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# Continuous-Time Overlapping Generations Models

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**Abstract**

Age structured populations are studied in economics through overlapping generations models. These models allow for a realistic characterization of life-cycle behaviors and display intertemporal equilibrium that are not necessarily efficient. This article uses the latest developments in continuous time overlapping generations models to show the influence of the vintage structure of the population on the volatility of intertemporal prices. Permanent cycles can be found on the neighborhood of steady-states while the transitional dynamics are generically governed by short run fluctuations.

**JEL Classification:** D5, D9, E2.

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## 1 Introduction

In a recent article, Boucekkine et al. (2004) convincingly argue that the analysis of the relationship between demographic and economic trends should

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be built on a rigorous modeling of the specific decisions of the various cohorts that compose a population. Overlapping generations models, which feature an explicit age structure of the population, are hence the natural tool for studying the impact of demographic changes on individual decisions and aggregate economic variables.

The first model with overlapping generations developed in literature (Allais 1947, Samuelson, 1958 and Diamond, 1965) only involves two coexisting generations at each point of time, meaning that the length of a period is about thirty years. The comparison with real world data is therefore difficult and, moreover, the heterogeneity yield by a demographic structure is implicitly eliminated. An alternative to this representation was proposed by Blanchard (1985). His approach has been celebrated since it allows for more than two periods within a life cycle and leads to simple analytics. However, this tractability hinges on the assumption of a constant hazard death rate, which is unable to capture the life cycle aspect of life. As a result, these two frameworks have been extensively used in theoretical works but not in applied ones. To measure the consequences of changes in the age structure of a population, Auerbach and Kotlikoff (1987) propose a deterministic framework and Rios-Rull (1996) a stochastic one, which include many generations. Recently, these models have been notably successful when applied to the evaluation of the impact of population aging on capital accumulation (Lau, 2009), public debt (Bullard and Russel, 1999), economic growth (Azoumahou et al., 2009), and pension systems (Bloom et al., 2007, Heijdra and Romp, 2009). From a theoretical point of view, recent contributions, such as those of Kehoe et al. (1991) and Azariadis et al. (2004), have thus been aimed at developing a fruitful analysis of large-square models and the setting out of specific conditions on the dynamic properties of equilibrium paths. However, a limitation with regard to the widespread application of such models springs from the difficulties that emerge in the course of their analytical resolution.

In the present work, we survey the latest developments of overlapping generations models with a realistic age structure of the population and introduce a simple method that allows for solving these models. We propose a complete resolution in a specific case that provides new results on the existence and the convergence properties of equilibrium paths. In models with many overlapping generations, the dynamics of endogenous variables depend on a finite number of their past and future realizations. This dependency is what gives rise to the analytical difficulty of these models. In discrete-time frameworks, the analysis of the dynamics requires the study of polynomials, whose order increases with the number of generations, such an analysis is not tractable in general. We believe that this difficulty may however be circumvented using the continuous-time framework initially developed by Cass and Yaari (1967): it eases the mathematical resolution without affecting the qualitative analysis. However, in continuous-time frameworks, the dynamics are characterized by a functional differential equation of mixed type, i.e. the dynamics are affected by distributed delays and advances. In this work, we propose an integrated treatment of such a structure in a pure exchange economy, where preferences and endowments are stationary.

The basic framework of the model is presented in the first two sections. In section 2, we propose a complete characterization of life cycle behavior in continuous-time with uncertain lifetime. We derive the optimal profile of consumption under a general utility function and discuss the implications of the usual assumptions made on these functions. In section 3, we define the intertemporal equilibrium of a pure exchange economy. We demonstrate how demographic assumptions affect the dynamics of aggregate variables. The equilibrium conditions on the asset markets are then presented, underlining the role of monetary creation. Section 4 presents the results on the volatility of intertemporal prices. In the case of a balanced equilibrium, where the aggregate value of assets is zero, we show that there exist sets of parameters such that certain solutions of the linearized functional differential equation take on Hopf bifurcation values. Contrary to the initial intuitions of Aiyagari (1989) and Simonovits (1999), permanent long run fluctuations may hence appear as a solution of the price dynamics. We then concentrate on the logarithmic preferences case, for which the functional differential equation happens to be linear. First of all, we demonstrate the existence of an intertemporal equilibrium by studying the existence of a strictly positive and bounded solution to the functional differential equation. This result points out the importance of initial conditions: not all initial wealth distributions among generations are compatible with the existence of an equilibrium. Some distributions may indeed generate short run fluctuations whose magnitude is so great that they would lead to negative prices. Finally, we propose numerical computations of the equilibrium and evaluate the influence of the initial distribution of assets.

# 2 Life cycle behavior

This section describes an individual's optimal behavior along the life cycle. Any variable related to an individual is indexed by both the calendar time and his or her birth date. Time is denoted by $t$ and satisfies $t \geq \tau$, where $\tau \in \mathbb{R}$ is the initial date of the economy. The birth date is denoted by $s \geq \tau - \omega$, where $\omega \in \mathbb{R}^{*+}$ is the upper bound of the individual's lifespan.

## 2.1 Assumptions

The livespan is uncertain and bounded. The age at death, denoted by $\alpha$, is a random variable over the interval $[0, \omega]$. Let us denote its density function, conditional to be alive at age $t - s$ by $f(\alpha, t - s)$ where $t \in [s, s + \omega]$. Let us define the survival function, i.e. the probability at birth of reaching age $\alpha$, as

$S(\alpha) = \int_\alpha^\omega f(u,0)\,du$. Thus, $f(\alpha, t-s) = f(\alpha,0)/s(t-s)$ or, equivalently,

$$f(\alpha, t-s) = \frac{-S'(\alpha)}{S(t-s)}. \tag{1}$$

Various survival functions have been proposed in the literature. As it will be explained in section 3.2, a crucial assumption lies in the upper bound of the lifespan. For instance, Boucekkine et al. (2002) propose a survival function given by:

$$S(t-s) = \frac{\left(1 - e^{(t-s-\omega)\mu}\right)}{\left(1 - e^{-\omega\mu}\right)}, \tag{2}$$

with $\mu > 0$. However, many authors assume that $\omega \to +\infty$. The Gompertz law indeed implies the following function:

$$S(t-s) = e^{\frac{1-(t-s)\mu}{\mu}}, \tag{3}$$

while Faruqee (2003) uses:

$$S(t-s) = e^{-\mu(t-s)} \left( \frac{e^{-\kappa} + e^{\kappa}}{e^{\kappa-(t-s)} + e^{\alpha-\kappa}} \right)^{\lambda}, \tag{4}$$

where $\lambda > 0$ and $\kappa > 0$.

Preferences are assumed to be additively separable with respect to time and to depend (provided that the individual is alive) on consumption only. The utility derived from consumption at age $t-s$ is denoted by $u(c(s,t))$, where $u(.)$ is increasing, concave and $C^3$. Moreover, $\partial(u'(x)/u''(x))/\partial x$ is assumed to be bounded. Finally, $c(s,t)$ is $\mathcal{C}^1((\tau-\omega,\infty)\times(\tau,\infty))$. If the individual is not alive, the utility is normalized to zero.

At time $t$, the utility of an individual living until age $\alpha \geq t-s$ is:

$$v(\alpha) = \int_t^{s+\alpha} e^{-\rho(z-t)} u(c(s,z))\, dz, \tag{5}$$

where $\rho$ is the pure time discount rate. The individual is assumed to be Fisherian, meaning that he or she is not altruistic towards his or her children. The utility when dead is thus assumed to be a finite constant that is not affected by the individual's decisions and can be neglected[1]. The expected utility is given by:

$$\int_{t-s}^{\omega} f(\alpha, t-s) \int_t^{s+\alpha} e^{-\rho(z-t)} u(c(s,z))\, dz d\alpha. \tag{6}$$

Integrating (6) by parts using (1) allows the expected utility to be rewritten as follows:

$$\int_t^{s+\omega} \frac{S(z-s)}{S(t-s)} e^{-\rho(z-t)} u(c(s,z))\, dz, \tag{7}$$

provided that $\lim_{z \to s+\omega} S(z-s)\, v(z-s) < \infty$. This latter condition requires that the utility of living until the maximal age $\omega$ be bounded, which is what is assumed. The expected utility, as expressed by (7), hence appears as a discounted integral of the instantaneous utility of consumption over the individual's maximal planning horizon. The discount rate here is the sum of the pure time discount rate and the hazard death rate given by $-S'(z-s)/S(z-s)$. Since $\omega < +\infty$, there is no need for a lower bound on

[1] For the impact of parental altruism in Blanchard's framework see d'Albis and Decreuse (2009).

