Prime Forms in Possibilistic Logic

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Abstract

Possibilistic logic is a weighted logic used to represent uncertain and inconsistent knowledge. Its semantics is often defined by a possibility distribution, which is a function from a set of interpretations to a totally ordered scale. In this paper, we consider a new semantic characteristics of knowledge bases in possibilistic logic (or possibilistic knowledge bases) by a generalized notion of propositional prime implicant, which we call prioritized prime implicant. We first consider several desirable properties of a prioritized prime implicant for characterizing possibilistic knowledge bases. Some examples show that existing generalizations of prime implicant in possibilistic logic do not satisfy all of these properties. We then provide a novel definition of prioritized prime implicant, which is a set of weighted literals that may be inconsistent. We show that the prioritized prime implicants satisfy all the desirable properties. Finally, we discuss the problem of computing prioritized prime implicants of a possibilistic knowledge base.

1 Introduction

Possibilistic logic [Dubois *et al.*, 1994] or possibility theory offers a convenient tool for handling uncertain or prioritized formulas and coping with inconsistency. At the syntactic level, it is a weighted logic which attaches to each formula with a weight belonging to a totally ordered scale, such as $[0, 1]$, where the weight is interpreted as the certainty level of the formula. A *possibilistic knowledge base* is a set of weighted formulas. At the semantic level, it is based on the notion of a *possibility distribution*, which is a mapping from the set of interpretations Ω to interval [0,1]. In the last 20 years, possibilistic logic plays an important role in knowledge representation and reasoning. It has been shown in [Dubois and Prade, 1991] that necessity measures defined over formulas in possibilistic logic is actually the numerical counterparts of epistemic entrenchment relations defined in belief revision [Gardenfors, 1988]. Because of this correspondence, many revision operators or merging operators have been proposed in possibilistic logic, such as those in [Dubois and Prade, 1992; Benferhat *et al.*, 2002b; 2002a; Benferhat and Kaci, 2003; Qi *et al.*, 2004; Qi, 2007].

The notion of prime implicants (or its dual notion called prime implicates) has been widely investigated in classical logic. It plays an important role in automated reasoning, knowledge compilation [Cadoli and Donini, 1997] and consequence finding [Marquis, 2000]. Prime implicants have also been to define revision operators or merging operators in propositional logic [Marchi *et al.*, 2010; Marchi and Perrussel, 2011]. In [Qi *et al.*, 2010], the authors generalize the notion of prime implicants to possibilistic logic, called weighted prime implicants, and define measures of conflict and agreement between two possibilistic knowledge bases using weighted prime implicants. They also apply weighted prime implicants to define merging operators in possibilistic logic in [Liu *et al.*, 2006]. However, the definition of weighted prime implicants is problematic. It has been shown in [Qi and Wang, 2012] that weighted prime implicants cannot be used to characterize a possibilistic knowledge base. That is, it is not the case that a possibilistic formula can be inferred from a possibilistic knowledge base if and only if the formula can be inferred from the disjunction of all weighted prime implicants of the knowledge base. Even worse, there may exist a possibilistic knowledge base that does not have any weighted prime implicants. A modified definition of weighted prime implicants is then given in [Qi and Wang, 2012] that satisfies some desirable properties falsified by the previous definition. However, the weighted prime implicants defined in [Qi and Wang, 2012] still cannot be used to characterize a possibilistic knowledge base.

In this paper, we provide a novel definition of prime forms in possibilistic logic, called prioritized prime implicants. Unlike existing definitions of a (weighted) implicant, a prioritized implicant is a set of weighted literals that may be inconsistent. To define a prioritized implicant, a paraconsistent consequence relation for a set of weighted literals is given. A prioritized prime implicant is then defined as a prioritized implicant that is minimal w.r.t. a lexicographic order. We show that the prioritized prime implicants can be used to characterize a possibilistic knowledge base and satisfy some other desirable properties. Finally, we discuss the problem of computing prioritized prime implicants of a possibilistic knowledge base. Our method for computing prioritized prime implicants is a modification of the method for computing weighted prime implicants given in [Qi *et al.*, 2010]. The main difference is that the modified method computes prioritized prime implicants, which may be sets of inconsistent weighted literals but the original method does not generate a set of inconsistent weighted literals. We show that the modified method is syntax-independent and can compute all the prioritized prime implicants of a possibilistic knowledge base.

The rest of the paper is organized as follows. In Section 2, we introduce some basic notions of propositional logic and possibilistic logic. We then discuss the problems of existing definitions of prime forms in possibilistic logic in Section 3. After that, we give a new definition of prime forms in possibilistic logic in Section 4. Finally, we conclude this paper in Section 5.

2 Preliminaries

2.1 Propositional logic

We consider a propositional language $\mathcal{L}_{PS}$ defined from a finite set of propositional variables (also called atoms) PS and the usual connectives. Formulas are denoted by Greek letters ϕ, ψ, ... The classical consequence relation is denoted as $\vdash_{\mathcal{L}}$. An interpretation is a total function from PS to $\{true, false\}$. The definition of an interpretation can be extended to formulas in a standard way. An interpretation is a model of a formula if it assigns truth value $true$ to the formula. A *knowledge base K* is a finite set of propositional formulas. An interpretation is a model of a knowledge base if it satisfies all the formulas in it. K is consistent if it has a model. Two knowledge bases K_1 and K_2 are equivalent, denoted $K_1 \equiv K_2$, if they have the same set of models.

