are negative in case of failure. Hence, to avoid negative payoffs to outside investors, the firm must hold cash reserves. Denote the period n cash reserves by M_n. The change in the level of these reserves is equal to the sum of the interest received on them and of the operating cash flow, minus the compensation to the agent and the payments to the investors. Thus M_n corresponds to initial cash holdings plus accumulated earnings until period n. Second, we impose that the firm be liquidated when it runs out of cash. It should be noted that these two restrictions on the implementation of the optimal mechanism do not affect in any way the efficiency of the outcome: we *are* implementing the optimal contract.

In the implementation of the optimal contract, the project is undertaken within a firm. The initial balance sheet of this firm is depicted in Table 1.

Assets	Liabilities
Current assets M_0	Debt D_0
Tangible fixed assets X_0c	Equity S_0
Intangible assets NPV_0	

Table 1 The initial balance sheet of the firm.

On the liability side is the market value of the securities issued by the firm: debt, D_0, held by the investors, and equity, S_0, held in proportion k by the agent, and $1-k$ by the investors. On the other side of the balance sheet, one must distinguish between tangible and intangible assets. Tangible assets are the sum of current assets, M_0, and of tangible fixed assets, X_0c. Intangible assets are the difference between the total market value of the firm, $S_0 + D_0$, and its book value $X_0c + M_0$. This difference is equal to the net present value of the project, $NPV_0 \equiv \mathbf{E}\left[\sum_{k=1}^{\infty} \frac{X_k R_k}{(1+r)^{k-1}}\right] - X_0c$.[11]

[10] This is in line with the assumption of full commitment to the contract, because coordination problems make renegotiation difficult for dispersed claim holders.

[11] To the extent that it reflects the gap between the market value and the book value, this difference can be interpreted in terms of goodwill.

Dividing current assets, M_n, by tangible fixed assets, $X_n c$, one obtains the liquidity ratio

$$m_n \equiv \frac{M_n}{X_n c}.$$

To characterize the implementation of the optimal contract, we define two thresholds for this liquidity ratio, in line with (13) and (14):

$$\begin{aligned} m^l &\equiv \frac{w^l}{kc} = \frac{p\Delta R}{c}, \\ m^p &\equiv \frac{w^p}{kc} = \frac{1+r}{r}\frac{p\Delta R}{c}, \end{aligned}$$

where k is defined by (16). The following result, a proof of which can be found in Biais, Mariotti, Plantin and Rochet (2004, Proposition 4), shows how to implement the optimal contract with cash reserves and securities.

Proposition 2 *The optimal contract can be implemented with cash reserves, stocks, and bonds, in such a way that the liquidity ratio is equal to*

$$m_n = \frac{w_n}{kc} \tag{18}$$

in any period n. Specifically,

(i) The firm initially issues stocks and bonds, and grants a fraction k of the stocks to the agent, who is prohibited from selling them. With the proceeds from the issuance, the firm acquires its assets and hoards cash corresponding to an initial liquidity ratio m_0.

(ii) The stock distributes a size-adjusted dividend $\phi_n \equiv \max\{c(m_n - m^p) + \Delta R, 0\}$ in case of success in period n. The bonds are consol bonds distributing in any period n a constant size-adjusted coupon $\psi \equiv pR^+ + (1-p)R^-$.

(iii) When $m_n \in (0, m^l)$, the firm is not liquid enough to meet its short-term commitments in period n. It is therefore downsized, $x_n = \frac{m_n}{m^l}$, after which the implementation starting with a size-adjusted liquidity ratio m^l is immediately executed. When $m_n = 0$, the firm is liquidated.

In the dynamic optimal contract characterized in Proposition 1, one of the key state variables is the rent of the agent, w_n. In the implementation of the contract, payout and downsizing decisions are contingent on the liquidity ratio m_n, which, according to (18), perfectly co-moves with w_n.

The roles of inside and outside equity and the nature of dividends in Proposition 2 are similar to those obtained by DeMarzo and Fishman (2007b). There is, however, a difference between their implementation and the one presented above, in that they consider a credit line instead of cash reserves. The mechanism is then as follows: when cash flows are not high enough to cover costs, the firm draws on its credit line. If the firm repeatedly draws on its credit line and eventually exhausts it, it is liquidated.

2.5 Empirical Implications and Stylized Facts

The implementation of the optimal contract emphasizes the role of the firm's cash reserves. In particular, it shows that the firm must maintain a minimum liquidity ratio, and be downsized when its liquidity ratio falls below this threshold. This endogenizes why firms enter financial distress and are partially liquidated when they run out of cash, even if their projects still have positive net present value.[12]

Unlike the optimal securities obtained in one-period models, which are defined as claims on a single cash flow, the securities implementing the optimal contract are defined as claims on streams of cash flows. Debt is a claim on a steady stream of coupons, equal to expected operating cash flows. Equity is a claim on a more irregular stream of cash flows, paid only when cash reserves reach a contractually specified threshold. As pointed out by Brealey, Myers, and Allen (2008, Chapter 25), bond covenants typically include clauses restricting the set of circumstances in which dividends can be paid. In particular, contractual clauses often preclude dividend payments when the firm does not have a sufficiently large ratio of liquid assets to total assets, as predicted by the model. The above theoretical results are also consistent with the empirical findings of DeAngelo, DeAngelo, and Stulz (2006), who observe a significant positive relationship between the distribution of dividends and the level of retained earnings, which amount to cash reserves in the model. In line with our emphasis on the liquidity ratio, firms that consistently paid dividends in their sample displayed a large ratio of cash balances to total assets. The model also implies that dividends are paid only after the firm has established a sufficient performance record. This is in line with the stylized fact that dividends are paid by large and mature firms, whereas young firms, especially in the high-tech industry, pay no cash to shareholders for long periods of time (Bulan, Subramaniam, and Tanlu (2007)).

In line with stylized facts on executive compensation (Dial and Murphy (1995), Murphy (1999)), the agent who manages the project is compensated with restricted stocks, which he

[12] Denis and Shome (2005) offer empirical evidence on the downsizing of financially distressed firms.

cannot sell; if he could, this would curb his incentives to exert effort. This is consistent with the empirical findings of Kaplan and Strömberg (2003) that financial contracts typically specify that managers receive cash compensation only when performance milestones are reached. Another implication of the model is that the agent's equity share k increases with the severity of the moral-hazard problem, as measured by $\frac{B}{\Delta p}$. This is consistent with the empirical findings of Kaplan and Strömberg (2004) that the use of performance benchmarks increases with asymmetric information about the operations of the firm.

3 The Discrete-Time Model when the Agent Is More Impatient than the Principal

When the principal and the agent are equally impatient, the optimal contract allows for several optimal compensation policies. One of them, stated in Proposition 1, is to pay the agent as soon as his size-adjusted continuation utility reaches w^p. Equivalently, one could delay payments further, while capitalizing them at rate r. Because this is the discount rate of the agent as well as of the principal, and because both of them are risk neutral, they would be indifferent to such a variation. By contrast, when the agent is more impatient than the principal, this multiplicity problem does not arise, and the agent's optimal compensation is uniquely determined, as we will see below.

Now, in practice, it is plausible that the agent will indeed be more impatient to consume than the principal. Consider a large population of potential principals, with different discount rates, some equal to r, and some higher. The most impatient of them, with discount rate higher than r, will incline towards consuming immediately and borrowing on the market at rate r, rather than investing in a project. The most patient of them, with discount rate equal to r, are the best placed to fund a project. They will naturally constitute the population of investors serving as principals in the model. By contrast, in presence of moral hazard, an agent managing an investment project cannot freely borrow to consume early. Delaying his consumption until the cumulative performance of his project reaches a milestone is necessary to ensure incentive compatibility. Moreover, to preserves incentives, the optimal contract must preclude him from borrowing from third parties and consuming early. Therefore, high impatience is a key consideration for the agent.

In this section, we study a dynamic moral-hazard model that is identical to the model considered in the previous section, except that the discount rate of the agent, ρ, is strictly greater than the discount rate of the principal, r. This will pave the way to the analysis of the different continuous-time limits of the model that we explore in Section 4.