$\rho$. Alternatively, in the limit case where $\omega \rightarrow +\infty$, we shall assume that:

$$\rho > \lim_{z+\infty} \frac{S'(z-s)}{S(z-s)}. \tag{8}$$

Finally, we note that the assumption of additively separable intertemporal preferences implies that the expected utility (7) is linear with respect to the survival probabilities. Bommier (2006) points out that the individual being considered is consequently neutral with respect to the risk of death.

Let $a(s,t)$ and $e(s,t) \geq 0$ denote the real wealth and real endowment of an individual of age $t-s$ respectively. Prices at time $t$ are denoted by $p(t)$. We assume that $a(s,t)$ is $\mathcal{C}^1((\tau-\omega,\infty)\times(\tau,\infty))$, while $p(t)$ is continuous for all $t \in (\tau,\infty)$. A fair annuity market, as described by Yaari (1965), is assumed to exist. An annuity is a particular actuarial note that makes an insurance company the legatee in case of death. Annuities return the regular risk-free interest rate, which is given here by the opposite of the inflation rate, plus the hazard rate of death. They are therefore the most lucrative asset if the individual's wealth is positive; furthermore due to the uncertainty on longevity, annuities are the only way to borrow. The selling of annuities is then interpreted as a demand for pure life insurance contracts (Bernheim, 1991). Hence, the individual holds his or her entire wealth in the form of annuities, and the flow budget constraint is given by:

$$\frac{\partial a(s,t)}{\partial t} = \left[-\frac{p'(t)}{p(t)} - \frac{S'(t-s)}{S(t-s)}\right] a(s,t) + e(s,t) - c(s,t). \tag{9}$$

Moreover, the following no-Ponzi game condition is imposed:

$$\lim_{t \to s+\omega} p(t) S(t-s) a(s,t) \geq 0. \tag{10}$$

The growth rate of the individual's debt should be greater than the asset's rate of return. Provided that the nominal value of wealth is finite, the condition (10) reduces to an equality. In that case, forward integration of equation (9) yields the intertemporal budget constraint at time $t$:

$$\int_t^{s+\omega} \frac{S(u-s)}{S(t-s)} \frac{p(u)}{p(t)} c(s,u)\, du = a(s,t) + \int_t^{s+\omega} \frac{S(u-s)}{S(t-s)} \frac{p(u)}{p(t)} e(s,u)\, du. \tag{11}$$

At time $t$, the difference between the expected values of nominal consumptions and endowments should equal the nominal wealth.

## 2.2 Optimal behavior

The problem can be solved using standard techniques of the calculus of variations. Let us first define:

$$L\left(a(s,z), \frac{\partial a(s,z)}{\partial z}, z\right) = \frac{S(z-s)}{S(t-s)} e^{-\rho(z-t)} u(c(s,z)), \tag{12}$$

where $c(s,z)$ should be replaced using (9). The problem of the individual is then described by the following:

$$\max_{a(.,.)} \int_t^{s+\omega} L\left(a(s,z), \frac{\partial a(s,z)}{\partial z}, z\right) dz,$$

$$s.t. \left| \begin{array}{l} \left[\frac{p'(z)}{p(z)} - \frac{S'(z-s)}{S(z-s)}\right] a(s,z) + e(s,z) - \frac{\partial a(s,z)}{\partial z} \geq 0, \\ \lim_{t \to s+\omega} p(t) S(t-s) a(s,t) \geq 0, \\ a(s,t) \ \text{ given.} \end{array} \right. \tag{13}$$

**Lemma 1.** *There exists a unique solution to problem (13). For all* $z \in [t, s+\omega]$*, the optimal consumption profile is:*

$$\frac{\partial \hat{c}(s,z)}{\partial z} = \frac{u'(c(s,z))}{u''(c(s,z))}\left[\frac{p'(z)}{p(z)} + \rho\right], \tag{14}$$

*the intertemporal budget constraint is (11) and* $a(s, s+\omega) = 0$*.*

<u>Proof.</u> The proof extends d'Albis (2007). Since $L$ is a $C^2$-function of $\left(a, \frac{\partial a}{\partial z}, z\right)$, the necessary conditions for a $C^2$-function $a(s,z)$ to be a solution of program (13) are the Euler equation (14) and the terminal condition:

$$-\frac{S(\omega)}{S(t-s)} e^{-\rho(s+\omega-t)} u'(c(s, s+\omega)) \leq 0, \tag{15}$$

which is satisfied since $p(s+\omega) S(\omega) a(s, s+\omega) \geq 0$. Given our assumptions on $u(.)$, the right-hand side of (14) is Lipschitz with respect to $c(s,z)$. Hence, there exists a unique solution to (14), denoted by $\hat{c}(s,z)$, which can be written as follows:

$$\hat{c}(s,z) = \hat{c}(s,t)\,\phi(z,t) \tag{16}$$

where $\phi(z,t)$ is a function that satisfies $\phi(t,t) = 1$. Substitute (16) into (11) to derive the initial consumption (i.e. at age $t-s$):

$$\hat{c}(s,t) = \frac{a(s,t) + \int_t^{s+\omega} \frac{S(u-s)}{S(t-s)} \frac{p(u)}{p(t)} e(s,u)\, du}{\int_t^{s+\omega} \frac{S(u-s)}{S(t-s)} \frac{p(u)}{p(t)} \phi(u,t)\, du}. \tag{17}$$

Consequently, there exists a unique consumption profile that satisfies the necessary conditions. Substituting this profile into (11) computed at time $z$

yields a unique wealth profile denoted by $\hat{a}(s,z)$ that satisfies:

$$\begin{aligned}\hat{a}(s,z) &= \left[a(s,t)+\int_t^{s+\omega}\frac{S(u-s)}{S(t-s)}\frac{p(u)}{p(t)}e(s,u)\,du\right]\phi(z,t) \\ &\quad\times\frac{\int_z^{s+\omega}\frac{S(u-s)}{S(z-s)}\frac{p(u)}{p(z)}\phi(u,z)\,du}{\int_t^{s+\omega}\frac{S(u-s)}{S(t-s)}\frac{p(u)}{p(t)}\phi(u,t)\,du}-\int_z^{s+\omega}\frac{S(u-s)}{S(z-s)}\frac{p(u)}{p(z)}e(s,u)\,du.\end{aligned} \tag{18}$$

We note that $\hat{a}(s,s+\omega)=0$. To conclude this proof, let us show that the necessary conditions are sufficient, by proving that for all $z\in[t,s+\omega]$, the function $L\left(a,\frac{\partial a}{\partial z},z\right)$ is concave with respect to $\left(a,\frac{\partial a}{\partial z}\right)$. The Hessian matrix is written as:

$$H=\frac{S(z-s)}{S(t-s)}e^{-\rho(z-t)}u''(c(s,z))\begin{bmatrix}\left[-\frac{p'(t)}{p(t)}-\frac{S'(t-s)}{S(t-s)}\right]^2 & \left[\frac{p'(t)}{p(t)}+\frac{S'(t-s)}{S(t-s)}\right] \\ \left[\frac{p'(t)}{p(t)}+\frac{S'(t-s)}{S(t-s)}\right] & 1\end{bmatrix} \tag{19}$$

As a consequence, $L$ is concave, and the unique solution of the Euler equation is globally maximal. □

This approach can easily be extended to more sophisticated problems. For instance, Boucekkine et al. (2002), d'Albis and Augeraud-Véron (2008), and Hritoenko and Yatsenko (2008) analyze education and retirement issues with individuals who choose the optimal length of their working lifetime.

## 2.3 Specific utility functions

It is common in the literature of continuous-time overlapping generations models to assume a specific utility function in order to explicitly derive the optimal consumption and wealth profiles. In most articles, the utility function is a power function:

$$u(c) = \left| \begin{array}{l} \frac{c^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}} \text{ if } \sigma \in (0,1) \cup (1,+\infty), \\ \\ \ln c \text{ if } \sigma = 1, \end{array} \right. \tag{20}$$

where $\sigma$ denotes the elasticity of intertemporal substitution. In this case, the Euler equation is written as:

$$\frac{\partial \hat{c}(s,z)}{\partial z} = -\sigma \left[ \frac{p'(z)}{p(z)} + \rho \right] \hat{c}(s,z). \tag{21}$$

The function (16) is thus $\phi(z,t) = e^{-\sigma\rho(z-t)} [p(z)/p(t)]^{-\sigma}$ which, based on the proof of Lemma 1, makes simple the derivation of the optimal consumption profile:

$$\hat{c}(s,z) = \frac{\frac{p(t)}{p(z)} a(s,t) + \int_t^{s+\omega} \frac{S(u-s)}{S(t-s)} \frac{p(u)}{p(z)} e(s,u)\, du}{\int_t^{s+\omega} \frac{S(u-s)}{S(t-s)} \left[ \frac{p(u)}{p(z)} \right]^{1-\sigma} e^{-\sigma\rho(u-t)} du} e^{-\sigma\rho(z-t)}. \tag{22}$$

With power utility functions, the consumption growth rate, given by (21), is independent of the hazard death rate. Due to the fair pricing of annuities, the increase in the discount rate implied by the uncertain longevity is canceled out by the increase in the annuities return (Davis, 1981). Consequently, the optimal consumption profile, given by (22), is proportional to the sum of the initial wealth and the human wealth in nominal terms.