A literal is either an atom or the negation of an atom. Let l be a literal, we denote by l^c the complement of l. A *clause* C is a *disjunction* of literals: $C = l_1 \vee ... \vee l_n$ and its dual clause, or *term* D, is a *conjunction* of literals: $D = l_1 \wedge ... \wedge l_n$. A term D is an implicant of formula ϕ iff $D \vdash_{\mathcal{L}} \phi$ and D does not contain two complementary literals. A *prime implicant* of knowledge base K is an implicant D of K such that for every other implicant D' of K, $D \not\vdash_{\mathcal{L}} D'$ [Quine, 1959] (or equivalently $D' \not\subset D$).

2.2 Possibilistic logic

We introduce the syntax of possibilistic logic [Dubois *et al.*, 1994]. A possibilistic formula is a pair $(\phi,\ a)$, where ϕ is a propositional formula and $a \in [0,1]$. A possibilistic literal is a pair (l, a), where l is a literal and $a \in [0,1]$; a possibilistic term is a set of possibilistic literals. In this paper, we assume that there do not exist two pairs (l, a) and (l, b) such that $a \neq b$ in a possibilistic term. The uncertain or prioritized pieces of information can be represented by a *possibilistic knowledge base* which is a finite set of possibilistic formulas of the form $B = \{(\phi_i,\ a_i) : i = 1, ..., n\}$. The classical base associated with B is $B^* = \{\phi_i | (\phi_i,\ a_i) \in B\}$. A possibilistic knowledge base B is consistent iff its classical base B^* is consistent.

The semantics of possibilistic logic is based on the notion of a *possibility distribution* π which is a mapping from the set of interpretations to interval [0,1]. The possibility degree $\pi(\omega)$ represents the degree of compatibility (resp. satisfaction) of ω with the available beliefs about the real world. From a *possibility distribution* π, the necessity degree of formula ϕ is defined as $N_\pi(\phi) = 1 - \Pi_\pi(\neg\phi)$, where $\Pi_\pi(\phi) = max\{\pi(\omega) : \omega \in \Omega, \omega \models \phi\}$. The interpretation of a possibilistic formula (ϕ, a) is that the necessity degree of ϕ is at least equal to a, i.e. $N(\phi) \geq a$.

Definition 1. *Let B be a possibilistic knowledge base, and $a \in [0,1]$. The a-cut (resp. strict a-cut) of B is $B_{\geq a} = \{\phi_i \in B^* | (\phi_i, b_i) \in B$ and $b_i \geq a\}$ (resp. $B_{>a} = \{\phi_i \in B^* | (\phi_i, b_i) \in B$ and $b_i > a\}$).*

There are two entailment relations in possibilistic logic.

Definition 2. *Let B be a possibilistic knowledge base. A possibilistic formula (ϕ, a) is a weak possibilistic consequence of B, denoted by $B \vdash (\phi, a)$, if $a > Inc(B)$, where $Inc(B) = max\{a_i : B_{\geq a_i}$ is inconsistent$\}$ and $B_{\geq a} \vdash_{\mathcal{L}} \phi$. A possibilistic formula (ϕ, a) is a possibilistic consequence of B, denoted $B \vdash_\pi (\phi,\ a)$, if (i) $B_{\geq a}$ is consistent; (ii) $B_{\geq a} \vdash_{\mathcal{L}} \phi$; (iii) $\forall b > a$, $B_{\geq b} \not\vdash_{\mathcal{L}} \phi$.*

Two possibilistic knowledge bases B and B' are said to be equivalent, denoted by $B \equiv_s B'$, iff $\forall\, a \in [0,1]$, $B_{\geq a} \equiv B'_{\geq a}$.

The disjunction of two possibilistic terms is defined in [Qi and Wang, 2012] as follows: $D_1 \vee D_2 = \{(l_i \vee l'_j, min(a_i, b_j)) | (l_i, a_i) \in D_1, (l'_j, b_j) \in D_2\}$. Since $\vee$ is associative and commutative, the disjunction of more than two possibilistic terms can be easily defined.

3 Problems with Existing Definitions

In this section, we introduce the definitions of weighted prime implicants of a possibilistic knowledge base defined in [Qi *et al.*, 2010] and [Qi and Wang, 2012] and discuss their problems.

Let $B = \{(\phi_1, a_1), ..., (\phi_n, a_n)\}$ be a possibilistic knowledge base where each ϕ_i is a clause[1]. In [Qi *et al.*, 2010], a weighted implicant of B is a possibilistic term $D = \{(\psi_1, b_1), ..., (\psi_k, b_k)\}$, such that $D \vdash_\pi B$, where ψ_i are literals such that no two complementary literals exist. Let D and D' be two weighted implicants of B, D is said to be *subsumed* by D', denoted as $D \prec_s D'$, iff $D \neq D'$, $D'^* \subseteq D^*$ and for all $(\psi_i, a_i) \in D$, there exists $(\psi_i, b_i) \in D'$ with $b_i \leq a_i$ (b_i is 0 if $\psi_i \in D^*$ but $\psi_i \notin D'^*$). The relation $\prec_s$ is used to compare two weighted implicants. A weighted prime implicant of B is a weighted implicant B that is not subsumed by any other weighted implicant of B. Formally, we have the following definition.