3.1 The Optimal Contract

Like in the previous section, the optimal contract hinges on two thresholds. For simplicity, we still denote them by w^l and w^p. But, unlike when the principal and the agent are equally impatient, we no longer have explicit expressions for w^l and w^p. In addition, it is useful to define a third threshold $w^r \equiv (1+\rho)\big(w^p - p\,\frac{B}{\Delta p}\big)$, which turns out to be lower than w^p in the optimal contract. Under parameter restrictions spelled out in Biais, Mariotti, Plantin, and Rochet (2004, Proposition 6), the optimal contract is as follows.[13]

Proposition 3 *Suppose that $\rho > r$. Then, in the optimal contract, there is no downsizing as long as $w \geq w^l$, and the agent receives no payment in case of failure as long as $w \leq w^p$. The optimal contract is as follows*:

(i) When $w \in [w^l, w^p)$, the agent is paid $u^+ = \max\big\{w - \big(w^p - \frac{B}{\Delta p}\big), 0\big\}$ in case of success and his continuation utility then moves up to $w^+ = \min\big\{(1+\rho)\big[w + (1-p)\frac{B}{\Delta p}\big], w^r\big\}$, whereas in case of failure he is not paid, $u^- = 0$, and his continuation utility moves down to $w^- = (1+\rho)\big(w - p\,\frac{B}{\Delta p}\big)$.

(ii) When $w \in (0, w^l)$, the firm is downsized, $x = \frac{w}{w^l}$, after which the optimal contract starting with a continuation utility w^l is immediately executed. When $w = 0$, the firm is liquidated.

There are similarities between Proposition 1 and Proposition 3. The value function f corresponding to the optimal contract stated in Proposition 3 is continuous, concave, and vanishes at $w = 0$. Moreover, it is linear over $[0, w^l]$, and affine with slope -1 over $[w^p, \infty)$. A further similarity is that, over the range $\big[w^l, w^p - \frac{B}{\Delta p}\big)$, the size-adjusted continuation utility of the agent is a ρ–discounted martingale,

$$w_{n+1} = (1+\rho)(w_n + k\varepsilon_n), \tag{19}$$

in analogy with (15).

The main difference between Proposition 1 and Proposition 3 is that, whereas, in the former, w^p acted as an absorbing boundary for the agent's size-adjusted continuation utility, in the latter, $w^r < w^p$ acts as a reflecting boundary. When w^r is hit, after a success, the agent receives an immediate payment. Then, in case of a further success, the agent's size-adjusted continuation utility remains at w^r, whereas, in case of a failure, it is reflected downward. The

[13] These parameter restrictions ensure the existence of an intermediary region where the project is not downsized, but the agent is compensated solely by promises of future payments.

intuition is the following. Suppose the upper bound on the agent's size-adjusted continuation utility were absorbing. This would mean that the principal would have to wait for a long string of successes until the agent would receive any payment, after which the project would be insulated from downsizing risk. When the principal and the agent are equally impatient, it is costless to wait that long. By contrast, when the agent is more impatient than the principal, it is efficient to trade off increased downsizing risk for earlier consumption. As a result, the reflecting boundary w^r is lower than the value of a perpetual annuity paying $\frac{B}{\Delta p}$ with probability p in every period, and capitalized at rate ρ.

Proposition 3 implies that, when $\rho > r$, the firm is never insulated from the risk of downsizing. This has dramatic consequences on the asymptotic size of the firm, as stated in the next proposition, which is proven in the appendix.

Proposition 4 *When $\rho > r$, the size of the firm shrinks to zero in the long run,*

$$\lim_{n\to\infty} X_n = 0,$$

P–*almost surely.*

The intuition is the following. By Proposition 3, w_n follows a Markov chain that is reflected at w^r. In particular, from any initial state w_1, w_n reaches the downsizing region with strictly positive probability. Now, two cases must be distinguished. If $w^l = p\,\frac{B}{\Delta p}$, the zero absorbing boundary can be reached from any initial state w_1. As a result, the unique ergodic distribution for w_n has all its mass at zero, reflecting that the firm is liquidated in finite time with probability one. By contrast, if $w^l > p\,\frac{B}{\Delta p}$, the zero absorbing boundary will never be reached, unless of course the initial state w_1 is zero. As a result, the firm will never be liquidated. Yet one can show that, in this case, with probability one, w_n visits infinitely often the downsizing region, driving the size of the firm to zero. Note from Proposition 1 that things are qualitatively different when the principal and the agent are equally impatient. In that case, there are *two* absorbing boundaries for w_n, namely, zero and w^p. Thus, whereas the ergodic distribution of w_n still puts mass on zero, the firm may have strictly positive size in the long run. This happens when the size-adjusted continuation utility of the agent reaches w^p in period n, after which the size of the firm forever remains at X_n.

That in the long run the continuation utility of the agent, $W_n = X_{n-1}w_n$, goes to its zero lower bound is reminiscent of the classical immiserization result of Thomas and Worall (1990). The economic forces at play in the two models are different, however. In Thomas and Worall (1990), the period utility function of the agent is concave and unbounded below.

Therefore, providing incentives is cheaper, the lower the agent's continuation utility, which gives the principal an incentive to let it drift downwards. By contrast, in the present model, both the principal and the agent are negatively affected when the latter's continuation utility becomes very low, because this raises the cost of incentives, and, consequently, the risk of an inefficient liquidation. Yet the combined effect of impatience and incentive compatibility is that the size of the firm, and, as a result, the agent's continuation utility, drift down to zero in the long run.

3.2 Implementation

As shown in Biais, Mariotti, Plantin and Rochet (2007, Proposition 2), one can implement the optimal contract when $\rho > r$ in a similar fashion as when $\rho = r$. The main difference is that, when the agent is more impatient than the principal, debt service is no longer constant per unit of size of the firm. Rather, the size-adjusted coupon decreases with size-adjusted cash reserves, which reflect cumulative firm performance. Specifically, the size-adjusted coupon in period n is now given by

$$\psi_n \equiv pR^+ + (1-p)R^- - (\rho - r)\,\frac{M_n}{X_n}. \tag{20}$$

This is equal to expected size-adjusted operating cash flows, minus the opportunity cost of holding cash for the agent. This implementation is consistent with clauses observed in practice in financial contracts. Kaplan and Strömberg (2003) find that venture capitalists often hold preferred shares, which are similar to bonds in that they deliver contractually specified revenues, to be paid before any dividend. As observed by Kaplan and Strömberg (2003, Table 3), the contracts defining these claims typically include clauses stating that their revenue is reduced if performance goals are attained, or that their owner is entitled to additional compensation if the performance of the firm lies below a certain threshold. The payments on the bonds when $\rho > r$ are also in line with those of step-up bonds, which have been issued in large amounts over the recent years, especially in the European telecom industry (see, for instance, Lando and Mortensen (2004), and Bielecki, Vidozzi, and Vidozzi (2008)). Such bonds have provisions stating that the coupon payments increase as the credit rating of the issuer deteriorates, which in the model corresponds to a decline in cash reserves. The bonds in the implementation are also similar to performance-pricing loans, for which the interest is tied to some pre-specified measure of the performance of the borrower.[14] Asquith, Beatty, and Weber (2005) document the prevalence of such clauses in bank loans.

[14]Tchistyi (2006) also obtains performance-pricing loans in a dynamic-contracting model.

3.3 Investment

So far, we have restricted our attention to the case where investment is ruled out after the initial date and firm size changes only when there is downsizing. Thus $X_n \leq X_{n-1}$ in any period n, where the inequality is strict when there is downsizing. We now turn to the more general case where investment is possible throughout the life of the project, as in DeMarzo and Fishman (2007a). Denote by g_n the rate at which the firm grows, that is,

$$g_n = \frac{X_n - X_{n-1}}{X_{n-1}}.$$

Positive growth $g_n > 0$ entails a per unit investment cost, which we denote by $c(g_n)$. A convenient formulation of this cost function is used in Biais, Mariotti, Rochet, and Villeneuve (2010), who assume that, for some constants $(c, \gamma) \in [0, \infty) \times (0, r)$,

$$c(g) = \begin{cases} cg & \text{if} \quad g \leq \gamma, \\ \infty & \text{if} \quad g > \gamma. \end{cases} \tag{21}$$

Alternatively, one may consider a quadratic specification of the investment cost function, as DeMarzo, Fishman, He, and Wang (2010) do. Under these specifications, firm growth is limited, as large growth rates are very costly. This is in line with the macroeconomic literature emphasizing the delays and costs associated with investment, such as time-to-build constraints (Kydland and Prescott (1982)) or convex adjustment costs (Hayashi (1982)).