These properties crucially depend on the power utility function. Let us suppose conversely that the utility is, for instance, described by an exponential function such that: $u(c) = -e^{-\alpha c}/\alpha$ with $\alpha > 0$. The growth rate of consumption is then:

$$\frac{\frac{\partial \hat{c}(s,z)}{\partial z}}{\hat{c}\left(s,z\right)}=\frac{-\alpha\left[\frac{p'(z)}{p(z)}+\rho\right]}{-\alpha\ln\frac{p(z)}{p(t)}-\alpha\rho\left[z-t\right]+\hat{c}\left(s,t\right)}, \tag{23}$$

where $\hat{c}\left(s,t\right)$ is the consumption at the initial age $t-s$, such that:

$$\begin{aligned}\hat{c}\left(s,t\right) &= \frac{p\left(t\right)a\left(s,t\right)+\int_t^{s+\omega}\frac{S(u-s)}{S(t-s)}p\left(u\right)e\left(s,u\right)du}{\int_t^{s+\omega}\frac{S(u-s)}{S(t-s)}p\left(u\right)du} \\ &+\frac{\alpha\int_t^{s+\omega}\frac{S(u-s)}{S(t-s)}p\left(u\right)\left[\ln\frac{p(u)}{p(t)}+\rho\left(u-t\right)\right]duz}{\int_t^{s+\omega}\frac{S(u-s)}{S(t-s)}p\left(u\right)du}.\end{aligned} \tag{24}$$

Consumption growth is no longer independent of the hazard death rate.

In the remaining sections of the paper, the utility function will be assumed to satisfy (20). In this case, the optimal wealth profile is:

$$\begin{aligned}\hat{a}\left(s,z\right) &= \left[a\left(s,t\right)+\int_t^{s+\omega}\frac{S\left(u-s\right)}{S\left(t-s\right)}\frac{p\left(u\right)}{p\left(t\right)}e\left(s,u\right)du\right]e^{-\sigma\rho(z-t)}\left[\frac{p\left(z\right)}{p\left(t\right)}\right]^{-\sigma} \\ &\times\frac{\int_z^{s+\omega}\frac{S(u-s)}{S(z-s)}\left[\frac{p(u)}{p(z)}\right]^{1-\sigma}e^{-\sigma\rho(u-z)}du}{\int_t^{s+\omega}\frac{S(u-s)}{S(t-s)}\left[\frac{p(u)}{p(t)}\right]^{1-\sigma}e^{-\sigma\rho(u-t)}du}-\int_z^{s+\omega}\frac{S\left(u-s\right)}{S\left(z-s\right)}\frac{p\left(u\right)}{p\left(z\right)}e\left(s,u\right)du.\end{aligned} \tag{25}$$

# 3 Intertemporal equilibrium

This section presents the dynamic system whose solution constitutes the intertemporal equilibrium. Firstly, the behaviors of individuals are aggregated and the production side of the economy is described. Conditions for the existence of an intertemporal equilibrium are then presented.

## 3.1 Demographic structure and aggregation

The demographic structure is based on Lotka's (1939) stable population theory. His theory demonstrates that the relative size of each cohort in the population converges to a steady-state when the birth and death rates have been constant over a sufficiently long period of time. We shall assume that the demographic structure has reached its stationary distribution.

Each individual belongs to a large cohort of identical individuals. Therefore, even though each individual's lifespan is stochastic, there is no aggregate uncertainty. The law of large numbers is supposed to apply, and thus, the size of each cohort is decreasing at the hazard death rate. Let $\beta > 0$ be the birth rate. The density of individuals of age $(t - s)$ is $\beta S(t-s)\, e^{-n(t-s)}$, where $n$ is the demographic growth rate, which is the solution, for all $t \geq \tau$, of:

$$\int_{t-\omega}^{t} \beta S(t-s)\, e^{-n(t-s)} ds = 1. \tag{26}$$

The left-hand side of (26) is a positive, decreasing function of $n$, whose limits are $+\infty$ and 0 when $n$ tends to $-\infty$ and $+\infty$ respectively. Hence, there exists a unique real root whose sign depends on the difference between the birth rate and the inverse of the life expectancy at birth. Strictly speaking, $n \geq 0$ if and only if $\beta \geq 1/\int_0^{\omega} S(u)\, du$. Such a population is often named "Malthusian".

The aggregate counterpart per capita of any individual variable $x(s, t)$ is denoted by $x(t)$ and satisfies:

$$x(t) = \int_{t-\omega}^{t} \beta S(t-s)\, e^{-n(t-s)} x(s,t)\, ds. \tag{27}$$

Moreover, it is useful to define $\alpha_x(t)$, the average age calculated cross-sectionally, as:

$$\alpha_x(t) = \frac{\int_{t-\omega}^{t} (t-s)\, S(t-s)\, e^{-n(t-s)} x(s,t)\, ds}{\int_{t-\omega}^{t} S(t-s)\, e^{-n(t-s)} x(s,t)\, ds}. \tag{28}$$

In the case where $x(s,t) = 1$, $\alpha_1(t)$ is the standard definition of the average age of the population. If $x(s,t) = c(s,t)$, $\alpha_c(t)$ is the average age of consumers weighted by their consumption.

## 3.2 Dynamics of aggregate consumption

In the case of a power utility function, the aggregate consumption, denoted by $c(t)$, satisfies (using (21)) the following differential equation:

$$c'(t) = \beta \hat{c}(t,t) - \left[n + \sigma\left(\frac{p'(z)}{p(z)} + \rho\right)\right] c(t) + \int_{t-\omega}^{t} \beta S'(t-s)\, e^{-n(t-s)} \hat{c}(s,t)\, ds, \tag{29}$$

where $\hat{c}(t,t)$ is the optimal initial consumption of an individual born at time $t$. Using (22), one can decompose the influence of prices on the dynamics of the aggregate consumption, $c'(t)$. First, the expectations of prices over the interval $[t, t+\omega]$ are reflected through the consumption by newborns $\hat{c}(t,t)$. Second, past prices, and more precisely those defined over the interval $[t-\omega, t]$ are reflected through the last term of the right-hand side of equation

(29), which represents the aggregate consumption of individuals who died at time $t$. Hence, prices which were relevant for these individuals also influence the aggregate dynamics. The assumption concerning the upper bound of the lifespan is important here, since the dynamics of the aggregate consumption depend on prices over the interval $[\tau, \infty)$, if $\omega \to +\infty$.

At equilibrium, the dynamics induced by (29) are characterized by a mixed-type delay-differential equation: discrete advances and lags influence the dynamics. As pointed out by Boucekkine et al. (2002) and (2004), lags appear as resulting from the demographic structure. Conversely, d'Albis and Augeraud-Véron (2004) argue that advances are yielded by the assumption of perfect forecasting. This can be demonstrated by supposing logarithmic utility and endowments distributed at birth only: $e(s,t) = e(t)$ if $s = t$, and zero otherwise. Then, using (22), the initial consumption of individuals born at time $t$ does not depend on expectations and can be written as:

$$\hat{c}(t,t) = \frac{e(t)}{\int_t^{t+\omega} S(u-s)\, e^{-\rho(u-t)} du}. \tag{30}$$

Differentiating (30) with respect to time yields an ordinary differential equation. As a consequence, equation (29) is a delay-differential equation: the absence of expectations on future prices at the individual level leads to aggregate dynamics that are only governed by the vintage structure of the population. Let us note that by combining various assumptions and notably a linear utility, Boucekkine et al. (2002) also obtain a delay-differential equation for the aggregate dynamics they consider.