Definition 3. *A weighted prime implicant (WPI) of B is a weighted implicant of B such that there does exist another weighted implicant D' of B such that D is subsumed by D'.*

It has been shown in [Qi and Wang, 2012] that WPIs do not satisfy the following two desirable properties:

Property 1: for any consistent possibilistic knowledge base, it has at least one consistent WPI.

Property 2: for any consistent possibilistic knowledge base B and any formula ϕ, $B^* \vdash \phi$ iff $\bigvee_{D_i \in \mathsf{WPI}(B)} D_i^* \vdash \phi$.

To see why Property 1 is violated, let us consider $B = \{(p \vee q, 0.8), (p, 0.7), (q, 0.7)\}$. To infer $(p \vee q, 0.8)$, any WPI D under previous definition should contain either $(p, 0.8)$ or $(q, 0.8)$. If it contains $(p, 0.8)$, then we do not have $D \vdash_\pi$

[1] A possibilistic formula of the form $(\phi_1 \wedge ... \wedge \phi_n, a)$ can be equivalently decomposed into a set of formulas (ϕ_1, a),...,(ϕ_n, a) due to the min-decomposability of necessity measures.

$(p, 0.7)$. If it contains $(q, 0.8)$, we do not have $D \vdash_\pi (q, 0.7)$. Thus, there is no WPI for B. This example also shows that Property 2 is violated.

In [Qi and Wang, 2012], the authors redefine the notion of weighted implicant based on the weak possibilistic inference. A weak weighted implicant of a possibilistic knowledge base B is a possibilistic term $D = \{(l_1, b_1), ..., (l_k, b_k)\}$, such that $D \vdash B$ such that no two complementary literals exist. To define their notion of WPIs, they simply replace weighted implicants in the definition of WPI given in [Qi *et al.*, 2010] by weak weighted implicant. The following proposition from [Qi *et al.*, 2010] shows that Property 2 holds for the new definition.

Proposition 1. $B^* \vdash \phi$ *iff* $\bigvee_{D_i \in \mathsf{WPI}(B)} D_i^* \vdash \phi$.

Unfortunately, this definition does not satisfy the following property, which states that WPIs should be used for compiling a possibilistic knowledge base [Qi and Wang, 2012].

Property 3: for any consistent possibilistic knowledge base B, $B \equiv_s \bigvee_{D_i \in \mathsf{WPI}(B)} D_i$.

Example 1. *[Qi and Wang, 2012] Consider* $B = \{(q \vee r, 0.9), (\neg r, 0.8)\}$. *Any WPI* D *of* B *must include* $(\neg r, 0.8)$, *thus* q *must appear in it. Since* D *can infer* $(q \vee r, 0.9)$, *it must contain* $(q, 0.9)$. *Thus* $D \vdash_\pi (q, 0.9)$ *and* D *is the only WPI of* B. *However, we can check that* $B \vdash_\pi (q, 0.8)$.

4 Prime Forms in Possibilistic Logic

In this section, we define our notion of prime implicants in possibilistic logic and discuss its properties.

4.1 Prioritized prime implicant

As previously shown, none of the existing definitions of weighted prime implicants can be used to compile a possibilistic knowledge base. Consider again Example 1: r is not allowed to be in any WPI of B because $\neg r$ must belong to all "reasonable" WPIs of B. This enforces that $(q, 0.9)$ is inferred. Complying to Property 3 entails that there must exist a WPI of B which contains $(r, 0.9)$. This will then force us to consider $D = \{(r, 0.9), (\neg r, 0.8)\}$ as a WPI of B. Now, the question is, since $\{(r, 0.9), (\neg r, 0.8)\}$ is inconsistent, what will be its logical consequences? It is clear that we cannot use possibilistic inference as $(\neg r, 0.8)$ will be blocked. Based on our previous discussion, we know that both $(r, 0.9)$ and $(\neg r, 0.8)$ should be inferred from D. Thus, a paraconsistent semantics should be considered here. This semantics should lead to infer $(q \vee r, 0.9)$ and also $(\neg r, 0.8)$ from term $D = \{(r, 0.9), (\neg r, 0.8)\}$. Clearly, it means that as we use pair $(\neg r, 0.8) \in D$ to infer $\neg r$ with value 0.8, we should not consider $(r, 0.9)$. In more formal terms, the paraconsistent semantics is defined by a *paraconsistent a-cut* as follows.

Definition 4. *Let* D *be a possibilistic term. The paraconsistent a-cut of* D *is*

$$D_{\succeq a} = \{l | (l, b) \in D, b \geq a, \nexists\, (l^c, b') \in D \text{ s.t. } a \leq b' < b\}$$

A possibilistic formula (ϕ, a) *is a consequence of* D, *denoted as* $D \vdash_P (\phi, a)$ *if* $D_{\succeq a} \vdash (\phi, a)$.