With investment, the recursive formulation of the value function of the principal is

$$F(X_{n-1}, W_n) = \max \left\{ \mathbf{E}\left[X_n(R_n - u_n) + \frac{F(X_n, W_{n+1})}{1+r} \,|\, \mathcal{H}_n \right] - c(g_n)X_{n-1} \right\}, \tag{22}$$

subject to the limited-liability, promise-keeping, and incentive-compatibility constraints. The expectation in (22) is taken anticipating that the agent will exert effort in period n and conditional on $\mathcal{H}_n$. Note also that the maximization is over the controls (g_n, u_n, W_{n+1}), where $u_n = u^+$ after success and $u_n = u^-$ after failure, $W_{n+1} = X_n w^+$ after success and $W_{n+1} = X_n w^-$ after failure, and $X_n = (1 + g_n)X_{n-1}$.

When there is no moral hazard, the social value generated by the firm does not depend on the rent left to the agent, but only on the size of the project. Denoting the first-best social value function by V^*, we have

$$F^*(X_{n-1}, W_n) = V^*(X_{n-1}) - W_n.$$

Here V^* satisfies the following Bellman equation:

$$V^*(X_{n-1}) = \max_{g_n \geq 0} \left\{ (1 + g_n)X_{n-1}\,\mathbf{E}[R_n] + \frac{V^*((1 + g_n)X_{n-1})}{1+r} - c(g_n)X_{n-1} \right\}, \tag{23}$$

where the first term on the right-hand side of (23) is the current cash flow, the second term is the discounted expected continuation value, and the third term is the cost of investment. Because of constant returns to scale, the first-best social value function is linear in the size of the firm; that is, the size-adjusted social value is a constant v^*:

$$V^*(X) = v^* X \tag{24}$$

for all $X \geq 0$. Substituting (24) into (23), and focusing on the linear specification (21), it is optimal to invest as soon as

$$\mathbf{E}[R_n] + \frac{v^*}{1+r} \geq c, \tag{25}$$

where the right-hand side is the marginal cost of investment, whereas the left-hand side is its marginal benefit. If this inequality holds, then it is optimal to invest in each period, and the size-adjusted social value satisfies

$$\frac{v^*}{1+r} = \frac{(1+\gamma)\mathbf{E}[R_n] - c\gamma}{r-\gamma},$$

which is the value of a perpetual annuity discounted at the appropriate rate, $r-\gamma$. In that case, (25) becomes

$$\mathbf{E}[R_n] + \frac{(1+\gamma)\mathbf{E}[R_n] - c\gamma}{r-\gamma} \geq c, \tag{26}$$

or, equivalently, $\frac{1+r}{r}\,\mathbf{E}[R_n] \geq c$. Condition (26) only involves exogenous parameters. When (26) holds, it is indeed optimal to continuously invest at rate γ. By contrast, if parameters are such that (26) does not hold, it is optimal to never invest.[15]

Remark. In the continuous-time limit (which we analyze more precisely in Section 4.2.4 below), the terms $\mathbf{E}[R_n]$, r, and γ are of the order of the length h of a period. Neglecting terms of order h, (26) then lead to

$$v^* = \frac{\mu - c\gamma}{r-\gamma} \geq c,$$

or equivalently $\frac{\mu}{r} \geq c$, where μ is the expected cash flow per unit of time if the agent exerts effort. Note that, according to (24), v^* is the ratio of the value of the firm to its assets, which, as noted by DeMarzo, Fishman, He, and Wang (2010), can be interpreted as Tobin's q. Thus, without moral hazard, investment takes place as soon as Tobin's q is greater than or equal to the marginal cost of investment.

[15]Whereas our linear specification generates a bang-bang optimal policy, the quadratic specification in Hayashi (1982) and De Marzo, Fishman, He, and Wang (2010) yields interior solutions.

With this first-best benchmark in mind, let us come back to the analysis of the second-best case. In size-adjusted terms, under the linear specification in (21), the maximand in (22) rewrites as

$$X_{n-1}\left\{\mathbf{E}\left[(1+g_n)(R_n-u_n)+\frac{(1+g_n)f\left(\frac{W_{n+1}}{(1+g_n)X_{n-1}}\right)}{1+r}\,|\mathcal{H}_n\right]-cg_n\right\}. \tag{27}$$

Taking the derivative of (27) with respect to g_n, we obtain that the optimal contract entails investment if and only if

$$\mathbf{E}\left[R_n-u_n+\frac{f\left(\frac{W_{n+1}}{X_n}\right)-\frac{W_n}{X_n}f'\left(\frac{W_{n+1}}{X_n}\right)}{1+r}\,|\mathcal{H}_n\right]\geq c. \tag{28}$$

Equations (26) and (28) have a similar structure. In both cases, the first term on the left-hand side is the per-period expected cash flow from operating the project, whereas the term on the right-hand side is the marginal cost of investment. However, whereas the second term on the left-hand side (26) reflects Tobin's average q, its counterpart in (28) can be interpreted in terms of Tobin's marginal q. Thus, as pointed out by DeMarzo, Fishman, He and Wang (2010), moral hazard induces a wedge between marginal q and average q, in contrast with the neo-classical case analyzed by Hayashi (1982). As a result, under moral hazard, empirical approaches relating investment to average q are misspecified.

In the present discrete-time, infinite-horizon setting, it is difficult to push the analysis much further so as to get more explicit results. As shown below, the continuous-time limit is more tractable, and we provide a fuller characterization of the optimal investment policy in Sections 4.2.4 and 4.2.5. It should be noted, however, that the discrete-time finite-horizon model of DeMarzo and Fishman (2007a) delivers rich qualitative insights. In particular, they show that investment is lower with moral hazard than in the first-best case, and that the growth in firm size is increasing in accumulated performance. This leads to the interesting empirical implication that investment should be increasing in current and past cash flows, and therefore should be positively serially correlated over time.

4 The Continuous-Time Case

We now turn to the continuous-time limit of the discrete-time model. This is a useful step for the analysis, because the continuous-time model is more tractable, which enables one to obtain additional results, and also because asset-pricing models are often formulated in continuous time. One therefore obtains a natural framework to bridge the gap between the design of securities, studied by corporate finance, and the dynamics of their valuations, studied by asset pricing.

To conduct this analysis, we need to slightly adjust our notation. Instead of indexing variables by the number of periods since the beginning of the contract, n, we index them by the time elapsed since the beginning of the contract, t. Denoting by h the length of a period, we then have $t = nh$. The primitives of the model are also functions of h. Thus we denote the probability of success under effort by p_h, and its counterpart under no effort by $p_h - \Delta p_h$. Similarly, we denote the size-adjusted cash flow in case of success by R_h^+, and its counterpart in case of failure by R_h^-. Finally, the size-adjusted private benefit from shirking and the discount rates over one period are linear in h, and denoted by Bh, rh, and ρh. We assume throughout that $\rho > r$, so that the agent is more impatient than the principal. For simplicity, and without loss of generality, we focus on the case where $X_0 = 1$.

We shall consider two alternative limits of the discrete-time model as h goes to zero, corresponding to Brownian and Poisson processes for cumulative cash flows. In this section, we only offer a heuristic sketch of the continuous-time analysis, which aims at providing economic intuitions. That is, we start from the discrete-time optimal contract characterized in Proposition 3, taking for granted its convergence, as well as that of the value function of the principal and the continuation utility of the agent. We refer to the original papers for rigorous derivations of these results.[16]

4.1 The Brownian Limit

4.1.1 Cash Flows

The Brownian limit corresponds to the case where the expectation and the variance of cash flows are linear in time. We also impose that the difference in expected returns between effort and shirking be linear in time and that the variance of the cash flows be unaffected by shirking. Therefore, we are looking for a specification of $(p_h, \Delta p_h, R_h^+, R_h^-)$ for which one has $\mathbf{E}[R_{nh} \,|\, e_{nh} = 1] = \mu h$, $\mathbf{E}[R_{nh} \,|\, e_{nh} = 1] - \mathbf{E}[R_{nh} \,|\, e_{nh} = 0] = \Delta\mu h$, and $\mathbf{Var}[R_{nh} \,|\, e_{nh} = 1] = \mathbf{Var}[R_{nh} \,|\, e_{nh} = 0] = \sigma^2 h$, where $(\mu, \Delta\mu, \sigma)$ is a triplet of positive real numbers. There exists a unique solution to this system of four equations in four unknowns. For this solution, when h is small, we have

$$p_h \simeq \frac{1}{2}\left(1 + \frac{\Delta\mu}{2\sigma}\sqrt{h}\right), \quad \Delta p_h \simeq \frac{\Delta\mu}{2\sigma}\sqrt{h}, \tag{29}$$