The representation using a mixed-type delay-differential equation is almost generic in continuous-time overlapping generations models and there are few cases for which it does not hold true. One such case is the celebrated model by Blanchard (1985) which reduces to a system of ordinary differential equations[2]. One of the key assumptions is that the random age at death follows a Poisson process. This is convenient because it implies that all individuals have the same life expectancy regardless of their age. Then, equation (29) may be rewritten as:

$$c'(t) = \beta c(t,t) - \left[n + \sigma\left(\frac{p'(z)}{p(z)} + \rho\right) - \mu\right] c(t), \tag{31}$$

where $\mu$ is the Poisson parameter featuring the hazard death rate.

Moreover, if the endowment grows at a constant rate over the life span, using (22), the initial consumption of individual born at time $t$ is given by:

$$\hat{c}(t,t) = \frac{e(t,t)\int_t^\infty e^{(\gamma-\mu)(u-t)} p(u)\, du}{\int_t^\infty e^{-(\mu+\sigma\rho)(u-t)} p^{1-\sigma}(u)\, du} p^{-\sigma}(t), \tag{32}$$

where $\gamma$ is the endowment growth rate. Differentiating (32) with respect to time gives an ordinary differential equation. The reader may wish to refer to De la Croix and Licandro (1999) who point out the importance of the endowment distribution over the life cycle with regard to the aggregate dynamics.

[2] See also Buiter (1988) and Weil (1989).

Combining a constant hazard death rate and a constant endowment growth rate drastically reduces the degree of heterogeneity in the economy: all individuals have indeed the same propensity to consume. Moreover, with these assumptions, models are unable to capture the life cycle features of asset accumulation (see d'Albis, 2007).

To conclude the setup of the model, it is necessary to establish the initial conditions that are derived from the distribution of initial individual wealth. We assume that:

$$\begin{aligned} &a(s,\tau) \text{ are given for all } s \in [\tau-\omega,\tau], \\ &a(s,s) = 0 \text{ for all } s > \tau. \end{aligned} \tag{33}$$

There are consequently two classes of individuals, depending on whether they belong to cohorts born before or after the initial date of the economy $\tau$. In the former case, the wealth at age $\tau - s$, namely $a(s,\tau)$, is given. In the latter case, the wealth at birth is assumed to be zero, since there is no intergenerational altruism. The dynamics of the aggregate consumption are then piecewise: for all $t \in [\tau, \tau+\omega]$, they depend on prices over $[\tau, t+\omega]$ and on the initial distribution of wealth, while for all $t \geq \tau+\omega$, they depend on prices over $[t-\omega, t+\omega]$.

## 3.3 Equilibrium

We now consider the computation of equilibrium prices. We extend the pioneering works of Samuelson (1958) and Gale (1973) to a continuous-time framework. There is one goods market on which is traded non-storable goods whose aggregate endowment is given and whose distribution among cohorts is stationary. On the asset market, consumption loans are available at a risk-free interest rate that equals the opposite of the inflation rate. Annuities are also proposed to ensure the individual against the uncertainty on longevity. Moreover, there may exist a durable but intrinsically worthless asset, called money for convenience, which can be distributed by a "Central Bank" to individuals. Money returns as much as consumption loans and can also be covered by annuities.

Let $m(s,t) \geq 0$ denote the quantity of money distributed at time $t$ to the individual born at time $s$. The real endowment in terms of non-storable goods received by the same individual is consequently $e(s,t) - m(s,t)/p(t)$, and the aggregate real endowment, denoted by $Y$, is:

$$Y = \int_{t-\omega}^{t} \beta e^{-n(t-s)} S(t-s) \left[ e(s,t) - \frac{m(s,t)}{p(t)} \right] ds. \tag{34}$$

Thus, the equilibrium condition on the goods market at time $t \geq \tau$ is given by:

$$c(t) = Y. \tag{35}$$

On the asset market, the aggregate assets should be equal to the total amount of money (evaluated in real terms) distributed since the initial date of the economy. Indeed, by definition, the aggregate value of consumption loans is zero. Thus, the equilibrium condition on the asset market at time $t \geq \tau$ is given by:

$$
\begin{aligned}
a(t) &= \frac{\beta}{p(t)}\int_{\tau}^{t} e^{-n(t-u)}\int_{u-\omega}^{u} e^{-n(u-s)}s(u-s)\,m(s,u)\,dsdu \\
&\quad +\frac{\beta e^{-n(t-\tau)}}{p(t)}\int_{\tau-\omega}^{\tau} e^{-n(\tau-s)}s(\tau-s)\,m(s,\tau)\,ds. \qquad (36)
\end{aligned}
$$

Monetary policy is thus important. For instance, the Central Bank may want to set the growth rate of the quantity of money to a constant, $\gamma$. This policy can notably be achieved by assuming that the distribution is such that for all $s \geq \tau$, $m(s,u) = m(u) > 0$ and $m'(u) = \gamma m(u)$. The equilibrium condition (36) can then be expressed as:

$$
a(t) = \frac{e^{-n(t-\tau)}}{p(t)}\left[\frac{e^{(n+\gamma)(t-\tau)}-1}{(n+\gamma)}+1\right]m(\tau). \qquad (37)
$$

Alternatively, we may wish for the money to be distributed at the initial date only. In which case, we may write:

$$
a(t) = \frac{e^{-n(t-\tau)}}{p(t)}\int_{\tau-\omega}^{\tau} \beta e^{-n(\tau-s)}s(\tau-s)\,m(s,\tau)\,ds. \qquad (38)
$$

In the particular case in which the population growth is equal to the opposite of the inflation rate, the right-hand side of the above equality is a constant (see d'Albis and Augeraud-Véron, 2008). Such a constant is nevertheless indeterminated since $p(0)$ is unknown.

The dynamics of the aggregate assets at equilibrium can be equivalently obtained by differentiating the demand or the supply side with respect to time. Thus, differentiating the left-hand side of (36) and using the individual's flow budget constraint (9), the initial conditions (33), and the equilibrium condition on the goods market (35), we obtain:

$$a'(t) = -\left[\frac{p'(t)}{p(t)} + n\right] a(t) + \int_{t-\omega}^{t} \beta e^{-n(t-s)} S(t-s) \frac{m(s,t)}{p(t)} ds \qquad (39)$$

for all $t \geq \tau$. In the case where money is distributed at the initial date only, the second term of the left-hand side of (39) equals zero for all $t > \tau$ and the aggregate assets per capita reach a steady-state under one of only two conditions: either when $p'(t)/p(t) = -n$ and the aggregate assets may be a positive constant, or when $p'(t)/p(t) > -n$ and the aggregate assets converge to zero.

Let us refer to the definition of an equilibrium path and that of a particular solution of it, namely the stationary one.

**Definition 1.** *An equilibrium with perfect foresight is a function* $F(t) = (c(t), a(t), p(t))$, $F : [\tau, +\infty) \rightarrow \mathbb{R}_+^3$, $C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, *such that (i) individuals maximize their utility subject to the budget constraint, (ii) the aggregate consumption equals the aggregate endowment (i.e.* $c(t) = Y$*), and (iii) the aggregate assets are non-negative (i.e.* $a(t) \geq 0$*). The equilibrium is said to be monetary if* $a(t) > 0$ *and balanced if* $a(t) = 0$.

**Definition 2.** *A stationary solution is an equilibrium such that* $p'(t)/p(t)$ *is constant.*

Among stationary solutions, the Golden Rule path, such that $p'(t)/p(t) =$

$-n$, has been extensively studied. Since the work of Samuelson (1958), it is customary to name this solution the "biological interest rate". The properties of this solution in continuous-time were initially studied by Arthur and McNicoll (1978) and Lee (1980), while recent advances have been proposed by Bommier and Lee (2003) and Brito and Dilão (2006).

# 4 Dynamics of prices

In this section, the dynamics of intertemporal prices are studied. The local dynamics in the vicinity of an existing steady-state are first analyzed in a nonlinear model. Then, the global dynamics of a linear model are considered.