Our definition of a paraconsistent a-cut modifies the definition of a a-cut by considering inconsistent terms. The idea is that, suppose l and l^c both appear in $D_{\geq a}$ with $(l, b) \in D$ and $(l^c, b') \in D$, where $b' > b$, then l^c should not be included in the paraconsistent a-cut $D_{\succeq b}$. Note that l^c is included $D_{\succeq b'}$, so it is used to infer formulas whose weights are equal to b'. Consider again Example 1 and term $D = \{(r, 0.9), (\neg r, 0.8)\}$. It is the case that $D \vdash_P (q \vee r, 0.9)$ since $D_{\succeq 0.9} = \{(r, 0.9)\}$ and $D \vdash_P (\neg r, 0.8)$ since $D_{\succeq 0.8} = \{(\neg r, 0.8)\}$.

We then get a new definition of implicant form of a possibilistic knowledge base by generalizing the definition of implicant in propositional logic. To avoid confusion of the notations, we call it a *prioritized implicant*.

Definition 5. *A* prioritized implicant *of a possibilistic knowledge base* B *is a possibilistic term* $D = \{(l_1, b_1), ..., (l_k, b_k)\}$, *such that* $D \vdash_P (\phi, a)$ *for all* $(\phi, a) \in B$, *such that there does not exist two complementary literals with the same weight.*

According to the definition, a prioritized implicant can be an inconsistent set of possibilistic literals. However, there does not exist a pair of conflicting literals in the form of l and l^c with the same weight. This requirement is important for two reasons. First, if there are two conflicting literals with the same weight a, then they both will be included in the paraconsistent a-cut set. Thus, the inference of the a-cut is trivialized. Second, if this requirement is violated, then the notion of prioritized implicant is not reduced to the notion of implicant.

Example 2. *(originally from [Qi and Wang, 2012]) Suppose there are four atoms* p, q, r *and* s, *where*

- p *represents "red light is on"*
- q *represents "green light is off"*
- r *represents "press the button"*
- s *represents "yellow light is on"*

Suppose we have a possibilistic knowledge base $B = \{(\neg q \rightarrow r, 0.8), (p \rightarrow \neg r, 0.7), (q, 0.7), (\neg s \rightarrow \neg r, 0.6)\}$ *that consists of three uncertain rules. Then* $D = \{(q, 0.8), (\neg p, 0.7), (s, 0.6)\}$ *and* $D' = \{(q, 0.8), (\neg p, 0.7), (\neg r, 0.7)\}$ *are two prioritized implicants of* B.

We now generalize the notion of prime implicant. To define the notion of prioritized prime implicant, we need to take into account of the weights associated with literals. We first stratify the prioritized implicants, and define a lexicographic ordering over prioritized implicants: the intuitive idea is that literals with the highest values should be considered at first. A stratified set A is the union of sets $A_1 \cup .... \cup A_n$ such that any two elements in A_i have the same priority and every element in A_i has higher priority than every element in A_j with $i < j$. For two stratified sets $A = A_1 \cup ... \cup A_n$ and $B = B_1 \cup ... \cup B_n$, lexicographic ordering is defined as:

> $A \prec_{lex} B$ iff (i) there exists i such that $A_i \subset B_i$ and (ii) for all $1 \leq j < i$, $A_j = B_j$. We write $A \preceq_{lex} B$ iff $A \prec_{lex} B$ or $A = B$.

We consider set inclusion instead of cardinality to define the lexicographic ordering. This is because set inclusion is used to compare two implicants in propositional logic.

Let us now stratify prioritized implicants. Suppose b_i, $i = 1, ..., m$ are all the distinct weights appearing in B such that $b_1 > b_2 > ... > b_m$ and D a prioritised implicant of B. Then $D = S_1 \cup S_2 \cup ... \cup S_m$ where $S_i = \{l : (l, b_i) \in D\}$. Based on this stratification step, we now define prioritized prime implicant (PPI).

Definition 6. *A* prioritized prime implicant *D of B is a prioritized implicant of B such that there exists no other prioritized implicant D' of B, $D' \prec_{lex} D$. We denote by $PPI(B)$ the set of all the PPIs of B.*

Example 3. *(Example 2 continued) D is stratified as $A_1 \cup A_2 \cup A_3$, where $A_1 = \{(\neg q, 0.8)\}$ $A_2 = \{(\neg p, 0.7), (q, 0.7)\}$ and $A_3 = \{(s, 0.6)\}$ and D' is stratified as $B_1 \cup B_2 \cup B_3$, where $B_1 = \{(\neg q, 0.8)\}$, $B_2 = \{(\neg p, 0.7), (q, 0.7), (\neg r, 0.7)\}$ and $B_3 = \varnothing$. Since $A_1 = B_1$ and $A_2 \subset B_2$, we have $D \prec_{lex} D'$. In fact, D is a prioritized prime implicant of B.*

4.2 Properties of prioritized prime implicants

Let us characterize the behaviour of prioritized prime implicant. The first proposition shows that PPIs reduced to propositional prime implicants when the knowledge base is flat, i.e., every formula in it has weight 1.