[16] Biais, Mariotti, Plantin, and Rochet (2004, 2007) establish the convergence of discrete-time optimal contracts to their continuous-time counterparts, whereas DeMarzo and Sannikov (2006), Sannikov (2008), and Biais, Mariotti, Rochet, and Villeneuve (2010) directly characterize continuous-time optimal contracts.

and

$$R_h^+ \simeq \left(\mu - \frac{\Delta\mu}{2}\right)h + \sigma\sqrt{h}, \quad R_h^- \simeq \left(\mu - \frac{\Delta\mu}{2}\right)h - \sigma\sqrt{h}. \tag{30}$$

As h goes to zero, conditional on high effort being exerted in all periods and no downsizing taking place, the corresponding cumulative cash-flow process converges in law to a Brownian motion with drift

$$Y_t \equiv \mu t + \sigma Z_t, \tag{31}$$

where $\{Z_t\}$ is a standard Brownian motion. One can see from (29) why we need to assume that $\rho > r$ when characterizing the Brownian limit of the discrete-time optimal contract. Indeed, for small values of h, substituting from (29) into (14), we obtain that the absorbing boundary w_h^p when $\rho = r$ in the discrete-time optimal contracting model is

$$\frac{1 + rh}{rh} \frac{p_h Bh}{\Delta p_h} \simeq \frac{B\sigma}{r\Delta\mu\sqrt{h}},$$

which goes to infinity when h goes to zero. As a result, when $\rho = r$, there exists no optimal contract in the limit.

4.1.2 The Value Function

An important preliminary result is that, when h goes to zero, the discrete-time downsizing threshold w_h^l goes to zero (Biais, Mariotti, Plantin, and Rochet (2004, Proposition 7, 2007, Lemma 4)). This reflects that, in the Brownian limit, no size adjustments need to take place before the firm is liquidated, unlike in the discrete-time framework, so that the size of the firm remains constant until it is liquidated. Intuitively, this is because, in the Brownian limit, incentives can be provided to the agent through infinitesimal changes in his continuation utility, unlike in the discrete-time model. Correspondingly, in the implementation of the optimal contract, there will be no need for the firm to maintain a minimum liquidity ratio, below which the firm must be downsized. In view of the evidence that financially distressed firms are often downsized when they run short of liquid assets, this can be seen as a limitation of the Brownian limit.

Now, consider the discrete-time value function f_h over the range $\left[w_h^l, w_h^p - \frac{Bh}{\Delta p_h}\right)$ in which the agent receives no current compensation under the discrete-time optimal contract. Given (19) and the specification (29)–(30), a Taylor–Young approximation yields

$$f_h(w_{(n+1)h}) \simeq f_h(w_{nh}) + [\rho h w_{nh} + k_h(R_{nh} - \mu h)]f_h'(w_{nh}) + \frac{k_h^2}{2}(R_{nh} - \mu h)^2 f_h''(w_{nh}), \tag{32}$$

where $k_h \equiv \frac{Bh}{\Delta p_h \Delta R_h}$. Substituting (32) into (10), and letting h go to zero, one formally obtains the following ordinary differential equation for the continuous-time value function:

$$rf(w) = \mu + \rho w f'(w) + \frac{k^2\sigma^2}{2} f''(w) \tag{33}$$

for each w in the range in which the agent receives no current compensation. Here

$$k \equiv \lim_{h\to 0} \frac{Bh}{\Delta p_h \Delta R_h} = \frac{B}{\Delta\mu} < 1.$$

In line with these heuristic remarks, Biais, Mariotti, Plantin, and Rochet (2004, Proposition 8, 2007, Proposition 3) prove the following result.

Proposition 5 *In the Brownian limit of the discrete-time optimal contract characterized in Proposition 3, the value function f_h converges uniformly to the unique solution f to the free-boundary problem characterized by (33) for $w \in [0, w^p)$ and $f(w) = f(w^p) + w^p - w$ for $w \in [w^p, \infty)$, with boundary conditions*

$$f(0) = 0, \tag{34}$$

$$f'(w^p) = -1, \tag{35}$$

$$f''(w^p) = 0. \tag{36}$$

Whereas Biais, Mariotti, Plantin and Rochet (2004, 2007) obtain this result by studying the continuous-time limit of the discrete-time Bellman equation (10), DeMarzo and Sannikov (2006, Proposition 1) derive it directly in a continuous-time model where cumulative cash flows evolve according to (31). Condition (34) reflects that the firm is liquidated when the agent's continuation utility drops to zero. Condition (35) is a smooth-pasting condition expressing the fact that w^p is the payment boundary, at which the marginal cost for the principal of an increase in the continuation utility promised to the agent is equal to the marginal cost of an immediate payment of one dollar. Condition (36) is a super-contact condition reflecting the optimality of the payment boundary w^p.[17]

4.1.3 The Continuation Utility of the Agent

For any time t prior to liquidation, denote by w_t the continuation utility of the agent at time t and by U_t the cumulative payment to the agent up to time t in the Brownian limit of the discrete-time optimal contract. Following Biais, Mariotti, Plantin, and Rochet (2007, Proposition 5), one can prove the following result.

[17] Essentially, (36) ensures that the function f is maximal among the class of solutions to (33)–(34) the derivative of which attains -1. See Dumas (1991) for an insightful discussion of the super-contact condition as an optimality condition for singular control problems.

Proposition 6 *In the Brownian limit of the discrete-time optimal contract characterized in Proposition 3, the process* $\{w_t, U_t\}$ *is the unique solution to the Skorokhod problem*

$$dw_t = \rho w_t dt + k\sigma dZ_t - dU_t, \tag{37}$$

$$w_t \leq w^p, \tag{38}$$

$$U_t = \int_0^t 1_{\{w_s = w^p\}} \, dU_s \tag{39}$$

for all $t \in [0, \tau]$, *where* $\tau \equiv \inf\{t \geq 0 \,|\, w_t = 0\}$ *and* $w_t = 0$ *for all* $t > \tau$.

(39) reflects that U_t increases only when w_t hits the payment boundary w^p, whereas (37) and (38) express the fact that this causes w_t to be reflected downwards. Hence w_t evolves according to (37) between a reflecting boundary, w^p, and an absorbing boundary, zero. Before w_t drops to zero, the firm is never downsized, but, as soon as it does, the firm is liquidated. It follows from standard results on the reflected Ornstein–Uhlenbeck process that $\tau < \infty$, $\mathbf{P}$–almost surely.[18] Thus the firm always ends up liquidated, consistent with Proposition 4. In line with (19), (37) shows that, as long as the agent is not paid, the continuation utility of the agent is a ρ–discounted martingale, the volatility of which is equal to the underlying volatility of cash flows, σ, multiplied by the severity of the moral-hazard problem, k. This multiplicative factor reflects the incentive-compatibility constraint: the more severe the moral-hazard problem, the higher the sensitivity to performance of the agent's continuation utility, and thus the higher its volatility.

From the joint dynamics (37)–(39) of w_t and U_t, one can derive using Itô's formula a probabilistic representation of the agent's and of the principal's utilities. Specifically, denote by $\mathbf{E}[\cdot \mid \mathcal{H}_t]$ the expectation operator conditional on public information available at time t, $\mathcal{H}_t$, when the agent is expected to always exert effort. Then, the following holds (Biais, Mariotti, Plantin, and Rochet (2007, Proposition 4)).

Proposition 7 *In the Brownian limit of the discrete-time optimal contract characterized in Proposition 3, the continuation utility of the agent and the continuation value of the principal admit the following representations*:

$$w_t = \mathbf{E}\left[\int_t^\tau e^{-\rho(s-t)} \, dU_s \,|\, \mathcal{H}_t\right],$$

$$f(w_t) = \mathbf{E}\left[\int_t^\tau e^{-r(s-t)} (dY_s - dU_s) \,|\, \mathcal{H}_t\right].$$

[18]See, for instance, Ward and Glynn (2003).