Some additional assumptions are made. Considering the individual, it is assumed that instantaneous utility is represented by a power function as in (20). Moreover, the endowment over the life cycle satisfies a uniform distribution. Let $\alpha$ and $\beta$ be such that $0 \leq \alpha < \beta \leq \omega$, then let us assume that:

$$e(s,t) = \begin{cases} \frac{\int_0^\omega e^{-nu} du}{\int_\alpha^\beta e^{-nu} du} \text{ if } \alpha < t - s < \beta, \\ \\ 0 \text{ otherwise.} \end{cases} \tag{40}$$

The aggregate endowment per capita is thus normalized to 1. Finally, the lifespan is deterministic:

$$S(t-s) = \begin{cases} 1 \text{ if } t \in [s, s+\omega), \\ \\ 0 \text{ if } t = s + \omega, \end{cases} \quad \text{and thus} \quad \int_{-\omega}^{0} e^{ns} ds = \frac{1}{\beta}. \tag{41}$$

Using this last assumption, the optimal consumption profiles can be derived from (22) and satisfy:

$$\hat{c}(s,t) = \begin{cases} \frac{p(\tau)a(s,\tau)+\int_\tau^{s+\omega} p(u)e(s,u)du}{\int_\tau^{s+\omega} p^{1-\sigma}(u)e^{-\sigma\rho(u-\tau)}du} e^{-\sigma\rho(t-\tau)} p^{-\sigma}(t) \text{ if } s \in [\tau-\omega,\tau), \\ \\ \frac{\int_s^{s+\omega} p(u)e(s,u)du}{\int_s^{s+\omega} p^{1-\sigma}(u)e^{-\sigma\rho(u-s)}du} e^{-\sigma\rho(t-s)} p^{-\sigma}(t) \text{ if } s \geq \tau. \end{cases} \tag{42}$$

Concerning the asset market, it is assumed that there is no money in the economy. The equilibrium conditions on the goods and asset markets on the goods and asset markets are expressed as:

$$\int_{t-\omega}^{t} \beta e^{-n(t-s)} \hat{c}(s,t)\, ds = 1, \tag{43}$$

$$\int_{t-\omega}^{t} \beta e^{-n(t-s)} \hat{a}(s,t)\, ds = 0, \tag{44}$$

for all $t \geq \tau$. These assumptions do not qualitatively affect the results that will be obtained since the dynamics of the aggregate consumption are nevertheless given by a mixed-type delay-differential equation.

## 4.1 Long run fluctuations

First of all, we consider the price fluctuations in the vicinity of a steady-state The initial trajectory can thus be ignored since only the dynamics for $t \geq \tau + \omega$ are relevant. Replacing (40) and (42) for $s \geq \tau$ in (43) gives the condition on the goods market for $t \geq \tau + \omega$:

$$\int_{t-\omega}^{t} \frac{e^{-(n+\sigma\rho)(t-s)} \int_{s+\alpha}^{s+\beta} \frac{p(u)}{p(t)} du}{\int_\alpha^\beta e^{-nu} du \int_s^{s+\omega} \left[\frac{p(u)}{p(t)}\right]^{1-\sigma} e^{-\sigma\rho(u-s)} du} ds = 1. \tag{45}$$

According to Definition 2, a stationary solution is a real root, denoted by $r$, that solves $\eta(r) = 1$, where $\eta(.)$ is obtained by replacing $p'(t)/p(t) = -r$

for all $t$ in (45). We obtain:

$$\eta(r) = \frac{\int_\alpha^\beta e^{-ru}du \int_0^\omega e^{-[n+\sigma(\rho-r)]u}du}{\int_\alpha^\beta e^{-nu}du \int_0^\omega e^{-[(1-\sigma)r+\sigma\rho]u}du}. \tag{46}$$

Let us now characterize the existence of a stationary solution. Firstly, we note that the biological interest rate (i.e. $r = n$) always appears as a candidate since $\eta(n) = 1$. However, the condition on the asset market (44) must also be satisfied. Hence, the following result may be established.

**Lemma 2.** *The biological interest rate is a stationary solution if and only if the average age of consumers equals the average age of endowment earners.*

Proof. The proof is based on the fact that aggregate assets per capita, namely $a(t)$, should be equal to zero. For any stationary solution $p'(t)/p(t) = -r$, constant aggregate assets per capita should satisfy:

$$a(t)[r-n] + 1 - \eta(r) = 0 \tag{47}$$

Then, aggregate assets are equal to zero if and only if: $\lim_{r=n} \frac{\eta(r)-1}{r-n} = 0$, or equivalently, by applying l'Hôpital's rule, if and only if: $\eta'(r)|_{r=n} = 0$. Using (46), we obtain the following condition:

$$\frac{\int_0^\omega ue^{-[(1-\sigma)n+\sigma\rho]u}du}{\int_0^\omega e^{-[(1-\sigma)n+\sigma\rho]u}du} = \frac{\int_\alpha^\beta ue^{-nu}du}{\int_\alpha^\beta e^{-nu}du}. \tag{48}$$

Let us note that $\sigma[n-\rho]$ is the growth rate of individual consumption when $p'(t)/p(t) = -r$ and conclude using (28). $\square$

Lemma 2 implies that when there is no money in the economy, the biological interest rate has little chance of being a valid equilibrium. The exception is nongeneric and is named the "coincidental equilibrium" in literature, for which the monetary and balanced equilibria are the same. Nevertheless, the interpretation given in Lemma 2 in terms of average ages is meaningful. By forbidding positive aggregate assets, the difference between the average ages of the consumer and the endowment earner should be zero. If the difference is positive, the "average worker may transfer endowments to the "average consumer", in which case the aggregate assets would have to be positive, which is not allowed. In the latter case, money has a positive value in the long run (see d'Albis and Augeraud-Véron, 2008).

Since $p'(t)/p(t) = -n$ does not constitute a valid steady-state, we are now looking for other steady-states. We now refer to the work d'Albis and Augeraud-Véron (2007).

**Lemma 3.** *Besides the biological interest rate, there exists* $(n, \rho, \sigma, \alpha, \beta, \omega)$ *such that there is no steady-state. If* $\alpha = 0$ *and* $\beta = \omega$*, there always exists a steady-state, namely* $p'(t)/p(t) = -\rho$*, such that the equilibrium is characterized by the absence of trade on the asset market.*

Proof. Firstly, we assume that $\rho = n = 0$. We demonstrate that there is no $r \neq 0$ such that $\eta(r) = 1$ if $\alpha = 0$ and $\beta \in (0, 1-\sigma]$. We define $\psi(r; \beta)$ such that:

$$\psi(\beta) = \frac{\int_0^1 e^{\sigma r z} dz}{\beta} \frac{\int_0^\beta e^{-rz} dz}{\int_0^1 e^{-(1-\sigma) r z} dz}. \tag{49}$$

We observe that $\psi(r; 1-\sigma) > 1 \Leftrightarrow r > 0$ and $\partial \psi(r; \beta) / \partial \beta > 0 \Leftrightarrow r < 0$. The existence of at least one steady-state is simple. Let us concentrate on a particular case that satisfies $\alpha = 0$ and $\beta = \omega$. Using (46) and (48), we may immediately conclude that $r = \rho$ is a valid steady-state. $\square$

The no-trade steady-state is an equilibrium such that there are no consumption loans sold on the market, and consequently no trading among generations: individuals consume their instantaneous endowments throughout their life. This is a particular solution, which is easily obtained. In general, however, when there are more than two generations, and obviously when one considers a continuum, the stationary solution of a balanced equilibrium does involve intergenerational trading[3].

Let us now refer to the study of long run fluctuations. To prove their existence, we demonstrate that some solutions of the linearized counterpart of equation (45), which characterizes the local motion of our economy, have Hopf bifurcation values. This proof follows Rustichini (1989b) and Benhabib and Rustichini (1991) who have also proposed economic models whose dynamics are described by nonlinear functional equations that may generate Hopf bifurcations. Their models are, however, quite different and describe

[3] See Kim (1983), Aiyagari (1988) and Ghiglino and Tvede (1995).

the behavior of an infinitely-lived representative agent.

**Lemma 4.** *There exists* $(n,\rho,\sigma,\alpha,\beta,\omega)$ *such that the characteristic function associated with the linearized dynamics has pure imaginary roots that have Hopf bifurcation values.*

<u>Proof.</u> Assuming that $\omega > 1$, $\rho = n = 0$, $\alpha = 0$ and $\beta = 1$, we prove that a Hopf bifurcation exists if $(\omega - 1)^{-1} \notin \mathbb{N}$. For these parameters, equation (45) may be simplified to:

$$\int_{t-\omega}^{t} \frac{\int_s^{s+\omega} \frac{p(u)}{p(t)} du}{\int_s^{s+\omega} \left[\frac{p(u)}{p(t)}\right]^{1-\sigma} du} ds = \omega. \tag{50}$$

The local dynamics in the vicinity of steady-state $r^*$ correspond to $x(t)$ defined such that $r(t) = r^* + \epsilon x(t)$. We replace $\frac{p(u)}{p(t)} = e^{-\int_t^u [r^* + \epsilon x(j)]dj}$ in (50) and carry out a Taylor expansion in the vicinity of $\epsilon = 0$. Given Lemma 3, we know that $r^* = 0$. Let us use the definition of $X(t) = \int_0^t x(u)\, du$ that can be rearranged to obtain:

$$X(t) = \frac{1}{\omega^2} \int_{t-\omega}^{t} \int_s^{s+\omega} X(z) dz ds. \tag{51}$$

Let $\Delta(\lambda, \sigma) = 0$ be the characteristic function, obtained by the following change of variable: $X(t) = e^{\lambda t}$ in (51):

$$\Delta(\lambda, \sigma) = \omega^2 - \int_0^{\omega} \int_0^{\omega} e^{\lambda(u-s)} du ds. \tag{52}$$

We then refer to Hupkes et al. (2008) who prove the existence of an infinite sequence of pairs $(\sigma_k, q_k)$, with $\sigma_k > 0$ and $q_k > 0$, such that:

*i)* $\sigma_k \to 0$ and $q_k \to \infty$ as $k \to \infty$.