Proposition 2. *Suppose $B = \{(\phi_i, 1) : i = 1, ..., n\}$ then for any $D \in PPI(B)$, D^* is a prime implicant of B^*.*

Proof. Since all formulas have the same weight, each prioritized implicant cannot contain two complementary literals. Thus, the definition of prioritized implicant is reduced to the definition of implicant of a classical knowledge base. Similarly, since all formulas have the same weight, the lexicographic ordering is reduced to the ordering defined by set inclusion. Thus, the definition of prioritized prime implicant is reduced to the definition of prime implicant. □

The next proposition shows that for any consistent possibilistic knowledge base, there exists at least one consistent prioritized prime implicant. It enforces Property 1. This property is a key one since PPI definition tolerates inconsistency.

Proposition 3. *Let B be possibilistic knowledge base. There exists a consistent prioritized prime implicant $A \in PPI(B)$ iff B is consistent.*

Proof. Suppose A is a prime implicant of B^*. Let $B^{=k} = \{(\phi, a) \in B : a = b_k\}$. We first find a minimal subset A_1 of A such that $(B^{=1})^*$ is inferred. We then attach weight b_1 to all literals in A_1. We then find a minimal subset A_2 of $A \setminus A_1$ such that $A_1 \cup A_2$ can entail $(B^{=2})^*$, and we attach weight b_2 to literals in A_2, and so on. It is easy to show that D obtained in this way is a prioritized prime implicant of B.

Conversely, suppose B is consistent. By Proposition 2, there is a prime implicant of B^* and from it we can construct a prioritized prime implicant of B which is consistent. □

Finally, we show that Property 2 and Property 3 also hold: PPI can be used to compile a possibilistic knowledge base. We first show that a-cut concept behaves soundly for PPI.

Lemma 1. *For any possibilistic knowledge base B, we have $B_{\geq a} \equiv \vee\{(D_i)_{\geq a} | D_i \in PPI(B)\}$ for any $a \in [0, 1]$.*

Proof. For any $D_i \in PPI(B)$, we show $(D_i)_{\geq a} \vdash (B_{\geq a})^*$. Suppose $(D_i)_{\geq a}$ is inconsistent, then this trivially holds. Otherwise, since $D_j \vdash_P (\phi, b)$ for all $(\phi, b) \in B$, and $(D_i)_{\geq a} = (D_i)_{\prec a}$, we have $(D_i)_{\geq a} \vdash B_{\geq a}$. So $\vee\{(D_i)_{\geq a} | D_i \in PPI(B)\} \vdash B_{\geq a}$.

Conversely, for any prime implicant A of $B_{\geq a}$, by Proposition 1 and Proposition 3, we can construct a prioritized prim implicant D of B such that $A = D_{\geq a}$. Thus $A \vdash D_{\geq a}$. It follows that $A \vdash \vee\{(D_i)_{\geq a} | D_i \in PPI(B)\}$. This completes the proof. □

The consequence is that any conclusion from a possibilistic KB B can also be inferred from its prime form (and vice-versa).

Theorem 1. *For any consistent possibilistic knowledge base B, $B \equiv_s \bigvee_{D_i \in PPI(B)} D_i$.*

Proof. We need to show that $B_{\geq a} \equiv (\bigvee_{D_i \in PPI(B)} D_i)_{\geq a}$ for any $a \in [0, 1]$. By Lemma 1, we only need to show that for $(\bigvee_{D_i \in PPI(B)} D_i)_{\geq a} = \vee\{(D_i)_{\geq a} | D_i \in PPI(B)\}$ for any $a \in [0, 1]$. Assume that $PPI(B) = \{D_1, ..., D_n\}$.

Suppose $\phi \in (\bigvee_{D_i \in PPI(B)} D_i)_{\geq a}$, then there exists $(l_i, a_i) \in D_i$ such that $\phi = l_1 \vee ... \vee l_n$ and $min(a_1, ..., a_n) \geq a$. So $a_i \geq a$ for all i. Thus $l_i \in (D_i)_{\geq a}$ and $\phi \in \vee\{(D_i)_{\geq a} | D_i \in PPI(B)\}$.

Conversely, suppose $\phi \in \vee\{(D_i)_{\geq a} | D_i \in PPI(B)\}$, then there exists $l_i in (D_i)_{\geq a}$ such that $\phi = l_1 \vee ... \vee l_n$, $(l_i, a_i) \in D_i$ and $a_i \geq a$. So $min(a_1, ..., a_n) \geq a$. It follows that $\phi \in (\bigvee_{D_i \in PPI(B)} D_i)_{\geq a}$. □

4.3 Computing prioritized prime implicants

Let us now detail how we can transform a possibilistic KB in a set of PPIs. The key idea is to proceed in an incremental way as in [Qi *et al.*, 2010]. Given a possibilistic KB $B = \{(\phi_i, a) : i = 1, ..., n\}$ and a possibilistic term D such that the weight of any literal in D is greater than a, a possibilistic term $D' = \{(l_i, a) : i = 1, n\}$ is said to be a D-extended prioritized implicant of B if $D \cup D' \vdash_P (\phi_i, a)$, for any $(\phi_i, a) \in B$. We further say that D' is a D-extended PPI of B if D' is a D-extended implicant of B, and there does not exist another D-extended implicant of B such that $D'' \subset D$.