4.1.4 Asset-Pricing Implications

As in the discrete-time model, the optimal contract can be implemented with cash reserves, stocks, and bonds. The dynamics of cash reserves are

$$dM_t = dY_t + rM_t dt - d\Phi_t - d\Psi_t,$$

where dY_t is the operating cash flow, $rM_t dt$ is the interest payment on cash reserves, $d\Phi_t$ is the dividend payment, and $d\Psi_t$ is the coupon payment. As in the discrete-time model, the liquidity ratio

$$m_t \equiv \frac{M_t}{c} = \frac{w_t}{kc}. \tag{40}$$

serves the role of state variable.[19]

Stock Prices The stocks are owned in proportion $1-k$ by the investors and in proportion k by the agent. Thus the payments to the agent are a fraction k of the dividends,

$$dU_t = kd\Phi_t. \tag{41}$$

The market value of the stock at time t is therefore

$$S_t = \mathbf{E}\left[\int_t^\tau e^{-r(s-t)}\, d\Phi_s \,|\, \mathcal{H}_t\right] = \mathbf{E}\left[\int_t^\tau e^{-r(s-t)}\, \frac{1}{k}\, dU_s \,|\, \mathcal{H}_t\right]. \tag{42}$$

This is the present value of the dividend flow $d\Phi_t$, discounted at rate r, the relevant rate for investors who can trade the stock in the market, unlike the agent, who is prohibited from selling his stock holdings. The stock price S_t is a deterministic function $\mathcal{S}(m_t)$ of the state variable m_t. According to (37) and (40), the liquidity ratio evolves according to

$$dm_t = \rho m_t dt + \frac{\sigma}{c}\, dZ_t - \frac{1}{kc}\, dU_t,$$

where U_t is given by (39). Because of (38), $m_t \in \left[0, \frac{w^p}{kc}\right]$ at any time t. Using Itô's formula, one obtains that the function $\mathcal{S}$ is the solution over $\left[0, \frac{w^p}{kc}\right]$ to the boundary problem

$$r\mathcal{S}(m) = \rho m \mathcal{S}'(m) + \frac{\sigma^2}{2c^2}\, \mathcal{S}''(m), \tag{43}$$

$$\mathcal{S}(0) = 0, \tag{44}$$

$$\mathcal{S}'\left(\frac{w^p}{kc}\right) = c. \tag{45}$$

[19] Recall that, in the Brownian limit, the firm remains constant in size until it is liquidated, and that we have normalized its size to one.

One can check from (43)–(45) that the function $\mathcal{S}$ is strictly increasing and strictly concave over $\left[0, \frac{w^p}{kc}\right]$ (Biais, Mariotti, Plantin, and Rochet (2007, Proposition 7)). Along with Itô's formula, (39) and (43)–(45) imply that the stock price evolves according to the following stochastic differential equation:

$$dS_t = rS_t dt + S_t \sigma^{\mathcal{S}}(S_t) dZ_t - \frac{1}{k}\, dU_t, \tag{46}$$

where $S\sigma^{\mathcal{S}}(S) \equiv \frac{\sigma}{c}\, \mathcal{S}'(\mathcal{S}^{-1}(S))$ is the volatility of the stock price. Equation (46) is reminiscent of the stock price dynamics postulated by Black and Scholes (1973) and Merton (1973), but differs from them in three major ways. First, the stock price is reflected downwards each time dividends are paid, which happens when S_t hits $\mathcal{S}\left(\frac{w^p}{kc}\right)$. Second, its volatility $S\sigma^{\mathcal{S}}(S)$ is bounded away from zero, so that the stock price can drop to zero, which occurs when the firm is liquidated. The third difference reflects the concavity of the function $\mathcal{S}$ and is stated in the following proposition.

Proposition 8 *The volatility of the stock price, $S\sigma^{\mathcal{S}}(S)$, and hence the volatility of the stock return, $\sigma^{\mathcal{S}}(S)$, are decreasing in the stock price S.*

This result is in line with the leverage effect pointed out by Black (1976), Christie (1982), and Nelson (1991): stock volatility tends to rise in respond to bad news and to fall in response to good news. The characterization (43)–(45) of stock prices also implies the following comparative statics result (Biais, Mariotti, Plantin, and Rochet (2007, Proposition 13)).

Proposition 9 *The stock price is decreasing in k.*

Intuitively, an increase in the severity of the moral-hazard problem raises the risk of an early liquidation, which lowers the value of the firm and hence its stock price. An implication or Proposition 9 is that the magnitude of agency costs should be negatively correlated with price-earnings ratios.

Bond Prices, Leverage, and Default Risk In line with (20), the coupon payment at time t is given by

$$d\Psi_t = [\mu - (\rho - r)cm_t]dt.$$

The market value of the bond at time t is therefore

$$D_t = \mathbf{E}\left[\int_t^{\tau} e^{-r(s-t)}\, d\Psi_s \,|\, \mathcal{H}_t\right] = \mathbf{E}\left[\int_t^{\tau} e^{-r(s-t)} [\mu - (\rho - r)cm_s]\, ds \,|\, \mathcal{H}_t\right]. \tag{47}$$

Like the stock price, the bond price is a deterministic function $\mathcal{D}(m_t)$ of the state variable m_t. Using Itô's formula, one obtains that the function $\mathcal{D}$ is the solution over $\left[0, \frac{w^p}{kc}\right]$ to the boundary problem

$$r\mathcal{D}(m) = \mu - (\rho - r)cm + \rho m \mathcal{D}'(m) + \frac{\sigma^2}{2c^2}\mathcal{D}''(m), \tag{48}$$

$$\mathcal{D}(0) = 0, \tag{49}$$

$$\mathcal{D}'\left(\frac{w^p}{kc}\right) = 0. \tag{50}$$

Dividing (47) by (42) yields the leverage ratio, $\frac{D_t}{S_t} = \frac{\mathcal{D}(m_t)}{\mathcal{S}(m_t)}$, expressed in market values. Using (43)–(45) along with (48)–(50) leads to the following result (Biais, Mariotti, Plantin, and Rochet (2007, Proposition 8)).

Proposition 10 *The leverage ratio is decreasing in the stock price.*

The intuition for this result is as follows. When the firm is consistently successful, the value of its stocks and that of its bonds increase. However, the sensitivity to performance of the former is greater than that of the latter, in line with the standard intuition that coupons are less risky than dividends. Proposition 10 implies that performance shocks and stock price movements induce persistent changes in the leverage of the firm. Such persistent changes are empirically well documented. In discussing these findings, Welch (2004) questions as to why firms do not issue or repurchase debt or equity to counterbalance the impact of stock price movements on their capital structure. This could sound puzzling if one were to rely upon a one-period model, such as, for instance, the tradeoff theory, according to which there exists an optimal leverage ratio to which the firm should endeavor to revert. By contrast, in the implementation of the optimal contract, the financial structure of the firm optimally adjusts through changes in the market values of its securities, without requiring further issuing activities. Observe from Proposition 8 and 10 that an increase in leverage is tied to an increase in the volatility of the stock, in line with Black's (1976) interpretation of the leverage effect.

As a measure of default risk, consider the credit yield spread Δ_t on a consol bond paying one dollar at each instant until default. At any time t prior to default, it is defined by

$$\int_t^{\infty} e^{-(r+\Delta_t)(s-t)}\, ds = \mathbf{E}\left[\int_t^{\tau} e^{-r(s-t)}\, ds \,|\, \mathcal{H}_t\right]. \tag{51}$$

Rearranging (51), we have

$$\Delta_t = \frac{rT_t}{1 - T_t}, \tag{52}$$

where $T_t \equiv \mathbf{E}[e^{-r(\tau-t)} \,|\, \mathcal{H}_t]$. Using Itô's formula, one obtains that $T_t = \mathcal{T}(m_t)$, where $\mathcal{T}$ is the solution over $\left[0, \frac{w^p}{kc}\right]$ to the boundary problem

$$r\mathcal{T}(m) = \rho m \mathcal{T}'(m) + \frac{\sigma^2}{2c^2}\mathcal{T}''(m), \tag{53}$$

$$\mathcal{T}(0) = 1, \tag{54}$$

$$\mathcal{T}'\left(\frac{w^p}{kc}\right) = 0. \tag{55}$$

One can check from (53)–(55) that the function $\mathcal{T}$ is strictly positive, strictly decreasing, and strictly convex over $\left[0, \frac{w^p}{kc}\right]$ (Biais, Mariotti, Plantin, and Rochet (2007, Proposition 10)). The first two properties reflect that the firm is never insulated from the risk of default, and that, if it is not successful, it runs out of cash, which increases its default risk. The intuition for the convexity of $\mathcal{T}$ stems from the fact that the impact of cash-flow news on default risk differs according to whether the liquidity ratio is high or low. When m_t is close to $\frac{w^p}{kc}$, positive cash-flow realizations do not further reduce default risk, because they are distributed as dividends. By contrast, when m_t is close to zero, positive cash-flow realizations have a strong impact on default risk, because they move the firm away from the liquidation boundary. Along with (52), the characterization (53)–(55) of the function $\mathcal{T}$ also implies the following comparative statics result (Biais, Mariotti, Plantin, and Rochet (2007, Proposition 12)).