*ii)* The characteristic function $\Delta(z, \sigma) = 0$ has two simple roots $z = \pm i q_k$.

*iii)* For all $k \in \mathbb{N}$ and $m \in \mathbb{Z} - \{\pm 1\}$, the inequality $\Delta(\operatorname{Im} q_k, \sigma) \neq 0$.

*iv)* $\operatorname{Re} \frac{\Delta_{\sigma_k}(iq_k, \sigma_k)}{\Delta_1(iq_k, \sigma_k)} \neq 0$. $\square$

In order to investigate the role of the initial conditions, we now consider the global dynamics of the model.

## 4.2 Global dynamics

In order to analyze the global dynamics of the model, we confine our attention to a linear version of the integral equation being considered. As in Demichelis and Polemarchakis (2007) and Edmond (2008), this can be done by supposing logarithmic preferences (i.e. $\sigma = 1$). Furthermore, to avoid the proofs being too lengthy, we moreover assume that endowments are uniformly distributed among generations: $\alpha = 0$ and $\beta = \omega$.

In what follows, we begin by defining the piecewise functional differential equations that characterize the equilibrium prices for $t \geq \tau$. Next, we proceed in two steps: first of all, we analyze the dynamic behavior for $t \geq \omega + \tau$. Then, we study the existence of a solution using the dynamics over $[\tau, \tau + \omega]$, as carried out by d'Albis and Augeraud-Véron (2004).

**Property 1.** *For $\sigma = 1$, $\alpha = 0$ and $\beta = \omega$, the dynamics are given by:*

$$\begin{aligned} p(t) &= f(t) + \int_0^t p(v)\,\delta_1(t,v)\,dv \\ &\quad + \int_t^{\omega} p(v)\,\delta_2(t,v)\,dv + \int_t^{t+\omega} p(\nu)\,\delta_3(v-t)\,dv, \end{aligned} \tag{53}$$

*for $\tau < t < \tau + \omega$ and:*

$$p(t) = \frac{\beta}{\int_0^{\omega} e^{-\rho z} dz}\left[\int_{t-\omega}^{t} p(z)\,\varphi_1(t-z)\,dz + \int_t^{t+\omega} p(z)\,\varphi_2(t-z+\omega)\,dz\right], \tag{54}$$

*for $t > \tau + \omega$, with:*

$$f(t) = \int_{t-\omega}^{\tau} \beta e^{-(n+\rho)(t-s)} \frac{a(s,\tau)}{\left(\int_{\tau}^{s+\omega} e^{-\rho(\nu-s)} dv\right)} ds, \tag{55}$$

$$\delta_1(t,\nu) = \beta\left(\int_{t-\omega}^{0} \frac{e^{-(n+\rho)(t-s)}}{\left(\int_0^{s+\omega} e^{-\rho(\nu-s)} dv\right)} ds + \left(\frac{\int_0^{v} e^{-(n+\rho)(t-s)} ds}{\int_0^{\omega} e^{-\rho z} dz}\right)\right), \tag{56}$$

$$\delta_2(t,\nu) = \beta\int_{z-\omega}^{0} \frac{e^{-(n+\rho)(t-s)}}{\left(\int_0^{s+\omega} e^{-\rho(\nu-s)} dv\right)} ds, \tag{57}$$

$$\delta_3(v) = \beta\frac{\int_0^{\omega-v} e^{-(n+\rho)s} ds}{\int_0^{\omega} e^{-\rho z} dz}, \tag{58}$$

$$\varphi_1(u) = \frac{e^{-(n+\rho)u} - e^{-(n+\rho)\omega}}{n+\rho} \text{ and } \varphi_2(u) = \frac{1 - e^{-(n+\rho)u}}{n+\rho}. \tag{59}$$

Proof. Refer to the Appendix. □

The system defined in Property 1 includes, via the variable $f(t)$, the initial distribution of wealth among generations that constitutes the initial condition of the system. The price dynamics, given by (53), are characterized by an algebraic functional equation with advance. When all the individuals who were alive at the initial date are dead, a second equation then characterizes the dynamics for an infinite future. The price dynamics, which are then given by (54), are characterized by an algebraic functional equation of mixed type, where both delays and advances have an influence. It is easy to check that the two equations are the same at $t = \tau + \omega$. According to Edmond (2008), we may rewrite (54) as follows:

$$p(t) = \int_0^\infty p(z)\, k(t, z)\, dt, \tag{60}$$

where the computation of the integral kernel $k(t, z)$ is given in the Appendix.

In the following Lemma, we analyze the roots of the characteristic equation. Due to the delays and the advances, this equation is transcendental. The number of roots is thus infinite.

**Lemma 5.** *The characteristic equation associated with the system of equations (53) and (54) has: 1) two real roots $-\rho$ and $-n$; 2) no complex roots with real parts that belong to the closed interval:* $[\min\{-\rho, -n\}, \max\{-\rho, -n\}]$, *except $-\rho$ and $-n$; 3) an infinity of complex roots with real parts greater than* $\max\{-\rho, -n\}$*; 4) an infinity of complex roots with real parts smaller than* $\min\{-\rho, -n\}$.

<u>Proof.</u> The characteristic equation is given by:

$$\Delta(\lambda) = \int_0^\omega e^{-\rho z} dz \int_0^\omega e^{-nz} dz - \int_0^\omega e^{-(n+\rho)s} \int_0^\omega e^{\lambda(z-s)} dz ds. \tag{61}$$

Result 1 is immediate. Result 2 is obtained by contradiction: assume that a root that satisfies $\lambda = p + iq$, where $p \in [-\min\{\rho, n\}, -\max\{\rho, n\}]$, and $q \neq 0$, exists. Let $g(\lambda) = \int_0^\omega e^{-(n+\rho)s} \int_0^\omega e^{\lambda(z-s)} dz ds$. Then,

$$|g(p+iq)| < |\int_0^\omega e^{-(n+\rho)s} \int_0^\omega e^{p(z-s)} dz ds| = g(q), \tag{62}$$

as $q \neq 0$, while the convexity of $g$ implies that $g(q) \leq \int_0^\omega e^{-\rho z} dz \int_0^\omega e^{-nz} dz$, for all $p \in [-\min\{\rho, n\}, -\max\{\rho, n\}]$, which implies that the considered root does not exist. Results 3 and 4 can be derived using the geometrical method proposed by Bellman and Cooke (1963, p. 410). □

The existence of solutions of mixed-type functional equations is not easy to establish. For instance, Rustichini (1989a) has proposed some simple counterexamples for which functional equations of mixed type may fail to have a solution. More generally, Burke (1996) proved that the existence of an equilibrium is not always ensured in continuous-time overlapping generations models even under rather standard and simple assumptions.

Let us now refer to Rustichini (1989a) and Mallet Paret and Verduyn Lunel (2008). They prove, using different techniques, that a decomposition of $C[\tau, \tau + 2\omega]$ can be obtained such that $C[\tau, \tau + 2\omega] = P \oplus Q$, where $P \subset C[\tau, \tau + 2\omega]$ (resp. $Q \subset C[\tau, \tau + 2\omega]$) is the set of initial conditions such that solutions of the functional equation satisfy $\lim_{t \to -\infty} p(t) e^{-rt} = 0$ (resp. $\lim_{t \to \infty} p(t) e^{-rt} = 0$). Using the results of Mallet Paret and Verduyn Lunel (2008), we rewrite the solutions $p(t)$, for $t \in [\tau + \omega, \infty)$ such that $\lim_{t \to \infty} p(t) e^{-rt} = 0$, as solutions of a delay equation with initial conditions in $[\tau, \tau + \omega]$. The characteristic polynomial of such a delay equation has the characteristic roots:

$$\{\lambda \text{ such that } \Delta(\lambda) = 0 \text{ and } \operatorname{Re}(\lambda) \leq \rho\}. \tag{63}$$

Thus, according to the localization of the real roots, in the vicinity of the no-trade steady-state, oscillations can survive if $\rho < n$ while they are all damped if $\rho > n$. Interestingly, we have seen that if $\rho + n = 0$, the characteristic equation may be written as:

$$\Delta(\lambda) = \omega^2 - \int_0^\omega \int_0^\omega e^{\lambda(z-s)} dz ds. \tag{64}$$

Then, if $\lambda$ is a characteristic root, $-\lambda$ is also a characteristic root, in accordance with the findings of Demichelis and Polemarchakis (2007). In this particular case, an analysis of the local indeterminacy can be done using a discrete counterpart of the model and this symmetry. For all other cases, indeterminacy in continuous-time overlapping generations models is a difficult problem that has yet to be solved.