Let $B^{\preceq k} = \{(\phi, b_k) \in B : a \geq b_k\}$ and $B^{=k} = \{(\phi, a) \in B : a = b_k\}$. We give a method for computing all the PPIs of B. The procedure works as follows. We first compute all the $\varnothing$-extended PPIs of prime implicants of $B^{=1}$. This is achieved by computing all the prime implicants of $(B^{=1})^*$ and attach weight b_1 to every literal in any of such prime implicant. Then, for each obtained $\varnothing$-extended PPI D of $B^{=1}$, we compute all the D-extended PPIs D' of $B^{\preceq 2}$, and so on. Let $B^{\leq k} = \{(\phi, a) \in B : a \geq b_k\}$. We use $PI\text{-}Ext(B^{\leq k})$ to denote the set of all potential prioritized prime implicants of $B^{\leq k}$ for $k \leq m$ obtained by the k step of the above method. Formally, we define $PI\text{-}Ext(B^{\leq k})$ as follows.

Definition 7. *$PI\text{-}Ext(B^{\leq k})$ is defined by induction as:*

1. *$PI\text{-}Ext(B^{=1}) = \{D | D^*$ is a prime implicant of $B^{=1}$ and every literal in D is attached with weight $b_1\}$;*
2. *$PI\text{-}Ext(B^{\preceq k}) = \{D_1 \cup D_2 | D_1 \in PI\text{-}Ext(B^{\preceq k-1})$ and D_2 is a D_1-extended prime implicant of $B^{\preceq k}\}$ for $k > 1$.*

Note that it is necessary to consider $B^{\preceq k-1}$ instead of $B^{=k-1}$ when we define $PI\text{-}Ext(B^{\preceq k})$. Otherwise the method is not syntax-independent. Consider $B = \{(q \vee r, 0.9), (\neg r, 0.8)\}$ and $B' = \{(q \vee r, 0.9), (\neg r, 0.8), (q, 0.8)\}$. It is easy to check that B and B' are equivalent according to the possibilistic inference. There are two $\varnothing$-extended PPIs of $B^{=1}$, i.e., $D_1 = \{(q, 0.9)\}$ and $D_2 = \{(r, 0.9)\}$. So $PI\text{-}Ext(B^{=1}) = \{D_1, D_2\}$. Clearly, $PI\text{-}Ext(B'^{=1}) = \{D_1, D_2\}$. The D_1-extended PPI of $B^{=2}$ (resp. $B'^{=2}$) is $\{(\neg r, 0.8)\}$ (resp. $\{(\neg r, 0.8)\}$) and the D_2-extended PPI of $B^{=2}$ (resp. $B'^{=2}$) is $\{(\neg r, 0.8)\}$ (resp. $\{(\neg r, 0.8), (q, 0.8)\}$). That is, the D_2-extended PPI of $B^{=2}$ and the D_2-extended PPI of $B'^{=2}$ are different. Instead, the D_2-extended PPI of $B^{\preceq 2}$ and the D_2-extended PPI of $B^{\preceq 2}$ are both $\{(\neg r, 0.8), (q, 0.8)\}$. Formally, we have the following result showing that $PI\text{-}Ext(B^{\preceq k})$ is syntax-independent.

Proposition 4. *Suppose $B \equiv_s B'$ and there are n distinct weights appearing in B. For each $1 \leq k \leq n$, we have $PI\text{-}Ext(B^{\preceq k}) = PI\text{-}Ext(B'^{\preceq k})$.*

The proof of Proposition 4 is easy to see because $B \equiv_s B'$ infers that $B^{\preceq k} \equiv_s B'^{\preceq k}$.

The following proposition shows that the above method actually computes all the prioritized implicants of B. We denote by $PPI(B^{\preceq k})$ the set of all the prioritized prime implicants of $B^{\preceq k}$.

Proposition 5. *For any possibilistic term D, $D \in PI\text{-}Ext(B^{\preceq k})$ iff $D \in PPI(B^{\preceq k})$.*

Proof. For the "Only if" direction. We show it by induction over k.

When $k = 1$. Suppose $D \in PI\text{-}Ext(B^{=1})$. Then D^* is a prime implicant of $(B^{=1})^*$. Thus, D is a prioritized prime implicant of $B^{=1}$.