Proposition 11 *The credit yield spread is increasing in k.*

Intuitively, when the moral-hazard problem becomes more severe, it is necessary to lower the threshold liquidity ratio at which the agent is paid, $\frac{w^p}{kc}$, in order to strengthen his incentives to exert effort. Such a generous compensation policy, however, increases the risk that the firm will run out of cash and end up liquidated earlier.

4.1.5 High-Water Marks

The compensation of the agent can be interpreted in terms of high-water marks. Consider the total flow of cash into the firm, net of debt service, and before dividends:

$$M_0 + Y_t + \int_0^t rM_s\,ds - \Psi_t. \tag{56}$$

Here Y_t is the total cash flow generated by the project, $\int_0^t rM_s\,ds$ are the cumulative interests on cash reserves, and Ψ_t is the sum of the coupons paid out to bondholders. Combining Proposition 6 with equations (40), (41), and (56), we obtain the following result, the proof of which is a direct implication of Skorokhod's equation (Karatzas and Shreve (1991, Chapter 3, Lemma 6.14).

Proposition 12 *At any time t prior to liquidation, cumulative dividends satisfy*

$$\Phi_t = \sup_{s\in[0,t]} \left\{ \max \left\{ M_0 + Y_t + \int_0^t rM_s\, ds - \Psi_t - \frac{w^p}{k}\,, 0 \right\} \right\}. \tag{57}$$

The intuition for formula (57) is the following. Take as a measure of the cumulative performance of the firm the total flow of cash into the firm, net of debt service and before dividends, as defined by (56). Then, as long as cumulative performance remains below the threshold $\frac{w^p}{k}$, there is no dividend distribution; in particular, the agent receives no compensation. After that, dividends are distributed, and the agent is compensated, but only when cumulative performance reaches a new maximum. Thus, in the implementation of the optimal contract, managerial compensation is structured along lines similar to those of high-water-mark contracts in the hedge-fund industry.[20] Specifically, if the manager accumulates good performance, he gets paid, but, after that, if performance is poor, losses must be recouped before he gets paid again. The above analysis thus provides an agency-theoretic rationale for this aspect of the compensation of hedge-fund managers.

4.2 The Poisson Limit

This section, which is drawn from Biais, Mariotti, Rochet, and Villeneuve (2010), considers an alternative continuous-time limit of the discrete-time model, in which cash flows are subject to large but infrequent losses that occur according to a Poisson process.[21]

4.2.1 Cash Flows

Fix a quadruple $(\mu, \lambda, \Delta\lambda, C)$ of positive real numbers, and, for h small enough, consider the following specification of the parameters of cash-flow dynamics:

$$p_h = 1 - \lambda h, \quad \Delta p_h = \Delta\lambda h, \quad R_h^+ = \mu h, \quad R_h^- = \mu h - C. \tag{58}$$

In every period, the probability that a loss C occurs is proportional to the length h of the period. It is equal to λh if the agent exerts effort, and $(\lambda + \Delta\lambda)h$ if he shirks. When h goes

[20]See Goetzmann, Ingersoll, and Ross (2003) for a description and asset-pricing analysis of such contracts.

[21]Myerson (2010) also studies optimal contracting with a Poisson noise structure, but the focuses of our analyses differ, as he considers a political economy problem, whereas we consider a corporate finance setting. Abreu, Milgrom, and Pearce (1991) and Sannikov and Skrzypacz (2010) also rely on Poisson processes to study repeated games with imperfect monitoring. Our focus differs from theirs in that we consider a full commitment environment, in which we explicitly characterize the optimal contract.

to zero, this probability goes to zero, but the size of the loss does not, as $\Delta R_h = C$. The limit process of cumulative cash flows is therefore discontinuous, and satisfies

$$dY_t = X_t(\mu dt - CdN_t),$$

where $\{N_t\}$ is a Poisson process of intensity λ when the agent exerts effort and $\lambda + \Delta\lambda$ when he shirks, and $\{X_t\}$ is a nonincreasing process describing the size of the firm. As in the Brownian case, one can see from (58) why we need to assume that $\rho > r$ when characterizing the Poisson limit of the discrete-time optimal contract. Indeed, for small values of h, substituting from (58) into (14), we obtain that the absorbing boundary w_h^p when $\rho = r$ in the discrete-time optimal contracting model is

$$\frac{1+rh}{rh}\frac{p_h Bh}{\Delta p_h} \simeq \frac{B}{r\Delta\lambda h},$$

which goes to infinity when h goes to zero. As a result, when $\rho = r$, there exists no optimal contract in the limit.

4.2.2 The Continuation Utility of the Agent

When h goes to zero, the optimal contract characterized in Proposition 3 converges to the optimal contract of the Poisson model. Biais, Mariotti, Rochet, and Villeneuve (2010, Propositions 1 and 3) show that, in this contract, the size-adjusted continuation utility of the agent remains in some interval $[k, w^p]$ and evolves according to

$$dw_t = \rho w_t dt - k(dN_t - \lambda dt) \tag{59}$$

as long as the agent is not paid and no downsizing takes place, where

$$k \equiv \lim_{h\to 0} \frac{Bh}{\Delta p_h \Delta R_h} = \frac{B}{\Delta\lambda C} < 1.$$

Formula (59) is similar to (19) and (37). First, as the process $\{N_t - \lambda t\}$ is a martingale when the agent exerts effort, his size-adjusted continuation utility is a ρ–discounted martingale as long as he is not paid and no downsizing takes place. Second, the size-adjusted continuation utility of the agent decreases each time he incurs a loss. However, unlike in the Brownian case, this decrease is discontinuous, and takes the form of a downward jump of size k for w_t each time $dN_t = 1$. Thus kdN_t in (59) is the lumpy analogue of the infinitesimal term $k\sigma dZ_t$ in (37), and represents the minimum penalty that gives the agent incentives to exert effort. This, together with the necessity of preserving the agent's limited liability, implies that w_t cannot stay below k. As a result, each time the agent incurs a loss while his before-loss size-adjusted continuation utility w_{t^-} is in the interval $(k, 2k)$, his after-loss size-adjusted continuation utility must be immediately reset at the level k, so that, if a new loss occurs soon after, it is again possible to subtract k from it. But, while resetting w_t from $w_{t^-} - k$ to k, it is necessary to maintain the continuity of $W_t = w_t X_{t^+}$ in order to satisfy the promise-keeping constraint after the loss. The size of the project must therefore be reduced by an amount that exactly offsets the resetting of the agents' size-adjusted continuation utility, so that $x_t \equiv \frac{X_{t+}}{X_t} = \frac{w_{t^-}-k}{k}$. By contrast, if a loss occurs while the agent's before-loss size-adjusted continuation utility w_{t^-} is in the interval $[2k, w^p]$, no such downsizing is needed. The following formula summarizes the dynamics of firm size:

$$X_{t^+} = X_t \min\left\{\frac{w_{t^-} - k}{k}, 1\right\} 1_{\{dN_t=1\}}.$$

Thus, unlike in the Brownian case, the downsizing region does not shrink to zero in the continuous-time limit. This is because the liquidation threshold w_h^l converges to $k > 0$ in the Poisson limit, and not to zero as in the Brownian limit. Another difference is that the process $\{U_t\}$ which describes the cumulative payments to the agent is singular in the Brownian case, but absolutely continuous in the Poisson case:

$$dU_t = X_t(\rho w^p + \lambda k) 1_{\{w_t = w^p\}} dt. \tag{60}$$

Like in the Brownian case, the agent is not paid until w_t reaches the payment boundary w^p, and there payments are interrupted after the first loss. But, contrary to the Brownian case, it can take a while for this first loss to occur, during which the agent receives a constant payment flow. Note that (60) expresses that the amount of this payment is exactly what is needed to satisfy the promise-keeping constraint, while maintaining w_t at w^p.