The existence of a solution is established in the following lemma.

**Lemma 6.** *For any given distribution of initial assets $a(s, \tau)$, there exists a unique solution to equations (53) and (54). Such a solution does not always constitute an equilibrium since it may imply $p(t) \leq 0$ for some $t \in [\tau, \infty)$.*

Proof. Let us rewrite equation for $t > \omega$ as:

$$p'(t) = \frac{\beta}{\int_0^{\omega} e^{-\rho z} dz} \left[ \int_t^{t+\omega} p(z)\, dz - e^{-(n+\rho)\omega} \int_{t-\omega}^{t} p(z)\, dz \right] - (n+\rho)\, p(t). \tag{65}$$

Rewriting this in a compact form, $p'(t) = \int_{-\omega}^{\omega} p(z+t)\, d\eta(z)$, with $\eta(z)$ being of dimension 1. The characteristic equation can then be rewritten as $\Delta(\lambda) = 0$, with:

$$\Delta(\lambda) = \lambda - \int_{-\omega}^{\omega} e^{\lambda z} d\eta(z). \tag{66}$$

Let $\lambda_0 \in \mathbb{C}$. According to Theorem 5.2. of Mallet Paret and Verduyn Lunel (2008), there exists $d\eta_-(z)$ and $d\eta_+(z)$ of finite Lebesgue-Stieltjes measures on $[\tau - \omega, \tau]$ and on $[\tau, \tau + \omega]$, such that the following decomposition of $\Delta(\lambda)$ can be written:

$$\Delta(\lambda)(\lambda - \lambda_0) = \Delta_-(\lambda)\, \Delta_+(\lambda) \tag{67}$$

with:

$$\Delta_-(\lambda) = \lambda - \int_{-\omega}^{0} e^{\lambda z} d\eta_-(z) \text{ and } \Delta_+(\lambda) = \lambda - \int_0^{\omega} e^{\lambda z} d\eta_+(z) \tag{68}$$

According to Theorem 5.3, if $\Delta_+(\lambda_0) = 0$, then every solution of $p'(t) - \int_{-\omega}^{0} p(z+t)\, d\eta_-(z)$ is also a solution of $p'(t) = \int_{-\omega}^{\omega} p(z+t)\, d\eta(z)$. Let $p(t) = e^{\lambda_1 t}$, with $\lambda_1 \in \{-n, -\rho\}$, be the stationary solution. If not, we can exchange roots from $\Delta_-$ to $\Delta_+$ such that all the roots in the quotient function $\Delta\Delta_-^{-1}$ satisfy $\operatorname{Re}\lambda > \lambda_1$. Theorem 5.4 gives the reverse of Theorem 5.3: any solution on $[\tau, \infty]$ of $p'(t) = \int_{-\omega}^{\omega} p(z+t)\, d\eta(z)$ for which

$p(t)\,e^{-\lambda_1 t}$ is bounded is also a solution of $p'(t) = \int_{-\omega}^{0} p(z+t)\,d\eta_-(z)$. Similarly, the dynamics over the interval $[\tau, \tau+\omega]$ can be written as: $p'(t) = \int_0^\omega p(z+t)\,d\eta_-(z,t) + f(t)$.

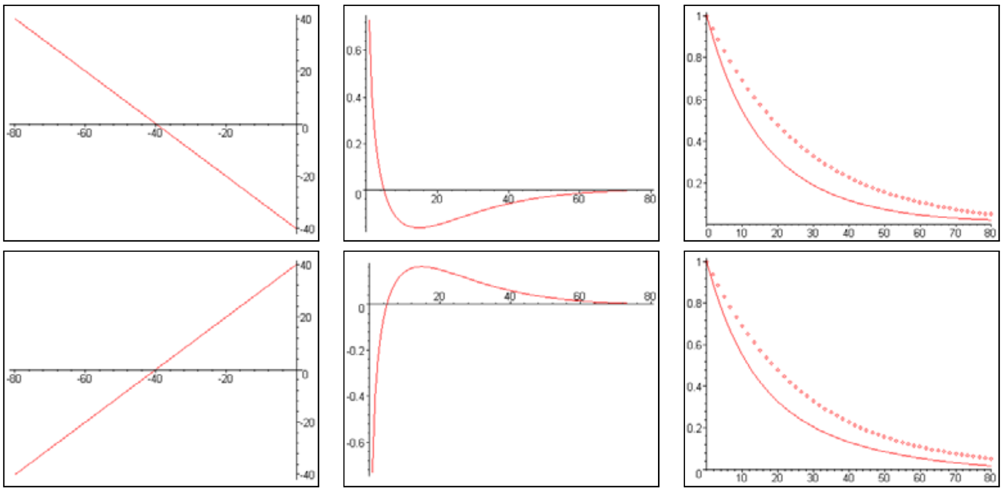
In order to obtain the existence of an equilibrium, we need to find an initial condition for $p'(t) = \int_{-\omega}^{0} p(z+t)\,d\eta_-(z)$, which is a function defined over the interval $[\tau, \tau+\omega]$, and which satisfies both the advance and the delay equations. This initial condition is a solution of the following integral equation:

$$\begin{aligned} p_0(t) &= f(t) + \int_0^t p_0(v)\,\delta_1(t,v)\,dv + \int_\omega^{t+\omega} \varphi(v, p_0(v))\,\delta_3(v-t)\,dv \\ &\quad + \int_t^\omega p_0(v)\,(\delta_2(t,v) + \delta_3(v-t))\,dv, \end{aligned} \tag{69}$$

where $p_0(0)$ and $\varphi(\omega, p_0(\omega)) = p_0(\omega)$ are given. Indeed, let $p_0(t)$ for $t \in [\tau, \tau+\omega]$. Integrating equation $p'(t) = \int_{-\omega}^{0} p(z+t)\,d\eta_-(z)$ over $t \in [\tau+\omega, \tau+2\omega]$ with initial condition $p_0(t)$, we obtain $p_1(t) = \varphi(t, p_0(t))$ for $t \in [\tau+\omega, \tau+2\omega]$. Let the solution of the advanced equation $p'(t) = \int_0^\omega p(z+t)\,d\eta_-(z,t) + f(t)$ with initial condition $p_1(t)$ in the interval $[\tau, \tau+\omega]$ be $p_2(t)$. Thus, we look for $p_0$ such that $p_0(t) = p_2(t)$. $\square$

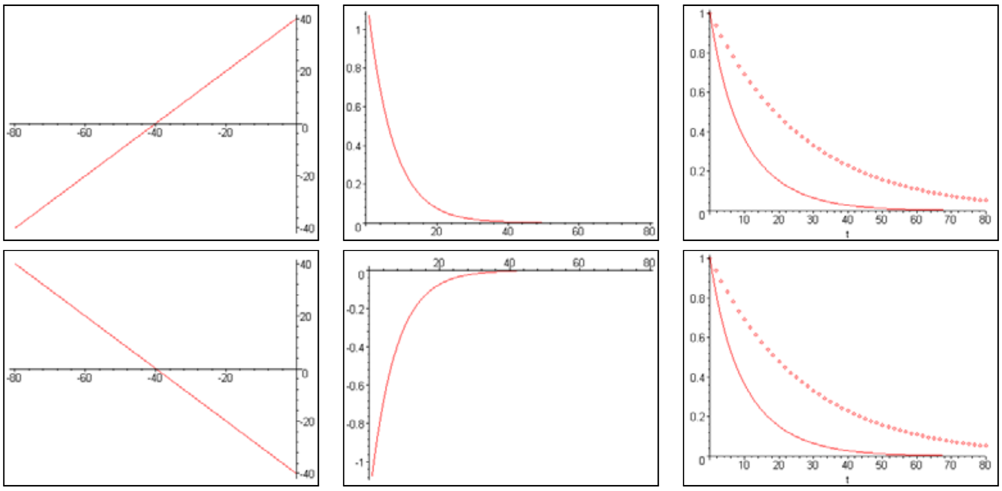
We use two preliminary results to prove the existence of a bounded solution. First, the dynamics are described by functional equations with discrete advances over the interval $[\tau, \tau+\omega]$ and discrete delays over $[\tau+\omega, \infty)$. The delay-differential equation may then be initialized for any set of initial conditions; consequently, there exists a unique solution in $[\tau, \infty)$. This solution is characterized by oscillations whose magnitude decreases with time. However, these oscillations may lead to negative prices for some finite interval of time, even with a positive price for times $t = \tau$ and $t \to \infty$. Such solutions are, of course, not a valid equilibrium.