Assume that the proposition holds for $k - 1$. Suppose $D \in PI\text{-}Ext(B^{\preceq k})$. Then $D = D_1 \cup D_2$, where $D_1 \in PI\text{-}Ext(B^{\preceq k-1})$ and D_2 is a D_1-extended prioritized prime implicant of $B^{\preceq k}$. Since $D_1 \in PI\text{-}Ext(B^{\preceq k-1})$, by assumption, $D_1 \in PPI(B^{\preceq k-1})$. Thus, $D_1 \vdash_P (\phi, a)$ for all $(\phi, a) \in B^{\preceq k-1}$. Since $D \in PI\text{-}Ext(B^{\preceq k})$, $D \vdash_P (\phi, b_k)$ for all $(\phi, b_k) \in B^{=k}$. So $D \vdash_P (\phi, a)$ for all $(\phi, a) \in B^{\preceq k}$. That is, D is a prioritized implicant of $B^{\preceq k}$. Assume $D \notin PPI(B^{\preceq k})$. Then there exists $D' \in PPI(B^{\preceq k})$, such that $D' \prec_{lex} D$. Suppose D (resp. D') is stratified as $S_1 \cup \ldots \cup S_n$ ($S'_1 \cup \ldots \cup S'_n$) as in Definition 5. Then there exists i such that $S'_i \subset S_i$ and $S'_j = S_j$ for all $j < i$. i can only be k as $D_1 \in PPI(B^{\preceq k-1})$. Thus $S'_k \subset S_k$ and $S_j = S'_j$ for all $j < k$. This means $D' = D_1 \cup D'_2$, where $(D'_2)^* = S'_k$ and the weight of literals in D'_2 is equal to the weight of literals in D_2. We have $D'_2 \subset D_2$. However, $D' \in PPI(B^{\preceq k})$ implies that $D' \vdash_P (\phi, b_k)$ for all $(\phi, b_k) \in B^{\preceq k}$. This contradicts the fact that $D \in PI\text{-}Ext(B^{\preceq k})$ as $D'_2 \subset D_2$.

For the "If direction". We also show it by induction over k.

For $k = 1$. Suppose $D \in PPI(B^{=1})$. Then D^* is a prime implicant of $(B^{=1})^*$. Thus, D is a $\varnothing$-extended prioritized prime implicant of $B^{=1}$.

Assume that the proposition holds for $k - 1$. Suppose $D \in PPI(B^{\preceq k})$. Let $D = D_1 \cup D_2$, where $D_1 = \{(l, a) \in D | a \geq b_{k-1}\}$ and $D_2 = \{(l, a) \in D | a = b_k\}$. We can show that D_1 is a prioritized prime implicant of $B^{\preceq k}$ by induction over k. Since $D \in PPI(B^{\preceq k})$, $D \vdash_P (\phi, a)$ for all $(\phi, a) \in B^{\preceq k}$. So D_2 is a D_1-extended implicant of $B^{=k}$. Suppose on the contrary that $D \notin PI\text{-}Ext(B^{\preceq k})$. Then we can find a D_1-extended prime implicant D'_2 of $B^{\preceq k}$ such that $D'_2 \subset D_2$. Then $D_1 \cup D'_2 \vdash_P (\phi, a)$ for all $(\phi, a) \in B^{\preceq k}$. This contradicts with the fact that $D \in PPI(B^{\preceq k})$. as $D \prec_{lex} D'$. □

This proposition also stresses up that the set of prioritized prime implicants is unique.

Example 4. *(Example 2 continued) There are three distinct weights in B. We have $B^{=1} = \{(q \vee r, 0.8)\}$, $B^{\preceq 2} = \{(q \vee r, 0.7), (\neg p \vee \neg r, 0.7), (q, 0.7)\}$ and $B^{\preceq 3} = \{(q \vee r, 0.6), (\neg p \vee \neg r, 0.6), (q, 0.6), (s \vee \neg r, 0.6)\}$. It is easy to see that $PI\text{-}Ext(B^{=1}) = \{D_1, D_2\}$, where $D_1 = \{(q, 0.8)\}$ and $D_2 = \{(r, 0.8)\}$. D_1-extended prime implicants of $B^{\preceq 2}$ are $\{(\neg p, 0.7)\}$ and $\{(\neg r, 0.7)\}$ and D_2-extended prime implicants of $B^{\preceq 2}$ are $\{(\neg p, 0.7), (q, 0.7)\}$ and $\{(\neg r, 0.7), (q, 0.7)\}$. So $PI\text{-}Ext(B^{\preceq 2}) = \{D_3, D_4, D_5, D_6\}$, where*

$D_3 = \{(q, 0.8), (\neg p, 0.7)\}$,
$D_4 = \{(q, 0.8), (\neg r, 0.7)\}$,
$D_5 = \{(r, 0.8), (\neg p, 0.7), (q, 0.7)\}$ *and*
$D_6 = \{(r, 0.8), (\neg r, 0.7), (q, 0.7)\}$.

Finally, D_3-extended prime implicants of $B^{\preceq 3}$ are $\{(s, 0.6)\}$ and $\{(\neg r, 0.6)\}$, D_5-extended prime implicants of $B^{\preceq 3}$ are $\{(s, 0.6)\}$ and $\{(\neg r, 0.6)\}$, there is no D_4-extended prime implicant of $B^{\preceq 3}$ and D_6-extended prime implicant of $B^{\preceq 3}$. Thus $PPI(B)$=$PI\text{-}Ext(B^{\preceq 3}) = \{D'_3, D''_3, D_4, D'_5, D''_5, D_6\}$, where

$D'_3 = \{(q, 0.8), (\neg p, 0.7), (s, 0.6)\}$,
$D''_3 = \{(q, 0.8), (\neg p, 0.7), (\neg r, 0.6)\}$,
$D'_5 = \{(r, 0.8), (\neg p, 0.7), (q, 0.7), (s, 0.6)\}$ *and*
$D''_5 = \{(r, 0.8), (\neg p, 0.7), (q, 0.7), (\neg r, 0.6)\}$.