4.2.3 The Value Function

Over the interval (k, w^p), the size-adjusted value function of the principal, f, satisfies a first-order delay-differential equation analogous to the second-order differential equation (33) in the Brownian case:

$$rf(w) = \mu - \lambda C + (\rho w + \lambda k) f'(w) - \lambda[f(w) - f(w-k)]. \tag{61}$$

This equation is obtained by noting that, over the interval (k, w^p), the principal does not make any payment to the agent. Thus, over an infinitesimal time period of length dt, the expected increase in the size-adjusted continuation value of the principal is just equal to $rf(w)dt$, minus the expected size-adjusted cash flow $(\mu - \lambda C)dt$ she receives from the project. Equation (61) is then obtained by applying the analogue of Itô's formula to f to compute the expected increase in her continuation value. By (59), this is given by

$$\mathbf{E}[df(w_t)\,|\,\mathcal{H}_t] = (\rho w_t + \lambda k)f'(w_t)dt - \lambda[f(w_t) - f(w_t - k)]dt.$$

For equation (61) to be well defined over the whole interval (k, w^p), we need to define $f(w-k)$ for $w \in (k, 2k)$. According to the above discussion, this is precisely the region where the assets of the firm are downsized by a factor $\frac{w_{t-}-k}{k}$ after a loss, after which w_t is reset at the level k. Because of our assumptions of constant returns to scale and of a zero liquidation value for productive assets, the continuation value of the principal after a loss in the interval $(k, 2k)$ is then exactly $\frac{w-k}{k} f(k)$. Defining $f(w-k)$ by this value in the interval $(k, 2k)$ then completes the determination of the delay-differential equation (61) over the whole interval (k, w^p). The only unknowns left are the values of $f(k)$ and w^p. A simple way to find them is first to set the function f equal to a linear function with slope $\frac{f(k)}{k}$ over the interval $[0, k)$, and then to solve (61) on each interval of the form $[mk, (m+1)k)$ by a simple recursion on the integer m. The induction is stopped at the first value for which the derivative of f attains -1. The optimal solution is then obtained by choosing the initial slope $\frac{f(k)}{k}$ so that the value function of the principal is maximum (Biais, Mariotti, Rochet, and Villeneuve (2010, Proposition 2)).

Remark. Sannikov (2005) also analyzes a dynamic moral-hazard model in a framework with Poisson risk. In his analysis, however, jumps correspond to positive cash-flow shocks, the probability of which is raised by managerial effort. The optimal contract arising in this context is qualitatively different from that presented above. Specifically, there is no need for downsizing, and, after a long period without positive cash-flow realization, the firm is liquidated. Thus liquidation is a predictable event, in contrast with the downsizing episodes in Biais, Mariotti, Rochet, and Villeneuve (2010).

4.2.4 Investment

So far in our continuous-time analysis, we have assumed away investment after the initial date. We now relax this assumption and consider the continuous-time counterpart of the discrete-time analysis of investment offered in Section 3.3. The firm can invest at rate $g_t \geq 0$, so that $dX_t = g_t X_t dt$. In line with (21), the marginal cost of such investment is constant and equal to c, but the growth rate is bounded above by some parameter $\gamma \in (0, r)$. In the context of the Poisson model presented above, Biais, Mariotti, Rochet, and Villeneuve (2010) show that it is optimal to invest if and only if the size-adjusted continuation utility

w_t of the agent has reached some threshold w^i. The reason is that increasing the size of the project is an indirect way of rewarding the agent. This is because the volume of the payments to the agent, once the payment boundary w^p has been reached, is increasing in the size of the project. Therefore, just as payments are optimally delayed to improve incentives, so are investments.

To better understand the economic forces at play, consider the dynamics of the size-adjusted continuation utility of the agent when the project grows at rate g_t, focusing on the region where the agent receives no immediate payment and there is no downsizing:

$$dw_t = (\rho - g_t)w_t dt - b(dN_t - \lambda dt).$$

As in the case without investment, the law of motion of w_t is obtained by dividing the state equation for the non-size-adjusted continuation utility of the agent, W_t, by the size of the project, X_t, taking into account that the latter grows at rate g_t. We see that increasing the size of the project is akin to making the agent less impatient, by reducing the discount factor from ρ to $\rho - g_t$. In this context, the Bellman equation becomes

$$rf(w) = \mu - \lambda C + (\rho w + \lambda k)f'(w) - \lambda[f(w) - f(w-k)] + \max_{g \in [0,\gamma]} \{g[f(w) - wf'(w) - c]\}.$$

By linearity of the maximand, the solution is bang-bang: the optimal investment rate is equal to γ if $f(w) - wf'(w) > c$, and to zero otherwise. Note that the term $f(w) - wf'(w)$, which is the marginal sensitivity of the value of the firm to new investment, is the exact counterpart of the term identified in the discrete case, see (28).[22]

By concavity of f, $f(w) - wf'(w)$ is a nondecreasing function of w, which reaches its maximum at w^p. The investment region is thus empty when $c > f(w^p) - w^p f'(w^p) = f(w^p) + w^p$. Otherwise, the investment region is an interval $(w^i, w^p]$, and, in that case, the optimal contract is characterized by three thresholds: the liquidation threshold k, below which the project is downsized, the investment threshold w^i, below which no new investment takes place, and the payment boundary w^p, below which the agent is not paid. When c is small enough, it can be shown that $w^i < w^p$ and that all three regions are nonempty.

4.2.5 Firm Size Dynamics

When new investments are possible, the dynamics of firm size are more complex than when the firm can only be downsized. As in the latter case, the firm is downsized by a factor $x_t \equiv \min\left\{\frac{w_{t-} - k}{k}, 1\right\}$ each time there is a loss. But firm size now grows at rate γ as long

[22]Note that the same term arises in the Brownian motion case analyzed by DeMarzo, Fishman, He, and Wang (2008).

as w_t stays above w^i. As a result, the time-averaged growth rate of the firm is equal to the difference between two terms. The first term corresponds to new investments, the second to downsizing:

$$\frac{\ln(X_t) - \ln(X_0)}{t} = \frac{\gamma}{t} \int_0^t 1_{\{w_s > w^i\}} \, ds - \frac{1}{\lambda t} \sum_{m=1}^{N_{t-}} \lambda \max \left\{ \ln\left(\frac{b}{w_{T_m^-} - b} \right), 0 \right\}, \tag{62}$$

where T_m it the time at which the m^{th} loss occurs. When t goes to infinity, the application of an appropriate law of large numbers for Markov ergodic processes allows one to compute the long-run growth rate of the firm. The time-averaged integral in the first term on the right-hand side of (62) converges to the average time spent above w^i by the process $\{w_t\}$. The time-averaged sum in the second term on the right-hand side of (62) converges to the expectation over $(b, 2b)$ of the function $\lambda \ln\left(\frac{b}{w-b}\right)$ under the unique invariant measure associated with the agent's before-loss size-adjusted continuation utility. When γ is small, the second term dominates, and the size of the firm converges to zero. By contrast, when γ is large enough, the size of the firm goes to infinity. Specifically, Biais, Mariotti, Rochet, and Villeneuve (2010, Proposition 5) establish the following proposition.

Proposition 13 *Fix some c such that $f(k) - kf'_+(k) \geq c$. Then, if γ is close to zero,*

$$\lim_{t \to \infty} X_t = 0,$$

$\mathbf{P}$*–almost surely, whereas if $\gamma > \frac{\lambda^2}{\rho - \gamma + \lambda}$,*

$$\lim_{t \to \infty} X_t = \infty,$$

$\mathbf{P}$*–almost surely.*

The condition $f(k) - kf'_+(k) \geq c$ ensures that $w^i = k$, so that always investing at the maximum rate γ is optimal. It is satisfied provided c is close enough to zero because the function f has a kink at k, with $f'_+(k) < f'_-(k) = \frac{f(k)}{k}$.

The asymptotic behavior described in Proposition 13 differs from that arising in Clementi and Hopenhayn (2006). These authors study a discrete-time model similar to the one we considered in Section 2. In particular, they assume that the agent and the principal are equally impatient. As a result, the agent is paid when his size-adjusted utility reaches an absorbing boundary, like in Proposition 1. In the long run, with some probability the firm is liquidated, whereas with the complementary probability it becomes insulated from the risk of liquidation. By contrast, in the Poisson limit of the discrete-time model where the agent is more impatient than the principal, the probability that the size of the firm eventually shrinks to zero is either zero or one, and so is the probability that the firm grows without bounds, depending on the values of the parameters (Biais, Mariotti, Rochet, and Villeneuve (2010, Proposition 6)).

4.2.6 Empirical Implications

In DeMarzo and Fishman (2007a), Biais, Mariotti, Rochet, and Villeneuve (2010), and DeMarzo, Fishman, He, and Wang (2010), firms invest only after a long enough record of good performance. This is consistent with the empirical findings of Kaplan and Strömberg (2004) that venture capital funding for new investment is contingent on both financial and nonfinancial milestones. Kaplan and Strömberg (2004) also find that such conditioning is more frequent when their proxy for agency problems is higher.