Let us conclude this section by some numerical computations of the solution. To do this, we use the same parameters as Edmond (2008). The lifetime is set to $\omega = 80$ (years) and the initial date to $\tau = 0$. The aggregate endowment is constant and the distribution of individual endowments is uniform within the population. The steady-state growth rate of prices is thus the opposite of the discount rate, which is set to $\rho = 0.04$. Figures 1 and 2 show the initial asset distribution $a(s, \tau)$, the function $f(t)$ defined in Property 1, and the equilibrium prices $p(t)$ represented by the solid line and compared to the benchmark $e^{-0.05t}$. The initial equilibrium price is not determined by the model and is thus set to 1. The top row of Figure 1 corresponds to the casein which the young have initial liabilities, whereas in the bottom row, it is the old who have initial liabilities. Finally the population growth rate is set to $n = 0.01$ in Figure 1, whereas $n = 0.1$ in Figure 2.



Asset distribution Function $f(t)$ Prices $p(t)$

Figure 1

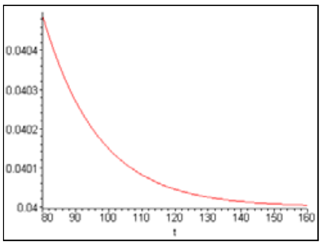


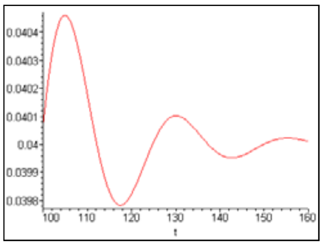
Asset distribution Function $f(t)$ Prices $p(t)$

Figure 2

At first sight, these figures are similar to those obtained by Edmond (2008). The distribution At first sight, these figures are similar to those obtained by Edmond (2008). The distribution of initial assets barely affects the intertemporal prices profile. It strongly modifies the function $f(t)$, but not the prices, which are much more sensitive to the population growth rate.

The higher the value of $n$, the wider the gap between pure exponential discounting and equilibrium prices. We also note that prices do not display large fluctuations. This is explained by the fact that the complex roots of the solution have real parts that are smaller than $\min\{-\rho, n\}$ (see Lemma 5), and hence the trajectory is dominated by $e^{-\rho t}$. This, however, does not mean that these fluctuations do not exist. Indeed, in Figure 3, the dynamics of the interest rate (i.e. $-p'(t)/p(t)$) are represented for both $n = 0.01$ and $n = 0.1$.




$-p'(t)/p(t)$ when $n = 0.01$ $-p'(t)/p(t)$ when $n = 0.1$

Figure 3

The interest rate converges to its long run value given by $\rho = 0.04$, but the dynamics depend on the sign of the difference between the discount rate $\rho$ and the population growth rate $n$. A direct result from Lemma 5 is that if $\rho > n$, complex roots that enter the solution have real parts that are much smaller than $-\rho$ and consequently, the dynamics are monotonic (picture shown on the left). Conversely, if $\rho < n$, there exist complex roots with real parts that are very close to $-\rho$ and thus, the dynamics are oscillatory.

# 5 Conclusion

In this article, we have presented the latest developments in continuous-time overlapping generations models. The aggregation of the behaviors of individuals allows us to obtain a functional differential equation of mixed type, whose solution represents the equilibrium prices. Over time, prices display fluctuations that may be permanent.

Several issues remain open in the theory of overlapping generations models in continuous-time. One of the most difficult issues is the determinacy of the equilibrium. It is well known that in traditional models, the solution does not determine the level of prices $p(\tau)$. Moreover, Demichelis and Polemarchakis (2007) argue that the degree of indeterminacy could be greater when considering continuous-time frameworks. A promising method of studying this issue could be inspired from the work of Mallet Paret and Verduyn Lunel (2008). The idea is that for a given mixed-type delay-differential equation, we can determine a delay-differential equation whose characteristic roots are those of $\Delta(\lambda) = 0$. It is, however, very difficult to arrange this equation in an explicit form, even in the simplest of cases. It is nevertheless equivalent to finding $\lambda_0$, $\Delta_-(\lambda)$, and $\Delta_+(\lambda)$ that appear in the factorization $\Delta(\lambda)(\lambda - \lambda_0) = \Delta_-(\lambda)\Delta_+(\lambda)$, where $\Delta_-(\lambda)$ has delays only and $\Delta_+(\lambda)$ has advances only. Mallet Paret and Verduyn Lunel (2008) go on to prove, in problem of dimension one, the existence of an invariant $n_\#$ of the factorization that satisfies $n_{\#} \geq 2$. In this case, the indeterminacy is of degree $n_{\#} - 1$.

# 6 Appendix

Proof of Property 1. Replacing (40) and (42) for $s \geq \tau$, in (43) yield:

$$p(t) = \frac{\beta}{\int_0^{\omega} e^{-\rho z} dz} \int_{t-\omega}^{t} e^{-(n+\rho)(t-s)} \int_{s}^{s+\omega} p(z)\, dz ds,$$

for $t > \tau + \omega$ and:

$$\begin{aligned} p(t) &= \int_{t-\omega}^{\tau} \beta e^{-(n+\rho)(t-s)} \frac{\left(a(s,\tau) + \int_{\tau}^{s+\omega} p(\nu)\, dv\right)}{\left(\int_{\tau}^{s+\omega} e^{-\rho(\nu-s)} dv\right)} ds \\ &\quad + \int_{\tau}^{t} \beta e^{-(n+\rho)(t-s)} \frac{\int_{s}^{s+\omega} p(z)\, dz}{\int_{s}^{s+\omega} e^{-\rho(z-s)} dz} ds, \end{aligned}$$

for $\tau < t < \tau + \omega$. Concerning the dynamics for $t > \tau + \omega$, using Fubini's Theorem and $\beta = n/\left(e^{-n\omega} - 1\right)$, we obtain:

$$\begin{aligned} p(t) &= \frac{e^{-n\omega} - 1}{n \int_0^{\omega} e^{-\rho z} dz} \left[ \int_{t-\omega}^{t} p(z) \frac{e^{-(n+\rho)\omega} - e^{-(n+\rho)(t-z)}}{n+\rho} dz \right. \\ &\quad \left. + \int_{t}^{t+\omega} p(z) \frac{e^{-(n+\rho)(t-z+\omega)} - 1}{n+\rho} dz \right] \end{aligned}$$

Concerning the dynamics for $\tau < t < \tau + \omega$, we use the fact that:

$$\begin{aligned} &\int_{t-\omega}^{\tau} \beta e^{-(n+\rho)(t-s)} \frac{\int_{\tau}^{s+\omega} p(\nu)\, dv}{\int_{\tau}^{s+\omega} e^{-\rho(\nu-s)} dv} ds \\ = &\int_{\tau}^{t} \beta p(\nu) \frac{\int_{t-\omega}^{\tau} e^{-(n+\rho)(t-s)} ds}{\int_{\tau}^{s+\omega} e^{-\rho(\nu-s)} dv} dv + \int_{t}^{t+\omega} \beta p(\nu) \frac{\int_{v-\omega}^{\tau} e^{-(n+\rho)(t-s)} ds}{\int_{\tau}^{s+\omega} e^{-\rho(\nu-s)} dv} dv, \end{aligned}$$

and:

$$\int_{\tau}^{t} \beta e^{-(n+\rho)(t-s)} \frac{\int_{s}^{s+\omega} p(z)\, dz}{\int_{s}^{s+\omega} e^{-\rho(z-s)} dz} ds$$
$$= \int_{\tau}^{t} \beta p(z) \frac{\int_{\tau}^{z} e^{-(n+\rho)(t-s)} ds}{\int_{s}^{s+\omega} e^{-\rho(z-s)} dz} dz + \int_{t}^{t+\omega} \beta p(z) \frac{\int_{z-\omega}^{t} e^{-(n+\rho)(t-s)} ds}{\int_{s}^{s+\omega} e^{-\rho(z-s)} dz} dz.$$

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