5 Conclusion and Future Work

In this paper, we considered the problem of defining prime forms of a possibilistic knowledge base. Existing definitions of prime forms of a possibilistic knowledge base are not desirable because they cannot be used to recover the possibilistic knowledge base. Our study shows that we have to drop the common assumption of a prime form of a formula or a knowledge base, i.e., a prime implicant or its generalization is a consistent formula. We defined a prioritized implicant of a possibilistic knowledge base as a set of weighted literals that may be inconsistent and provided a paraconsistent semantics for it. The notion of a prioritized prime implicant is then defined by considering a lexicographic ordering. We show that our new definition is desirable by showing that prioritized prime implicants show some desirable properties. Finally, we presented a method for computing prioritized prime implicants of a possibilistic knowledge base.

Prime forms of a possibilistic knowledge base play an important role in knowledge management in possibilistic logic. The have been used to define revision operators and merging operators in possibilistic logic (see [Qi and Wang, 2012] and [Liu *et al.*, 2006]). However, it is nontrivial to apply our new definition of prime forms of a possibilistic knowledge base to define a revision operator a merging operator in possibilistic logic. The revision operators defined in [Qi and Wang, 2012] is based on a distance function between two weighted prime implicants. However, a prioritized prime implicant can be an inconsistent set of possibilistic literals, and it is not clear how to define a distance function between two inconsistent set of possibilistic literals. We will leave this problem as future work.

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References

[Benferhat and Kaci, 2003] Salem Benferhat and Souhila Kaci. Fusion of possibilistic knowledge bases from a postulate point of view. *Int. J. Approx. Reasoning*, 33(3):255–285, 2003.

[Benferhat *et al.*, 2002a] Salem Benferhat, Didier Dubois, Souhila Kaci, and Henri Prade. Possibilistic merging and distance-based fusion of propositional information. *Ann. Math. Artif. Intell.*, 34(1-3):217–252, 2002.

[Benferhat *et al.*, 2002b] Salem Benferhat, Didier Dubois, Henri Prade, and Mary-Anne Williams. A practical approach to revising prioritized knowledge bases. *Studia Logica*, 70(1):105–130, 2002.

[Cadoli and Donini, 1997] Marco Cadoli and Francesco M. Donini. A survey on knowledge compilation. *AI Communication*, 10(3-4):137–150, 1997.

[Dubois and Prade, 1991] Didier Dubois and Henri Prade. Epistemic entrenchment and possibilistic logic. *Artif. Intell.*, 50(2):223–239, 1991.

[Dubois and Prade, 1992] Didier Dubois and Henri Prade. Belief change and possibility theory. In Peter Gärdenfors, editor, *Belief revision*, pages 142–182. 1992.

[Dubois *et al.*, 1994] Didier Dubois, Jérôme Lang, and Henri Prade. Possibilistic logic. In Handbook of Logic in Aritificial Intelligence and Logic Programming, *Volume 3*, pages 439–513. Oxford University Press, 1994.

[Gardenfors, 1988] Peter Gardenfors. *Knowledge in Flux-Modeling the Dynamic of Epistemic States*. The MIT Press, Cambridge, 1988.

[Liu *et al.*, 2006] Weiru Liu, Guilin Qi, and David A. Bell. Adaptive merging of prioritized knowledge bases. *Fundam. Inform.*, 73(3):389–407, 2006.

[Marchi and Perrussel, 2011] Jerusa Marchi and Laurent Perrussel. Prime normal forms in belief merging. In *Proc. of FLAIRS'11*, 2011.

[Marchi *et al.*, 2010] Jerusa Marchi, Guilherme Bittencourt, and Laurent Perrussel. Prime forms and minimal change in propositional belief bases. *Ann. Math. Artif. Intell.*, 59(1):1–45, 2010.

[Marquis, 2000] Pierre Marquis. Consequence finding algorithms. In Handbook on Defeasible Reasoning and Uncertainty Management Systems, *Volume 5*, pages 41–145. Kluwer, 2000.

[Qi and Wang, 2012] Guilin Qi and Kewen Wang. Conflict-based belief revision operators in possibilistic logic. In *Proc. of AAAI'12*, pages 800–806, 2012.

[Qi *et al.*, 2004] Guilin Qi, Weiru Liu, and David H. Glass. A split-combination method for merging inconsistent possibilistic knowledge bases. In *KR'04*, pages 348–356. AAAI Press, 2004.

[Qi *et al.*, 2010] Guilin Qi, Weiru Liu, and David A. Bell. Measuring conflict and agreement between two prioritized knowledge bases in possibilistic logic. *Fuzzy Sets and Systems*, 161(14):1906–1925, 2010.

[Qi, 2007] Guilin Qi. A model-based approach for merging prioritized knowledge bases in possibilistic logic. In *Proc. of AAAI'07*, pages 471–476, 2007.

[Quine, 1959] W. V. Quine. On cores and prime implicants of truth functions. *American Mathematics Monthly*, 66:755–760, 1959.