In Biais, Mariotti, Rochet, and Villeneuve (2010), small firms tend to be below the investment threshold. They are thus more likely to be exposed to financial constraints on investment, as documented by Beck, Demirgüç-Kunt, and Maksimovic (2005). Furthermore, small firms are relatively more fragile, because a few negative shocks are enough to drive them into the zone where further losses trigger downsizing. Conversely, large firms that have enjoyed long periods of sustained investment are more likely to have long records of good performance, which pushes them away from that zone. Overall, the probability of downsizing is decreasing in firm size. This is in line with the empirical findings of Dunne, Roberts, and Samuelson (1989) that failure rates decline with increases in firm or plant size. A further testable implication of the model is that downsizing decisions should typically be followed by relatively long periods during which no investment takes place, corresponding to the time it takes for the firm to reach the investment threshold again and resume growing.

The model also sheds light on the relationship between CEO compensation and firm size. Gabaix and Landier (2008) note that different theoretical explanations have been offered for variations in CEO pay. Whereas some analyses emphasize incentive problems, Gabaix and Landier (2008) propose to focus on firm size. Empirically, they find that CEO pay increases with firm size. Consistent with these results, dynamic moral-hazard models imply that the size of the firm and the compensation of the agent should be positively correlated: after a long stream of good performance, the scale of operations is large and so are the payments to the agent. Dynamic moral-hazard models also suggest that explanations based on size should not be divorced from explanations based on incentives, and that investment and managerial compensation are complementary incentive instruments, in line with the empirical findings of Kaplan and Strömberg (2003).

5 Conclusion

This paper surveyed studies of dynamic financial contracting when managers must exert unobservable effort to reduce downside risk. Optimal contracts imply that managers should be paid and firms should invest only after sufficiently high cumulative performance, and that, after poor performance, firms should be downsized or liquidated. These results suggest that maintaining large bonuses and large scale operations after severe losses could generate inappropriate incentives, such as, for instance, in the financial industry. Tighter capital requirements and bonus regulation could be called for, to reduce the gap between optimal contracts and actual ones, if, for reasons outside the models we surveyed, contracts were to diverge in practice from information-constrained optima.

The implications from dynamic-contracting models are in line with a wide spectrum of empirical findings and stylized facts, ranging from stock price dynamics to venture capital contracts. A promising avenue for future research would be to take a structural approach to confront the theory to the data. Dynamic-contracting models imply that compensation and investment decisions are contingent on a state variable, the agent's continuation utility, which increases when cash flows exceed their expectation, and decreases otherwise. Whereas this variable is not observable by the econometrician, it is a function of the parameters of the model (to be estimated) and of the cash flows (which can be observed). Furthermore, the models impose restrictions on how this variable should correlate with decisions. For instance, managers should receive bonuses only when their continuation utility reaches a milestone, whereas investment should be positively correlated with it. Combining these restrictions with firm-level data to estimate the parameters and test these models could significantly enhance our understanding of dynamic financial contracting.

Appendix

Proof of Proposition 4. We can assume without loss of generality that the Markov process $\{w_n\}$ lives in the interval $I \equiv \big[(1+\rho)\big(w^l - p\,\frac{B}{\Delta p}\big), w^r\big]$. Let $P : I \times \mathcal{B}(I) \to [0,1]$ denote the associated transition function, where $\mathcal{B}(I)$ is the Borel σ–field over I. Because $\{w_n\}$ is reflected downward at w^r, and the optimal contract starting with a size-adjusted continuation utility w^l is executed each time the firm is downsized, which results with probability $1-p$ in a size-adjusted continuation utility $\inf I = (1+\rho)\big(w^l - p\,\frac{B}{\Delta p}\big)$, there exists an integer $N \geq 1$ and a number $\varepsilon > 0$ such that

$$P^N(w, \{\inf I\}) \geq \varepsilon$$

for all $w \in I$. Hence the transition function P satisfies Condition M in Stokey and Lucas, with Prescott (1989, Chapter 11, Section 4). Specifically, for each $A \in \mathcal{B}(I)$, the following holds: either $\inf I \in A$, and thus $P^N(w, A) \geq \varepsilon$ for all $w \in I$, or $\inf I \notin A$, and thus $P^N(w, I \setminus A) \geq \varepsilon$ for all $w \in I$. Let $\Delta(I)$ be the set of Borel probability measures over I, and let $T^* : \Delta(I) \to \Delta(I)$ be the adjoint operator associated with P, defined by

$$(T^*\boldsymbol{\mu})(A) = \int_I P(w, A)\,\boldsymbol{\mu}(dw), \quad (\boldsymbol{\mu}, A) \in \Delta(I) \times \mathcal{B}(I).$$

Condition M stated above implies that T^{*N} is a contraction of modulus $1-\varepsilon$ over the space $\Delta(I)$ endowed with the total variation norm $\|\cdot\|_{\mathrm{TV}}$ (Stokey and Lucas, with Prescott (1989, Lemma 11.11)). Because this is a complete metric space, it follows from the contraction mapping theorem that T^* has a unique invariant measure $\boldsymbol{\mu}^*$. Using the fact that Condition M is stronger than Doeblin's condition (Condition D in Stokey and Lucas, with Prescott (1989, Chapter 11, Section 4)), one can deduce from the uniqueness of the invariant measure $\boldsymbol{\mu}^*$ that there exists a unique ergodic set.

If $w^l = p\,\frac{B}{\Delta p}$, $\inf I = 0$ is an absorbing state for $\{w_n\}$ that can be reached from any state in I; hence the unique invariant measure $\boldsymbol{\mu}^*$ is a Dirac mass at zero, and $\{0\}$ is the unique ergodic set. In that case, $\{w_n\}$ eventually drops to zero, and thus the firm is liquidated in finite time, $\mathbf{P}$–almost surely.

Suppose now that $w^l > p\,\frac{B}{\Delta p}$, so that $\inf I > 0$. For each $\boldsymbol{\mu} \in \Delta(I)$,

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} T^{*k}\boldsymbol{\mu} = \boldsymbol{\mu}^*$$

in the total variation norm (Stokey and Lucas, with Prescott (1989, Theorem 11.10)), and therefore, a fortiori, in the topology of weak convergence. Moreover, the transition function P

satisfies the Feller property. The strong law of large numbers for Markov processes (Stokey and Lucas, with Prescott (1989, Theorem 14.7)) then implies that, for any continuous function $g : I \to \mathbb{R}$,

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} g(w_k) = \int g\, d\boldsymbol{\mu}^*, \tag{63}$$

$\mathbf{P}$–almost surely. It is easy to verify that $\inf I$ belongs to the support of $\boldsymbol{\mu}^*$. Along with (63), this implies that, for each $\eta > 0$, $w_k \in [\inf I, \inf I + \eta)$ infinitely often, $\mathbf{P}$–almost surely, that is, $\mathbf{P}\left[\overline{\lim}_{k\to\infty}\{w_k \in [\inf I, \inf I + \eta)\}\right] = 1$. To see why, fix $\eta > 0$, and suppose that g in (63) is strictly positive over $[\inf I, \inf I + \eta)$ and vanishes elsewhere over I, so that $\int g\, d\boldsymbol{\mu}^* > 0$ as $\inf I$ belongs to the support of $\boldsymbol{\mu}^*$. Now, if in some state there exists some $k_0 \geq 1$ such that $w_k \notin [\inf I, \inf I + \eta)$ for all $k \geq k_0$, the limit on the left-hand side of (63) is zero as g vanishes outside this interval. Because $\int g\, d\boldsymbol{\mu}^* > 0$, this can only happen on a set of states of measure zero, and the claim follows. To conclude the proof, choose $\eta \in (0, w^l - \inf I)$, and observe that, for each $n \geq 1$,

$$X_n = X_0 \prod_{k=1}^{n} x_k = X_0 \prod_{k=1}^{n} \min\left\{\frac{w_k}{w^l}, 1\right\} \leq X_0 \left(\frac{\inf I + \eta}{w^l}\right)^{\kappa(n)},$$

where $\kappa(n) \equiv \#\{k \leq n \,|\, w_k \in [\inf I, \inf I + \eta)\}$. By the above reasoning, $\lim_{n\to\infty} \kappa(n) = \infty$, $\mathbf{P}$–almost surely. Because $\frac{\inf I + \eta}{w^l} \in (0, 1)$ by construction, $\lim_{n\to\infty} X_n = 0$, $\mathbf{P}$–almost surely. Hence the result. ∎

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